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Periodic Solutions to Stochastic Reaction-Diffusion Neural Networks With S-Type Distributed Delays

QI YAO[1], YANGFAN WANG[2], AND LINSHAN WANG[2], (Member, IEEE)
[1]School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China
[2]College of Marine Life Science, Ocean University of China, Qingdao 266100, China

Corresponding authors: Yangfan Wang (yfwang@ouc.edu.cn) and Linshan Wang (wangls@ouc.edu.cn)

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ABSTRACT In this paper, the existence and stability of mild periodic solutions to the stochastic reaction–diffusion neural networks (SRDNNs) with S-type distributed delays are studied. First, the key issues of the Markov property of mild solutions to the SRDNNs with S-type distributed delays in C^b-space are investigated. Next, the existence of mild periodic solutions is discussed by the dissipative theory and the operator semigroup theory. Then, some sufficient conditions ensuring the stability of mild periodic solutions are derived by the Lyapunov method. To overcome the difficulties created by the special features possessed by S-type distributed delays, the truncation method is applied. Finally, a numerical example is given to illustrate the feasibility of our results.

INDEX TERMS Existence and stability, mild periodic solutions, reaction-diffusion, stochastic neural networks, S-type distributed delays.

I. INTRODUCTION

Stochastic neural networks (SNNs) with delays have been extensively studied in the past few years because of their wide applications in many fields and a great many works about the dynamical behaviors have recently been reported [1]–[6]. As we know, the reaction-diffusion phenomena are inevitable once electrons are moving in asymmetric electromagnetic fields, so they usually exist in neural networks [7]–[9]. On the other hand, the networks with S-type distributed delays typically contain the ones with discrete time delays and distributed delays [10], [11]. Hence, the SRDNNs with S-type distributed delays are now being recognized to represent more natural frameworks for mathematical modeling of many real-world phenomena and it is of great theoretical and practical importance to study the dynamical behaviors of them [12]–[14].

It is noted that the periodicity of neural networks is significant in the learning theory due to the fact that learning generally requires repetitions [15], [16]. As a result, many researchers focus on the topic of the periodic solutions to neural networks and some interesting results have been reported in the literature [11], [17]–[21]. In particular, Xu *et al.* [17] presented some conditions ensuring the existence of periodic solutions to the SNNs with finite delays and discussed the global attractiveness of periodic solutions, which extended J. K. Hale's work in [22]. Then, based on this work, in [19], Li and Xu generalized the existence theorem of periodic solutions for the SNNs with finite delays to the SNNs with infinite delays. However, to the best of our knowledge, few have considered periodic solutions to the SRDNNs with S-type distributed delays.

Inspired by the aforementioned discussions, this paper addresses the existence and exponential stability of mild periodic solutions to the SRDNNs with S-type distributed delays. Highlights: (i) The key issues of the Markov property of mild solutions $\boldsymbol{u}_t$ to the SRDNNs with S-type distributed delays in C^b-space are discussed (in general, $\boldsymbol{u}_t$ is not a Markov process in $\mathbb{R}^n$). (ii) The existence of mild periodic solutions to the networks is investigated, which extended some of the results in [17] and [19]. (iii) The issues caused by infinite time delays are handled by the truncation method and the approximation method. (iv) Some easy-to-test algebraic criteria of the existence and exponential stability of mild periodic solutions to the SRDNNs with S-type distributed delays are presented by virtue of the dissipative theory and the operator

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semigroup techniques. (v) The effectiveness of the proposed results is evidenced by illustrative simulations.

Notations: $L^2(\mathcal{O})^n$ is a Hilbert space with the norm $\|\boldsymbol{u}\| = (\int_{\mathcal{O}}|\boldsymbol{u}(t,\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x})^{1/2}$, and $(\cdot,\cdot)$ denotes the inner product. $H_0^1(\mathcal{O})$ is a Hilbert space with the norm $|||\boldsymbol{u}||| \triangleq ||\boldsymbol{u}||_{H_0^1(\mathcal{O})} = ||\nabla \boldsymbol{u}||$. $C^b \triangleq C^b((-\infty,0]\times\mathcal{O}, L^2(\mathcal{O})^n)$ is the Banach space of all bounded and continuous functions from $(-\infty,0]\times\mathcal{O}$ to $L^2(\mathcal{O})^n$, with the norm $\|\boldsymbol{\varphi}\|_C = \sup_{\theta\in(-\infty,0]}\|\boldsymbol{\varphi}(\theta)\|$. $(\Omega,\mathscr{F},\mathbb{P})$ is a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$. $C^b_{\mathscr{F}_0}$ denotes the family of $\mathscr{F}_0$-measurable bounded $C^b((-\infty,0]\times\mathcal{O},\mathbb{R}^n)$-valued stochastic variables $\boldsymbol{\phi}$ with $E||\boldsymbol{\phi}||_C < \infty$. $||\boldsymbol{B}||_F = [tr(\boldsymbol{B}\boldsymbol{B}^T)]^{1/2}$ is the Frobenius norm of $\boldsymbol{B}\in\mathbb{R}^{n\times m}$, where tr is the trace operator. $\mathfrak{L}_2^0(\Re, L^2(\mathcal{O})^n)$ is the space of all Hilbert-Schmidt operators from $\Re \triangleq Q^{\frac{1}{2}}(L^2(\mathcal{O})^m)$ into $L^2(\mathcal{O})^n$, where Q is a positive definite, self-adjoint, Hilbert-Schmidt operator with a finite trace. It is a Hilbert space with the norm $||\Phi||_{F^*} \triangleq \sqrt{tr(\Phi Q\Phi^*)}$, where Φ^* is the adjoint of Φ.

