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Reliability Assessment of Hierarchical Hypercube Networks

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ABSTRACT The g-good neighbor connectivity $\kappa^g(G)$ and g-good-neighbor diagnosability $t_g(G)$ are two important parameters to evaluate the reliability and fault tolerance for an interconnection network G. So far, almost all known results about $\kappa^g(G)$ and $t_g(G)$ are about special g except the hypercubes, the star graphs, the k-ary n-cubes, and so on. In this paper, we focus on $\kappa^g(HHC_n)$ and $t_g(HHC_n)$ for the n-dimensional hierarchical hypercube network HHC_n for $1 \leq g \leq m-1$ and $m \geq 2$, where $n = 2^m + m$. We show that $\kappa^g(HHC_n) = 2^g(m+1-g)$ for $1 \leq g \leq m-1$. In addition, we show that $t_g(HHC_n) = 2^g(m+2-g)-1$ under the PMC model and MM^* model for $1 \leq g \leq m-1$.

INDEX TERMS Fault tolerance, connectivity, diagnosability, PMC model, MM^* model.

I. INTRODUCTION

For multiprocessor systems, they usually take interconnection networks as underlying topology. An interconnection network is usually modeled by a connected graph $G = (V, E)$, where vertices represent processors and edges represent communication links between processors. The connectivity $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose removal disconnects the graph G and the edge connectivity $\lambda(G)$ is defined as the minimum number of edges whose deletion disconnects the graph G. They are two important parameters to evaluate the reliability of a network. Xu [26] showed that the higher these parameters are, the reliable the network is. However, these parameters always underestimate the resilience of a network. To overcome the shortcoming, Esfahanian [2] introduced the concept of restricted connectivity, which is a parameter to evaluate the fault tolerance of the network in terms of vertex failure. Later, Latifi *et al*. [7], Oh and Choi [12] generalized the parameter to g-good neighbor connectivity $\kappa^g(G)$.

For a connected graph $G = (V, E)$, a subset $S \subseteq V(G)$ is called a g-good neighbor vertex cut of G if $G - F$ is disconnected and any vertex in $G-F$ has at least g neighbors in $G - F$. The g-good neighbor connectivity is the size of the minimum g-good neighbor vertex cut and denoted by $\kappa^g(G)$. There are some results about g-good neighbor connectivity $\kappa^g(G)$ of networks G, one can refer [17], [28]–[30].

In addition, as the processors may fail and create faults in the large multiprocessor system. Hence, node fault identification is also of great importance for the system. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t. The diagnosability $t(G)$ of a system G is the maximum value of t such that G is t-diagnosable [1], [3], [6].

To identify the faulty processors, some diagnosis models were proposed. One of which was introduced by Preparata *et al*. [14] in 1967 and it is called the PMC diagnosis model. The diagnosis of the system is achieved through two linked processors testing each other. Another is the MM^* diagnosis model, which was proposed by Maeng and Malek [11] in 1981. For the MM^* model, to diagnose the system, a node sends the same task to two of its neighbors and then compares these responses. In 2005, Lai *et al*. [6] introduced the restricted diagnosability of a system, which is called conditional diagnosability. They consider the situation that any faulty set cannot contain all neighbors of any vertex in the system. In 2012, Peng *et al*. [13] proposed a

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new measurement for fault diagnosis of the system, that is, the g-good-neighbor diagnosability. This kind of diagnosis requires that every fault-free node contains at least g fault-free neighbors and they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the *PMC* model in [13]. In addition, there are some results about the g-good-neighbor diagnosability of other networks. For example, Wang *et al*. [24] studied the 1-good-neighbor connectivity and diagnosability of Cayley graphs generated by complete graphs; Wang and Han studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the MM^* model in [21]; Yuan *et al*. [27] studied the g-good-neighbor diagnosability of the k-ary n-cube under the *PMC* model and MM^* model; Wang *et al.* [22] studied the 1-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the *PMC* model and MM^* model; Wang *et al.* [23] studied the 2-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the *PMC* model and MM^* model; Xu *et al.* [25] studied the reliability of complete cubic networks under the condition of g-good-neighbor and Zhou *et al.* [32] studied the conditional fault diagnosis of hierarchical hypercubes etc..

In this paper, we focus on the g-good neighbor connectivity $\kappa^g(HHC_n)$ and the g-good neighbor diagnosability $t_g(HHC_n)$ for the n-dimensional hierarchical hypercube network HHC_n for $1 \le g \le m-1$ and $m \ge 2$, where $n = 2^m + m$. We show that $\kappa^g(HHC_n) = 2^g(m+1-g)$ for $1 \le g \le m-1$. In addition, we show that $t_g(HHC_n) = 2^g(m+2-g) - 1$ under the *PMC* model and MM^* model for $1 \le g \le m-1$.

II. PRELIMINARIES

In this section, we will introduce some definitions and notations needed for our discussion.

Let $G = (V, E)$ be a non-complete undirected graph, *the degree* of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of edges incident with v. The *minimum degree* of a vertex v in G is denoted by $\delta(G)$. For any subset $F \subseteq V$, the notation $G - F$ denotes a graph obtained by removing all vertices in F from G and deleting those edges with at least one end vertex in F, and the notation $V(G) \setminus F$ denotes deleting the vertex set F from $V(G)$. *The neighborhoods* of the vertex v in G is denoted by $N_G(v)$. Let $S \subset V(G)$, we use $N_G(S)$ to denote the vertex set $\bigcup_{v \in S} N_G(v) \setminus S$. Let $N_G[S] = N_G(S) \bigcup S$. If for any vertex $v \in V(G)$, $d_G(v) = k$, then the graph is called *k-regular*. Let $F_1, F_2 \subseteq V(G)$, $F_1 \triangle F_2 = (F_1 \bigcup F_2) \setminus (F_1 \bigcap F_2)$. The subgraph induced by $V \subseteq V(G)$, denoted by $G[V]$, is a graph whose vertex set is V and the edge set is the set of all the edges of G with both ends in V. A faulty set $F \subset V(G)$ is called a *g-good-neighbor faulty set* if for any vertex $v \in V(G) \setminus F$, $|N(v) \bigcap (V \setminus F)| \ge g$.

