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Agent-Based Models for Opinion Formation: A Bibliographic Survey

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ABSTRACT Agent-based models are now largely adopted to describe how opinions emerge in a group of people. This survey provides an analysis of the literature on the subject, highlighting the major characteristics of such models. Over the last decade, the number of papers has grown at an overall annual rate of 16%, though not continually. Two communities contribute to the research effort: physics and control systems. However, their mutual awareness and collaboration are rather low. The prevailing mechanism adopted to describe the interaction among the agents is bilateral, but not symmetric. In most cases, the opinion is described by a continuous variable. Just a few papers consider a utility function for the agents.

INDEX TERMS Agent-based modeling, decentralized control, multi-agent systems, opinion dynamics.

I. INTRODUCTION

The formation of opinions within a group of people has been a subject of interest in many areas, e.g. psychology, sociology, economics and finance. When group members' actions follow a set of rules, their behavior may be described by an agent-based model. Agent-based models may be employed to describe a variety of characteristics of the agents involved and the way they interact, allowing us to understand the evolution of the opinions of the individuals, and if and how they reach a final consensus or whether the agents polarize around a small number of different opinions.

This research area has spawned a large number of papers, with a growing interest over the years. Since a number of different research paths have been explored, it is now the time to try to get an overall view of the research landscape in this area. Although survey papers on the same research area have been published, some of them focus on a particular class of models or a specific application. For example, the review in [1] is focused in the classical kinetic theory approach, while that in [2] concerns bounded-confidence models. Instead, the survey in [3] is limited to opinion propagation in online social networks. Hence, a systematic review of the field seems to be missing.

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In this paper, we provide a bibliographic survey of the research area on agent-based models for opinion forming. Our aim is to classify the papers by their common modeling features. In addition to providing a guide to the research literature on the subject, we can also identify the most investigated cases and at the same time highlight the cases that have not received an adequate coverage.

In Section II, we first provide information about the literature evolution over time, its relevance as measured by the number of citations, and its scientific community collocation. In Section III, we introduce several features to classify the papers, and examine the presence of each feature in the overall set of papers. Sections IV and V are devoted respectively to get an atomic view of paper classes as resulting from the classification of Section III, and to aggregate those atomic classes into clusters of papers with similar modeling characteristics.

II. LITERATURE META-STATISTICS

We examined more than 150 papers. This final number was the result of a selection process arranged over three phases:

1) Search over major scientific databases;
2) Spanning through the citation tree;
3) Elimination of papers loosely related with the survey theme.

The process started with the acquisition of the papers indexed in Google Scholar, Web of Science and Scopus that were published from 1974 to 2018 on opinion dynamics. This initial search was carried out using the keywords *opinion dynamics*, *opinion forming*, *opinion formation*, and several combinations thereof. For each paper in this initial list we retrieved its citing papers and added them to the list. We stopped at the second level of citations (i.e., we also considered the papers citing a paper citing a paper in the initial list). Finally, we went through each paper in this list and examine its contents, eliminating those whose focus was not opinion formation and those that did not employ an agent-based model. Though we were primarily interested in documents in the *Article* or *Proceedings Paper* class, we also considered documents in the *Review* or *Book Chapter* class, whenever these also included original results.

Before delving into the contents of the papers, analyzing the characteristics of their opinion formation models, in this section we report some meta-statistics of theirs, i.e. those characteristics of the literature that do not explicitly concern their content. In particular we consider the following features:

- time evolution;
- number of citations;
- prolific authors;
- number of authors per paper;
- publishing venues.

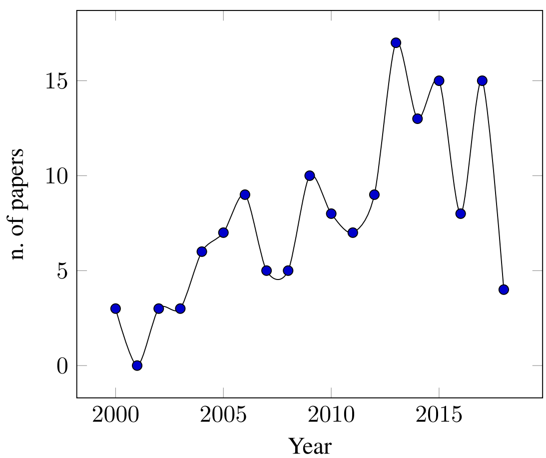

FIGURE 1. Number of papers per year.

A. TIME EVOLUTION

Though the seminal paper about the use of agents to describe how a consensus emerges around an opinion dates back to 1974 [4], scientific papers have started to appear in significant numbers after 2000. The number of papers appearing each year is an obvious measure of the interest for a topic. In FIGURE 1 we have plotted the number of papers appearing each year. The plot shows a significantly non-monotonic behavior, with several ups and downs. However, the general trend is positive. If we compute the Compound Annual Growth Rate (CAGR) over the period from 2003 to 2017 (we start with 2003, since the papers prior to 2003 were very few and scattered through the years), we get a 16% annual increase.

B. CITATIONS

Not all the papers published on the subject are equally important. A bibliometric measure of the relevance of a paper is the number of citations it receives. In order to get an overall view of the relevance of the papers examined in our survey, in FIGURE 2 we show the rank-size distribution for those papers. Though a long tail with few citations is present, there is a significant number of papers with hundreds of citations.