II. PRELIMINARY

Consider the following SRDNNs with S-type distributed delays:

$$\begin{cases}\mathrm{d}\boldsymbol{u} = [\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) - \boldsymbol{A}(t)\boldsymbol{u} + \boldsymbol{B}(t)\boldsymbol{f}(\boldsymbol{u}) \\ \qquad +\boldsymbol{C}(t)\boldsymbol{h}(\mathcal{S}(\boldsymbol{u})) + \boldsymbol{I}(t)]\mathrm{d}t + \boldsymbol{K}(t)\boldsymbol{G}(\mathcal{S}(\boldsymbol{u}))\mathrm{d}\boldsymbol{W}(t), \\ \boldsymbol{u}(t,\boldsymbol{x})\mid_{\boldsymbol{x}\in\partial\mathcal{O}} = 0, \quad t\geq 0, \\ \boldsymbol{u}(\theta,\boldsymbol{x}) = \boldsymbol{\phi}(\theta,\boldsymbol{x}) \in C^b_{\mathscr{F}_0}, \quad -\infty<\theta\leq 0, \ \boldsymbol{x}\in\mathcal{O},\end{cases} \tag{1}$$

where $\boldsymbol{x}\in\mathbb{R}^l$, $\boldsymbol{u} = (u_1(t,\boldsymbol{x},\omega), u_2(t,\boldsymbol{x},\omega),\cdots,u_n(t,\boldsymbol{x},\omega))^T$. $\boldsymbol{u}_t = \boldsymbol{u}(t+\theta,\boldsymbol{x},\omega)$, $\theta\in(-\infty,0]$, $t\geq 0$, $\boldsymbol{x}\in\mathcal{O}$, $\omega\in\Omega$. $\mathcal{S}(\boldsymbol{u}) = \int_{-\infty}^0 \mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta,\boldsymbol{x})$ is Lebesgue-Stieltjes integrable, and $\eta_i(\theta)$ $(i=1,2,\cdots,n)$ are non-decreasing bounded variation functions satisfying $\int_{-\infty}^0\mathrm{d}\eta_i(\theta) = N_i > 0$. $\boldsymbol{D}(\boldsymbol{x}) = (D_{ik}(\boldsymbol{x}))_{n\times l}$, $\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) = (\sum_{j=1}^l \frac{\partial(D_{1j}(\boldsymbol{x})\frac{\partial u_1}{\partial x_j})}{\partial x_j}, \sum_{j=1}^l \frac{\partial(D_{2j}(\boldsymbol{x})\frac{\partial u_2}{\partial x_j})}{\partial x_j},\cdots,\sum_{j=1}^l \frac{\partial(D_{nj}(\boldsymbol{x})\frac{\partial u_n}{\partial x_j})}{\partial x_j})^T$. $\boldsymbol{f}(\boldsymbol{u}) = (f_1(u_1(t,\boldsymbol{x})), f_2(u_2(t,\boldsymbol{x})),\cdots,f_n(u_n(t,\boldsymbol{x})))^T$, $\boldsymbol{h}(\mathcal{S}(\boldsymbol{u})) = (h_1(\int_{-\infty}^0\mathrm{d}\eta_1(\theta)u_1(t+\theta,\boldsymbol{x})), h_2(\int_{-\infty}^0\mathrm{d}\eta_2(\theta)u_2(t+\theta,\boldsymbol{x})),\cdots,h_n(\int_{-\infty}^0\mathrm{d}\eta_n(\theta)u_n(t+\theta,\boldsymbol{x})))^T$, $\boldsymbol{G} = (G_{ij})_{n\times m}\in M_2^{n,m}$ [23], and $\boldsymbol{W}(t)$ is a Q-Wiener process [24]. $\boldsymbol{A}(t) = \mathrm{diag}(a_1(t),a_2(t),\cdots,a_n(t))$, $\boldsymbol{B}(t) = (b_{ij}(t))_{n\times n}$, $\boldsymbol{C}(t) = (c_{ij}(t))_{n\times n}$, $\boldsymbol{I}(t) = (I_1(t),I_2(t),\cdots,I_n(t))^T$, and $\boldsymbol{K}(t) = (k_{ij}(t))_{n\times n}$ are all periodic continuous functions with period T for $t\in\mathbb{R}$. Besides, $\mathcal{O}$ is an open bounded and connected subset of $\mathbb{R}^l$ with a sufficiently regular boundary $\partial\mathcal{O}$ and $\boldsymbol{\phi}(\theta,\boldsymbol{x}) = (\phi_1(\theta,\boldsymbol{x}),\phi_2(\theta,\boldsymbol{x}),\cdots,\phi_n(\theta,\boldsymbol{x}))^T$ is the initial data. The parameters of (1) have the similar physical meanings with those in [7].

Throughout this paper, we make the following assumptions:

Assumption 1: There exists a constant $\alpha > 0$, such that $D_{ij}(\boldsymbol{x}) \geq \alpha/l$, $i=1,2,\cdots,n$, $j=1,2,\cdots,l$.

Assumption 2: $\underline{c_i}\leq|c_i(t)|\leq\overline{c_i}$, $\underline{a_{ij}}\leq|a_{ij}(t)|\leq\overline{a_{ij}}$, $\underline{b_{ij}}\leq|b_{ij}(t)|\leq\overline{b_{ij}}$, $\underline{k_{ij}}\leq|k_{ij}(t)|\leq\overline{k_{ij}}$, $\underline{I_i}\leq|I_i(t)|\leq\overline{I_i}$.

Assumption 3: $\boldsymbol{f},\boldsymbol{h},\boldsymbol{G}$ satisfy global Lipschitz conditions, which means there exist positive constants ρ_1 and ρ_2 such that $||\boldsymbol{f}(\boldsymbol{u})-\boldsymbol{f}(\boldsymbol{v})||\vee||\boldsymbol{h}(\boldsymbol{u})-\boldsymbol{h}(\boldsymbol{v})|| \leq \rho_1||\boldsymbol{u}-\boldsymbol{v}||$ and $||\boldsymbol{G}(\boldsymbol{u})-\boldsymbol{G}(\boldsymbol{v})||_{F^*}\leq\rho_2||\boldsymbol{u}-\boldsymbol{v}||$, where $\boldsymbol{u},\boldsymbol{v}\in L^2(\mathcal{O})^n$.

Remark 1: Assumption 2 holds directly because of the periodicity of (1), which is different from the similar assumptions in [25].