Let $[n] = \{1, 2, 3, \ldots, n\}$. The hypercube is one of the most fundamental interconnection networks. An n-dimensional hypercube, shortly *n-cube*, is an undirected graph $Q_n = (V, E)$ with $|V| = 2^n$ and $|E| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is an edge between two vertices whenever their binary string representation differs in only one bit position. Let m be an integer and $m = \sum_{i=0}^{s} 2^{t_i}$ be the decomposition of m such that $t_0 = [\log_2 m]$ and $t_i = [\log_2(m - \sum_{r=0}^{i-1} 2^{t_r})]$ for $i \ge 1$. Let X be a vertex set of Q_n and $|X| = m$. We denote by $\frac{ex_m}{2}$ the maximum size (the number of edges) of the subgraph (of Q_n) induced by m vertices, i.e., $ex_m = \max\{2|E(Q_n[X])| : X \subset V(Q_n)$ and $|X| = m\}$, is the maximum sum of the degrees of the subgraph (of Q_n) induced by m vertices.

For the hypercube network Q_n, it suffers from a practical limitation: as n increases, it becomes more difficult to design and fabricate the nodes of a Q_n network because of the large fanout. To remove the limitation, the cube-connected cycles (*CCC* for short) network [15] was designed as a substitute for the hypercube network. The node degree of a *CCC* network is restricted to three. However, this restriction degrades the performance of a *CCC* network at the same time. For example, a *CCC* network has a greater diameter than a hypercube network with the same number of nodes. Taking both the practical limitation and the performance into account, the hierarchical hypercube (HHC_n for short) network [8]–[10] was proposed as a compromise between the hypercube network and the *CCC* network. An HHC_n network, which has a two-level structure, takes hypercubes as basic modules and connects them in a hypercube manner. An HHC_n network has a logarithmic diameter, which is the same as a hypercube network. Since the topology of an HHC_n network is closely related to the topology of a hypercube network, it inherits some favorable properties from the latter.

An *n-dimensional hierarchical hypercube network* HHC_n, where $n = 2^m + m$ and $m \ge 1$ is an integer, can be obtained by replacing each vertex, say P, of Q_{2^m} with one Q_m, where each vertex of Q_m is uniquely connected to an adjacent vertex of P. Each vertex of HHC_n can be identified with a two-tuple (S, P), where $S = s_{n-m-1}s_{n-m-2}\ldots s_0$ is a binary sequence of length $n - m$ telling which Q_m the vertex is located in and $P = p_{m-1}p_{m-2}\ldots p_0$ is a binary sequence of length m giving the address of the vertex in the located Q_m.

For a binary sequence $X = x_{n-1}x_{n-2}\ldots x_0$, we let $X^l = x_{n-1}\ldots\overline{x_l}\ldots x_0$ and $dec(X)$ be the decimal value of X, where $0 \le l \le n-1$ and n denotes the length of X. Following, the definition of the n-dimensional hierarchical hypercube network HHC_n will be introduced.

Definition 1: An n-dimensional hierarchical hypercube network HHC_n with vertex set $\{(S, P) | S = s_{n-m-1} s_{n-m-2}\ldots s_0$ and $P = p_{m-1}p_{m-2}\ldots p_0$ are two binary sequences of lengths $n - m$ and m, respectively$\}$, where $n = 2^m + m$ and $m \ge 1$. A vertex (S, P) in HHC_n is linked to

(1) (S, P^l) for all $0 \le l \le m - 1$ or

(2) $(S^{(dec(P))}, P)$.

By the definition of the hierarchical hypercube network HHC_n, the edges defined by (1) are referred to as internal edges, and those defined by (2) are referred to as external edges. Each internal edge is contained in a Q_m, and each external edge connects two distinct Q_m's. Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n, then any vertex of H_i has a neighbor outside H_i, which is called the outside neighbor, where $i \in [2^{2^m}]$. The HHC_6 is shown in Fig.1, where $m = 2$.

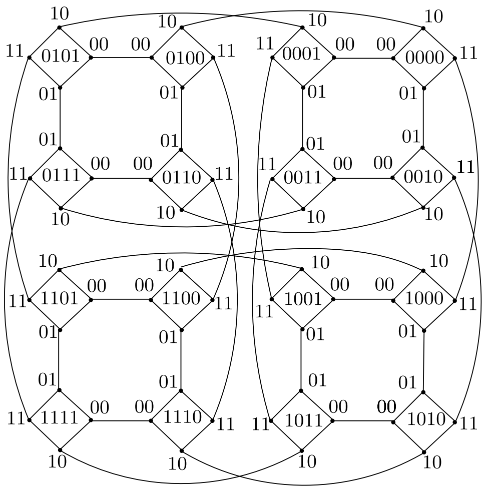

FIGURE 1. Illustration of the 6-dimensional hierarchical hypercube network HHC_6.

There are some results about hierarchical hypercube network, one can refer [8]–[10], [18], [19] etc. for the detail.

The paper is organized as follows. In section 3, the g-good-neighbor connectivity of HHC_n is determined. In section 4, the g-good-neighbor diagnosability of HHC_n under the *PMC* model is determined. In section 5, the g-good-neighbor diagnosability of HHC_n under the MM^* model is determined. In section 6, the paper is concluded.

III. THE g-GOOD NEIGHBOUR CONNECTIVITY OF HHC_n

The following results about the hierarchical hypercube networks HHC_n are useful.

Lemma 2 [8]–[10]: Let HHC_n be the n-dimensional hierarchical hypercube network and $H_1, H_2, \ldots, H_{2^{2^m}}$ be the 2^{2^m} m-cubes of HHC_n, where $n = 2^m + m$ and $m \geq 1$. Then the following results hold.

(1) HHC_n has 2^{2^m+m} vertices and it is $(m+1)$-regular.
(2) HHC_n has the vertex connectivity of $m+1$.
(3) HHC_n is a bipartite graph.
(4) Any vertex of H_i has exactly one outside neighbor and the outside neighbors of vertices in H_i belong to different copies of Q_m.
(5) There is at most one cross edge between H_i and H_j for $i \neq j$ and $i, j \in [2^{2^m}]$.

Observation 3: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. By contracting each of the m-cube of HHC_n as a vertex, then the resulting graph is isomorphic to a 2^m-cube Q_{2^m}.

As HHC_n is an invariant of the n-cube Q_n, some properties on the n-cube Q_n are very useful for the proofs of the main results.

Lemma 4 [31]: Any two vertices in $V(Q_n)$ have exactly two common neighbors for $n \geq 2$ if they have any.

