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A State Space Decomposition Filtering Method for a Class of Discrete-Time Singular Systems

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ABSTRACT This paper develops a state space decomposition method for a class of discrete-time singular systems. The singular matrix in this system is allowed to vary not only in the value but also in the dimension. An orthogonal factorization is used to rewrite the original space as a direct sum of its two subspaces. Correspondingly, the singular system is replaced by two reduced-order systems, whose states are then calculated respectively. Based on this construction, the prediction of part of the state, optimal filtering result, an optimal one step smoothing result is proposed. Lastly, a numerical example is presented to demonstrate the effectiveness of the new method.

INDEX TERMS Singular systems, space decomposition, state estimation, Kalman filter.

I. INTRODUCTION

Recently, a rich and growing literature has emerged on dynamic state estimation and widely applied in signal processing and control engineering [12]–[14]. However, only a small fraction has been concerned with the estimation problem for singular systems (Sometimes called implicit system or descriptor systems). Singular formulation arises naturally in most problems of non-causal phenomena and should be considered in many fields. Applications involving singular models have included image modeling, economical systems, electronic network model, robotics, and two manipulators grasping object model [10]. The filtering problem for singular system was first proposed by [8]. Dai considers the system with square singular matrix and presents the result by reducing the original system to a standard linear system [3]. Then, the research area was extended to some more general and complex cases. The various approaches taken to these systems mainly contain augmentation and least square methods for the system with rectangle singular matrix [6], a "9-block" form method for state noises and measurement noises being correlated system [7], a stochastic shuffle method for general singular system [9], and polynomial filtering methods for stochastic non-Gaussian system [4]. In [16], the stabilization of a class of fuzzy singular system is discussed by using of sliding-mode control. In [2], the stability of continuous-time positive singular systems subject to time-varying delays is discussed.

It should be pointed out that only a few works mentioned above argue the prediction of the state (or part of the state) and the smoothing problem. The major difficulty for the prediction problem lies in the fact that the coefficient matrix of current state is singular. Ishihara et al. presents the prediction of singular system [7]. However, the results are under the assumption that the singular matrix has full column rank, which limits its application. Zhang et al. turn singular system into a nonsingular one and develops the predictor by adding the measurement equation to the state equation [15]. The main shortcomings of this prediction algorithm are the need of using measurement, which means the plant cannot work if the measurement is delay or even missing. As for the smoothing problem, the difficulty in which is to find an equation that describes the relation between current measurement and the state at former instant.

In this paper, a new state space decomposition method for adapting the Kalman filter [1] algorithm is developed to solve the prediction and filtering problem without the assumption of singular matrix being full column rank. The filtering process is easily understood since it is similar to the standard Kalman filter. For the same singular system, we also establish the relation equation bridging the current measurement and the state at former instant to solve the one

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step smoothing problem. The space spanned by the singular matrix can be viewed as a subspace of the state space, which enables us to decompose the full state space into two parts and estimate two components respectively. The singular dynamic system then has a new framework, resulting in two nonsingular state space estimation problem of reduced dimension. When the state space is decomposed in this manner, one component is solved by using the singular dynamic equation, the other component is given by combining the system equation and measurement equation, and the initial parameterization can be recovered at any time.

The remainder of this paper is organized as follows. Section II formulates the singular systems and the problems to be solved. Section III contains description of state space decomposition, predictor of part of the state, and a new filtering approach. In section IV, a smoothing result based on space decomposition is developed. a numerical example is presented to illustrate the new method performance in section V and a conclusion is offered in Section VI.

II. PROBLEM FORMULATION

Consider a linear discrete-time stochastic singular system described by following model:

$$M\,(k)\,x\,(k) = A\,(k-1)\,x\,(k-1) + w\,(k-1) \tag{1}$$

$$z(k) = H(k)x(k) + v(k) \tag{2}$$

where $x(k) \in R^n$, $z(k) \in R^m$, represent the state, measurement output, respectively. $w(k)$ is the system noise, and $v(k)$ is the measurement noise. $M(k)$, $A(k-1)$, $H(k)$ are real matrices with appropriated dimensions.

Assumption 1: $M(k) \in R^{r(k)\times n}$ is a rectangular matrix, and $Rank\,[M(k)] = s(k) < r(k) \leq n$.