Similar to the proof of Theorem 1 in [12], we can prove the following result:

Lemma 1: If (1) satisfies Assumption 1– Assumption 3, then there is a unique globally mild solution $\boldsymbol{u}(t)$ to (1).

Definition 1 ([17]): A stochastic process $\boldsymbol{u}_t$ with values in Banach space C^b, defined on $(\Omega,\mathscr{F},\mathbb{P})$ is called a Markov process if, for all $A\in\mathcal{B}$, $0\leq v\leq t$, $\mathbb{P}\{\boldsymbol{u}_t\in A|\mathscr{F}_v\} = \mathbb{P}\{\boldsymbol{u}_t\in A|\boldsymbol{u}_v\}$, where $\mathscr{F}_v$ is the σ-algebra of events generated by all events of the form $\{\boldsymbol{u}_s\in A, s\leq v\}$ and $\mathcal{B}$ denotes the σ-algebra of Borel sets in C^b.

Definition 2 ([26]): A stochastic process $\boldsymbol{u}_t$ is said to be periodic with period T if, its finite dimensional distributions are periodic with period T, i.e., for any positive integer m and any moments of time $t_1,t_2,\cdots,t_m$, the joint distributions of the random variables $\boldsymbol{u}_{t_1+kT},\boldsymbol{u}_{t_2+kT},\cdots,\boldsymbol{u}_{t_m+kT}$ are independent of k $(k=\pm1,\pm2,\cdots)$.

Definition 3 ([17]): The mild solution $\boldsymbol{u}_t$ is said to be
(i) p-uniformly bounded if, for each $\alpha\geq 0$, there exists a constant $M = M(\alpha) > 0$ such that $E||\boldsymbol{\phi}||_C^p\leq\alpha$ implies

$$E||\boldsymbol{u}_t||_C^p\leq M, \quad t\geq t_0; \tag{2}$$

(ii) p-point dissipative if, there exists a bounded set $S\subset C^b_{\mathscr{F}_0}$ such that, for any initial data $\boldsymbol{\phi}\in C^b_{\mathscr{F}_0}$, $\boldsymbol{u}_t$ with $\boldsymbol{\phi}$ satisfies that

$$\rho(E||\boldsymbol{u}_t||^p, S)\to 0, \quad \text{as } t\to\infty, \tag{3}$$

where $\rho(\boldsymbol{\psi},S) = \inf_{\boldsymbol{\varphi}\in S}\sup_{s\in(-\infty,0]}||\boldsymbol{\psi}(s)-E\boldsymbol{\varphi}(s)||$, $\boldsymbol{\psi}\in C^b$.

Lemma 2 (Poincaré Inequality [27]): Let $\mathcal{O}$ be an open bounded domain in $\mathbb{R}^l$ with a smooth boundary, then

$$||\boldsymbol{u}||\leq\beta^{-1}|||\boldsymbol{u}|||, \quad \forall\boldsymbol{u}\in H_0^1(\mathcal{O}), \tag{4}$$

where the constant β depends on the domain $\mathcal{O}$.

Lemma 3 ([17]): If Markov families $\boldsymbol{u}_t$ with T-periodic transition functions are uniformly bounded, point dissipative and uniformly stochastically continuous, then there is a T-periodic Markov process.

From [28], we can easily deduce the following lemma.

Lemma 4 (Generalized Halanay Inequality [28]): Assume that $\Psi(t)\geq 0$, and

$$\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t}\leq -a\Psi(t)+b\overline{\Psi}(t)+c, \quad t\geq 0, \tag{5}$$

where $\overline{\Psi}(t) = \sup_{s\in[-\tau,0]}\Psi(t+s)$, $a>b>0$ and $c\geq 0$. Then for any $t\geq 0$, there exists a constant $\gamma>0$ such that

$$\Psi(t)\leq\overline{\Psi}(0)e^{-\gamma t}+\frac{c}{a-b}. \tag{6}$$

III. THE MARKOV PROPERTY OF THE MILD SOLUTIONS IN C^b-SPACE

Theorem 1: If, in addition to Assumption 1– Assumption 3, $\boldsymbol{\phi}$ is independent of $\boldsymbol{W}(t)$, $t \geq 0$ and $E||\boldsymbol{\phi}||_C < \infty$, then the mild solution $\boldsymbol{u}_t$ to (1) is a Markov process in C^b-space.

Proof: The mild solutions to (1) can also be written as

$$\begin{aligned} u_t &= S(t+\theta)\phi(0) + \int_0^{t+\theta} S(t+\theta-s)[-A(s)u \\ &\quad + B(s)f(u) + C(s)h(\mathcal{S}(u)) + I(s)]\mathrm{d}s \\ &\quad + \int_0^{t+\theta} S(t+\theta-s)K(s)G(\mathcal{S}(u))\mathrm{d}W(s) \\ &= S(t)\Big[S(\theta)\phi(0) + \int_0^{\theta} S(\theta-s)[-A(s)u \\ &\quad + B(s)f(u) + C(s)h(\mathcal{S}(u)) + I(s)]\mathrm{d}s \\ &\quad + \int_0^{\theta} S(\theta-s)K(s)G(\mathcal{S}(\boldsymbol{u}))\mathrm{d}\boldsymbol{W}(s) + \int_{\theta}^{t+\theta} \\ &\quad S(\theta-s)[-A(s)\boldsymbol{u} + B(s)f(u) + C(s)h(\mathcal{S}(u)) \\ &\quad + I(s)]\mathrm{d}s + \int_{\theta}^{t+\theta} S(\theta-s)K(s)G(\mathcal{S}(u))\mathrm{d}W(s)\Big] \\ &\triangleq S(t)\left[\Phi(\theta) + \int_{l_0}^{l} \boldsymbol{F}(s, \boldsymbol{u}_s)\,\mathrm{d}s + \int_{l_0}^{l} \boldsymbol{\sigma}(s, \boldsymbol{u}_s)\,\mathrm{d}\boldsymbol{W}(s)\right] \end{aligned} \tag{7}$$