Lemma 5 [4]: Let X be a vertex set of Q_n with size m. Then $ex_m = \sum_{i=0}^{s} t_i 2^{t_i} + \sum_{i=0}^{s} 2 \cdot i \cdot 2^{t_i}$, where $ex_m = \max\{2|E(Q_n[X])| : X \subset V(Q_n) \text{ and } |X| = m\}$.

Lemma 6 [20]: If X is a subgraph of Q_n and $\delta(X) \geq g$, then $|X| \geq 2^g$.

Lemma 7 [5]: If X is a subgraph of Q_n and $\delta(X) \geq g$, then $|X| + |N_{Q_n}(X)| \geq 2^g(n + 1 - g)$.

Lemma 8 [20]: $\kappa^g(Q_n) = 2^g(n - g)$ for $0 \leq g \leq n - 1$.

By Lemma 2(2), we obtain that $\kappa^0(HHC_n) = \kappa(HHC_n) = m+1$. Following, we determine the g-good neighbor connectivity of HHC_n for $1 \leq g \leq m - 1$ to avoid duplication.

Lemma 9: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then $\kappa^g(HHC_n) \leq 2^g(m + 1 - g)$ for $1 \leq g \leq m - 1$.

Proof: For convenience, let $x_1Q_m, x_2Q_m, \ldots, x_{2^{2^m}}Q_m$ be the m-cubes of HHC_n instead of $H_1, H_2, \ldots, H_{2^{2^m}}$, where x_i is a $n - m$ binary string for each $i \in [2^{2^m}]$. For a fixed g with $1 \leq g \leq m-1$, let x_1Q_g be a subgraph of x_1Q_m induced by the vertex set $\{(S, P)|S = x_1 \text{ and } P = p_{m-1}p_{m-2}\ldots p_0$, where $p_i = 0$ for each $0 \leq i \leq m-g-1\}$. Let $S = N_{HHC_n}(V(x_1Q_g))$, see Fig. 2, then $HHC_n - S$ is disconnected. By the choice of x_1Q_g, we obtain that $|N_{HHC_n}(V(x_1Q_g))| = 2^g(m - g) + 2^g = 2^g(m + 1 - g)$.

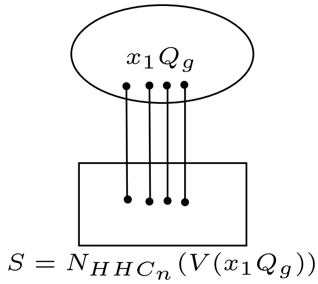

FIGURE 2. Illustration of the proof of Lemma 9.

Next, we show that S is a g-good neighbor vertex cut, that is, any vertex of $HHC_n - S$ has at least g neighbors. By Lemma 2(5), there is at most one cross edge between x_iQ_m and x_jQ_m for different i, j and $i, j \in [2^{2^m}]$. Thus, $|S \cap V(x_jQ_m)| \leq 1$ for $j \neq 1$.

For any $j \neq 1$, let $z \in V(x_jQ_m)$. As $|S \cap V(x_jQ_m)| \leq 1$ for $j \neq 1$, then z has at most one neighbor in $S \cap V(x_jQ_m)$. By Lemma 2(4), z has exactly one outside neighbor in HHC_n, say z', and it is possibly $z' \in S \cap V(x_lQ_m)$, where $l \in [2^{2^m}] \setminus \{1, j\}$. Thus, z has at least $m + 1 - 2 \geq g$ neighbors in $HHC_n - S$.

Let $S_1 = V(x_1Q_m) \cap S$ and $T_1 = x_1Q_m - (V(x_1Q_g) \cup S_1)$. Following, we need to show that any vertex in $x_1Q_m - S_1$ has at least g neighbors in $HHC_n - S$. Obviously, any vertex in x_1Q_g has at least g neighbors. If $T_1 = \emptyset$, then we are done.

Following, let $T_1 \neq \emptyset$ and let $w \in V(T_1)$, then $g \leq m-2$. If w has no neighbor in S_1, then it has at least $m \geq g$ neighbors in $HHC_n - S$ and we are done. Suppose w has one neighbor in S_1. By the definition of x_1Q_g, there is exactly one 1 for p_is of the vertices in S_1, where $S_1 = \{(S, P)|S = x_1$ and $P = p_{m-1}p_{m-2}\ldots p_0\}$ and $0 \leq i \leq m-g-1$. Let $T_1 = \{(S, P)|S = x_1$ and $P = p_{m-1}p_{m-2}\ldots p_0\}$, then there are exactly two 1s for p_is for $0 \leq i \leq m-g-1$, which implies that w has at most two neighbors in S_1. Thus, w has at least $m-2 \geq g$ neighbors in $HHC_n - S$.

Hence, for any vertex u of $HHC_n - S$, it has at least g neighbors in $HHC_n - S$. That is, S is an g-good neighbor vertex cut of HHC_n. Thus, $\kappa^g(HHC_n) \leq |S| = 2^g(m+1-g)$ for $1 \leq g \leq m-1$.

Theorem 10: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then $\kappa^g(HHC_n) = 2^g(m+1-g)$ for $1 \leq g \leq m-1$.

Proof: By Lemma 9, we just need to show $\kappa^g(HHC_n) \geq 2^g(m+1-g)$.

Let F be the minimum g-good neighbor vertex cut of HHC_n, X be the minimum connected component of $HHC_n - F$ and let $Y = V(HHC_n) \setminus (F \cup X)$. Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n, and let $X_i = X \cap V(H_i)$, $Y_i = Y \cap V(H_i)$ and $F_i = F \cap V(H_i)$. Let $J_X = \{i \in [2^{2^m}] | X_i \neq \emptyset\}$, $J_Y = \{i \in [2^{2^m}] | Y_i \neq \emptyset\}$ and $J_0 = J_X \cap J_Y$.

Clearly, if $J_0 \neq \emptyset$, then $X_i \neq \emptyset$ and $Y_i \neq \emptyset$ for each $i \in J_0$, which means that F_i is a vertex cut of H_i. As F is a g-good neighbor vertex cut of HHC_n, any vertex in $X_i \cup Y_i$ has at least g neighbors in $HHC_n - F$. Note that any vertex of HHC_n has exactly one outside neighbor, then F_i is a $(g-1)$-good neighbor vertex cut of H_i. As H_i is an m-cube, by Lemma 8, we have $|F_i| \geq 2^{g-1}(m+1-g)$ for each $i \in J_0$. By Lemma 6, we obtain that $|X_i| \geq 2^{g-1}$ and $|Y_i| \geq 2^{g-1}$ for each $i \in J_0$.

