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Fluctuation Analysis of Instantaneous Availability for the Parallel Repairable Systems

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ABSTRACT This paper is aimed to study the fluctuation of the instantaneous availability (IA) for the parallel repairable system. First, the Markov model of IA for the two-unit parallel system with two states (up and down) under exponential distribution is established. Next, by Laplace transformation, the analytical solution of IA is obtained. The fluctuation of IA is analyzed according to the fluctuation theories. Then, in the same way, the fluctuation of IA for the n-unit parallel system with two states is analyzed, where the n units of the system are all same. Finally, some numerical examples are provided to show the effectiveness of the results in this paper.

INDEX TERMS Fluctuation, instantaneous availability, Markov model, the parallel system.

I. INTRODUCTION

Nowadays, availability theory plays a more and more important role in many fields, such as information network [1], military field [2], traffic field [3] and some other basic fields [4], [5]. Steady-state availability as an important aspect of reliability theory, which representing the availability in case the operating time of the whole equipment system goes to infinity, has been studied for many years [6]–[8]. However, with the rapid development of science and technology, equipment updating and out becomes faster and the equipment service period may now be only a few years or less. Moreover, the complicated mechatronic systems nowadays always consists of a lot of small units. The interaction of units often leads to undulations of the availability in the early use of new equipment. Therefore, the availability of these complicated systems usually fluctuates heavily in the early stage, but this fluctuation is quite far from the steady-state availability to be adopted for a long period by system developers and users. This factor seriously affects the evaluation of the system performance of these complicated mechatronic systems. Thus the conventional steady-state availability must be replaced by another kind of research.

Under this circumstance, there has been tremendous interest in researching the instantaneous availability (IA) of complicated mechatronic system in recent years [9], [10]. The IA represents the availability level of the equipment at any operating point. With IA, the fluctuation in the initial stage can be observed and the real-time performance of a system can be evaluated. Currently, the mathematical analysis on the IA of complicated mechatronic systems is unfolded from two main aspects. The first one is field data statistics. In field data statistics, a conclusion may be drawn using the definition of availability based on the basic theories and methodology of mathematical statistics. However, by this method, the fluctuation of IA can only be observed but not explained. The second aspect is mathematical probability model, including a study of how to set up an IA model, solve the IA and analyze the IA fluctuation for systems. The Markov process is widely used when the system working time and repair time follow exponential distribution [11]. If the working time and repair time follow general distribution (eg. uniform, gamma, and Weibull distributions), the Markov renewal process and simulation analysis are often used [1], [12]. For one-unit system, the results in [13] show that there exists no fluctuation of IA for the system with two states (up and down) under exponential distribution. However, when the two states obey uniform distribution, IA has the fluctuation [14]. The fluctuation of IA for one-unit system with three states has been studied in [15] and [16] and the conclusions have been drawn that there exist fluctuations of IA with three states (up, repair-delay and repair) and three states (up, minor repair, repair). For multi-units systems, the authors in [17] have concluded that there exists no fluctuation of IA for the two-unit series system with

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two states. The mathematical models for the two-unit series repairable system and parallel repairable system with three states have been analyzed in [18] and the numerical solutions of IAs have been obtained.

However, among all the literatures which we can get hold of, none of them studies the fluctuation of IA for two-unit parallel repairable systems with two states. In this paper, we will analyze the mathematical model and IA fluctuations for the two-unit parallel repairable system with two states. Firstly, we establish the Markov model of IA for the two-unit parallel system with two states (up and down) under exponential distribution, where the two units are heterogeneous. Next, by Laplace transformation, we have solved the analytical solution of IA for two-unit parallel system and the fluctuation of IA is analyzed according to the fluctuation theories in [15]. Moreover, we have also solved the analytical solution of IA and analyzed the fluctuation of IA for the n-unit parallel system with two states, where the n units of the system are all same. Finally, some numerical examples are provided to show the effectiveness of the results in this paper.

The organization of remainder is as follows: In section 2, we analyze the fluctuation of IA for two-unit parallel system with two states. Section 3 shows the fluctuation analysis of IA for n-unit parallel system with two states. Some numerical examples are given in section 4.

II. FLUCTUATION ANALYSIS OF IA FOR TWO-UNIT PARALLEL SYSTEM WITH TWO STATES

In this section, we will analyze the fluctuation of IA for two-unit parallel system with two states. According to [19], for establishing the Markov model of IA for the two-unit parallel system with two states, we make the following assumptions.

Assumption 1: Each of the unit is independence and does not impact relatively. The two units will not break down simultaneously.

Assumption 2: Each unit has its own maintenance equipment. When the unit is down, its maintenance equipment will repair the unit immediately. After maintenance of the unit, it will be as new and we suppose that at the beginning all units are new.