Assumption 2: System is observable, i.e.

$$Rank\begin{bmatrix} zM(k) - A(k) \\ H(k) \end{bmatrix}, \quad Rank\begin{bmatrix} M(k) \\ H(k) \end{bmatrix} = n$$

where z is an arbitrary complex.

Assumption 3: $w(k)$ and $v(k)$ are independent with zero mean and positive covariance matrix $W(k)$ and $V(k)$, respectively, i.e.

$$E\left\{\begin{bmatrix} w(k) \\ v(k) \end{bmatrix}\begin{bmatrix} w^T(j)v^T(j) \end{bmatrix}\right\} = \begin{bmatrix} W(k) & 0 \\ 0 & V(k) \end{bmatrix}\delta_{k,j}$$

Assumption 4: The initial state $x(0)$ with mean x_0 and covariance p_0 is independent of $w(k)$ and $v(k)$.

In Assumption 1, we explicitly allow the singular matrix $M(k)$ to vary with time. In this framework, we even allow the variation not only in the value of $M(k)$, but in the dimension. However, change in the singular matrix requires matrix factorizations and state space decomposition to be computed at each step, at a potentially computation cost. Some of the cost and performance effects are reduced in the time-invariant system, i.e. $M(k) = M$, $A(k) = A$, and $H(k) = H$ for any k.

As we all know, when $M(k)$ is a nonsingular square matrix, the prediction $\hat{x}(k|k-1)$ of the state $x(k)$ based on measurement $z(1), z(2), \cdots, z(k-1)$ can be obtained [1]. Thus, the optimal estimate result can be obtained by using the standard Kalman filtering. However, with singular matrix $M(k)$, can we still gain some information about $x(k)$ from the same measurements? If the answer is yes, with measurement $z(k)$, can we still formulate the estimate process being similar to Kalman filtering? In this paper, we will give the answer to above two questions, and find the optimal filter and smoother of original state.

III. PREDICTION AND FILTERING

Singular dynamic equations (1) illustrates that only a part of the state $x(k)$ is related with the state $x(k-1)$, which means that it is impossible to solve the prediction problem of $x(k)$ based on $z(1), z(2), \cdots, z(k-1)$. In this section, we will decompose the state space into two parts according to the construction of $M(k)$, and put the estimate problem into these two subspaces. One part, which also can be treated as the prediction of part of the state, is estimated from the singular dynamic systems and measurements $z(1), z(2), \cdots, z(k-1)$. The other part will be estimated based on the measurement $z(k)$. Provided both the two components of the state estimate in two subspaces are determined, they can always be summed to produce an estimate in the original full state space. Next, we will turn to a classical idea of QR factorization to decompose the state space.

A. STATE SPACE DECOMPOSITION

Let us first consider the solution of an equation

$$Gx = b \tag{3}$$

where G is a $m \times n$ matrix and b is a m -dimensional vector. We assume that G has full row rank and that $m < n$. The solution of this equation is not unique, and we will use state space decomposition method to discuss the solution set.

Lemma 1 [11]: For an $k \times n$ full column rank matrix B with $k > n$, the QR factorization of B yields an $k \times k$ orthogonal matrix Q and an $k \times n$ upper triangular matrix R such that.

$$B = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

where Q_1 is a $k \times n$ matrix, Q_2 is a $k \times (k-n)$ matrix, and R_1 is a $n \times n$, nonsingular, upper triangular matrix.

From lemma 1, the QR factorization of G^T can be written as

$$G^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}\begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{4}$$

where Q_1 is a $n \times m$ matrix, Q_2 is a $n \times (n-m)$ matrix, and R_1 is a $m \times m$, nonsingular, upper triangular matrix.

In addition, since the columns of Q are a orthogonal vector bundle, the spaces respectively spanned by Q, Q_1, and Q_2, satisfy

$$span\,(Q) = span\,(Q_1) \bigoplus span\,(Q_2)$$

If $x \in span\,(Q)$ is a solution of (3), there will exist $x_1 \in span\,(Q_1)$ and $x_2 \in span\,(Q_2)$, satisfying

$$x = x_1 + x_2$$

On the other hand, if the value range of x_1 and x_2 are known, the solution set of (3) will be found. Even better, since Q_1 is a basis for $span\,(Q_1)$ and Q_2 is a basis for $span\,(Q_2)$, the solution can be written as

$$x = Q_1\xi + Q_2\eta \tag{5}$$

and the problem is changed into searching for two reduced dimension vector ξ and η.