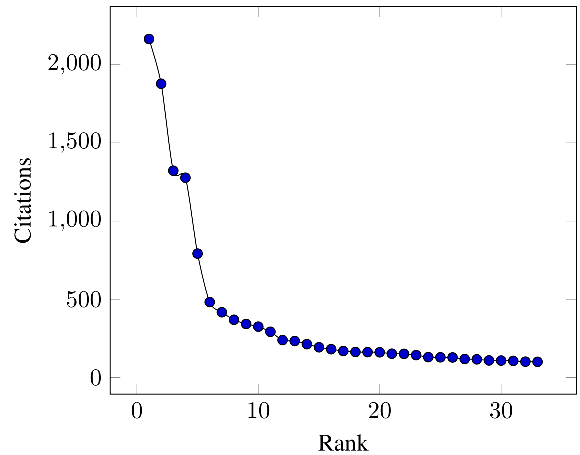

FIGURE 2. Rank-size distribution of papers by number of citations.

TABLE 1. Top 10 papers as to the number of citations (from Google Scholar).

Paper ref.	Year	No. of citations
[4]	1974	2164
[5]	2002	1878
[6]	2000	1322
[7]	2000	1277
[8]	1990	792
[9]	2002	482
[10]	2002	417
[11]	2002	369
[12]	2013	342
[13]	2000	325

In Table 1, we report the data for the 10 most cited papers. Though the most cited paper dates back to more than forty years ago, the most are millennials. Actually, just one has appeared within the latest 5 years, but that's expected since the oldest papers have taken advantage of a longer stretch of time to collect citations.

C. PROLIFIC AUTHORS

A number of authors contributed to the literature. However, some of them contributed most and can therefore be considered as reference authors (if we except the very rare case of a researcher publishing a seminal paper on the subject and then nothing else). In Table 2, we report the most prolific authors.

We can correlate the status of prolific authors with the list of most cited papers. Though the correspondence is not perfect, we see that three of them (Deffuant, Amblard, and Weisbuch) actually published 3 out of the 10 top papers,

TABLE 2. Number of papers by author (Top 8).

Author	n. of papers
N. E. Friedkin	7
F. Amblard	6
G. Deffuant	6
F. Bullo	5
S. Galam	5
J. M. Hendrickx	5
A. C. R. Martins	5
G. Weisbuch	5

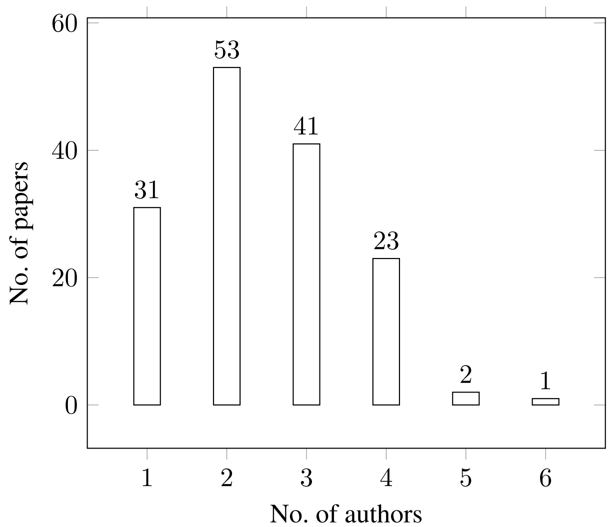

FIGURE 3. Distribution of papers by the number of authors.

while Friedkin and Galam each have 1 of their papers in the top list. Summing up, 5 among the top 8 prolific authors provided half of the top 10 papers.

D. NUMBER OF AUTHORS PER PAPER

In addition to identifying the authors who have contributed most to the subject so far, we may wish to see if the works appeared in the literature are the product of individuals or research teams instead.

For that purpose we look at the number of authors for each paper. The resulting distribution is shown in FIGURE 3. The distribution exhibits a mode at 2 authors, representing roughly one third of the papers, with the cases of 3 authors and a single author being less frequent. The cases of 5 or more authors are very rare. Actually, this is quite in line with the figures recorded for the papers both in general computer science and specific areas, as reported in [14].

E. PUBLISHING VENUES

The last issue we consider is where papers have appeared. In Table 3 the distribution of papers by journal or conference is shown. The publishing venues for this topic can be roughly divided into two categories, corresponding to two quite different communities: the physics community and the control systems community. The former gathers, e.g., Physica A, Physical Review, the International Journal of Modern Physics, while the latter includes the IEEE Transactions on Automatic Control, Automatica, and the SIAM Journal on Control on Optimization. The contributions of the two communities appear quite balanced.

TABLE 3. Most frequent publishing venues.

Journal/Conference	No. of papers
Physica A: Statistical Mechanics and its Applications	20
IEEE Transactions on Automatic Control	12
Journal of artificial societies and social simulation	11
Physical Review E	6
International Journal of Modern Physics C	6
Automatica	5
American Control Conference (ACC)	4
PLOS ONE	4
The European Physical Journal B	4
EPL (Europhysics Letters)	3
IEEE Conference on Decision and Control (CDC)	3
SIAM Journal on Control and Optimization	3

TABLE 4. Inbreed and cross-fertilization indices.