Assumption 3: Each unit i has working time X_i and repair time Y_i which obey the following exponential distributions respectively

$$X_i \sim F_i(t) = 1 - e^{-\lambda_i t}, \tag{1}$$

$$Y_i \sim G_i(t) = 1 - e^{-u_i t}, \tag{2}$$

where $i = 1, 2$.

Under these assumptions, we can get all states of the system:

State 0: unit 1 and unit 2 are up.

State 1: unit 1 is up and unit 2 is down.

State 2: unit 1 is down and unit 2 is up.

State 3: unit 1 and unit 2 are down.

The state transition diagram is shown in Fig.1.

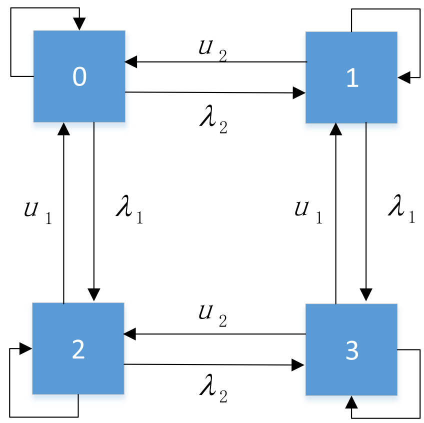

FIGURE 1. The state transition diagram of two-unit parallel system with two states.

According to [19], for this Markov process, we have the state transfer probability function as follows

$$\begin{cases} P_{00}(\Delta t) = 1 - (\lambda_1 + \lambda_2)\Delta t + o(\Delta t), \\ P_{01}(\Delta t) = \lambda_2 \Delta t + o(\Delta t), \\ P_{02}(\Delta t) = \lambda_1 \Delta t + o(\Delta t), \\ P_{10}(\Delta t) = u_2 \Delta t + o(\Delta t), \\ P_{11}(\Delta t) = 1 - (\lambda_1 + u_2)\Delta t + o(\Delta t), \\ P_{13}(\Delta t) = \lambda_1 \Delta t + o(\Delta t), \\ P_{20}(\Delta t) = u_1 \Delta t + o(\Delta t), \\ P_{22}(\Delta t) = 1 - (\lambda_2 + u_1)\Delta t + o(\Delta t), \\ P_{23}(\Delta t) = \lambda_2 \Delta t + o(\Delta t), \\ P_{31}(\Delta t) = u_1 \Delta t + o(\Delta t), \\ P_{32}(\Delta t) = u_2 \Delta t + o(\Delta t), \\ P_{33}(\Delta t) = 1 - (u_1 + u_2)\Delta t + o(\Delta t). \end{cases} \tag{3}$$

The elements m_{ij} of the transfer rate matrix M satisfies

$$\begin{cases} \lim\limits_{\Delta t \to 0} \dfrac{P_{ij}(\Delta t)}{\Delta t} = m_{ij} \quad i \neq j, \\ \lim\limits_{\Delta t \to 0} -\dfrac{1 - P_{ii}(\Delta t)}{\Delta t} = m_{ii}. \end{cases} \tag{4}$$

So the transfer rate matrix M is as follows

$$M = \begin{bmatrix} -\lambda_1 - \lambda_2 & \lambda_2 & \lambda_1 & 0 \\ u_2 & -\lambda_1 - u_2 & 0 & \lambda_1 \\ u_1 & 0 & -\lambda_2 - u_1 & \lambda_2 \\ 0 & u_1 & u_2 & -u_1 - u_2 \end{bmatrix}. \tag{5}$$

Let $P(t) = (P_0(t), P_1(t), P_2(t), P_3(t))$, where $P_i(t)$ ($i = 0, 1, 2, 3$) denotes the probability that state i occur at time t. So the ordinary differential equation of $P(t)$ can be obtained as follows

$$\begin{cases} P'(t) = P(t) M, \\ P(0) = (1, 0, 0, 0). \end{cases} \tag{6}$$

After solving the ordinary differential equation (6), we will get $P_i(t)$. When system is in state 0, state 1 or state 2, the system is up and when system is in state 3, the system is down. So the probability that the system is up at time t is as follows

$$A(t) = P_0(t) + P_1(t) + P_2(t) = 1 - P_3(t). \tag{7}$$

Next we will analyze the fluctuation of IA for two-unit parallel system with two states. First, we will give the following theorem.