Substituting (4) and (5) into (3) gives

$$\begin{aligned}&\left([Q_1 \quad Q_2]\begin{bmatrix}R_1\\0\end{bmatrix}\right)^T [Q_1\xi + Q_2\eta]\\ &= \begin{bmatrix}R_1^T & 0\end{bmatrix}\begin{bmatrix}Q_1^T\\Q_2^T\end{bmatrix}[Q_1\xi + Q_2\eta]\\ &= R_1^T Q_1^T [Q_1\xi + Q_2\eta]\\ &= R_1^T Q_1^T Q_1\xi + R_1^T Q_1^T Q_2\eta\\ &= R_1^T\xi\\ &= b\end{aligned}$$

i.e.

$$\xi = R_1^{-T} b$$

From above discussion, we find that ξ is determined and η is without any constraint. The solution set of equation (3) can be written as

$$x = Q_1 R_1^{-T} b + Q_2\eta$$

where η is arbitrary.

B. PREDICTION OF PART OF THE STATE

The decomposition method defined in previous subsection can be used to create a new state space model on which to operate. Consider the singular system (1) and (2). For each time index k, since $M(k)$ in (1) is not full row rank, there exist a nonsingular matrix $D(k) \in R^{r(k)\times r(k)}$, such that

$$D(k)M(k) = \begin{bmatrix}M_1(k)\\0\end{bmatrix} \tag{6}$$

where $M_1(k)$ is a $s(k) \times n$ full row matrix, and 0 is a $(n - s(k)) \times n$ zero matrix.

With the QR factorization of $M_1^T(k)$ and equation (5), the state $x(k)$ can be written as

$$x(k) = Q_1(k)\xi(k) + Q_2(k)\eta(k) \tag{7}$$

where $\xi(k) \in R^{s(k)}$ and $\eta(k) \in R^{n-s(k)}$ are two new and reduced vectors. We can see that if both the estimate of $\xi(k)$ and $\eta(k)$ can be found, the estimate of $x(k)$ also will be given.

Combining (6) and (7) gives

$$\begin{aligned}\begin{bmatrix}M_1(k)\\0\end{bmatrix}x(k) &= D(k)A(k-1)x(k-1) + D(k)w(k-1)\\ \begin{bmatrix}R_1^T(k)Q_1^T(k)\\0\end{bmatrix}&[Q_1(k)\xi(k) + Q_2(k)\eta(k)]\\ &= \bar{A}(k-1)x(k-1) + D(k)w(k-1)\\ \begin{bmatrix}R_1^T(k)\\0\end{bmatrix}\xi(k) &= \bar{A}(k-1)x(k-1) + D(k)w(k-1)\end{aligned} \tag{8}$$

where $\bar{A}(k-1) = D(k)A(k-1)$

Since $Rank\begin{bmatrix}R_1^T(k)\\0\end{bmatrix} = Rank\left[R_1^T(k)\right] = s(k) = dim[\xi(k)]$, the minimum-norm least squares solution, i.e. the prediction of $\xi(k)$ is

$$\hat{\xi}(k|k-1) = \Gamma(k)\bar{A}(k-1)\hat{x}(k-1|k-1) \tag{9}$$

where

$$\begin{aligned}\Gamma(k) &= \left([R_1(k) \quad 0]\,\bar{W}^{-1}(k)\begin{bmatrix}R_1^T(k)\\0\end{bmatrix}\right)^{-1}\\ &\quad\times [R_1(k) \quad 0]\,\bar{W}^{-1}(k)\\ \bar{W}(k) &= D(k)W(k-1)D^T(k)\\ &\quad+ \bar{A}(k-1)P(k-1|k-1)\bar{A}^T(k-1)\end{aligned}$$

The prediction error covariance is

$$\begin{aligned}&P_\xi(k|k-1)\\ &= E\left\{\left[\xi(k) - \hat{\xi}(k|k-1)\right]\left[\xi(k) - \hat{\xi}(k|k-1)\right]^T\right\}\\ &= \left([R_1(k) \quad 0]\,\bar{W}^{-1}(k)\begin{bmatrix}R_1^T(k)\\0\end{bmatrix}\right)^{-1}\end{aligned} \tag{10}$$

The prediction and prediction error covariance of $\xi(k)$ are presented by (9) and (10), respectively. Next, we will estimate based on measurement $z(k)$. It should be point out that, despite prediction contains only part of the full state, it is also important in some cases. For example, consider a vehicle moving on a road, whose state is 2-dimension composed of position and velocity, and the singular parameter is $M(k) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$. If the measurement at instant k is delay or even missing, most existing methods will cannot work. With our method, as expressed in (9) and (10), the prediction and corresponding covariance of position can be computed, which is critical to judge the position of the vehicle.

