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Passive Synthesis of Immittance for Fractional-Order Three-Element-Kind Circuit

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ABSTRACT As an interdisciplinary research area, fractional circuits and systems have attracted extensive attention of scholars and researchers for their superior performance and potential applications. The passive realization of the fractional-order (FO) immittance function plays an important role in fractional circuits' theory, which is useful in fractional circuit design and modeling. This paper deals with the passive synthesis of FO three-element-kind circuits. First, the method is given for judging the immittance functions of FO three-element-kind circuits. Then, by making use of impedance scaling and variable substitution, the synthesis method of such an FO immittance function is proposed, which is based on the bivariate reactance synthesis method. Finally, a procedure is proposed to realize such immittance functions using the three-element-kind circuit. And several examples are given to illustrate the proposed method.

INDEX TERMS Fractional circuits, passive synthesis, fractional-order impedance, impedance scaling, multivariable synthesis.

I. INTRODUCTION

In recent years, Fractional Calculus has been widely used in modeling the dynamics of many natural phenomena, which is because of its higher capability of providing accurate description than integer dynamical systems. Applications of Fractional Calculus have been reported in many areas [1]–[7], such as Physics, Biology, Biomedical Engineering, Financial Market, Signal Processing and so on. And in electrical engineering, the application of Fractional Calculus has also growing, such as modeling of electrical equipment [8]–[12] and wireless power transmission system design [13]. Furthermore, the fractional electrical circuits have been studied from diverse aspects, for instance: synthesis of filters [14]–[16], realization of oscillators [17], stability analysis [18]–[20], frequency domain analysis [21]–[27], time-frequency domain analysis [28]–[32], energy efficiency [33], sensitivity analysis [34], fractional-order (FO) reconfigurable filters [35], FO resonator [36], [37], FO general impedance converter (GIC) [38].

With the development of numerical calculations about fractional calculus equations [39], [40], it is feasible to simulate and analyze the FO systems using fractional models. Meanwhile, due to the progresses in fabrication of fractional capacitors and inductors [41]–[43], the FO systems can also be realized directly with real fractional elements [44]. At present, most of the developed FO impedances are fractional capacitors [41], while the floating or grounded fractional inductors can be realized by active devices like GIC [38], operational transconductance amplifier [42], and current carrying conveyor [43]. Obviously, both the fractional circuits modeling and design motivate the need to study the synthesis methods for fractional circuits. At present, several attempts into this goal have been done. In [45], [46], the necessary and sufficient conditions for the realization of FO impedance function with a passive RLC biport terminated one fractional element is found. By transforming the immittance function to multivariable positive-real function, a synthesis method for a class of FO immittance function is proposed in [47], where the n-variable positive-real function $Z(p_1, p_2, \cdots, p_n)$ contains a high-order variable and n-1 one-order variables $p_2, p_3, \cdots, p_n$, and for $i = 2, 3, \cdots, n$, $\partial Z(p_1, p_2, \cdots, p_n)/\partial p_i = \pm$ (a perfect square). The positive-real property and passivity of the FO circuits were discussed in [48]–[50].

In multivariable synthesis theory, Ozaki and Kasami [51] firstly introduced the concept of positive real functions of several variables in 1960, and then multivariable synthesis

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theory has made a great progress [52]–[54]. In the case of bivariate, [55] can realize lossless reciprocal circuit, and arbitrarily reactance matrix is realizable as shown in [56]. Furthermore, there are some k-variable theoretically reactance matrix synthesis methods [57], [58].

Due to the fact that the synthesis for fractional circuits is much more complex than that for classical circuits, the proposed synthesis methods for fractional circuits are only applicable to several kinds of specific immittance function (matrix) forms. [58] shows the synthesis of two-element-kind network, and this paper is a continue work of [58]. This paper mainly deals with the passive synthesis of FO immittance using three kinds of elements, where the order of fractional elements included can be any real number between 0 and 1, and the number of fractional elements is arbitrary.

The rest of the paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, the method for judging the immittance functions of FO three-element-kind circuits is obtained. And Section 4 elaborates the synthesis method and specific synthesis procedure for the fractional circuits composed of three kinds of element, and several examples are presented. The conclusions are drawn in Section 5.