Theorem 1: If working time X_i and repair time Y_i of the two-unit parallel system obey (1) and (2) respectively, $A(t)$ of the system is as follows

$$A(t) = \frac{\lambda_1 u_2 + \lambda_2 u_1 + u_1 u_2}{(\lambda_1 + u_1)(\lambda_2 + u_2)} - \frac{\lambda_1\lambda_2}{(\lambda_1 + u_1)(\lambda_2 + u_2)} \times [e^{-(\lambda_1+\lambda_2+u_1+u_2)t} - e^{-(\lambda_1+u_1)t} - e^{-(\lambda_2+u_2)t}]. \tag{8}$$

Proof: By applying Laplace transformation to (6), we can get

$$P^*(s)\begin{bmatrix} s+\lambda_1+\lambda_2 & -\lambda_2 & -\lambda_1 & 0 \\ -u_2 & s+\lambda_1+u_2 & 0 & -\lambda_1 \\ -u_1 & 0 & s+\lambda_2+u_1 & -\lambda_2 \\ 0 & -u_1 & -u_2 & s+u_1+u_2 \end{bmatrix} = (1, 0, 0, 0), \tag{9}$$

where $P^*(s)$ denotes the Laplace transformation of $P(t)$. Let D be

$$D := \begin{bmatrix} s+v\lambda_1+\lambda_2 & -\lambda_2 & -\lambda_1 & 0 \\ -u_2 & s+\lambda_1+u_2 & 0 & -\lambda_1 \\ -u_1 & 0 & s+\lambda_2+u_1 & -\lambda_2 \\ 0 & -u_1 & -u_2 & s+u_1+u_2 \end{bmatrix}. \tag{10}$$

The determinant of D is $|D| = s(s+\lambda_1+u_1)(s+\lambda_2+u_2)(s+\lambda_1+\lambda_2+u_1+u_2)$.

According to (7), for getting $A(t)$, we only need to get $P_3(t)$, which means that we only need to get the first row fourth columns of inverse matrix of matrix D for obtaining $P_3^*(s)$ which is the Laplace transformation of $P_3(t)$. The inverse matrix of matrix D is

$$D^{-1} := \begin{bmatrix} * & * & * & \lambda_1\lambda_2(2s+\lambda_1+\lambda_2+u_1+u_2) \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}, \tag{11}$$

where $*$ denotes the element in the matrix D^{-1} which does not need to be calculated out. Thus, we can work out $P_3^*(s)$ as

$$P_3^*(s) = \frac{\lambda_1\lambda_2(2s+\lambda_1+\lambda_2+u_1+u_2)}{s(s+\lambda_1+u_1)(s+\lambda_2+u_2)} \times \frac{1}{s+\lambda_1+\lambda_2+u_1+u_2}. \tag{12}$$

Then $P_3^*(s)$ can be rewritten as

$$P_3^*(s) = \frac{B_1}{s} + \frac{B_2}{s+\lambda_1+u_1} + \frac{B_3}{s+\lambda_2+u_2} + \frac{B_4}{s+\lambda_1+\lambda_2+u_1+u_2}, \tag{13}$$

where

$$B_1 = -B_2 = -B_3 = B_4 := \frac{\lambda_1\lambda_2}{(\lambda_1+u_1)(\lambda_2+u_2)}.$$

Applying inverse Laplace transformation to (13), we will get

$$P_3(t) = B_1 + B_2 e^{-(\lambda_1+u_1)t} + B_3 e^{-(\lambda_2+u_2)t} + B_4 e^{-(\lambda_1+\lambda_2+u_1+u_2)t}. \tag{14}$$

So the analytical expression of $A(t)$ is as follows

$$\begin{aligned} A(t) &= 1 - P_3(t) \\ &= \frac{\lambda_1 u_2 + \lambda_2 u_1 + u_1 u_2}{(\lambda_1+u_1)(\lambda_2+u_2)} + \frac{\lambda_1\lambda_2}{(\lambda_1+u_1)(\lambda_2+u_2)} \\ &\quad \times (e^{-(\lambda_1+u_1)t} + e^{-(\lambda_2+u_2)t} - e^{-(\lambda_1+\lambda_2+u_1+u_2)t}). \end{aligned} \tag{15}$$

The proof is completed. □

Next, we will analyze the fluctuation of IA. To analyze the fluctuation of IA, and the fluctuation definition and the fluctuation theory have been put forward in [15].

Definition 1 [15]: A continuous function $A(t)$ has fluctuation if there exit three point t_0, t_1, t_2 in domain and $t_0 < t_1 < t_2$, which make the inequality $(A(t_0) - A(t_1))(A(t_1) - A(t_2)) < 0$ hold.

Lemma 1 [15]: A continuous function $A(t)$ has no fluctuation if $A(t)$ is a monotone function.

According to the fluctuation theory, the fluctuation of IA can be analyzed. We can get the following theorem.

Theorem 2: IA of two-unit parallel system with two states has no fluctuation.