C. OPTIMAL FILTERING

Next, we will use the measurement equation to give the estimate result of $\eta(k)$ and update $\hat{\xi}(k|k-1)$. With the assumption 2 and $Rank\,[M(k)] = s(k)$, there will exist $n-s(k)$ rows of $H(k)$, denoted as $H_1(k)$, is full row rank and satisfies $Rank\begin{bmatrix}M(k)\\H_1(k)\end{bmatrix} = n$.

Without loss generalization, $H_1(k)$ is assumed to be composed of the first $n - s(k)$ rows of $H(k)$, which can be written as $H(k) = \begin{bmatrix} H_1(k) \\ H_2(k) \end{bmatrix}$. The initial measurement equation becomes

$$z_1(k) = H_1(k)x(k) + v_1(k) \tag{11}$$
$$z_2(k) = H_2(k)x(k) + v_2(k) \tag{12}$$

With (5), equation (11) can be written as

$$\begin{aligned} z_1(k) &= H_1(k)\left[Q_1(k)\xi(k) + Q_2(k)\eta(k)\right] + v_1(k) \\ &= \bar{H}_{12}(k)\eta(k) + \bar{H}_{11}(k)\xi(k) + v_1(k) \end{aligned} \tag{13}$$

where $\bar{H}_{11}(k) = H_1(k)Q_1(k)$, $\bar{H}_{12}(k) = H_1(k)Q_2(k)$

From the appendix, we know that $Rank\left[\bar{H}_{12}(k)\right] = n-s(k)$. With (9) and (13), the estimate value of $\eta(k)$, denoted as $\hat{\eta}_1(k|k)$, is

$$\hat{\eta}_1(k|k) = \bar{H}_{12}^{-1}(k)\left[z_1(k) - \bar{H}_{11}(k)\hat{\xi}(k|k-1)\right] \tag{14}$$

whose error covariance is

$$\begin{aligned} &P_{\eta,1}(k|k) \\ &= E\left\{\left[\eta(k) - \hat{\eta}_1(k|k)\right]\left[\eta(k) - \hat{\eta}_1(k|k)\right]^T\right\} \\ &= \bar{H}_{12}^{-1}(k)\left\{\bar{H}_{11}(k)P_{\xi}(k|k-1)\bar{H}_{11}^T(k) + V_1(k)\right\}\bar{H}_{12}^{-T}(k) \end{aligned}$$

where $V_1(k) = E\left\{v_1(k)v_1^T(k)\right\}$.

We can see that the prediction of $\xi(k)$ and estimate of $\eta(k)$ are described as (9) and (14), respectively. The corresponding error covariance is

$$\begin{aligned} P_{\xi,\eta,1}(k|k) &= E\left\{\begin{bmatrix} \xi(k) - \hat{\xi}(k|k-1) \\ \eta(k) - \hat{\eta}_1(k|k) \end{bmatrix}\right. \\ &\quad \left.\times \begin{bmatrix} \xi(k) - \hat{\xi}(k|k-1) \\ \eta(k) - \hat{\eta}_1(k|k) \end{bmatrix}^T\right\} \\ &= \begin{bmatrix} P_{\xi}(k|k-1) & Cov_{\xi,\eta,1}(k|k) \\ Cov_{\eta,\xi,1}(k|k) & P_{\eta,1}(k|k) \end{bmatrix} \end{aligned} \tag{15}$$

where

$$Cov_{\xi,\eta,1}(k|k) = P_{\xi}(k|k-1)\bar{H}_{11}^T(k)\bar{H}_{12}^{-T}(k)$$

and

$$Cov_{\eta,\xi,1}(k|k) = Cov_{\xi,\eta,1}^T(k|k)$$

Next, we use $z_2(k)$ to update $\hat{\xi}(k|k-1)$ and $\hat{\eta}_1(k|k)$, from (7) and (12), we have

$$z_2(k) = \bar{H}_{21}(k)\xi(k) + \bar{H}_{22}(k)\eta(k) + v_2(k) \tag{16}$$

where $\bar{H}_{21}(k) = H_2(k)Q_1(k)$, $\bar{H}_{22}(k) = H_2(k)Q_2(k)$.