II. PRELIMINARIES

This section is devoted to presenting some preliminaries.

Fractional Calculus is an extension of integral order calculus. At present, there are three widely used definitions of fractional differential [3], namely Riemann-Liouville (RL), Grünwald-Letnikov (GL), and Caputo definitions. More details can be found in [3].

The Laplace transform of fractional differential [3] at zero initial condition is given by

$$L\left\{{}_0D_t^{\alpha} f(t)\right\} = s^{\alpha} F(s) \tag{1}$$

where ${}_0D_t^{\alpha} = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}$ denotes the fractional derivative operator of order α.

In this paper, we assume that $\alpha \in [0, 1]$.

The common fractional elements are fractional capacitor and inductor [2].

The characteristic equation of a fractional capacitor is

$$i_{\alpha}(t) = C_{\alpha} \frac{\mathrm{d}^{\alpha} u_{\alpha}(t)}{\mathrm{d}t^{\alpha}} \tag{2}$$

where $\alpha \in (0, 1]$ is the order of the passive fractional capacitor, C_{α} with unit $\mathrm{Fs}^{\alpha-1}$ denotes the pseudo-capacitance of the fractional capacitor, $u_{\alpha}(t)$ and $i_{\alpha}(t)$ are respectively the voltage and current of the fractional capacitor. The impedance of the fractional capacitor in the Laplace domain equals $Z(s) = 1/C_{\alpha}s^{\alpha}$. The circuit symbol of the fractional capacitor [23] is depicted in Fig. 1(a).

The characteristic equation of a fractional inductor is

$$u_{\beta}(t) = L_{\beta} \frac{\mathrm{d}^{\beta} i_{\beta}(t)}{\mathrm{d}t^{\beta}} \tag{3}$$

where $\beta \in (0, 1]$ is the order of the passive fractional inductor, L_{β} with unit $\mathrm{Hs}^{\beta-1}$ denotes the pseudo-inductance

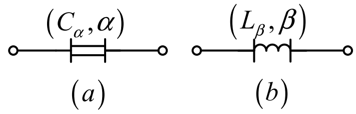

FIGURE 1. Symbols of fractional elements. (a) Fractional capacitor. (b) Fractional inductor.

of the fractional inductor, $u_{\beta}(t)$ and $i_{\beta}(t)$ are respectively the voltage and current of the fractional inductor. The impedance of the fractional inductor in the Laplace domain equals $Z(s) = L_{\beta}s^{\beta}$. The circuit symbol of fractional inductor [23] is depicted in Fig. 1(b).

Fractional capacitors and fractional inductors are collectively referred to as fractional reactance elements (fractances for short). Distinct with classical reactance elements, fractances are lossy when the orders of the fractional elements are in (0, 1).

Then, the definition of multivariable positive real function and multivariable reactance function are described below.

Definition 1 [51]: $Z(p_1, p_2, \cdots, p_n)$ is a n-variable positive real function, if

1) $Z(p_1, p_2, \cdots, p_n)$ is a rational real function of, and
2) $ReZ \geq 0$ in the domain $Rep_i > 0\,(i = 1, 2, \cdots, n)$.

Definition 2 [51]: $Z(p_1, p_2, \cdots, p_n)$ is a n-variable reactance function, if

1) $Z(p_1, p_2, \cdots, p_n)$ a n-variable positive real function, and
2) $Z(p_1, p_2, \cdots, p_n) = -Z(-p_1, -p_2, \cdots, -p_n)$.

In this paper, the order of fractional element referred to as the element-order. The network that the element-orders are all the same is called the fractional commensurate network (commensurate network for short). A network only composed of fractances is called a FO reactance network. Differently with the reactance network, the FO reactance network is lossy when the orders of the fractional elements are in (0, 1).

III. JUDGEMENT METHOD OF IMMITTANCE FOR FO THREE-ELEMENT-KIND CIRCUITS

This section discusses the method for judging the immittance functions of FO three-element-kind circuits. The method is obtained mainly in view of scaling impedance and variable substitution.