Proof: By taking derivative to (15) with respect to t, we can get

$$\begin{aligned} A'(t) = \frac{\lambda_1\lambda_2}{(\lambda_1+u_1)(\lambda_2+u_2)} &\times [(\lambda_1+\lambda_2+u_1+u_2) \\ &\times e^{-(\lambda_1+\lambda_2+u_1+u_2)t} - (\lambda_1+u_1)e^{-(\lambda_1+u_1)t} \\ &- (\lambda_2+u_2)e^{-(\lambda_2+u_2)t}\Big]. \end{aligned}$$

Since $f(x) = e^{-x}$ monotonically decreases, then we have $e^{-(\lambda_1+\lambda_2+u_1+u_2)t} < e^{-(\lambda_1+u_1)t}$ and $e^{-(\lambda_1+\lambda_2+u_1+u_2)t} < e^{-(\lambda_2+u_2)t}$ for all $t \in (0, +\infty)$. Thus we can come to the conclusion that $A'(t) < 0$. According to Lemma 1, IA has no fluctuation. The proof is completed. □

III. FLUCTUATION ANALYSIS OF IA FOR *n*-UNIT PARALLEL SYSTEM WITH TWO STATES

In this section, we will analyze the fluctuation of IA for n-unit parallel system with two states. Suppose the n units are the same, it means that working time X_i and repair time $Y_i (i = 1, 2, \cdots, n)$ of each unit are subject to the same distribution function

$$X_i \sim F_i(t) = 1 - e^{-\lambda t}, \tag{16}$$

$$Y_i \sim G_i(t) = 1 - e^{-ut}. \tag{17}$$

Each unit has its maintenance equipment, so there does not exit repair-waiting time.

$$M=\begin{bmatrix} -n\lambda & n\lambda & & & & 0 \\ u & -(n-1)\lambda-u & (n-1)\lambda & & & \\ & 2u & -(n-2)\lambda-2u & (n-2)\lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & (n-1)u & -\lambda-(n-1)u & \lambda \\ 0 & & & & nu & -nu \end{bmatrix}. \tag{18}$$

According to the Markov model in [19], the transfer rate matrix M is as (18), as shown at the top of this page. Let $P_i(t)$ be the rate that i-th unit are down at time t and $P(t) = (P_0(t), P_1(t), \cdots, P_n(t))$, so we can get ordinary differential equation

$$\begin{cases} P'(t) = P(t)M, \\ P(0) = (1, 0, \cdots, 0). \end{cases} \tag{19}$$

For n-unit parallel system, only when all n units are down, the system is down. So IA of the n-unit parallel system can be got as

$$A(t) = 1 - P_n(t). \tag{20}$$

Now we are going to solve the analytical expression of $A(t)$ and analyze its fluctuation. Firstly, for getting $A(t)$ of the n-unit parallel system with two states, we will introduce the following two lemmas.

Lemma 2:

$$S_n(s)=\begin{vmatrix} s+n\lambda & -n\lambda & 0 & 0 & 0 & 0 \\ -u & s+(n-1)\lambda+u & -(n-1)\lambda & 0 & 0 & 0 \\ 0 & -2u & s+(n-2)\lambda+2u & -(n-2)\lambda & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -(n-1)u & s+\lambda+(n-1)u & -\lambda \\ 0 & 0 & 0 & 0 & -nu & s+nu \end{vmatrix}$$
$$= s(s+\lambda+u)(s+2(\lambda+u))\cdots(s+n(\lambda+u)). \tag{21}$$

Proof: We will use mathematical induction to prove the lemma. When $n = 1$, we have

$$S_1(s) = \begin{vmatrix} s+\lambda & -\lambda \\ -u & s+u \end{vmatrix} = s(s+\lambda+u) \tag{22}$$

The result is consistent with the lemma. Now suppose the result is true when $n = k$, i.e.

$$S_k(s)=\begin{vmatrix} s+k\lambda & -k\lambda & 0 & 0 & 0 & 0 \\ -u & s+(k-1)\lambda+u & -(k-1)\lambda & 0 & 0 & 0 \\ 0 & -2u & s+(k-2)\lambda+2u & -(k-2)\lambda & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -(k-1)u & s+\lambda+(k-1)u & -\lambda \\ 0 & 0 & 0 & 0 & -ku & s+ku \end{vmatrix}$$
$$= s(s+\lambda+u)(s+2(\lambda+u))\cdots(s+k(\lambda+u)) \tag{23}$$