Following, we show that $|F| \geq 2^g(m+1-g)$ for $1 \leq g \leq m-1$ according to $|J_0|$ and the following two cases are considered.

Case 1. $|J_0| \geq 2$

As $|J_0| \geq 2$, then $|F| \geq \sum_{i \in J_0} |F_i| \geq 2 \cdot 2^{g-1}(m+1-g) = 2^g(m+1-g)$ and the result holds.

Case 2. $0 \leq |J_0| \leq 1$.

Let $|J_X \setminus J_0| = a$, $|J_Y \setminus J_0| = b$ and $|[2^{2^m}] \setminus (J_X \cup J_Y)| = c$.

If $c \geq 1$, then there exists some i such that $V(H_i) \subseteq F$ as F is a vertex cut. Then $|F| \geq 2^m \geq 2^g(m+1-g)$ for $1 \leq g \leq m-1$ and the result holds.

If $c = 0$, then $a + b + |J_0| = 2^{2^m}$. If $a \geq 1$ and $b \geq 1$, then for $j_1 \in J_X \setminus J_0, j_2 \in J_Y \setminus J_0$, if there is one cross edge between H_{j_1} and H_{j_2}, one of its end vertex must be in F for the cross edge as F is a vertex cut. Thus, by Observation 1 and Lemma 5, we obtain that

$$\sum_{i \in (J_X \cup J_Y) \setminus J_0} |F_i| \geq \sum_{i \in J_X \setminus J_0,\, j \in J_Y \setminus J_0} |E(H_i, H_j)| \geq a \cdot 2^m - ex_a - \sum_{i \in J_X \setminus J_0,\, j \in J_0} |E(H_i, H_j)|.$$

Next, we consider the following two subcases for $|J_0| = 0$ or $|J_0| = 1$.

Subcase 2.1. $|J_0| = 0$

In this case, it implies that $a \geq 1$. If $a \geq 2$, we obtain that $|F| \geq \sum_{i \in J_X \cup J_Y} |F_i| \geq a \cdot 2^m - ex_a \geq 2^g(m+1-g)$ for $1 \leq g \leq m-1$.

If $a = 1$, without loss of generality, let $J_X = \{1\}$. Then $X_1 \subseteq V(H_1)$, all vertices in $V(H_1) \setminus X_1$ and all outside neighbors of vertices in X_1 are contained in F. Recall that any vertex of X_1 has exactly one outside neighbor, then $|F| \geq |V(H_1) \setminus X_1| + |X_1| = 2^m \geq 2^g(m+1-g)$ for $1 \leq g \leq m-1$ and the result holds.

Subcase 2.2. $|J_0| = 1$

In this case, $a \geq 0$ and $b = 2^{2^m} - a - 1$. Without loss of generality, let $J_0 = \{1\}$. If $a \geq 1$, we obtain that

$$|F| \geq |F_1| + \sum_{i \in (J_X \cup J_Y) \setminus J_0} |F_i| \geq 2^{g-1}(m+1-g) + a \cdot 2^m - ex_a - 1 \geq 2^g(m+1-g).$$

If $a = 0$, then $X \subseteq H_1$ and $N_{HHC_n}(X) \subseteq F$. Note that $\delta(H_1[X]) \geq g$ and H_1 is an m-cube, by Lemma 7, we have $|F| \geq |X| + |N_{H_1}(X)| \geq 2^g(m+1-g)$ and the result holds.

IV. THE g-GOOD NEIGHBOUR DIAGNOSABILITY OF HHC_n UNDER THE *PMC* MODEL

First, we introduce the PMC model. Under the PMC model, a self-diagnosable system is often modeled as a directed graph $T = (V, L)$, where V represents the same set of vertices as in G and $L = \{(u, v)|(u, v) \in E(G)$ and the vertex u tests the vertex $v\}$. The outcome of a test (u, v) is denoted by $r(u, v)$. If a tester u evaluates a tester v as faulty, we have $r(u, v) = 1$; otherwise, $r(u, v) = 0$. If the tester u is faulty, then the testing result is unreliable. For this reason, some assignments are made: if $r(u, v) = 1$, at least one member of $\{u, v\}$ is faulty; otherwise, if $r(u, v) = 0$, both of u and v are fault-free.

The collection of all test results, defined as a function Ω: $L \rightarrow \{0, 1\}$, is called a syndrome. Suppose that a test syndrome σ is on the multiprocessor system $G = (V, E)$. A g-good-neighbor faulty set F is consistent with respect to σ under the *PMC* model if the following conditions (1) and (2) hold.

(1) $r(u, v) = 0$ for $u \in V \setminus F$ and $v \in V \setminus F$;

(2) $r(u, v) = 1$ for $u \in V \setminus F$ and $v \in F$.

A g-good-neighbor faulty set F may produce different syndromes because a faulty tester u may return an unreliable result. For each subset $F \subseteq V$, which is the set of all faulty vertices, let $\Omega(F)$ represent the set of syndromes that can be produced. Two distinct g-good-neighbor faulty subsets F_1 and F_2 of V are distinguishable if $\Omega(F1) \cap \Omega(F2) = \emptyset$; otherwise, F_1 and F_2 are said to be indistinguishable. That is, when F_1 and F_2 are distinguishable, for each syndrome σ in $\Omega(F_1) \cup \Omega(F_2)$, exactly one of F_1 and F_2 is the unique g-good-neighbor faulty set that is consistent with respect to σ.

Definition 11 [13]: A system $G = (V, E)$ is g-good-neighbor t-diagnosable if and only if for any two distinct g-good-neighbor faulty subsets F_1 and F_2 of V such that $|F_1| \leq t$ and $|F_2| \leq t$, the sets F_1 and F_2 are distinguishable.

Lemma 12 [14]: For any two distinct subsets F_1 and F_2 in a system $G = (V, E)$, the sets F_1 and F_2 are distinguishable if and only if there exists a vertex $u \in V \setminus (F_1 \bigcup F_2)$ and $v \in F_1 \Delta F_2$ such that $(u, v) \in E$.

By the proof of Theorem 10, the following result holds.