Fractional three-element-kind networks have up to three different element-orders, and thus can be classified into three kinds of networks: three element-orders networks, for example $L_{\beta1}L_{\beta2}C_{\alpha}$, $L_{\beta}C_{\alpha1}C_{\alpha2}$, $L_{\beta1}L_{\beta2}L_{\beta3}$, and $C_{\alpha1}C_{\alpha2}C_{\alpha3}$ networks, two element-orders networks such as $RL_{\beta}C_{\alpha}$, $RL_{\beta1}L_{\beta2}$, $RC_{\alpha1}C_{\alpha2}$, $L_{\beta}L_{\alpha}C_{\alpha}$ and $L_{\beta}C_{\beta}C_{\alpha}$ networks, and commensurate networks like $RL_{\beta}C_{\beta}$ networks.

Theorem: By specific impedance scaling and variable substitution, the impedance function of FO n-element-kind circuit $Z(s)$ can be converted into an reactance function of up to (n-1) variables.

Proof: For a passive circuit with n kinds of elements, without loss of generality, we set the elements' value to be 1, then the impedances of each kind of element are $s^{\gamma_1}, s^{\gamma_2}, \cdots, s^{\gamma_n}$ respectively, where $-1 \le \gamma_1 < \cdots < \gamma_i < \cdots < \gamma_j < \cdots < \gamma_n \le 1$.

With the scaling parameter s^{γ}, where

$$\gamma = -\left(\gamma_i + \gamma_j\right)/2, \quad (i, j = 1, 2, \cdots, n; i \neq j) \tag{4}$$

the elements whose impedance are s^{γ_i} and s^{γ_j} can be transformed into elements of order $\left(\gamma_j - \gamma_i\right)/2$, whose impedance are $s^{(\gamma_i - \gamma_j)/2}$ and $s^{(\gamma_j - \gamma_i)/2}$. For these two kinds of elements, we can substitute them with variable $p_i = s^{(\gamma_j - \gamma_i)/2}$. The other kinds of elements can be represented by up to n-2 variables. Thus, after scaling the impedance levels of the elements and variable substitutions, the impedance function of network is transformed into a reactance function up to (n-1) variables.

Remark: The impedance function of FO three-element-kind circuit can be converted into a bivariate reactance function by suitable impedance scaling and variable substitution. And there are three scaling parameters available.

Then we show the judgment method based on the remark.

The method contains the following three steps,

Step 1 Figure out scaling parameter based on the element-orders.

Step 2 Determine if the impedance function can be transformed to a bivariate reactance function.

Step 3 Determine whether it is a fractional three-element-kind network.

For a impedance $Z\left(s^{\tau_1}, s^{\tau_2}, s^{\tau_3}\right)$ of the three element-orders network, set $\gamma_i = \pm\tau_i,\ i = 1, 2, 3$, choosing the scaling parameter s^{γ} as $s^{-(\gamma_1+\gamma_2)/2}$, $s^{-(\gamma_1+\gamma_3)/2}$, and $s^{-(\gamma_2+\gamma_3)/2}$. Then using these scaling parameters and variable substitution, transform $s^{\gamma}Z\left(s^{\tau_1}, s^{\tau_2}, s^{\tau_3}\right)$ to a bivariate reactance function $Z\left(p_1, p_2\right)$. If there are three types of scaling parameter and variable substitution that can lead a bivariate reactance function, it should be a fractional three-element-kind network. Moreover, if the three scaling parameters are $\gamma_{01}, \gamma_{02}, \gamma_{03}$, then $\gamma_1, \gamma_2, \gamma_3$ can be represented as $\gamma_1 = -\gamma_{01} - \gamma_{02} + \gamma_{03}$,

$$\gamma_2 = -\gamma_{01} + \gamma_{02} - \gamma_{03}, \quad \gamma_3 = \gamma_{01} - \gamma_{02} - \gamma_{03}.$$

For two element-orders networks, let $\tau_3 = 0$, above method also works. As for commensurate networks, it can be determined just using variable substitution. After the variable substitution $p = s^{\beta}$, if the impedance of a $RL_{\beta}C_{\beta}$ network is positive real but is not a RL,RC, or LC function in p-plane, the network should be a fractional three-element-kind network.

It is worth mentioning that through the synthesis method in the next section, as long as we get a bivariate reactance function with suitable scaling impedance and variable substitution, the implementation of such function can be completed.