According to (23), we have

$$S_k(s+\lambda+u)$$
$$=\begin{vmatrix} s+(k+1)\lambda+u & -k\lambda & 0 & 0 & 0 & 0 \\ -u & s+k\lambda+2u & -(k-1)\lambda & 0 & 0 & 0 \\ 0 & -2u & s+(k-1)\lambda+3u & -(k-2)\lambda & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -(k-1)u & s+2\lambda+ku & -\lambda \\ 0 & 0 & 0 & 0 & -ku & s+\lambda+(k+1)u \end{vmatrix}$$
$$= (s+\lambda+u)(s+2(\lambda+u)) \times \cdots$$
$$\times (s+k(\lambda+u))(s+(k+1)(\lambda+u)). \tag{24}$$

Then when $n = k+1$,

$$S_{k+1}(s)=\begin{vmatrix} s+(k+1)\lambda & -(k+1)\lambda & 0 & 0 & 0 & 0 \\ -u & s+k\lambda+u & -k\lambda & 0 & 0 & 0 \\ 0 & -2u & s+(k-1)\lambda+2u & -(k-2)\lambda & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & -ku & s+\lambda+ku & -\lambda \\ 0 & 0 & 0 & 0 & -(k+1)u & s+(k+1)u \end{vmatrix} \tag{25}$$

Applying elementary column transformation to $S_{k+1}(s)$, we can get

$$S_{k+1}(s)=\begin{vmatrix} s & -(k+1)\lambda & 0 & 0 & 0 & 0\\ s & s+u & -k\lambda & 0 & 0 & 0\\ 0 & s & s+2u & -(k-1)\lambda & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & s & s+ku & -\lambda\\ s & 0 & 0 & 0 & s & s+(k+1)u \end{vmatrix} \tag{26}$$

Then extracting the first column of $S_{k+1}(s)$ and applying elementary row transformation, we have

$$S_{k+1}(s) = s\begin{vmatrix} s+(k+1)\lambda+u & -k\lambda & 0 & 0 & 0 & 0\\ -u & s+k\lambda+2u & -(k-1)\lambda & 0 & 0 & 0\\ 0 & -2u & s+(k-1)\lambda+3u & -(k-2)\lambda & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & -(k-1)u & s+2\lambda+ku & -\lambda\\ 0 & 0 & 0 & 0 & -ku & s+\lambda+(k+1)u \end{vmatrix}. \tag{27}$$

Combining with (25), we can obtain

$$\begin{aligned} S_{k+1}(s) &= sS_k(s+\lambda+u)\\ &= s(s+\lambda+u)\cdots(s+(k+1)(\lambda+u)). \end{aligned} \tag{28}$$

The result is consistent with the lemma. The proof is completed. □

Lemma 3:

$$\begin{aligned} H_n &= \frac{1}{s(s+d)(s+2d)\cdots(s+nd)}\\ &= \frac{1}{n!d^n}\sum_{i=0}^{n}(-1)^i\frac{C_n^i}{s+id}. \end{aligned} \tag{29}$$

Proof: We will use mathematical induction to prove the lemma. When $n=1$,

$$H_1=\frac{1}{s(s+d)}=\frac{1}{d}(\frac{1}{s}-\frac{1}{s+d}). \tag{30}$$

The result is consistent with the lemma. Then supposing $n=k$, lemma is true, i.e.

$$\begin{aligned} H_k &= \frac{1}{s(s+d)(s+2d)\cdots(s+kd)}\\ &= \frac{1}{k!d^k}\sum_{i=0}^{k}(-1)^i\frac{C_k^i}{s+id}. \end{aligned} \tag{31}$$

when $n=k+1$,

$$\begin{aligned} H_{k+1} &= \frac{1}{s(s+d)(s+2d)\cdots(s+kd)(s+(k+1)d)}\\ &= H_k\times\frac{1}{s+(k+1)d}. \end{aligned} \tag{32}$$

For getting H_{k+1}, we have

$$\begin{aligned} &\frac{C_k^i}{(s+id)(s+(k+1)d)}\\ &\quad= \frac{C_k^i}{(k+1-i)d}\times(\frac{1}{s+id}-\frac{1}{s+(k+1)d})\\ &\quad= \frac{C_{k+1}^i}{(k+1)d}\times(\frac{1}{s+id}-\frac{1}{s+(k+1)d}). \end{aligned} \tag{33}$$

So

$$\begin{aligned} H_{k+1} &= \frac{1}{(k+1)!d^{k+1}}\sum_{i=0}^{k}(-1)^iC_{k+1}^i(\frac{1}{s+id}-\frac{1}{s+(k+1)d})\\ &= \frac{1}{(k+1)!d^{k+1}}\left(\sum_{i=0}^{k}(-1)^i\times C_{k+1}^i\times\frac{1}{s+id}\right.\\ &\quad\left.-\sum_{i=0}^{k}(-1)^i\times C_{k+1}^i\times\frac{1}{s+(k+1)d}\right)\\ &= \frac{1}{(k+1)!d^{k+1}}\sum_{i=0}^{k+1}(-1)^i\times C_{k+1}^i\times\frac{1}{s+id}. \end{aligned} \tag{34}$$

The result is consistent with the lemma. The proof is completed. □

Next we will give the analytical expression of $A(t)$ for the n-unit parallel system.