		Citing	
		CSC	PC
Cited	CSC	56	7
	PC	16	91

However, we wish to see if the two communities interact in some way or not. In order to examine that issue, we have first selected the top 5 journals/conferences for each community in the list of Table 3, which are respectively:

Physics community (PC)

1) Physica A: Statistical Mechanics and its Applications;
2) International Journal of Modern Physics C;
3) Physical Review E;
4) PLOS ONE;
5) The European Physical Journal B.

Control systems community (CSC)

1) IEEE Transactions on Automatic Control;
2) Automatica;
3) American Control Conference;
4) IEEE Conference on Decision and Control;
5) SIAM Journal on Control and Optimization.

Then, we have computed the number of citations falling into either category, obtaining the following four figures:

- no. of PC papers cited in PC journals;
- no. of PC papers cited in CSC journals;
- no. of CSC papers cited in PC journals;
- no. of CSC papers cited in CSC journals.

We summarize the results in Table 4. In that table the main diagonal elements represent the inbreeding components, i.e., those pertaining to communities citing themselves. Instead, the antidiagonal represent the cross-fertilization component, i.e., communities citing each other. We can take the ratio of the cross-fertilization component to the overall number of papers as a cross-fertilization index $I \in [0, 1]$:

$$I = \frac{16 + 7}{56 + 7 + 16 + 91} = 0.135, \tag{1}$$

TABLE 5. Papers by category.

Opinion domain	continuous on $\mathbb{R}$	[4], [8], [12], [13], [15]–[64]
	continuous and bounded	[5], [6], [9], [11], [65]–[129]
	discrete	[7], [10], [130]–[154]
Interacting agents	pairwise	[6], [9], [11], [20], [46], [47], [61], [67]–[70], [72]–[75], [77], [79], [81]–[85], [87], [92], [94], [95], [103], [106], [107], [109]–[113], [115], [118], [119], [126]–[129], [139], [142]–[147], [149], [150], [152]–[155]
	any-to-any	[4], [8], [10], [13], [15]–[19], [23], [25], [27]–[30], [32], [37], [39], [41], [43]–[45], [49], [51], [58], [60], [63]–[66], [71], [86], [90], [91], [93], [96], [130], [132], [135]–[138], [140], [148]
	closest neighbours	[5], [7], [12], [21], [22], [24], [26], [31], [33]–[36], [38], [40], [42], [48], [50], [52]–[57], [59], [62], [76], [78], [80], [88], [89], [97]–[102], [104], [105], [108], [114], [116], [117], [120]–[125], [131], [133], [134], [141], [151]
Interaction direction	bilateral	[4]–[6], [8], [10]–[13], [15]–[45], [47]–[73], [75]–[77], [79]–[84], [86]–[88], [91]–[105], [107]–[123], [125]–[132], [134]–[141], [145], [146], [148]–[152]
	unilateral	[7], [9], [46], [74], [78], [85], [89], [90], [106], [124], [133], [142]–[144], [147], [148], [153], [154]
Interaction symmetry	symmetric	[6], [11], [38], [47], [61], [83], [84], [86], [92], [94], [95], [107], [110], [112], [113], [115], [118], [126]–[128], [130], [139], [146], [152]
	non symmetric	[4], [5], [7]–[10], [12], [13], [15]–[37], [39]–[46], [48]–[60], [62]–[82], [85], [87]–[91], [93], [96]–[106], [108], [109], [111], [114], [116], [117], [119]–[125], [129], [131]–[138], [140]–[145], [147]–[151], [153]–[155]
Updating function	linear	[4], [8], [16]–[19], [21], [23], [25], [27]–[32], [34]–[37], [39], [40], [45], [51], [63], [64], [67], [96], [105], [106], [108], [144]
	nonlinear	[5]–[7], [9]–[13], [15], [20], [22], [24], [26], [33], [38], [41]–[44], [46]–[50], [52]–[62], [65], [66], [68]–[95], [97]–[104], [107], [109]–[143], [145]–[155]
Updating frequency	periodic	[4], [5], [7], [8], [10], [12], [13], [15]–[19], [21]–[45], [47]–[60], [62]–[66], [71], [73], [74], [80], [82], [86], [88], [91], [93], [96]–[102], [104], [106], [108], [108], [112], [114], [116], [117], [120]–[123], [125], [131], [134], [135], [137], [138], [140], [141], [150], [151]
	non periodic	[6], [9], [11], [20], [46], [61], [67]–[70], [72], [75]–[79], [81], [83]–[85], [87], [89], [90], [92], [94], [95], [103], [107], [109]–[111], [113], [115], [118], [119], [124], [126]–[130], [132], [133], [136], [139], [142]–[149], [152]–[155]
Utility function	yes	[21], [26], [30], [32], [33], [35], [96], [105], [108], [110], [149]
	no	[4]–[13], [15]–[20], [22]–[25], [27]–[29], [31], [34], [36]–[95], [97]–[104], [106], [107], [109], [111]–[148], [150]–[155]

which is rather low. Cross-fertilization is poor, with the two communities working mostly parallel without crossing each other's path.