Example 1: Given the impedance function

$$Z\left(s\right) = \frac{s^{\tau_1+2\tau_2+\tau_3} + s^{\tau_2+\tau_3} + 2s^{\tau_1+\tau_2} + 1}{s^{2\tau_2+\tau_3} + 2s^{\tau_2}} \tag{5}$$

It can be observed that there are three types of scaling parameter and variable substitution to transform (5) into a bivariate reactance function if we set $\gamma_1 = \tau_1, \gamma_2 = -\tau_2, \gamma_3 = \tau_3$.

1) According to the Theorem, we scaling the impedance levels of the elements with parameter $s^{\gamma} = s^{-(\gamma_3+\gamma_2)/2}$. Then we have

$$s^{\gamma}Z\left(s\right) = \frac{s^{\tau_1+2\tau_2+\tau_3} + s^{\tau_2+\tau_3} + 2s^{\tau_1+\tau_2} + 1}{s^{\frac{3(\tau_2+\tau_3)}{2}} + 2s^{\frac{\tau_2+\tau_3}{2}}}$$

Let $p_1 = s^{(\gamma_3-\gamma_2)/2}, p_2 = s^{\gamma_1-(\gamma_3+\gamma_2)/2}$, we get a bivariate function from $s^{\gamma}Z\left(s\right)$,

$$Z\left(p_1, p_2\right) = \frac{p_1^3 p_2 + p_1^2 + 2p_1 p_2 + 1}{p_1^3 + 2p_1} \tag{6}$$

Obviously, $Z\left(p_1, p_2\right) = s^{\gamma}Z\left(s\right)$, and it is a bivariate reactance function of p_1 and p_2.

The other two types of scaling parameter and variable substitution are as follows,

2) $s^{\gamma} = s^{-(\gamma_1+\gamma_2)/2}, p_1 = s^{(\gamma_2-\gamma_1)/2}, p_2 = s^{\gamma_3-(\gamma_1+\gamma_2)/2}$
3) $s^{\gamma} = s^{-(\gamma_1+\gamma_3)/2}, p_1 = s^{(\gamma_3-\gamma_1)/2}, p_2 = s^{\gamma_2-(\gamma_1+\gamma_3)/2}$
and hence it is a fractional three-element-kind network.

IV. PASSIVE SYNTHESIS OF FO THREE-ELEMENT-KIND CIRCUITS

In this section, we shall first show the synthesis method for FO three-element-kind circuit. Then, the specific synthesis procedure for the fractional circuits composed of three kinds of elements is given.

A. SYNTHESIS METHOD

After getting the bivariate reactance function with suitable transformation, we synthesis the bivariate reactance function as a bivariate reactance network. Then, by inversely transformation, the passive fractional network is obtained.

The Lemma 1 and Lemma 2 show that a bivariate reactance function $Z\left(p_1, p_2\right)$ can be realized as a bivariate reactance network.

Lemma 1 [55]: A bivariate reactance function $Z\left(p_1, p_2\right)$ can be decomposed as

$$Z\left(p_1, p_2\right) = Z_1\left(p_1\right) + Z_2\left(p_2\right) + Z_0\left(p_1, p_2\right) \tag{7}$$

where Z_1 and Z_2 are reactance functions in p_1 and p_2, respectively, and Z_0 is a bivariate reactance function with no p-independent or s-independent poles.

Lemma 2 [56]: Every bivariate reactance function $Z_0\left(p_1, p_2\right)$ can be realized as the impedance seen at the first one-ports of a lossless (1+k)-port consisting of reactance in the p_1-plane terminated at its last k ports with unit inductors in p_2-plane. Furthermore, such a realization uses the minimum possible number of reactance.

The synthesis procedure of $Z\left(p_1, p_2\right)$ is shown as follows,

1) Decompose $Z\left(p_1, p_2\right)$ as (x). And realize the univariate reactance function Z_1 and Z_2 using classical synthesis method 59].
2) Synthesize $Z_0\left(p_1, p_2\right)$.