Theorem 3: If working time X_i and repair time $Y_i (i=1,\cdots,n)$ of the n-unit parallel system with two states obey (16) and (17) respectively, then $A(t)$ of the system is as follows

$$A(t)=1-\frac{\lambda^n}{(\lambda+u)^n}\sum_{i=0}^{n}(-1)^iC_n^ie^{-i(\lambda+u)t}. \tag{35}$$

Proof: Applying Laplace transformation to (19), we can get

$$sP^*(s)-(1,0,\cdots,0)=P^*(s)M, \tag{36}$$

where $P^*(s)$ denotes the Laplace transformation of $P(t)$ and M is the transform rate matrix in (18). So

$$P^*(s)=(1,0,\cdots,0)(sE-M)^{-1}, \tag{37}$$

where E is identity matrix. As only when n units are all down, the system will be down, so in order to get $A(t)$, we only need to work out $P_n(t)$. Next we will calculate out $P_n^*(s)$ which is the Laplace transformation of $P_n(t)$. According to (37), $P_n^*(s)$ is as follows

$$P_n^*(s)=\frac{n!\lambda^n}{|sE-M|}. \tag{38}$$

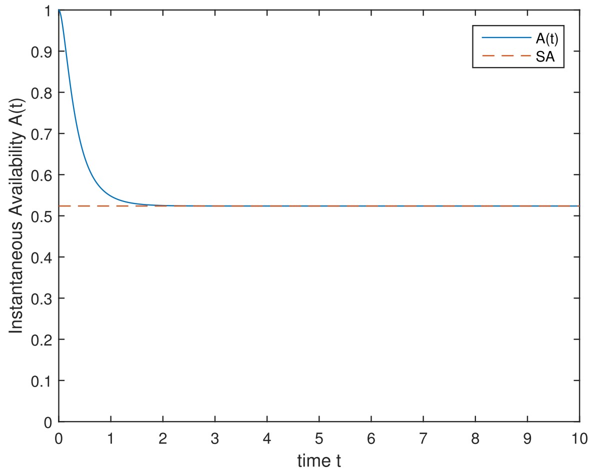

FIGURE 2. The curve of IA for case 1.

Based on Lemma 2, we have

$$|sE - M| = s(s+\lambda+u)(s+2(\lambda+u))\cdots(s+n(\lambda+u)). \tag{39}$$

So

$$P_n^*(s) = \frac{n!\lambda^n}{s(s+\lambda+u)(s+2(\lambda+u))\cdots(s+n(\lambda+u))}. \tag{40}$$

Then by Lemma 3, we can obtain

$$P_n^*(s) = \frac{\lambda^n}{(\lambda+u)^n}\sum_{i=0}^{n}(-1)^i\frac{C_n^i}{s+i(\lambda+u)}. \tag{41}$$

Applying inverse Laplace transformation to (41), the analytical expression of $P_n(t)$ is obtained as

$$P_n(t) = \frac{\lambda^n}{(\lambda+u)^n}\sum_{i=0}^{n}(-1)^i C_n^i e^{-i(\lambda+u)t}. \tag{42}$$

So the analytical expression of $A(t)$ is as follows

$$A(t) = 1 - \frac{\lambda^n}{(\lambda+u)^n}\sum_{i=0}^{n}(-1)^i C_n^i e^{-i(\lambda+u)t}. \tag{43}$$

The proof is completed. □

Finally, we will analyze the fluctuation of IA.

Theorem 4: IA of n-unit parallel system with two states has no fluctuation.

Proof: By taking derivative to (35), we have

$$A'(t) = \frac{\lambda^n}{(\lambda+u)^{(n-1)}}\sum_{i=1}^{n}(-1)^i C_n^i i e^{-i(\lambda+u)t}. \tag{44}$$

As $iC_n^i = nC_{n-1}^{i-1}$, we have

$$\begin{aligned} A'(t) &= \frac{n\lambda^n}{(\lambda+u)^{(n-1)}}\sum_{i=1}^{n}(-1)^i C_{n-1}^{i-1} e^{-i(\lambda+u)t} \\ &= -\frac{n\lambda^n}{(\lambda+u)^{(n-1)}} e^{-(\lambda+u)t}(1-e^{-(\lambda+u)t})^{n-1}. \end{aligned} \tag{45}$$