Lemma 13: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, and let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n, $X \subseteq V(H_i)$ and $HHC_n[X] \cong Q_g$. Then $|N_{HHC_n}(X)| = 2^g(m + 1 - g)$ and $\delta(HHC_n - N_{HHC_n}[X]) \geq g$ for $1 \leq g \leq m - 1$.

By Lemma 9 and the definition of the hierarchical hypercube network HHC_n, the following lemma holds.

Lemma 14: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$ and let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. Then for any two distinct vertices u and v of the hierarchical hypercube network HHC_n, one of the following conditions hold.

(1) If u and v belong to the same m cube and $N_{HHC_n}(u) \cap N_{HHC_n}(v) \neq \emptyset$, then $|N_{HHC_n}(u) \cap N_{HHC_n}(v)| = 2$.
(2) If u and v belong to different m cubes and $N_{HHC_n}(u) \cap N_{HHC_n}(v) \neq \emptyset$, then $|N_{HHC_n}(u) \cap N_{HHC_n}(v)| = 1$.
(3) If u and v belong to different m cubes and $N_{HHC_n}(u) \cap N_{HHC_n}(v) = \emptyset$, then $|N_{HHC_n}(u) \cap N_{HHC_n}(v)| = 0$.

Lemma 15: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$. If X is a subgraph of HHC_n and $\delta(X) \geq g$, then $|X| \geq 2^g$ for $1 \leq g \leq m - 1$.

Proof: Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. Let $I = \{i | V(X) \cap V(H_i) \neq \emptyset\}$ for $i \in [2^{2^m}]$. To prove the result, the following cases are considered.

Case 1. $|I| = 1$

Without loss of generality, let $V(X) \cap V(H_1) \neq \emptyset$. By Lemma 6, $|X| \geq 2^g$.

Case 2. $|I| \geq 2$

As $V(X) \cap V(H_i) \neq \emptyset$ for each $i \in I$. By the definition of HHC_n, any vertex of HHC_n has exactly one outside neighbor. Thus, $d_{H_i[V(X) \cap V(H_i)]}(u) \geq g - 1$ for any vertex $u \in V(X) \cap V(H_i)$ and $i \in I$. By Lemma 6, $|V(X) \cap V(H_i)| \geq 2^{g-1}$ for each $i \in I$. Thus, $|X| = | \cup_{i \in I} (V(X) \cap V(H_i))| \geq 2^{g-1} + 2^{g-1} = 2^g$. □

Following, we will determine the g-good neighbor diagnosability of HHC_n under the *PMC* model.

Theorem 16: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the *PMC* model satisfies $t_g(HHC_n) \leq 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

Proof: Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. Let $X \subseteq V(H_1)$ and $HHC_n[X] \cong Q_g$. Let $F_1 = N_{HHC_n}(X)$ and $F_2 = N_{HHC_n}[X]$. By Lemma 13, both F_1 and F_2 are g-good neighbor faulty sets. As $|N_{HHC_n}(X)| = 2^g(m + 1 - g)$, we have

$$|F_1| = 2^g(m + 1 - g) \leq 2^g(m + 2 - g)$$

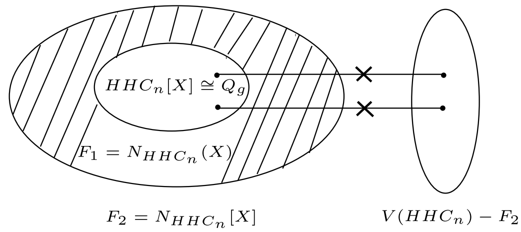

FIGURE 3. Illustration of the proof of Theorem 16 and Theorem 20.

and

$$\begin{aligned} |F_2| &= |F_1| + |X| \\ &= 2^g(m + 1 - g) + 2^g \\ &\leq 2^g(m + 2 - g) \end{aligned}$$

As $F_1 \Delta F_2 = X$, there is no cross edge between $HHC_n - (F_1 \cup F_2)$ and $F_1 \Delta F_2$ (see Fig. 3). By Lemma 12, the g-good neighbor faulty sets of F_1 and F_2 are indistinguishable. By Definition 11, then hierarchical hypercube network HHC_n is not g-good neighbor $2^g(m + 2 - g)$-diagnosable under the *PMC* model. Thus, $t_g(HHC_n) \leq 2^g(m + 2 - g) - 1$. □

Theorem 17: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the *PMC* model satisfies $t_g(HHC_n) \geq 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

Proof: Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. To prove the result, we just need to show that for any two distinct g-good neighbor faulty subsets F_1 and F_2 of HHC_n such that $|F_1| \leq 2^g(m+2-g)-1$ and $|F_2| \leq 2^g(m+2-g)-1$, the sets F_1 and F_2 are distinguishable. By Lemma 12, we need to show that there is an edge between $V(HHC_n) \setminus (F_1 \cup F_2)$ and $F_1 \bigtriangleup F_2$.

We prove the result by contradiction. That is, there are two distinct g-good neighbor faulty subsets F_1 and F_2 of HHC_n such that $|F_1| \leq 2^g(m+2-g)-1$ and $|F_2| \leq 2^g(m+2-g)-1$, but they are indistinguishable.

First, we show that $V(HHC_n) \neq F_1 \cup F_2$.

Suppose to the contrary, that is, $V(HHC_n) = F_1 \cup F_2$. For $1 \leq g \leq m - 1$, we have

$$\begin{aligned} |F_1 \cup F_2| &= |F_1| + |F_2| - |F_1 \cap F_2| \\ &\leq 2[2^g(m + 2 - g) - 1] \end{aligned}$$

Let $f(g) = 2^g(m+2-g)$, then $f'(g) = 2^g[(m+2-g) \cdot ln2 - 1]$. If $f'(g) = 0$, then $g = m + 2 - \frac{1}{ln2} > m$ and $f'(g) > 0$ for $1 \leq g \leq m - 1$. Thus, $f(g)$ is monotonically increasing for $1 \leq g \leq m - 1$. Thus,

$$\begin{aligned} f(g)_{max} &= f(m - 1) \\ &= = 2^{m-1}[m + 2 - (m - 1)] \\ &= 3 \cdot 2^{m-1}. \end{aligned}$$

Thus, we have

$$\begin{aligned} |F_1 \cup F_2| &\leq 2(3 \cdot 2^{m-1} - 1) \\ &= 3 \cdot 2^m - 2. \end{aligned}$$

As $|V(HHC_n)| = 2^{2^m+m}$. Obviously, $2^{2^m+m} > 3 \cdot 2^m - 2$ for $m \geq 2$, which is a contradiction.