III. CLASSIFICATION

Each paper has adopted several assumptions concerning the behavior of the agents and the description of the opinion. Some assumptions have found a wider acceptance, while others may be relatively unexplored, which does not imply that they are not valid. We have extracted the major features that have been employed for the purpose of defining the agent-based models. In this section, we describe those features and review the authors' preferences. The papers falling into each category are shown in Table 5.

A. CATEGORIES

After reading the large number of papers selected for our survey, we have extracted the following categories that allow to characterize each agent-based model in this context:

- Opinion domain;
- Interacting agents;
- Interaction direction;
- Interaction symmetry;
- Updating function;
- Updating frequency;
- Utility function.

In the following subsections, we consider separately each category in the above list and examine the choices made by the authors.

B. OPINION DOMAIN

The first choice that has to be made is the numeric value chosen to represent the opinion. Though an opinion is intrinsically a qualitative and potentially multi-faceted feature, its study through an agent-based model requires it to be described by a numeric variable. In addition, in all the papers considered, the opinion has been represented by a scalar variable.

The choices made in the papers as to the domain of the opinion can be categorized as follows (the numbers between brackets indicate the percentage of papers falling in that category):

- Discrete (18.5%);
- Continuous over a bounded interval (45.7%);
- Continuous over $\mathbb{R}$ (35.8%).

As can be seen there is a large prevalence of the continuous choice, since the latter two categories amount to 81.4% of the total.

In the discrete case, the agent may opt for an opinion within a limited set; though the opinion is anyway a numeric variable in this framework, its value can be easily mapped to a qualitative feature. For example, in the case of a binary opinion variable, we could use the two values to represent respectively a positive versus a negative opinion. Within the group of papers opting for a discrete set of opinions, the majority (15 out of 28, i.e., 53.5%) have employed a binary variable [7], [10], [130]–[141], [148], while 4 (14.2%) have chosen a ternary variable [142]–[145], and 9 (32.14%) have employed a variable with more than 3 states [146], [147], [149]–[155].

When the authors have instead opted for a continuous but bounded interval, the most frequent choices have been the [0, 1] interval, chosen in 37 papers (53.6%) [5], [6], [11], [96]–[129], and the [−1, 1] interval, chosen in 22 papers (31.8%) [9], [65]–[85].

C. INTERACTION DIRECTION AND SYMMETRY

Another aspect we wish to analyze is the direction of the influence that agents exert over each other. If any two agents always influence each other mutually, we talk about a bilateral interaction. Instead, if an agent may influence another agent without being influenced by it, we talk about a unilateral influence. The overwhelming majority of papers (88.1%) adopts a bilateral model, though this is mitigated by the possibility of having different weights in the opinion updating equations, so that the actual impact of agent A on agent B is different from that in the reverse direction: just 18% of the bilateral models assume symmetry.

D. INTERACTING AGENTS

In addition to considering the symmetry of interaction, the research has also differentiated as to the agents involved in the interaction at each step.

We have the following three categories:

- Pairwise;
- Any-to-any;
- Closest neighbors.

In the first case, at each step there is just one pair of agents interacting with each other; at the next time step it will be the turn of another pair, and so on. Of course, the pairs are not fixed, so that in the long term any agent has the chance of interacting (influencing or being influenced by) with any other agent.

In the any-to-any interaction case, instead, at each time step all the agents change their opinion by being influenced at once by the opinions of the other agents at the previous time step. If the interaction is actually any-to-any depends of course on the interaction rules as described in Section III-C. For example, if the agents are the set $\{A, B, C, D\}$ and a unilateral model of interaction is in place, and the agent A may be influenced just by agents C and D, at time step $t+1$ the opinion of A will change according to the opinions of agents C and D only (in addition to its own previous opinion, of course), though the interaction is potentially of the any-to-any type.

The third category (closest neighbors) may appear when we introduce a measure of distance among agents. This is quite natural when the agents are actually part of a social network. In that case, the measure of distance associated to the social network is used: this may be the number of edges on the shortest path connecting the two agents in a non-weighted network, or the sum of the weights on the least costly path in a weighted network. Even if the social network is not explicitly formed, a measure of distance may anyway be introduced, which in many cases is equivalent to embed a social network.

In the set of papers we have examined, the models are split nearly evenly among the three types. Precisely, the most employed interacting categories are the pairwise one (35.8%) and the closest neighbors one (35.1%), while the any-to-any type is slightly less used (29.1%).

E. UPDATING FUNCTION

Another important aspect concerns the mathematical nature of the function that relates the opinion of an agent to the opinions of the other agents. For that purpose, it is natural to classify the papers into two categories, adopting respectively a linear model or a nonlinear one.

In the linear class, the opinion of an agent is a linear combination of the opinions of the other agents.

The nonlinear class is actually rather broad.

The distribution of papers between the two classes is largely in favor of the nonlinear one, which includes 79.5% of the papers, with the linear class getting the remaining 20.5%.

If we consider a population $\mathcal{N}$, composed of n individuals; $x_i(t)$ is the opinion of agent $i \in \mathcal{N}$ at time $t \in \mathbb{N}$. Then, the opinion formation of agent i can be described by the following averaging equation:

$$x_i(t+1) = \sum_{j \in \mathcal{N}} a_{ij}\, x_j(t)\,, \quad i \in \mathcal{N}\,,\ t > 0\,, \tag{2}$$

(the same for continuous time version, i.e. having $\dot{x}_i$ instead of $x_i(t+1)$). The coefficient a_{ij} represents the influence that agent i exerts on agent j. That is, agent i takes a weighted average with weight a_{ij} of agents' opinion at time t to update his/her opinion in period $t+1$. Having these notations, we say that influence is *linear* if a_{ij} does not depend on the state x, $\forall i, j \in \mathcal{N}$.