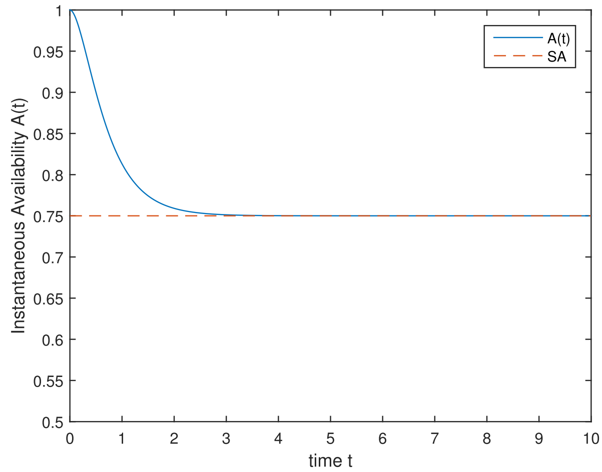

FIGURE 3. The curve of IA for case 2.

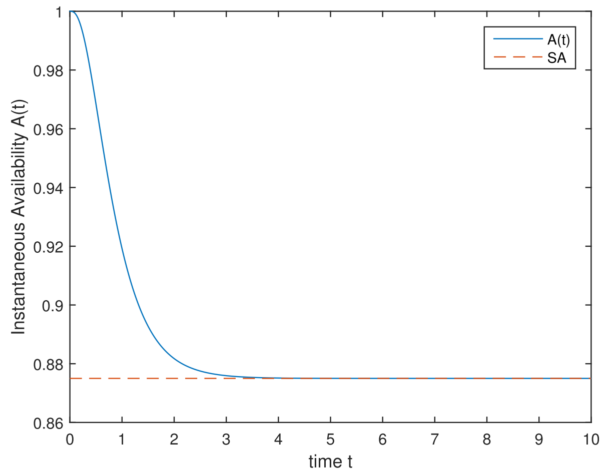

FIGURE 4. The curve of IA for case 3.

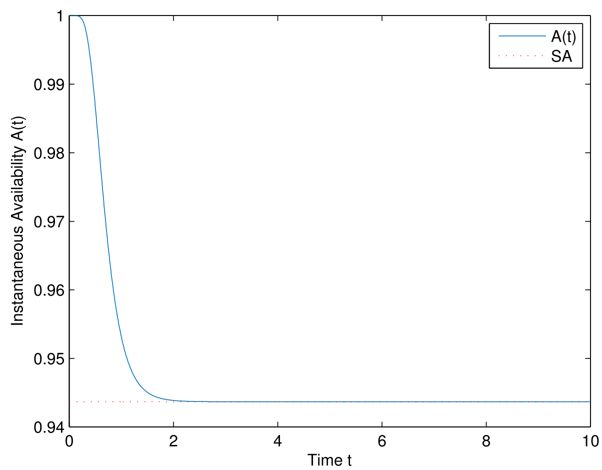

FIGURE 5. The curve of IA for case 4.

As $0 < e^{-(\lambda+u)t} < 1$ when $t > 0$ and $A'(t) < 0$ when $t \in (0, +\infty)$, combining with Lemma 1, we can draw the conclusion that the $A(t)$ of n-unit parallel system with two states has no fluctuation. The proof is completed. □

IV. NUMERICAL SIMULATION

In this section, we will give three examples to show the fluctuation of IA.

Case 1: $\lambda_1 = 2, u_1 = 1, \lambda_2 = 5, u_2 = 2, n = 2$.

FIGURE 2 shows that there is no fluctuation of IA for two-unit parallel system with two states. The results are coincided with Theorem 2.

Case 2: $\lambda = 1, u = 1, n = 2$.

FIGURE 3 shows that there is no fluctuation of IA for two-unit parallel system with two states, where the units of the system are the same. The results is coincided with Theorem 4.

Case 3: $\lambda = 1, u = 1, n = 3$.

Case 4: $\lambda = 3, u = 1, n = 10$.

FIGURE 4 and 5 show that there is no fluctuation of IA for three-unit parallel system with two states where the units of the system are the same. The results is coincided with Theorem 4. FIGURE 2-5 give four examples of four cases respectively, which show that there is no fluctuation of IA in each case. The results are coincided with Theorem 2 and Theorem 4.

V. CONCLUSIONS

In this paper, we have analyzed the fluctuation of IA for the two-unit parallel system with two states. By Laplace transformation, we have obtained the analytical expression of IA. With differential calculation, we have found that there does not exits fluctuation of IA. Moreover, we have analyzed the fluctuation of IA for the n-unit parallel system with the two states, where n units are all the same. Analyzing by the same way, we have found that the IA of n-unit parallel system has no fluctuation either. The results in this paper can be helpful when engineers are designing systems and hoping for little fluctuation of IA. However, there are still some deficiencies in the research. This paper has not touched on the fluctuation of IAs for two-unit parallel system under non-exponential distribution and for n-heterogeneous-unit parallel system. These problems will be researched in the future.