Second, we prove the main result. Without loss of generality, let $F_2 \setminus F_1 \neq \emptyset$. As F_1 is a g-good neighbor faulty set, then for any vertex u of $HHC_n - F_1$, $d_{HHC_n[V(HHC_n)\setminus F_1]}(u) \geq g$. As there is no cross edge between $HHC_n - (F_1 \cup F_2)$ and $F_1 \triangle F_2$, then $d_{HHC_n[V(HHC_n)\setminus F_1]}(u) = d_{HHC_n[F_2\setminus F_1]}(u) \geq g$. Thus, for any vertex of $u \in F_2 \setminus F_1$, $d_{HHC_n[F_2\setminus F_1]}(u) \geq g$. By Lemma 15, $|F_2 \setminus F_1| \geq 2^g$. As F_1 and F_2 are both g-good neighbor faulty sets, $F_1 \cap F_2$ is also a g-good neighbor faulty set. In addition, as there is no cross edge between $HHC_n - (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a g-good neighbor faulty cut. By Theorem 10, $|F_1 \cap F_2| \geq 2^g(m + 1 - g)$. Thus

$$\begin{aligned} |F_2| &= |F_2 \setminus F_1| + |F_1 \cap F_2| \\ &\geq 2^g + 2^g(m + 1 - g) \\ &= 2^g(m + 2 - g) \end{aligned}$$

By the hypothesis, $|F_2| \leq 2^g(m + 2 - g) - 1$, which is a contradiction. Thus, $t_g(HHC_n) \geq 2^g(m+2-g)-1$ for $m \geq 1$ and $1 \leq g \leq m - 1$. □

By Theorem 16 and Theorem 17, the following theorem can be obtained.

Theorem 18: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the *PMC* model is $t_g(HHC_n) = 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

V. THE g-GOOD NEIGHBOUR DIAGNOSABILITY OF HHC_n UNDER THE MM^* MODEL

In the MM model [11], [27], to diagnose a system, a vertex sends the same task to two of its neighbors, and then compares their responses. To be consistent with the MM model, we have the following assumptions. In this paper, for consistency with the MM^* model, we have the following assumptions.

(1) All faults are permanent.

(2) A faulty processor produces incorrect outputs for each of its given testing tasks.

(3) The output of a comparison performed by a faulty processor is unreliable.

(4) Two faulty processors given the same input and task do not produce the same output.

The comparison scheme of a system G is modeled as a multigraph, denoted by $M(V(G), L)$, where L is the labeled edge set. A labeled edge $(u, v)_w \in L$ represents a comparison in which two vertices u and v are compared by a vertex w, which implies $uw, vw \in E(G)$. The collection of all comparison results in $M(V(G), L)$ is called the syndrome, denoted by σ^*, of the diagnosis. If the comparison $(u, v)_w$ disagrees, then $\sigma^*((u, v)_w) = 1$; otherwise, $\sigma^*((u, v)_w) = 0$. Hence, a syndrome is a function from L to $\{0, 1\}$. The MM^* model is a special case of the *MM* model. In the MM^* model, all comparisons of G are in the comparison scheme of G, i.e., if $uw, vw \in E(G)$, then $(u, v)_w \in L$.

Lemma 19 [16]: Let $G = (V, E)$ be a system under the MM^* model. Two distinct subsets F_1 and F_2 of V are distinguishable if and only if at least one of the following conditions holds:

(1) There are two vertices $u, w \in V \setminus (F_1 \bigcup F_2)$ and there is a vertex $v \in F_1 \Delta F_2$ such that $uw \in E$ and $vw \in E$.

(2) There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \bigcup F_2)$ such that $uw \in E$ and $vw \in E$.

(3) There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \bigcup F_2)$ such that $uw \in E$ and $vw \in E$.

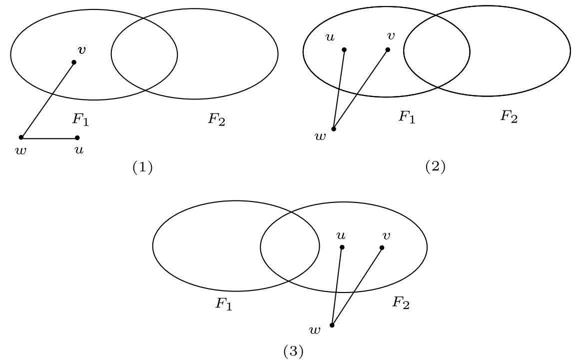

FIGURE 4. Illustration of distinguishable sets F_1 and F_2 under the MM^* model.

Theorem 20: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the MM^* model satisfies $t_g(HHC_n) \leq 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

Proof: Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the m-cubes of HHC_n. Let $X \subseteq V(H_1)$ and $HHC_n[X] \cong Q_g$. Let $F_1 = N_{HHC_n}(X)$ and $F_2 = N_{HHC_n}[X]$. By Lemma 13, both F_1 and F_2 are g-good neighbor faulty sets. As $|N_{HHC_n}(X)| = 2^g(m + 1 - g)$, we have

$$|F_1| = 2^g(m + 1 - g) \leq 2^g(m + 2 - g)$$

and

$$\begin{aligned} |F_2| &= |F_1| + |X| \\ &= 2^g(m + 1 - g) + 2^g \\ &\leq 2^g(m + 2 - g) \end{aligned}$$

As $F_1 \Delta F_2 = X$, there is no cross edge between $V(HHC_n) \setminus (F_1 \cup F_2)$ and $F_1 \Delta F_2$ (see Fig. 3). By Lemma 19, the g-good neighbor faulty sets of F_1 and F_2 are indistinguishable. By Definition 11, the hierarchical hypercube network HHC_n is not g-good neighbor $2^g(m + 2 - g)$-diagnosable under the MM^* model. Thus, $t_g(HHC_n) \leq 2^g(m + 2 - g) - 1$. □

Theorem 21: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the MM^* model satisfies $t_g(HHC_n) \geq 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

Proof: Let $H_1, H_2, \ldots, H_{2^{2^m}}$ be the 2^{2^m} m-cubes of HHC_n. To Prove the result, we just need to show that for any two distinct g-good neighbor faulty subsets F_1 and F_2 of HHC_n such that $|F_1| \leq 2^g(m + 2 - g) - 1$ and $|F_2| \leq 2^g(m + 2 - g) - 1$, the sets F_1 and F_2 are distinguishable.