The advantage of having a linear model is that the approach to the problem can be formulated in terms of powerful linear techniques such as matrix theory, Markov chains and graph theory. Actually, using matrix notation, a classical linear model of fixed weights is

$$x(t+1) = A\,x(t)\,, \quad t > 0\,, \tag{3}$$

where A is a fixed stochastic matrix and $x(t)$ is the column vector of opinions at time t. This model has been proposed to describe how a group of individuals who act together as a team or committee, might reach agreement by pooling their individual opinions [4], [63] (see also the popular generalization of this model made in [8]).

A most general form of model (3) can be compactly written as

$$x(t+1) = A\,(t, x(t))\; x(t)\,, \quad t > 0\,. \tag{4}$$

If the weights depend on the opinions themselves, then the model turns from linear to nonlinear, i.e. $A\,(t, x(t)) = A\,(x(t))$ and so it does not explicitly depend on time.

A widespread nonlinear model in literature incorporates bounded confidence among the agents: fixing an agent i, the set of agents that he/she takes into account is given by

$$I(i,x) = \{1 \le j \le n : |x_i - x_j| \le \epsilon_i\}\,, \tag{5}$$

i.e. those agents whose opinions differ from his/her own not more than a certain confidence level ϵ_i. Symbol $|\cdot|$ in (5) denotes the absolute value of a real number. The model with bounded confidence is given by

$$x_i(t+1) = |I\,(i, x(t))\,|^{-1} \sum_{j \in I\,(i,x(t))} x_j(t), \quad t > 0\,, \tag{6}$$

i.e. in the light of the general model (4) we have $a_{ij}(x) = 0$ for $j \notin I(i,x)$ and $a_{ij}(x) = |I\,(i,x)\,|^{-1}$ for $j \in I(i,x)$. Now symbol $|\cdot|$ in (6) denotes the cardinality of a set, i.e. the number of its elements. This model has been developed in [13] and the papers [5], [6], [11] can be considered to belong in the same class.

A more general approach does not require to restrict the influence weights $a_{ij}(x)$ to specific functional forms, but defines them through some general reasonable assumptions (e.g., increasing function, other conditions on derivatives and so on). According to this general framework, each opinion evolves following the dynamics:

$$\dot{x}_i(t) = \sum_j a_{ij}\,(x_j - x_i)\,, \quad a_{ij} = \phi\left(|x_i - x_j|\right)\,. \tag{7}$$

Here, ϕ is the so-called *influence function*. By assumption, it is equipped with certain properties depending on the problem framework, see for example [41], [88].

Another popular nonlinear model is that employing kinetic models of opinion formation, whose goal is to describe the evolution of opinions in a society by means of microscopic (usually binary) interactions among agents that exchange information ([68], [81] for example). From a microscopic view point, the binary interaction is described by the rules:

$$\begin{aligned} x' &= x - \gamma P\,(|x|)\,(x - x_*) + \eta D\,(|x|) \\ x'_* &= x_* - \gamma P\,(|x_*|)\,(x_* - x) + \eta_* D\,(|x_*|)\;, \end{aligned} \tag{8}$$

for a given constant γ and certain random variables η and η_*. In (8), the pair (x, x_*) denotes the opinions of two arbitrary individuals before the interaction and (x', x'_*) their opinions after exchanging information between them and with background; the functions $P(\cdot)$ and $D(\cdot)$ describe the local relevance of the compromise and diffusion for a given opinion.

F. UPDATING FREQUENCY

In addition to the mathematical function adopted to update the state (opinion) of each agent, a relevant parameter for opinion dynamics is the updating frequency.

In our classification we have defined two broad categories, including respectively those models where all the agents may change their opinion at each time step and those models where that's not the case. We call those categories respectively periodic and aperiodic updating. The second category encompasses several subcases, e.g., when just a couple of agents changes their opinion at each time step (and it is not known in advance when their turn comes again), when opinions are updated just if a triggering event takes place, or when the agents changing their opinion are a random selection of all the agents.

The major category in the papers we have examined is that of periodic updating, which accounts for 62.25% of all papers, while the aperiodic class contains the remaining 37.75%.

G. UTILITY FUNCTION

The last issue we consider concerns the adoption of a utility function, which would signal the interest of agents to update their opinion.

Among the papers considered here, there are few cases [21], [26], [30], [32], [33], [35], [96], [105], [108], [110], [149] where agents move maximizing an utility function or according to their expected payoff.

An interesting case is represented by [30]. Here, an agent $i \in \mathcal{N}$ takes an opinion x_i, which is compared with the opinion of his/her reference group, q_i, but expresses an opinion $s_i \in \mathbb{R}$ which need not coincides with her true opinion x_i. In [30] the utility of agent i depends on both the opinion distances, i.e. true opinion-stated opinion and stated opinion-group opinion:

$$u_i(s_i, x_i) := -(1-\delta_i)(s_i - x_i)^2 - \delta_i(s_i - q_i)^2$$

where $\delta_i \in (-1, 1)$ represents the relative importance of the preference for (counter-) conformity in relation to the preference for truthfulness.