REFERENCES

[1] K. Wang, Y. Liu, M. Yin, and H. Jiang, "Analysis on the instantaneous availability of series repairable systems in cyber warfare environment," in *Proc. 33rd Chin. Control Conf.*, Jul. 2014, pp. 3001–3005.

[2] J. Li, K. Teng, C. Yang, and B. Li, "Research on the SMTN-based instantaneous availability of military aircraft in the mission preparation period," in *Proc. 7th Int. Conf. Mech. Aerosp. Eng. (ICMAE)*, Jul. 2016, pp. 517–523.

[3] M. Siergiejczyk, "Reliability model of motorway emergency communication system," *Solid State Phenomena*, vol. 210, pp. 122–129, Oct. 2013.

[4] A. Xue *et al.*, "Instantaneous availability for the protection devices based on non-homogeneous Markov model and a case study," in *Proc. 33rd Chin. Control Conf.*, Jul. 2014, pp. 6524–6527.

[5] X.-B. Li, Z.-J. Yang, G. Li, H. Zhao, F. Chen, and X. Zhu, "Instantaneous availability modelling of CNC machine tool based on functional data analysis," *J. Jilin Univ.*, vol. 42, no. 5, pp. 1151–1156, 2012.

[6] B. Sericola, "Interval-availability distribution of 2-state systems with exponential failures and phase-type repairs," *IEEE Trans. Rel.*, vol. 43, no. 2, pp. 335–343, Jun. 1994.

[7] D. V. Raje, R. S. Olaniya, P. D. Wakhare, and A. W. Deshpande, "Availability assessment of a two-unit stand-by pumping system," *Rel. Eng. Syst. Saf.*, vol. 68, no. 3, pp. 269–274, Jun. 2000.

[8] J. Mi, "Limiting availability of system with non-identical lifetime distributions and non-identical repair time distributions," *Statist. Probab. Lett.*, vol. 76, no. 7, pp. 729–736, Apr. 2006.

[9] H. Sun and J. J. Han, "Instantaneous availability and interval availability for systems with time-varying failure rate: Stair-step approximation," in *Proc. Pacific Rim Int. Symp. Dependable Comput.*, Dec. 2001, pp. 371–374.

[10] Y. Yang, "General Probab. "Distribution system instantaneous availability discrete time Model. Anal. application," Ph.D. dissertation, Nanjing Univ. Sci. Technol., Nanjing, China, 2008.

[11] J. Sarkar and A. Biswas, "Availability of a one-unit system supported by several spares and repair facilities," *J. Korean Stat. Soc.*, vol. 39, no. 2, pp. 165–176, Jun. 2010.

[12] Y. Yang, R. Kang, and L. Wang, "Comparative analysis of the continuous-time model and discrete-time instantaneous availability model of the delay repairable system," *J. Converg. Inf. Technol.*, vol. 7, no. 13, pp. 258–266, 2012.

[13] S. Ren, "Fluctuation analysis of the instantaneous availability for a one-unit repairable system," M.S. thesis, Nanjing Univ. Sci. Technol., Nanjing, China, 2017.

[14] Y. Yang, S. Ren, and Y. Yu, "Theory analysis of system instantaneous availability under uniform distribution," *J. Beijing Univ. Aeronaut. Astronaut.*, vol. 42, no. 1, pp. 28–34, Jan. 2016.

[15] S. Ren, Y. Yang, and H. Xu, "Fluctuation analysis of instantaneous availability under delay-repair," in *Proc. 35th Chin. Control Conf. (CCC)*, Jul. 2016, pp. 6746–6750.

[16] S. Ren, Y. Yang, Y. Chen, and R. Kang, "Fluctuation analysis of instantaneous availability under minor repair," *J. Beijing Univ. Aeronaut. Astronaut.*, vol. 43, no. 3, pp. 602–607, 2017.

[17] Q. Li, Y. Yang, X. Chen, and H. Xu, "Fluctuation analysis of instantaneous availability for the series repairable system," in *Proc. 36th Chin. Control Conf.(CCC)*, Jul. 2017, pp. 7443–7447.

[18] Y. Yang, Q. Li, and X. Chen, "The mathematical model and numerical solution of instantaneous availabilities for the two-unit series repairable system and parallel repairable system with three states," *Appl. Sci.*, vol. 8, no. 10, p. 1759, Sep. 2018.

[19] J. Cao and K. Cheng, *A Mathematical Introduction to Reliability*. Beijing, China: Higher Education Press, 2012.

Authors' photographs and biographies not available at the time of publication.