We prove the result by contradiction. That is, there are two distinct g-good neighbor faulty subsets F_1 and F_2 of HHC_n with $|F_1| \leq 2^g(m+2-g)-1$ and $|F_2| \leq 2^g(m+2-g)-1$, but they are indistinguishable. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. To obtain a contradiction, the following fact and claim are useful.

Fact 1. $V(HHC_n) \neq F_1 \bigcup F_2$.

With a similar proof as Theorem 17, we obtain that $V(HHC_n) \neq F_1 \bigcup F_2$.

Claim 1. There is no isolated vertex in $HHC_n - (F_1 \bigcup F_2)$.

Proof of Claim 1. To prove the result, the following two cases are considered.

Case 1. $2 \leq g \leq m-1$.

Suppose to the contrary, that is, $HHC_n - (F_1 \bigcup F_2)$ has at least one isolated vertex, say u. Obviously, we have $d_{HHC_n-(F_1\bigcup F_2)}(u) = 0$ and $d_{HHC_n[F_2\setminus F_1]}(u) = d_{HHC_n[V(HHC_n)\setminus F_1]}(u)$. As F_1 is a g-good neighbor faulty set, then $d_{HHC_n[V(HHC_n)\setminus F_1]}(u) \geq g$ and $d_{HHC_n[F_2\setminus F_1]}(u) = d_{HHC_n[V(HHC_n)\setminus F_1]}(u) \geq g \geq 2$, which satisfies condition (3) of Lemma 19. Thus, the g-good neighbor faulty sets F_1 and F_2 are distinguishable, a contradiction.

Case 2. $g = 1$ and $m \geq 2$.

Suppose to the contrary. That is, $HHC_n - (F_1 \bigcup F_2)$ has at least one isolated vertex, say w. Let W be the set of all isolated vertices in $HHC_n - (F_1 \bigcup F_2)$ and $H = HHC_n - (F_1 \bigcup F_2 \cup W)$.

If $F_1 \setminus F_2 = \emptyset$, then $F_1 \subseteq F_2$. As F_2 is a 1-good neighbor faulty set, $d_{HHC_n[V(HHC_n)\setminus F_2]}(w) \geq 1$. As $F_1 \subseteq F_2$, then $d_{HHC_n-(F_1\cup F_2)}(w) = d_{HHC_n[V(HHC_n)\setminus F_2]}(w) \geq g \geq 1$, which contradicts with the fact that w is an isolated vertex in $HHC_n - (F_1 \bigcup F_2)$.

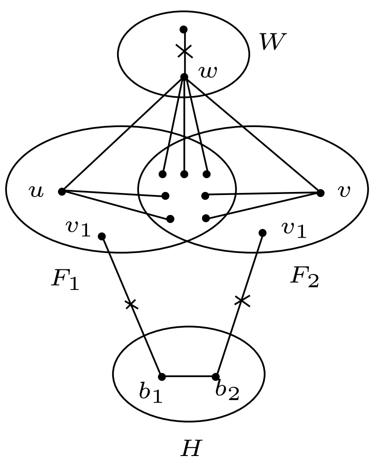

FIGURE 5. Illustration of the proof of Case 2 of Claim 1.

Now, suppose that $F_1 \setminus F_2 \neq \emptyset$ (see Fig. 5). Recall that w is an isolated vertex in $HHC_n - (F_1 \bigcup F_2)$. Obviously, we have $d_{HHC_n[V(HHC_n)\setminus(F_1\cup F_2)]}(w) = 0$ and $d_{HHC_n[F_1\setminus F_2]}(w) = d_{HHC_n[V(HHC_n)\setminus F_2]}(w)$. As F_2 is a 1-good neighbor faulty set, we have $d_{HHC_n[F_1\setminus F_2]}(w) = d_{HHC_n[V(HHC_n)\setminus F_2]}(w) \geq g = 1$.

If $d_{HHC_n[F_1\setminus F_2]}(w) \geq 2$, then it satisfies condition (2) of Lemma 19. Then the g-good neighbor faulty sets F_1 and F_2 are distinguishable, a contradiction.

Thus, $d_{HHC_n[F_1\setminus F_2]}(w) = 1$. Let $u \in F_1 \setminus F_2$ such that $uw \in E(HHC_n)$. Similarly, as F_1 is a 1-good neighbor faulty set, we have $d_{HHC_n[F_2\setminus F_1]}(w) = 1$. Let $v \in F_2 \setminus F_1$ such that $vw \in E(HHC_n)$. Thus, $d_{HHC_n[F_1\cap F_2]}(w) = (m+1-2)$ and $|F_1 \cap F_2| \geq (m+1)-2$. For $g = 1$, $|F_2| \leq 2^g(m+2-g)-1 = 2m+1$. Then we deduce that

$$\begin{aligned}\sum_{w\in W} |N_{HHC_n[F_1\cap F_2](w)}| &= |W|[(m+1)-2] \\ &\leq \sum_{v\in F_1\cap F_2} d_{HHC_n}(v) \\ &= |F_1 \cap F_2|(m+1) \\ &\leq (|F_2|-1)(m+1) \\ &= 2m(m+1)\end{aligned}$$

Thus, we have $|W| \leq 2m(m+1)/(m-1)$.

If $H = \emptyset$, then

$$\begin{aligned}|V(HHC_n)| &= 2^{2^m+m} \\ &= |F_1 \bigcup F_2| + |W| \\ &= |F_1| + |F_2| - |F_1 \cap F_2| + |W| \\ &\leq 2(2m+1) - (m+1-2) \\ &\quad + 2m(m+1)/(m-1) \\ &= 3m+3+2m(m+1)/(m-1)\end{aligned}$$

However, $2^{2^m+m} > 3m+3+2m(m+1)/(m-1)$ for $m \geq 2$, a contradiction. Thus, $H \neq \emptyset$.

For any vertex $b_1 \in H$, as H has no isolated vertex, then there exists some vertex $b_2 \in H$ such that $b_1b_2 \in E(HHC_n)$. If $b_1v_1 \in E(HHC_n)$ for some vertex $v_1 \in F_1 \triangle F_2$, then the 1-good neighbor faulty sets F_1 and F_2 are distinguishable by condition (1) of Lemma 19, which is a contradiction. Thus, $b_1v_1 \notin E(HHC_n)$ for any vertex $v_1 \in F_1 \triangle F_2$.