Another example, in which a quadratic utility function is considered, is provided by [35]. Here, an agent i has an internal opinion x_i, which remains unchanged due to external influences, and an expressed opinion s_i which is updated by choosing s_i to minimize

$$(s_i - x_i)^2 + \sum_{j \in I(i,x(t))} a_{ij}(s_j - s_i)^2 ,$$

where $a_{i,j} \geq 0$ (recall that $I\ (i, x)$ denotes the set of neighbors of i).

Both papers, [35] and [30], are inspired by classical models due to DeGroot [4] and Friedkin-Johnsen [8]. Also [96] consider a set of nonstrategic agents, $\mathcal{N}$, which update their opinion $x(t)$ at discrete time instances as in the model of DeGroot [4] (i.e. according the classical linear model of fixed weights 3). Then, [96] consider two strategic agents a_1 and a_2, each of which forms exactly one link with a nonstrategic agent in order to influence the opinion formation in the network. Each strategic agent a_i chooses a strategy g_i in $\mathcal{N}$ that represents the nonstrategic agent he/she decides to influence by creating a link with. The payoffs of agent a_1 is

$$\pi(g) = 1' \cdot (I - \Delta(g)A)^{-1} \frac{\lambda}{d_{g_1}} \Delta(g)e_{g_1}$$

while $n - \pi(g)$ is that for a_2 ($\mathcal{N}$ is composed of n individuals), where: A is the interaction matrix which appears in (3); $g = (g_1, g_2)$ is the strategy profile; d_i is the number of outgoing links of i; e_j denotes the unit vector with coordinate 1 at j; Δ is a diagonal matrix whose diagonal entries depend on g and on the impact of a_1 and a_2 on the network, which we denote by $\lambda > 0$ and $\mu > 0$, respectively.

IV. TAXONOMY

The categories proposed in Section III induce a partition on the set of papers. We can describe that partition by a taxonomy tree, whose leaves represent the atomic classes of papers, each class featuring a unique combination of category values. In this section, we report the resulting taxonomy

In building the tree we have defined the tree levels with the aim of minimizing the overall number of branches. At the same time, we have pruned the tree, cutting all the empty branches and leaves, i.e. those combinations of category values that are not present in any paper. In the pruned tree, if a node contains only one child node and is not a leaf of the tree, that node and its child are collapsed to a single node whose label is the concatenation of the labels of the constituent nodes.

The resulting tree is shown in FIGURE 4. Each circle represents a feature, and a child represents a feature adding to those of its father and ancestors. The same number in a circle can therefore appear in different branches. The meaning of labels is as follows:

- Opinion domain: 1 = continuous in $\mathbb{R}$; 2 = continuous and bounded; 3 = discrete.
- Interaction direction: 4 = bilateral; 5 = unilateral.
- Interacting agents: 6 = pairwise; 7 = any-to-any; 8 = closest neighbors.
- Interaction symmetry: 9 = symmetric; 10 = non symmetric.

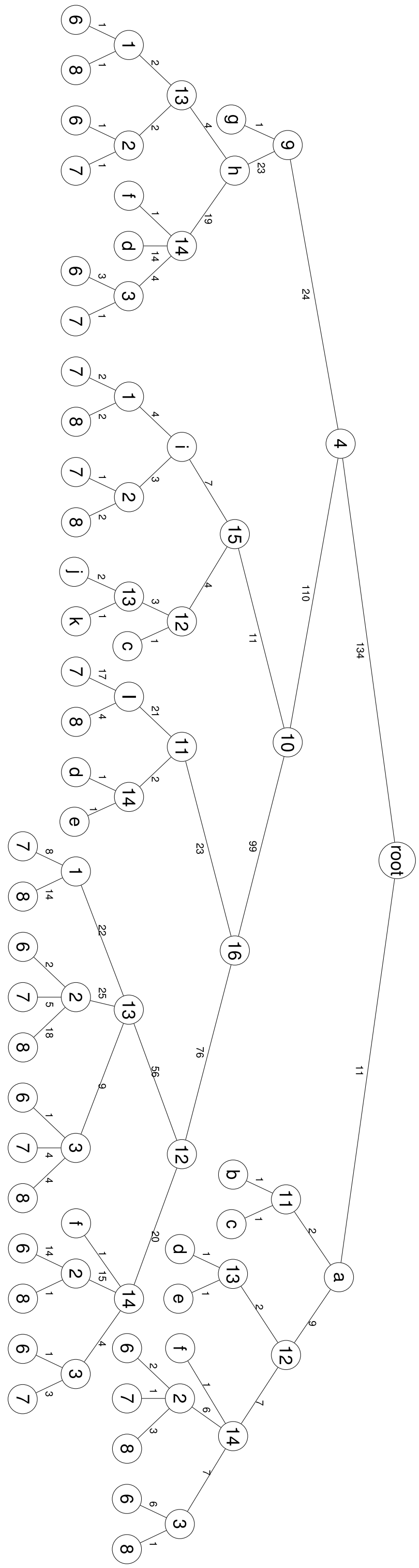

FIGURE 4. Taxonomy of agent-based models for opinion formation.