where $l_0 = \theta$, and $l = t + \theta$. Then, proceeding as the proof of Lemma 3.2 in [17], we obtain that $\boldsymbol{\Phi}(\theta) + \int_{l_0}^{l} \boldsymbol{F}(s, \boldsymbol{u}_s)\mathrm{d}s + \int_{l_0}^{l} \boldsymbol{\sigma}(s, \boldsymbol{u}_s)\mathrm{d}\boldsymbol{W}(s)$ is a Markov process in C^b-space. Because of the determinacy of $S(t)$, we can derive that $S(t)[\boldsymbol{\Phi}(\theta) + \int_{l_0}^{l} \boldsymbol{F}(s, \boldsymbol{u}_s)\mathrm{d}s + \int_{l_0}^{l} \boldsymbol{\sigma}(s, \boldsymbol{u}_s)\mathrm{d}\boldsymbol{W}(s)]$ is a Markov process in C^b-space. That is, $\boldsymbol{u}_t$ is a Markov process in C^b-space. □

Remark 2: The mild solution $\boldsymbol{u}_t$ is usually not a Markov process in $\mathbb{R}^n$-space because of the delays. But Theorem 1 indicates that $\boldsymbol{u}_t$ is a Markov process in C^b-space, and the Markov property can be used in C^b-space.

IV. THE EXISTENCE OF THE MILD PERIODIC SOLUTIONS

Theorem 2: Let Assumption 1– Assumption 3 hold. If the following assumption holds,

Assumption 4: $\alpha\beta^2 + a_{\min} - \frac{3}{2} - \rho_1^2||\overline{\boldsymbol{B}}||_F^2 - 2\rho_1^2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{N}||_F^2 - 2\rho_2^2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{N}||_F^2 > 0$, where $a_{\min} = \min\{\underline{a_1}, \underline{a_2}, \cdots, \underline{a_n}\}$, $\overline{\boldsymbol{B}} = (\overline{b_{ij}})_{n\times n}$, $\overline{\boldsymbol{C}} = (\overline{c_{ij}})_{n\times n}$, $\overline{\boldsymbol{K}} = (\overline{k_{ij}})_{n\times n}$, and $\boldsymbol{N} = \mathrm{diag}(N_1, N_2, \cdots, N_n)$, then there is a mild periodic solution to (1).

Proof: Let $\boldsymbol{u}(t)$ be a mild solution to (1), and define a function $V(t) = ||\boldsymbol{u}(t)||^2$. Then,

$$\begin{aligned} \mathrm{d}V(t) &= 2(\boldsymbol{u}, \mathscr{A}\boldsymbol{u})\mathrm{d}t - 2(\boldsymbol{u}, \boldsymbol{A}(t)\boldsymbol{u})\mathrm{d}t + 2(\boldsymbol{u}, \boldsymbol{B}(t)\boldsymbol{f}(\boldsymbol{u}))\mathrm{d}t \\ &\quad + 2(\boldsymbol{u}, \boldsymbol{C}(t)\boldsymbol{h}(\mathcal{S}(\boldsymbol{u})))\mathrm{d}t + 2(\boldsymbol{u}, \boldsymbol{I}(t))\mathrm{d}t \\ &\quad + tr((\boldsymbol{K}(t)\boldsymbol{G}(\mathcal{S}(\boldsymbol{u})))Q(\boldsymbol{K}(t)\boldsymbol{G}(\mathcal{S}(\boldsymbol{u})))^*)\mathrm{d}t \\ &\quad + 2(\boldsymbol{u}, \boldsymbol{K}(t)\boldsymbol{G}(\mathcal{S}(\boldsymbol{u})))\mathrm{d}\boldsymbol{W}(t). \end{aligned} \tag{8}$$

Integrating both sides of (8) from 0 to t, taking the expectation and taking the derivative yield

$$\begin{aligned} \frac{\mathrm{d}EV(t)}{\mathrm{d}t} &= 2E(\boldsymbol{u}, \mathscr{A}\boldsymbol{u}) - 2E(\boldsymbol{u}, \boldsymbol{A}(t)\boldsymbol{u}) + 2E(\boldsymbol{u}, \boldsymbol{B}(t)\boldsymbol{f}(\boldsymbol{u})) \\ &\quad + 2E(\boldsymbol{u}, \boldsymbol{C}(t)\boldsymbol{h}(\mathcal{S}(\boldsymbol{u}))) + 2E(\boldsymbol{u}, \boldsymbol{I}(t)) \\ &\quad + E\left\|\boldsymbol{K}(t)\boldsymbol{G}(\mathcal{S}(\boldsymbol{u}))\right\|_{F^*}^2 \\ &\triangleq L_1 + L_2 + L_3 + L_4 + L_5 + L_6. \end{aligned} \tag{9}$$

By Assumption 1 and Lemma 2, we get

$$L_1 \leq -2\alpha E|||\boldsymbol{u}|||^2 \leq -2\alpha\beta^2 EV(t). \tag{10}$$

In terms of Assumption 2, one can observe that

$$L_2 \leq -2a_{\min}E||\boldsymbol{u}||^2 = -2a_{\min}EV(t). \tag{11}$$

Applying the Young inequality, Assumption 2 and Assumption 3 leads to

$$L_3 \leq EV(t) + 2\rho_1^2||\overline{\boldsymbol{B}}||_F^2 EV(t) + 2||\overline{\boldsymbol{B}}||_F^2||\boldsymbol{f}(\boldsymbol{0})||^2, \tag{12}$$

$$L_5 \leq EV(t) + ||\overline{\boldsymbol{I}}||^2, \tag{13}$$

where $\overline{\boldsymbol{I}} = (\overline{I_1}, \overline{I_2}, \cdots, \overline{I_n})^T$. Similarly, in view of Assumption 3, it follows that

$$\begin{aligned} L_4 &\leq EV(t) + 2\rho_1^2||\overline{\boldsymbol{C}}||_F^2 E||\mathcal{S}(\boldsymbol{u})||^2 + 2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{h}(\boldsymbol{0})||^2, \\ L_6 &\leq 2\rho_2^2||\overline{\boldsymbol{K}}||_F^2 E||\mathcal{S}(\boldsymbol{u})||^2 + 2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{G}(\boldsymbol{0})||_{F^*}^2. \end{aligned}$$