Expand $Z_0(p_1,p_2)$ in a Laurent series:

$$Z_0(p_1,p_2)=A_{-1}(p_1)+\sum_{l=0}^{\infty}A_l(p_1)\,p_2^{-(k+1)},$$

Find $g(p_1,p_2)$, the least common denominator of elements in $Z_0(p_1,p_2)$ and express it in the form

$$g(p_1,p_2)=\sum_{k=0}^{r}a_k(p_1)\,p_2^{r-k}.$$

Form the $(r\times r)$ matrix $\boldsymbol{N}_{r-1}(p_1)$, defined by

$$\boldsymbol{N}_{r-1}=\begin{bmatrix} A_0 & \cdots & A_{r-1} \\ -A_1 & \cdots & -A_r \\ \vdots & \ddots & \vdots \\ (-1)^{r-1}A_{r-1} & \cdots & (-1)^{r-1}A_{2r-2}\end{bmatrix}.$$

Factor $\hat{\boldsymbol{N}}_{r-1}(p_1)=a_0^{2r}(p_1)N_{r-1}(p_1)$, a polynomial matrix, as

$$\hat{\boldsymbol{N}}_{r-1}(p_1)=\boldsymbol{M}(p_1)\boldsymbol{M}^T(-p_1)$$

Unless simultaneously, $a_0(p_1)=-a_0(-p_1)$ and r is odd, in which case factor $-\hat{\boldsymbol{N}}_{r-1}(P_1)$. The factorization must be such that $\boldsymbol{M}$ is a $(k\times r)$ polynomial matrix with k = rank of $\boldsymbol{N}_{r-1}(p_1)$ and $\tilde{\boldsymbol{M}}$, the left inverse of $\boldsymbol{M}$ analytic in the open right plane. The existence of such a factorization is guaranteed by [56].

Partition $\boldsymbol{M}(p_1)$ into $(1\times k)$ blocks,

$$\boldsymbol{M}(p_1)=\left[\boldsymbol{M}_0(p_1)\ \cdots\ \boldsymbol{M}_{r-1}(p_1)\right]$$

Form the $(r\times r)$ matrix $\boldsymbol{\Omega}(p_1)$, defined by

$$\boldsymbol{\Omega}(p_1)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\dfrac{a_r}{a_0} & -\dfrac{a_{r-1}}{a_0} & -\dfrac{a_{r-2}}{a_0} & \cdots & -\dfrac{a_1}{a_0}\end{bmatrix}.$$

It can be verified that with

$$z_{11}=A_{-1}(p_1)=Z_0(p_1,\infty),$$
$$z_{21}=\frac{\boldsymbol{M}_0(p_1)}{a_0^r(p_1)},\ z_{12}(p_1)=-z_{21}^T(-p_1),$$

and

$$z_{22}=\tilde{\boldsymbol{M}}^T(p_1)\,\boldsymbol{\Omega}(-p_1)\,\boldsymbol{M}^T(-p_1),$$

where $\tilde{\boldsymbol{M}}^T(p_1)$ denotes the left inverse of $\boldsymbol{M}^T$,

$Z_0(p_1,p_2)$ can be decomposed as follows:

$$Z_0(p_1,p_2)=z_{11}(p_1)-z_{12}(p_1)\left[z_{22}(p_1)+p_2\boldsymbol{E}_k\right]^{-1}z_{21}(p_1)$$

Then, $Z_0(p_1,p_2)$ is realizable as in Fig. 2.

(3) Connect all three networks in series as shown in Fig. 3.

The given $Z(p_1,p_2)$ is thus realized as a passive bivariate network.

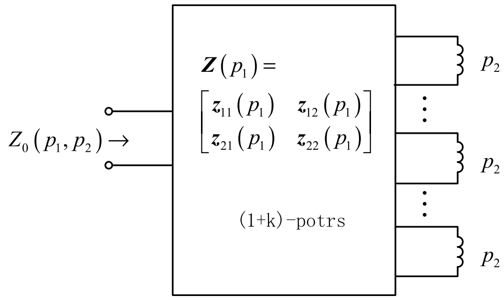

FIGURE 2. Realization of $Z_0(p_1,p_2)$.