- Updating function: 11 = linear; 12 = nonlinear.
- Updating frequency: 13 = periodic; 14 = non periodic.
- Utility function: 15 = yes; 16 = no.

- Moreover: a = unilateral, non symmetric, no; b = periodic, continuous and bounded, pairwise; c = non periodic, discrete, pairwise; d = continuous and bounded, pairwise; e = discrete, closest neighbors; f = continuous on $\mathbb{R}$, pairwise; g = yes, nonlinear, non periodic, continuous and bounded, pairwise; h = no, nonlinear; i = linear, periodic; j = continuous on $\mathbb{R}$, closest neighbors; k = discrete, any-to-any; l = periodic, continuous on $\mathbb{R}$.

For example, following through the leftmost branch, the uppermost node represents the models with a bilateral interaction domain (circle number 4), which has two child nodes, representing those models that adopt at the same time a bilateral interaction domain and either symmetric interaction (circle number 9) or non symmetric interaction (circle number 10).

As can be seen, the number of leaves, representing unique combination of features, does not cover the whole set of possible combinations. This is due in some cases to incompatibility, e.g. we cannot have an interaction that is symmetric and unilateral at the same time. But the absence of compatible combinations of features signals some cases that have not been explored yet in the literature.

The top 5 leaves, i.e. the combinations of features with the largest presence in the set of papers, are the following (we describe them by the set of model features with dashes in between):

1) Bilateral interaction direction - non symmetric interaction - no utility function - nonlinear updating function - periodic updating - continuous and bounded opinion domain - nearest agents interaction (18 papers);
2) Bilateral interaction direction - non symmetric interaction - no utility function - linear updating function - periodic updating - continuous opinion domain in $\mathbb{R}$ - any-to-any interaction (17 papers);
3) Bilateral interaction direction - symmetric interaction - no utility function - nonlinear updating function - non periodic updating - continuous and bounded opinion domain - pairwise interaction (14 papers);
4) Bilateral interaction direction - non symmetric interaction - no utility function - nonlinear updating function - periodic updating - continuous opinion domain in $\mathbb{R}$ - nearest agents interaction (14 papers);
5) Bilateral interaction direction - non symmetric interaction - no utility function - nonlinear updating function - non periodic updating - continuous and bounded opinion domain - pairwise interaction (14 papers).

As can be seen, the commonest features in this Top 5 list are

- Bilateral interaction direction (common to all);
- No utility function (common to all);
- Non symmetric interaction (common to 4 out of 5);
- nonlinear updating function (common to 4 out of 5).

V. FAMILIES OF MODELS

In Section IV, we have classified all the models presented in the literature, identifying the groups of papers that propose models with identical characteristics, as defined in Section III-A. Those groups are represented as leaves in the tree of FIGURE 4. That representation allows us to recognize papers that follow very closely the same modeling approach, but may return an extremely atomic view of the subject.

It may result interesting as well to identify families of models, which stem from a similar approach but differentiate for minor variations. In this section, we pursue that task by adopting a clustering approach.

In order to identify the clusters, we consider each paper (i.e., its model) as described by the attributes listed in Section III-A and reported in the first column of Table 5. For each of the n papers we have then 7 attributes. As we can see, all attributes are categorical in nature and take either two or three values. For example, the opinion domain is a ternary variable, while the utility function is binary. Without loss of generality we may identify the j-th attribute (following the order of the bullet list of Section III-A) of the i-th paper by the variable s_{ij} and adopt the categorical values it takes as listed in the second column of Table 5. For example, if we consider the *Interaction direction* attribute for the i-th paper, and the value of that attribute is *bilateral*, we have $s_{i,3} = 1$.

In order to carry out a cluster analysis of our sets of papers, we need to introduce a distance matrix between any two papers on the sets of attributes. An established metric for categorical variables is defined in Section 3.2 of [156]: the distance between any two papers i and m is the number of different attribute values they have

$$d(i, m) = \sum_{j=1}^{7} \delta(s_{ij}, s_{mj}), \tag{9}$$

where

$$\delta(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \tag{10}$$

After defining the distance metric, we have to choose a clustering algorithm. We have adopted the k-mode clustering algorithm, which is an extension of the well known k-means algorithm, as defined in [157]. The final aim is to get the n papers classified into c clusters. We introduce some notation to describe the algorithm. We identify the set of papers belonging to the i-th cluster as C_i and the whole set of papers as P. The set of clusters forms a partition of P, so that $P = \bigcup_i C_i$ and $C_i \cap C_j = \emptyset$ for $i, j \in \{1, 2, \ldots, c\}$ and $i \neq j$. We then define the subsets $X_{ijk} = \{x \in C_i | s_{xj} = k\}$. For the generic cluster C_i we introduce its medoid Q_i, whose generic j-th attribute is q_{ij}.

The papers are first assigned to the clusters, e.g. in a random fashion.

For each cluster a medoid is then computed. According to the k-mode algorithm, each attribute of the medoid is assigned the most frequent value in its cluster (i.e., its mode, hence the *k-mode* name). For example, if the attribute values in the *Utility function* category for the papers of a cluster are such that 30 paper employ a utility function (attribute value = *Yes*) and 20 do not (attribute value = *No*), the most frequent value is *Yes*, and we set the attribute value of the medoid for the *Utility function* category as *Yes*. We have therefore the following medoid attribute assignment rule

$$q_{ij} = \underset{k}{\text{argmax}} \quad |X_{ijk}| \tag{11}$$

with $i = 1, 2, \ldots, c$ and $j = 1, 2, \ldots, 7$.