Noting that $\eta_i(\theta)$ $(i = 1, 2, \cdots, n)$ are non-decreasing bounded variation functions satisfying $\int_{-\infty}^{0} \mathrm{d}\eta_i(\theta) = N_i > 0$, we obtain that for any $\varepsilon > 0$, there exists $\tau \geq t$ such that

$$\begin{aligned} &\int_{-\infty}^{-\tau} \mathrm{d}\eta_i(\theta) < \varepsilon, \\ &\int_{-\tau}^{0} \mathrm{d}\eta_i(\theta) \leq \int_{-\infty}^{0} \mathrm{d}\eta_i(\theta) = N_i. \end{aligned}$$

Then,

$$\begin{aligned} E||\mathcal{S}(\boldsymbol{u})||^2 &\leq 2E||\int_{-\tau}^{0} \mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta)||^2 \\ &\quad + 2E||\int_{-\infty}^{-\tau} \mathrm{d}\boldsymbol{\eta}(\theta)\boldsymbol{u}(t+\theta)||^2 \\ &\leq 2||\boldsymbol{N}||_F^2 E(\sup_{s\in[t-\tau,t]} V(s)) + 2n\varepsilon^2 E||\boldsymbol{\phi}||_C^2. \end{aligned}$$

It follows immediately that

$$\begin{aligned} L_4 &\leq EV(t) + 4\rho_1^2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{N}||_F^2 E(\sup_{s\in[t-\tau,t]} V(s)) \\ &\quad + 4\rho_1^2||\overline{\boldsymbol{C}}||_F^2 n\varepsilon^2 E||\boldsymbol{\phi}||_C^2 + 2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{h}(\boldsymbol{0})||^2, \end{aligned} \tag{14}$$

$$\begin{aligned} L_6 &\leq 4\rho_2^2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{N}||_F^2 E(\sup_{s\in[t-\tau,t]} V(s)) \\ &\quad + 4\rho_2^2||\overline{\boldsymbol{K}}||_F^2 n\varepsilon^2 E||\boldsymbol{\phi}||_C^2 + 2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{G}(\boldsymbol{0})||_{F^*}^2. \end{aligned} \tag{15}$$

Thus, according to (9)–(15), we deduce that

$$\begin{aligned} \frac{\mathrm{d}EV(t)}{\mathrm{d}t} &\leq -(2\alpha\beta^2 + 2a_{\min} - 3 - 2\rho_1^2||\overline{\boldsymbol{B}}||_F^2)EV(t) \\ &\quad + (4\rho_1^2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{N}||_F^2 + 4\rho_2^2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{N}||_F^2)E\overline{V}(t) \\ &\quad + ||\overline{\boldsymbol{I}}||^2 + 2(||\overline{\boldsymbol{B}}||_F^2||\boldsymbol{f}(\boldsymbol{0})||^2 + ||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{h}(\boldsymbol{0})||^2 \end{aligned}$$

$$+||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{G}(\boldsymbol{0})||_{F^*}^2)+4\varepsilon^2nE||\boldsymbol{\phi}||_C^2(\rho_1^2||\overline{\boldsymbol{C}}||_F^2+\rho_2^2||\overline{\boldsymbol{K}}||_F^2)$$
$$\triangleq -k_1EV(t)+k_2E\overline{V}(t)+k_3+k_4\varepsilon^2, \tag{16}$$

where $E\overline{V}(t) = E\sup_{s\in[-\tau,0]}V(t+s)$. From Lemma 4, we can infer that there exists a constant $\gamma > 0$ such that

$$EV(t) < E||\boldsymbol{\phi}||_C^2 e^{-\gamma t}+\frac{k_3+k_4\varepsilon^2}{k_1-k_2},\quad t>0.$$

That is, $E||\boldsymbol{u}_t||_C^2 \le E||\boldsymbol{\phi}||_C^2+\frac{k_3+k_4\varepsilon^2}{k_1-k_2}$. Let $\varepsilon\to 0$, then

$$E||\boldsymbol{u}_t||_C^2 \le E||\boldsymbol{\phi}||_C^2+\frac{k_3}{k_1-k_2}. \tag{17}$$

Therefore, the mild solution $\boldsymbol{u}_t$ to (1) is uniformly bounded in the mean-square sense. Proceeding as the proof of Theorem 3 in [29], we can derive that $\boldsymbol{u}_t$ is uniformly stochastically continuous.

Next, we will prove that the mild solution $\boldsymbol{u}_t$ to (1) is point dissipative.

Since $\boldsymbol{u}_t$ is uniformly bounded, there is $\sigma > 0$ such that $\overline{\lim_{t\to\infty}}\, EV(t)=\sigma$. In other words, for each $\varepsilon>0$, there exists $t_1^*>0$ such that for every $t>t_1^*$, $E\overline{V}(t)\le\sigma+\varepsilon$. We assert

$$(-k_1+k_2)\sigma+k_3\ge 0. \tag{18}$$

In fact, if this is not true, that is

$$(-k_1+k_2)\sigma+k_3< 0, \tag{19}$$

then there is a sufficiently small constant $0<\varepsilon<\sigma$ such that $-k_1(\sigma-\varepsilon)+k_2(\sigma+\varepsilon)+k_3<0$. Now, we argue that there exists $t_2\ge t_1^*$ such that for any $t\ge t_2$,

$$\frac{\mathrm{d}EV(t)}{\mathrm{d}t}\le 0. \tag{20}$$

Suppose (20) is not true, then there exists $t_2^*\ge t_1^*$ satisfying

$$\frac{\mathrm{d}EV(t_2^*)}{\mathrm{d}t}> 0, \tag{21}$$
$$EV(t_2^*)\ge\sigma-\varepsilon. \tag{22}$$