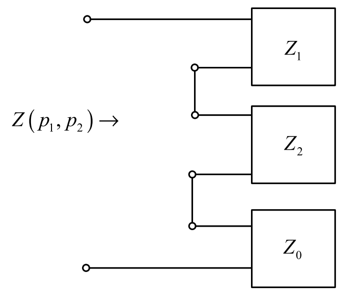

FIGURE 3. Realization of $Z(p_1,p_2)$.

Combining the Remark, Lemma 1, and Lemma 2, the immitance function of a fractional three-element-kind network can always be realized as a passive bivariate reactance network. Then, by inversely transformation, a fractional three-element-kind network can be obtained. And it is passive.

B. SYNTHESIS PROCEDURE

The specific implementation steps of the FO three-element-kind circuit are given below,

1) According to the element-order of $Z(s)$, find possible scaling parameters and variable substitutions.
2) Transform $Z(s)$ into $Z(p_1,p_2)$ by suitable impedance scaling and variable substitution, and $Z(p_1,p_2)=s^{\gamma}Z(s)$.
3) Realize $Z(p_1,p_2)$ as a passive bivariate reactance network.
4) Obtain the passive FO three-element-kind circuit corresponding to $Z(s)$ by inversely transformation for the bivariate reactance network.

And the process of synthesizing FO three-element-kind network is shown in Fig. 4.

Continue to Example 1, we finish the realization of (5).

Example 2: Consider the bivariate reactance function,

$$Z(p_1,p_2)=\frac{p_1^3p_2+p_1^2+2p_1p_2+1}{p_1^3+2p_1}\tag{8}$$

Realize (8) as a passive bivariate reactance network, as shown in Fig. 5.

By using $p_1=s^{(\tau_3+\tau_2)/2}$, $p_2=s^{\tau_1-(\tau_3-\tau_2)/2}$, we get a network as Fig. 6. Then by means of $s^{-\gamma}==s^{-(\tau_2-\tau_3)/2}$,

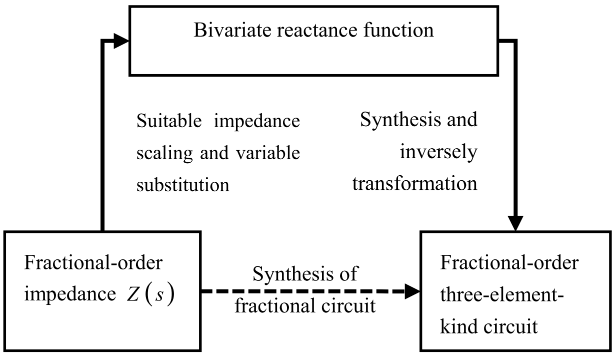

FIGURE 4. Process of the synthesis method.

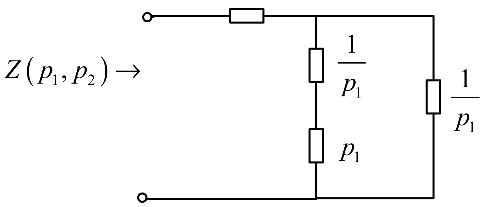

FIGURE 5. Realization of (8) in p-plane.

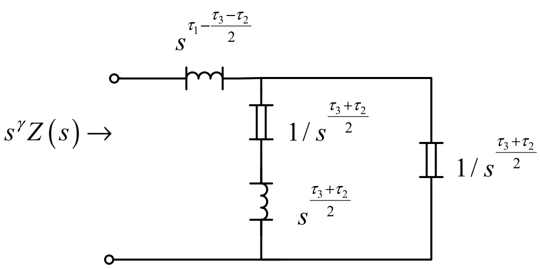

FIGURE 6. Realization of (8) in s-plane.

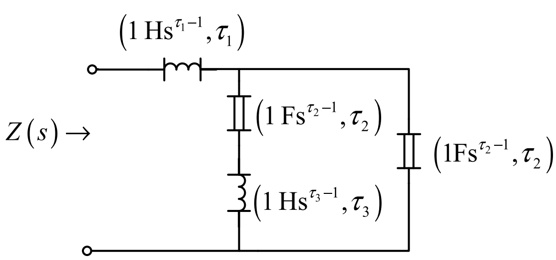

FIGURE 7. Realization of (5) in s-plane.

the passive FO three-element-kind circuit corresponding to $Z(s)$ is obtained. The result is Fig. 7.