The papers are then reassigned to the cluster whose medoid is the closest, i.e.

$$\text{paper } m \in C_i \text{ if } d(m, Q_i) \leq d(m, Q_j) \qquad j \neq i \tag{12}$$

The algorithm then goes iteratively through the medoid computation and paper assignment phases till the cluster composition does not change anymore.

There are two issues with the k-mode algorithm. First, the solution provided depends on the initial assignment of nodes (papers) to clusters. Second, the k-mode algorithm provides a clustering partition where the number of clusters is set at the beginning, but does not provide the optimal number of clusters by itself. In order to deal with the first issue, we have considered 1000 different starting assignment, obtaining 1000 clustering solutions, among which we have selected the best one. In order to select the best clustering solution for a given number of cluster and the best number of clusters, we have adopted a clustering goodness metric as suggested in [156]. In that book, two different metrics are considered to measure the (lack of) homogeneity and the separation properties, aiming at minimizing the lack of homogeneity within a cluster and maximizing the separation between the clusters. The metrics proposed in Section 5.2 of [156] for those purposes for the generic i-th cluster are respectively

$$\begin{aligned} h(i) &= \sum_{l=1}^{n_i} \sum_{\substack{v=1 \\ v \neq l}}^{n_i} d_{il,iv} \\ s(i) &= \sum_{l=1}^{n_i} \sum_{k \neq i} \sum_{v=1}^{n_k} d_{il,kv} \end{aligned} \tag{13}$$

We can now sum over the number c of clusters to get the overall metrics H and S. Since the values of the sums in Equation (13) depend also on the number of terms in the sum, we have introduced a normalization factor to take into account the number of terms involved in the computation of h and s, so that the overall metrics for the lack of homogeneity and separation are

$$\begin{aligned} H(c) &= \sum_{i=1}^{c} \frac{h(i)}{n_i(n_i - 1)} \\ S(c) &= \sum_{i=1}^{c} \frac{s(i)}{n_i(n - n_i)} \end{aligned} \tag{14}$$

In order to consider both aims, we could define a single metric that takes into account both H and S, such as their difference or their ratio. However, the values of those two metrics appear to differ by orders of magnitude, so that their direct combination into a single metric is not very useful. We prefer to consider their behavior separately. In FIGURE 5 we see that a reasonable trade-off choice between the different aims for H and S is $c = 3, 4$.

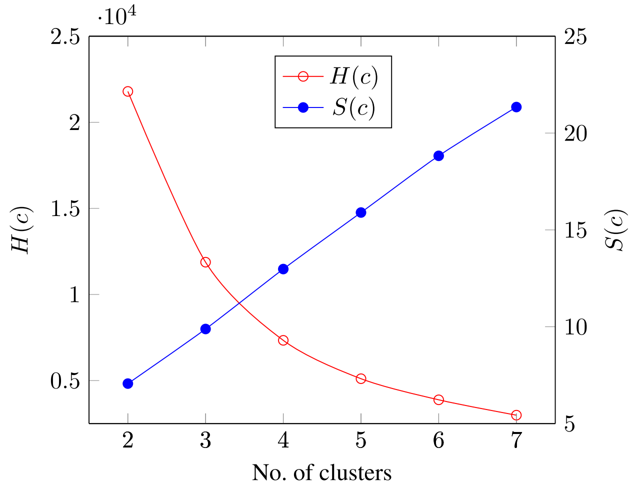

FIGURE 5. **Clustering metrics for lack of homogeneity and separation.**

As to the case of 3 clusters, the first cluster contains all the papers that adopt a periodic updating; the second cluster includes instead the papers that adopt an aperiodic updating but where the interaction concerns either pairs of agents or all the agents; the third cluster takes all the remaining papers, with no specific common feature.

In the case of four clusters, all papers adopting a unilateral interaction direction make up Cluster 1, so that the papers considering a bilateral interaction direction, which are the majority, are distributed among the three other clusters. Cluster 2 and Cluster 4 contain the papers with a bilateral symmetric pairwise interaction and aperiodic updating. Cluster 3 contains instead the papers adopting a bilateral nonsymmetric interaction and a linear updating function, but not those considering a pairwise interaction.

VI. CONCLUSIONS

Our survey has included over 150 papers, i.e., a large body of research literature.

The interest in the topic is growing at an aggregate yearly rate of 16%, though most papers among the most cited ones were published around the year 2000, with the notable exception of a paper published in 2013 and appearing in the Top 10.

Two scientific communities appear to constantly working on the subject, though the interaction between them is very limited. However, the development in the area could benefit from a mutual awareness of their work and a tighter interaction.

As to the models, not all the combination of features have been equally explored. The most adopted models feature a bilateral, but non symmetric, interaction direction; most studies do not employ a utility function, and adopt a nonlinear updating function.

Probably, the most interesting feature that could be explored in the future is the adoption of a utility function, which would signal the interest of agents to update their opinion. The survey may also be employed to highlight other features that have not been sufficiently explored so far.

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