Using (16), we obtain

$$\frac{\mathrm{d}EV(t_2^*)}{\mathrm{d}t}\le -k_1EV(t_2^*)+k_2E\overline{V}(t_2^*)+k_3$$
$$\le -k_1(\sigma-\varepsilon)+k_2(\sigma+\varepsilon)+k_3<0, \tag{23}$$

which is in contradiction with (21). Hence (20) follows. With the monotone convergence theorem, one can deduce that

$$\lim_{t\to\infty}EV(t)=\sigma. \tag{24}$$

Furthermore, for any given $E||\boldsymbol{\phi}||_C^2\le(k_1-k_2)^{-1}k_3$, we claim that $EV(t)\le(k_1-k_2)^{-1}k_3$, where $t\ge 0$. Otherwise, there exists $t_1>0$ for which

$$\frac{\mathrm{d}EV(t_1)}{\mathrm{d}t}>0, \tag{25}$$
$$EV(t_1)=(k_1-k_2)^{-1}k_3, \tag{26}$$
$$EV(t)\le(k_1-k_2)^{-1}k_3,\quad t\in[0,t_1). \tag{27}$$

Together with (16), we have

$$\frac{\mathrm{d}EV(t_1)}{\mathrm{d}t}\le -k_1EV(t_1)+k_2E\overline{V}(t_1)+k_3\le 0, \tag{28}$$

which contradicts (25). Therefore, for all $t\ge 0$,

$$EV(t)\le(k_1-k_2)^{-1}k_3. \tag{29}$$

Noting that $k_1-k_2>0$, (19) together with (29) leads to

$$EV(t)\le(k_1-k_2)^{-1}k_3<\sigma, \tag{30}$$

which is a contradiction. Hence, (18) holds, i.e., the mild solutions to (1) is point dissipative.

Since the coefficients of (1) are T-periodic in t, the transition functions of (1) are T-periodic. It follows from Lemma 3 and Theorem 1 that (1) has a mild periodic solution. □

If

$$\eta_i(\theta)=\begin{cases}-1, & \theta\le -r,\\ 0, & -r<\theta\le 0,\quad i=1,2,\cdots,n,\end{cases}$$

then (1) becomes the following neural networks with discrete time delays:

$$\begin{cases}\mathrm{d}\boldsymbol{u}=[\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u})-\boldsymbol{A}(t)\boldsymbol{u}+\boldsymbol{B}(t)\boldsymbol{f}(\boldsymbol{u})\\ \qquad +\boldsymbol{C}(t)\boldsymbol{h}(\boldsymbol{u}(t-r,\boldsymbol{x}))+\boldsymbol{I}(t)]\mathrm{d}t\\ \qquad +\boldsymbol{K}(t)\boldsymbol{G}(\boldsymbol{u}(t-r,\boldsymbol{x}))\mathrm{d}\boldsymbol{W}(t),\\ \boldsymbol{u}(t,\boldsymbol{x})\mid_{\boldsymbol{x}\in\partial\mathcal{O}}=0,\quad t\ge 0,\\ \boldsymbol{u}(\theta,\boldsymbol{x})=\boldsymbol{\phi}(\theta,\boldsymbol{x})\in C^b_{\mathscr{F}_0},\quad -r\le\theta\le 0,\ \boldsymbol{x}\in\mathcal{O},\end{cases} \tag{31}$$

so we have the following result.

Theorem 3: Suppose that Assumption 1– Assumption 3 hold, then there is a mild periodic solution to (31), if $\alpha\beta^2+a_{\min}-\frac{3}{2}-\rho_1^2||\overline{\boldsymbol{B}}||_F^2-2n\rho_1^2||\overline{\boldsymbol{C}}||_F^2-2n\rho_2^2||\overline{\boldsymbol{K}}||_F^2>0$.

Remark 3: Let $\boldsymbol{D}(\boldsymbol{x})=0$, Theorem 3 becomes the existence of periodic solutions to the SNNs with delays, which has been discussed in [17]. Also, [19] is a special case of Theorem 2 in this paper. Compared with those conditions, our results are much easier to check since they are algebraic criteria.

V. THE EXPONENTIAL STABILITY OF THE MILD PERIODIC SOLUTIONS

Theorem 4: Let Assumption 1– Assumption 3 hold, and $\boldsymbol{u}^*(t,\boldsymbol{x})$ be a mild periodic solution to (1) with initial data $\boldsymbol{\phi}^*(t,\boldsymbol{x})$. Suppose further that

Assumption 5: $2\alpha\beta^2+2a_{\min}-2-\rho_1^2||\overline{\boldsymbol{B}}||_F^2-2\rho_1^2||\overline{\boldsymbol{C}}||_F^2||\boldsymbol{N}||_F^2-2\rho_2^2||\overline{\boldsymbol{K}}||_F^2||\boldsymbol{N}||_F^2>0$,

then the mild periodic solution is exponentially stable in the mean-square sense.

Proof: Denoting $z(t,\boldsymbol{x})=\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{u}^*(t,\boldsymbol{x})$ and $Z(t)=||z(t,\boldsymbol{x})||^2$, we obtain

$$\frac{\mathrm{d}EZ(t)}{\mathrm{d}t}=2E(z,\mathscr{A}z)-2E(z,\boldsymbol{A}(t)z)$$
$$+2E(z,\boldsymbol{B}(t)(\boldsymbol{f}(\boldsymbol{u})-\boldsymbol{f}(\boldsymbol{u}^*)))$$
$$+2E(z,\boldsymbol{C}(t)(\boldsymbol{h}(\mathcal{S}(\boldsymbol{u}))-\boldsymbol{h}(\mathcal{S}(\boldsymbol{u}^*))))$$
$$+E\left\|\boldsymbol{K}(t)(\boldsymbol{G}(\mathcal{S}(\boldsymbol{u}))-\boldsymbol{G}(\mathcal{S}(\boldsymbol{u}^*)))\right\|_{F^*}^2$$
$$\triangleq L_7+L_8+L_9+L_{10}+L_{11}. \tag{32}$$