In [60], the approximation result of some supercapacitors requires a fractional-order three-element-kind network to describe [60]–[62]. The synthesis for such supercapacitors are shown in Example 3.

Example 3: Given a two element-order impedance function

$$Z(s) = \frac{3s^{\alpha+0.5} + s^{\alpha} + 4s^{0.5} + 1}{s^{\alpha+0.5} + s^{0.5}} \tag{9}$$

For the sake of simplicity, set $\alpha = 0.9$. Then, choosing $s^{\gamma} = s^{0.45}$ and $p_1 = s^{0.45}, p_2 = s^{0.05}$, we have a bivariate reactance function,

$$Z(p_1, p_2) = \frac{3p_1^3 p_2 + 4p_1 p_2 + p_1^2 + 1}{p_1^2 p_2 + p_2} \tag{10}$$

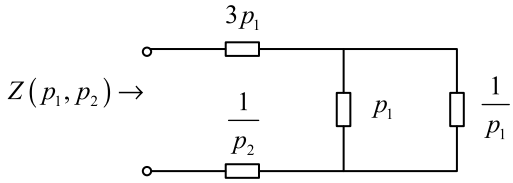

FIGURE 8. Realization of (10) in p-plane.

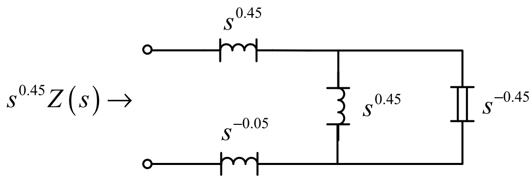

FIGURE 9. Realization of (10) in s-plane.

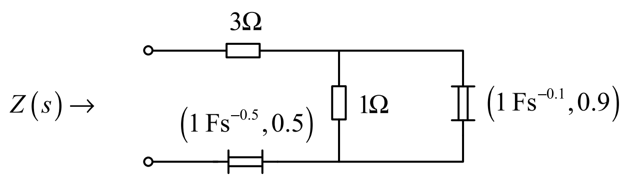

FIGURE 10. Realization of (9) in s-plane.

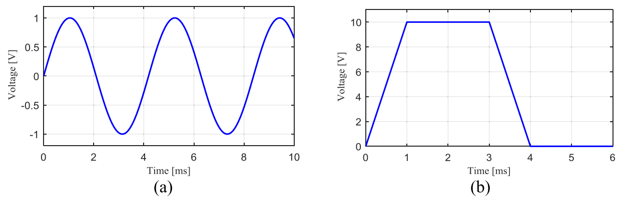

FIGURE 11. Voltage excitation. (a) Sinusoidal steady-state voltage. (b) Transient voltage.

The realization of (10) is shown in Fig. 8. By using $p_1 = s^{0.45}, p_2 = s^{0.05}$, we get the network drawn in Fig. 9. Then, with the scaling parameter $s^{-\gamma} = s^{-0.45}$, the passive network of (9) is obtained, which is shown in Fig. 10.

In order to valid the results of the paper, we set $\gamma_1 = 0.2$, $\gamma_2 = 0.3, \gamma_3 = 0.7$ in Example 2, and two voltage excitations that shown in Fig. 11 are applied to fractional order impedance function and fractional order network respectively, then gets two voltages by frequency domain analysis.

The simulation of circuits in Example 2 and 3 are shown in Fig. 12 and 13 respectively. The "Function Calculation" means mathematical calculation of fractional order impedance function $Z(s)$ based on $I(s) = U(s)/Z(s)$; "Circuit Simulation" means fractional order network port-voltage obtained by modified nodal approach. In Fig. 12 and 13, "Function Calculation" and "Circuit Simulation" are consistent, the results of two synthesis networks are correct.