By the arbitrariness of v_1 and b_1, there is no edge between H and $F_1 \triangle F_2$.

As both F_1 and F_2 are 1-good neighbor faulty sets, $F_1 \cap F_2$ is a 1-good neighbor faulty set. As there is no edge between H and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a 1-good neighbor faulty cut. By Theorem 10, $|F_1 \cap F_2| \geq 2(m+1-1) = 2m$.

As $|F_1| \leq 2m+1, |F_2| \leq 2m+1, F_1 \setminus F_2 \neq \emptyset$ and $F_2 \setminus F_1 \neq \emptyset$, we have $|F_1 \setminus F_2| = 1$ and $|F_2 \setminus F_1| = 1$.

Let $F_1 \setminus F_2 = \{u\}$ and $F_2 \setminus F_1 = \{v\}$. By Lemma 14, u and v have at most two common neighbors. Thus, $|W| \leq 2$ as any vertex of W is adjacent to both u and v.

If $|W| = 2$, let $W = \{w_1, w_2\}$. By Lemma 14 and Lemma 2(3), no two vertices of u, v, w_1 and w_2 have a common neighbor in $F_1 \cap F_2$. Thus,

$$\begin{aligned}|F_1 \cap F_2| &\geq |N_{HHC_n}(u) \setminus \{w_1, w_2\}| + |N_{HHC_n}(v) \setminus \{w_1, w_2\}| \\ &\quad + |N_{HHC_n}(w_1) \setminus \{u, v\}| + |N_{HHC_n}(w_2) \setminus \{u, v\}| \\ &= 4(m+1-2) \\ &= 4m-4.\end{aligned}$$

Then we have

$$|F_2| = |F_1 \cap F_2| + |F_2 \setminus F_1|$$

$$\begin{aligned} &\geq 4m - 4 + 1 \\ &> 2m + 1 \\ &\geq |F_2| \quad \text{for } m \geq 2, \end{aligned}$$

which is a contradiction.

If $|W| = 1$, say $W = \{w\}$, then $uv \notin E(HHC_n)$ by Lemma 2(3). Then we have

$|N_{HHC_n[F_1 \cap F_2]}(u, v, w)| \geq |N_{HHC_n}(u) \setminus \{w\}| + |N_{HHC_n}(v) \setminus \{w\}| + |N_{HHC_n}(w) \setminus \{u, v\}| - |(N_{HHC_n}(u) \cap N_{HHC_n}(v)) \setminus \{w\}| - |N_{HHC_n}(u) \cap N_{HHC_n}(w)| - |N_{HHC_n}(v) \cap N_{HHC_n}(w)| + |N_{HHC_n}(u) \cap N_{HHC_n}(v) \cap N_{HHC_n}(w)| = 2m + (m + 1 - 2) - 1 + 0 = 3m - 2$.

Thus, $|F_1 \cap F_2| \geq |N_{HHC_n[F_1 \cap F_2]}(u, v, w)| \geq 3m - 2$ and

$$\begin{aligned} |F_2| &= |F_1 \cap F_2| + |F_2 \setminus F_1| \\ &\geq 3m - 2 + 1 \\ &> 2m + 1 \\ &\geq |F_2| \quad \text{for } m \geq 2, \end{aligned}$$

which is a contradiction.

The proof of Claim 1 is complete.

By Fact 1 and Claim 1, for any vertex $u \in HHC_n - (F_1 \cup F_2)$, there exists some vertex $v \in HHC_n - (F_1 \cup F_2)$ such that $uv \in E(HHC_n)$. If $uw \in E(HHC_n)$ for $w \in F_1 \triangle F_2$, it satisfies condition (3) of Lemma 19. Thus, the g-good neighbor faulty sets F_1 and F_2 are distinguishable, a contradiction. That is to say, $uw \notin E(HHC_n)$. By the arbitrariness of u, w, there is no edge between $HHC_n - (F_1 \cup F_2)$ and $F_1 \triangle F_2$, we have $d_{HHC_n[V(HHC_n) \setminus F_1]}(w) = d_{HHC_n[F_2 \setminus F_1]}(w) \geq g$. Thus, by Lemma 15, $|F_2 \setminus F_1| \geq 2^g$.

As both F_1 and F_2 are g-good neighbor faulty sets, $F_1 \cap F_2$ is a g-good neighbor faulty set. In addition, as there is no edge between $HHC_n - (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a g-good neighbor faulty cut. By Theorem 10, $|F_1 \cap F_2| \geq 2^g(m+1-g)$. Thus,

$$\begin{aligned} |F_2| &= |F_1 \cap F_2| + |F_2 \setminus F_1| \\ &\geq 2^g(m + 1 - g) + 2^g \\ &= 2^g(m + 2 - g) \end{aligned}$$

which contradicts with $|F_2| \leq 2^g(m + 2 - g) - 1$.

The proof of the theorem is complete. □

By Theorem 20 and Theorem 21, the following theorem can be obtained.

Theorem 22: Let HHC_n be the n-dimensional hierarchical hypercube network for $n = 2^m + m$ and $m \geq 2$, then the g-good neighbor diagnosability of HHC_n under the MM^* model is $t_g(HHC_n) = 2^g(m + 2 - g) - 1$ for $1 \leq g \leq m - 1$.

VI. CONCLUDING REMARKS

As the hierarchical hypercube network HHC_n has some attractive properties to design interconnection networks. In this paper, we focus on the g-good neighbor connectivity $\kappa^g(HHC_n)$ and the g-good neighbor diagnosability $t_g(HHC_n)$ for the n-dimensional hierarchical hypercube network HHC_n for $1 \leq g \leq m - 1$, where $n = 2^m + m$ and $m \geq 2$. We show that $\kappa^g(HHC_n) = 2^g(m+1-g)$ for $1 \leq g \leq m-1$. In addition, we show that $t_g(HHC_n) = 2^g(m + 2 - g) - 1$ under the *PMC* model and MM^* model for $1 \leq g \leq m-1$. In the future work, we would like to study g-good neighbor connectivity and the g-good neighbor diagnosability of the balanced hypercubes BH_n under the *PMC* model and MM^* model. This problem could be meaningful and worthy of further investigation.

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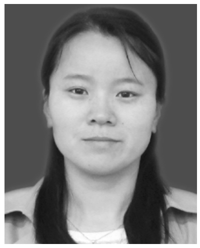

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