Like the proof in Theorem 2, we deduce that

$$L_7 \le -2\alpha\beta^2 EZ(t), \quad (33)$$

$$L_8 \le -2a_{\min} EZ(t), \quad (34)$$

$$L_9 \le EZ(t) + \rho_1^2 ||\overline{\boldsymbol{B}}||_F^2 EZ(t), \quad (35)$$

$$L_{10} \le EZ(t) + 2\rho_1^2 ||\boldsymbol{N}||_F^2 ||\overline{\boldsymbol{C}}||_F^2 E\overline{Z}(t) + 2n\rho_1^2 ||\overline{\boldsymbol{C}}||_F^2 \varepsilon^2 E||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_C^2, \quad (36)$$

$$L_{11} \le 2\rho_2^2 ||\boldsymbol{N}||_F^2 ||\overline{\boldsymbol{K}}||_F^2 E\overline{Z}(t) + 2n\rho_2^2 ||\overline{\boldsymbol{K}}||_F^2 \varepsilon^2 E||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_C^2. \quad (37)$$

Then, combining (32)–(37), it follows that

$$\frac{\mathrm{d}EZ(t)}{\mathrm{d}t} \le -(2\alpha\beta^2 + 2a_{\min} - 2 - \rho_1^2 ||\overline{\boldsymbol{B}}||_F^2) EZ(t) + 2(\rho_1^2 ||\overline{\boldsymbol{C}}||_F^2 + \rho_2^2 ||\overline{\boldsymbol{K}}||_F^2) ||\boldsymbol{N}||_F^2 E\overline{Z}(t) + 2\varepsilon^2 n(\rho_1^2 ||\overline{\boldsymbol{C}}||_F^2 + \rho_2^2 ||\overline{\boldsymbol{K}}||_F^2) E||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_C^2 \triangleq -k_5 EZ(t) + k_6 E\overline{Z}(t) + k_7 \varepsilon^2, \quad (38)$$

From Assumption 5, $k_5 > k_6 > 0$, $k_7 > 0$. In view of Lemma 4, one can derive that there is a constant γ^*, such that

$$E||\boldsymbol{u}(t) - \boldsymbol{u}^*(t)||^2 \le E||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_C^2 e^{-\gamma^* t} + \frac{k_7 \varepsilon^2}{k_5 - k_6}. \quad (39)$$

Let $\varepsilon \to 0$,

$$E||\boldsymbol{u}(t) - \boldsymbol{u}^*(t)||^2 \le E||\boldsymbol{\phi} - \boldsymbol{\phi}^*||_C^2 e^{-\gamma^* t}. \quad (40)$$

Therefore, $\boldsymbol{u}^*(t)$ is exponentially stable, which completes the proof. □

Like Theorem 3, we can also obtain the exponential stability in the mean-square sense of mild periodic solutions to (31) as follows:

Theorem 5: Suppose Assumption 1– Assumption 3 hold and $\boldsymbol{u}^*(t, \boldsymbol{x})$ is a mild periodic solution to (31) with initial data $\boldsymbol{\phi}^*(t, \boldsymbol{x})$, then $\boldsymbol{u}^*(t, \boldsymbol{x})$ is exponentially stable in the mean-square sense, if $2\alpha\beta^2 + 2a_{\min} - 2 - \rho_1^2 ||\overline{\boldsymbol{B}}||_F^2 - 2n\rho_1^2 ||\overline{\boldsymbol{C}}||_F^2 - 2n\rho_2^2 ||\overline{\boldsymbol{K}}||_F^2 > 0$.

Due to the fact that an equilibrium can be regarded as a periodic solution with an arbitrary period, we now consider the following SRDNNs with S-type distributed delays:

$$\begin{cases} \mathrm{d}\boldsymbol{u} = [\nabla \cdot (\boldsymbol{D}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}) - \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{f}(\boldsymbol{u}) \\ \qquad + \boldsymbol{C}\boldsymbol{h}(\mathcal{S}(\boldsymbol{u})) + \boldsymbol{I}]\mathrm{d}t + \boldsymbol{G}(\mathcal{S}(\boldsymbol{u}))\mathrm{d}\boldsymbol{W}(t), \\ \boldsymbol{u}(t, \boldsymbol{x}) \mid_{\boldsymbol{x} \in \partial\mathcal{O}} = 0, \quad t \ge 0, \\ \boldsymbol{u}(\theta, \boldsymbol{x}) = \boldsymbol{\phi}(\theta, \boldsymbol{x}) \in C_{\mathscr{F}_0}^b, \quad -\infty < \theta \le 0, \ \boldsymbol{x} \in \mathcal{O}, \end{cases} \quad (41)$$

where $\boldsymbol{A} = \mathrm{diag}(a_1, a_2, \cdots, a_n)$, $\boldsymbol{B} = (b_{ij})_{n\times n}$, $\boldsymbol{C} = (c_{ij})_{n\times n}$, and $\boldsymbol{I} = (I_1, I_2, \cdots, I_n)^T$ are constant matrices. The meanings of other symbols are similar with their counterparts in (1). According to Theorem 4, it is easy to get the following corollary.

Corollary 1: Suppose that Assumption 1 and Assumption 3 hold, then the equilibrium of (41) is exponentially stable in the mean-square sense if $2\alpha\beta^2 + 2A_{\min} - 2 - \rho_1^2 ||\boldsymbol{B}||_F^2 - 2\rho_1^2 ||\boldsymbol{C}||_F^2 ||\boldsymbol{N}||_F^2 - 2\rho_2^2 ||\boldsymbol{N}||_F^2 > 0$, where $A_{\min} = \min\{a_1, a_2, \cdots, a_n\}$.