Example 4: Consider the impedance function

$$Z(s) = \frac{s^{2\tau_1+2\tau_2} + s^{\tau_1+2\tau_2} + s^{\tau_1} + 1}{s^{\tau_1+2\tau_2} + s^{\tau_1+\tau_2} + s^{\tau_2} + 1} \tag{11}$$

By choosing $s^{\gamma} = s^{-\tau_1/2}$ and $p_1 = s^{\tau_1/2}, p_2 = s^{\tau_2+(\tau_1/2)}$, we can get the bivariate reactance function,

$$Z(p_1, p_2) = s^{\gamma} Z(s) = \frac{(p_1^2+1)(p_2^2+1)}{(p_1+p_2)(p_1 p_2+1)} \tag{12}$$

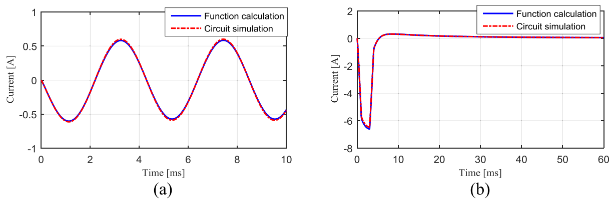

FIGURE 12. Port-current response of Example 2. (a) Sinusoidal steady-state voltage. (b) Transient voltage.

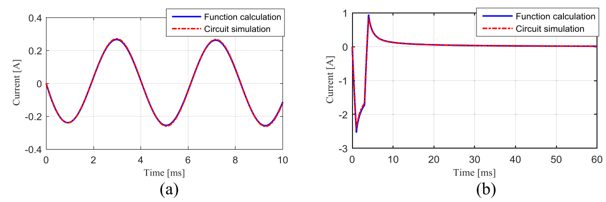

FIGURE 13. Port-current response of Example 3. (a) Sinusoidal steady-state voltage. (b) Transient voltage.

Then we synthesis $Z(p_1, p_2)$. The impedance matrix $\mathbf{Z}(p_1)$ is obtained using the method in section 4.

The $\mathbf{Z}(p_1)$ can be verified to be lossless. According to Lemma 2, $Z(p_1, p_2)$ in (12) can be realized as a impedance seen at the first one port and terminated at its last two ports

$$\mathbf{Z}(p_1) = \begin{bmatrix} \frac{p_1^2+1}{2p_1} & \frac{p_1^2+1}{\sqrt{2}p_1} & -\frac{p_1^2+1}{\sqrt{2}p_1} \\ \frac{p_1^2+1}{\sqrt{2}p_1} & \frac{p_1^2+1}{2p_1} & -\frac{(p_1+1)^2}{2p_1} \\ -\frac{p_1^2+1}{\sqrt{2}p_1} & -\frac{(p_1-1)^2}{2p_1} & \frac{p_1^2+1}{2p_1} \end{bmatrix}$$

by unit p_2-plane inductors.

After that, by making use of $p_1 = s^{\tau_1/2}, p_2 = s^{\tau_2+(\tau_1/2)}$, the passive network of (12) is obtained. Finally, with the scaling parameter $s^{-\gamma} == s^{\tau_1/2}$, we get the passive FO circuit of (11).

As for the commensurate networks composed of three kinds of elements, their immittance function can be realized with classical synthesis method [59].

V. CONCLUSIONS

Passive synthesis of the FO three-element-kind circuits is discussed in this paper. The FO three-element-kind circuits are more diverse than integer-order three-element-kind circuits. It can be classified into 3 kinds of networks: three element-orders networks, for example $L_{\beta 1}L_{\beta 2}C_\alpha$, $L_\beta C_{\alpha 1}C_{\alpha 2}$, $L_{\beta 1}L_{\beta 2}L_{\beta 3}$, and $C_{\alpha 1}C_{\alpha 2}C_{\alpha 3}$ networks, two element-orders networks such as $RL_\beta C_\alpha$, $RL_{\beta 1}L_{\beta 2}$, $RC_{\alpha 1}C_{\alpha 2}$, $L_\beta L_\alpha C_\alpha$ and $L_\beta C_\beta C_\alpha$ networks, and commensurate network like $RL_\beta C_\beta$ network.

A method for judging the immittance functions of FO three-element-kind circuits is given. Then, a passive synthesis method is proposed for fractional circuits contain three kinds of elements. Also, a specific synthesis procedure is given. The investigation in this paper further enriched the synthesis methods of the fractional electrical networks.

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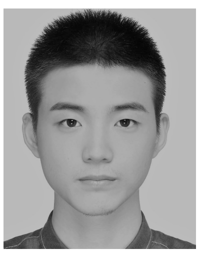

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