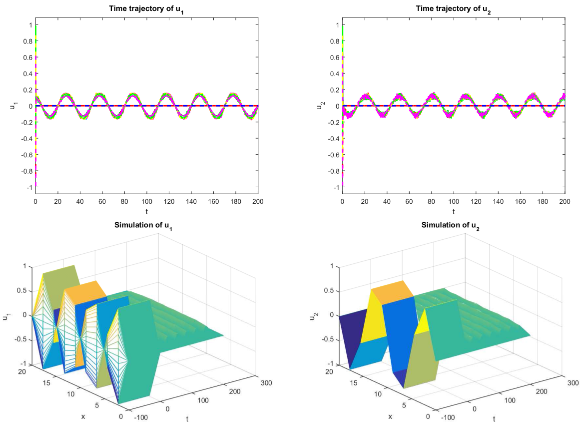

FIGURE 1. The state trajectory and simulation of (42).

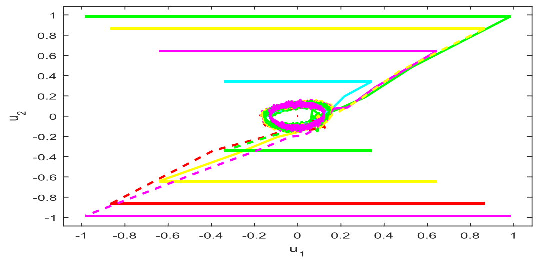

FIGURE 2. The phase graph of 30-periodic solutions to (42) on [0, 200].

Remark 4: The stability of equilibrium of (41) has already been studied in [12], and Corollary 1 contains some of the results.

VI. A NUMERICAL SIMULATION

Consider the following neural networks:

$$\begin{cases} \mathrm{d}u_1 = [10\Delta u_1 - (6.5 + \sin(\frac{\pi t}{15}))u_1 + 0.5\sin(\frac{\pi t}{15}) \\ \quad \tanh u_1 + 0.5\sin(\frac{\pi t}{15}) \tanh u_2 + \sin(\frac{\pi t}{15}) \\ \quad +0.5\sin(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_1(t+\theta, x)) \\ \quad +0.5\cos(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_2(t+\theta, x))]\mathrm{d}t \\ \quad +\cos(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_1(t+\theta, x))\mathrm{d}W, \\ \mathrm{d}u_2 = [10\Delta u_2 - (6.5 + \sin(\frac{\pi t}{15}))u_2 + 0.5\sin(\frac{\pi t}{15}) \\ \quad \tanh u_1 + 0.5\sin(\frac{\pi t}{15}) \tanh u_2 + \cos(\frac{\pi t}{15}) \\ \quad +0.5\sin(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_1(t+\theta, x)) \\ \quad +0.5\sin(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_2(t+\theta, x))]\mathrm{d}t \\ \quad +\cos(\frac{\pi t}{15}) \tanh(\int_{-\infty}^0 \mathrm{d}\eta(\theta) u_2(t+\theta, x))\mathrm{d}W, \\ u_i \mid_{x \in \partial\mathcal{O}} = 0, \quad t \ge 0, \quad i = 1, 2, \\ (u_1, u_2) = (\cos(\theta\pi) sin(0.2\pi x), sin(0.2\pi x))^T, \\ \quad -\infty < \theta \le 0, \quad x \in \mathcal{O}, \end{cases} \quad (42)$$

where $\mathcal{O} = (0, 20)$, $\eta(\theta) = 0.5e^{\theta}$, and W is a standard Brownian motion. It is easy to verify that $\alpha = 10$, $\beta \ge 0.05$, $a_{\min} = 5.5$, $\rho_1 = \rho_2 = 1$, $||\overline{\boldsymbol{B}}||_F^2 = ||\overline{\boldsymbol{C}}||_F^2 = 1$, $||\overline{\boldsymbol{K}}||_F^2 = 2$, and $||N||_F^2 = 0.5$. So Assumption 1–Assumption 5 are satisfied. From Theorem 2 and Theorem 4, (42) has a 30-periodic solution which is exponentially stable. In fact, FIGURE 1 demonstrates the state trajectory and simulation of (42) which implies the existence and stability of mild periodic solutions. In order to give a clear description, the phase graph is depicted in FIGURE 2.

VII. CONCLUSION

In this paper, we discussed the existence and exponential stability of mild periodic solutions to the SRDNNs with S-type distributed delays. As a result, some easy-to-test criteria of the existence and stability of mild periodic solutions are presented, which generalized some of the works in the literature. However, when the reaction-diffusion terms are nonautonomous, the semigroup theory used in this paper is invalid. Note that the recent work by Wei et al. [14] considered the existence-uniqueness and stability of mild solutions to the nonautonomous SRDNNs with delays, which is based on the evolution system theory. Hence, a new problem arises: is it possible to investigate mild periodic solutions to the nonautonomous SRDNNs with delays by virtue of the evolution system theory. Further research will be done on this topic.

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QI YAO was born in Rizhao, China. She received the B.S. degree in mathematics and applied mathematics from Shandong Normal University, Jinan, China, in 2015. She is currently pursuing the Ph.D. degree with the School of Mathematical Sciences, Ocean University of China, Qingdao, China. Her current research interests include stochastic partial differential equations and neural networks.

YANGFAN WANG was born in Liaocheng, China. He received the Ph.D. degree in computer science from the Ocean University of China, Qingdao, China, in 2010. He was a Joint Ph.D. Student with the University of Dundee, Dundee, U.K, where he held a postdoctoral position, from 2010 to 2011. He was a Lecturer with the Ocean University of China, from 2011 to 2014, where he has been an Associate Professor, since 2014. His current research interests include image processing and neural networks, and finite-element analysis and its application in mathematical biology.

LINSHAN WANG was born in Shenxian, China. He received the Ph.D. degree in operational research and control theory from Sichuan University, Chengdu, China, in 2002. He was a Lecturer with the Shandong University of Technology, Zibo, China, from 1977 to 1980. From 1981 to 1999, he was an Assistant, a Lecturer, an Associate Professor, and a Professor with Liaocheng University, Liaocheng, China. In 2002, he joined the College of Mathematics, Ocean University of China, Qingdao, China, as a Professor. From 2009 to 2011, he was a Visiting Professor with the University of Dundee, Dundee, U.K., and Harvard University, Cambridge, MA, USA. His current research interests include dynamical systems, stochastic partial differential equation, and its application in neural networks.