Using the well-known projection theory [1], the update equation is

$$\begin{aligned} \begin{bmatrix} \hat{\xi}(k|k) \\ \hat{\eta}(k|k) \end{bmatrix} &= \begin{bmatrix} \hat{\xi}(k|k-1) \\ \hat{\eta}_1(k|k) \end{bmatrix} + K(k)\left\{z_2(k)\right. \\ &\quad \left. - \left[\bar{H}_{21}(k), \bar{H}_{22}(k)\right]\begin{bmatrix} \hat{\xi}(k|k-1) \\ \hat{\eta}_1(k|k) \end{bmatrix}\right\} \end{aligned} \tag{17}$$

where

$$\begin{aligned} K(k) &= P_{\xi,\eta,1}(k|k)\left[\bar{H}_{21}(k), \bar{H}_{22}(k)\right]^T \\ &\quad \times \left\{\left[\bar{H}_{21}(k), \bar{H}_{22}(k)\right]\right. \\ &\quad \left.\times P_{\xi,\eta,1}(k|k)\left[\bar{H}_{21}(k), \bar{H}_{22}(k)\right]^T + V_2(k)\right\}^{-1} \\ V_2(k) &= E\left\{v_2(k)v_2^T(k)\right\} \end{aligned}$$

The estimate error covariance is

$$P_{\xi,\eta}(k|k) = \left\{I(k) - K(k)\left[\bar{H}_{21}(k), \bar{H}_{22}(k)\right]\right\}P_{\xi,\eta,1}(k|k) \tag{18}$$

Equations (9), (14), (15), (17) and (18) construct the estimate process of $\begin{bmatrix} \hat{\xi}(k) \\ \hat{\eta}(k) \end{bmatrix}$, which is similar to the standard Kalman fitler. With equation (7), the estimate result of initial state is

$$\hat{x}(k|k) = \left[Q_1(k) \quad Q_2(k)\right]\begin{bmatrix} \hat{\xi}(k|k) \\ \hat{\eta}(k|k) \end{bmatrix}$$

and the corresponding estimate error covariance is

$$P(k|k) = \left[Q_1(k) \quad Q_2(k)\right]P_{\xi,\eta}(k|k)\begin{bmatrix} Q_1^T(k) \\ Q_2^T(k) \end{bmatrix}$$

Remark 1: When $M(k)$ is a full rank square matrix, the system will be nonsingular. There is no need to decompose the state space, since $Q_1(k) = I_n$, $\xi(k) = x(k)$, $Q_2(k)$ and $\eta(k)$ will disappear. Similarly, $z_2(k) = z(k)$, $H_2(k) = H(k)$, $z_1(k)$ and $H_1(k)$ will disappear. In this situation, the prediction will contain full state, and the singular filter collapses to the standard Kalman filter.

IV. OPTIMAL ONE STEP SMOOTHER

The key to solve a smoothing problem is to find an equation to describe the relation between current measurement $z(k+1)$ and the former instant state $x(k)$. Using the state space decomposition method, we will present the desired relation and develop the optimal one step smoother of the singular system in this subsection. At instant $k+1$, state equation (8) can be written as following two new equations.

$$R_1^T(k+1)\xi(k+1) = \bar{A}_1(k)x(k) + D_1(k+1)w(k)$$

i.e.

$$\xi(k+1)=R_1^{-T}(k+1)\left[\bar{A}_1(k)x(k)+D_1(k+1)w(k)\right] \tag{19}$$

and

$$0 = \bar{A}_2(k)x(k) + D_2(k+1)w(k) \tag{20}$$

where $\bar{A}_1(k)$ and $\bar{A}_2(k)$ are the first $s(k)$ rows of $\bar{A}(k)$ and the last $n-s(k)$ rows of $\bar{A}(k)$, respectively. $D_1(k+1)$ is the first $s(k)$ rows of $D_1(k+1)$ and $D_2(k+1)$ is the last $n-s(k)$ rows of $D_1(k+1)$.

At instant $k+1$, equation (13) gives

$$\begin{aligned} \eta(k+1) = \bar{H}_{12}^{-1}(k+1)&\left[z_1(k+1)\right. \\ &\left. - \bar{H}_{11}(k+1)\xi(k+1) - v_1(k+1)\right] \end{aligned} \tag{21}$$

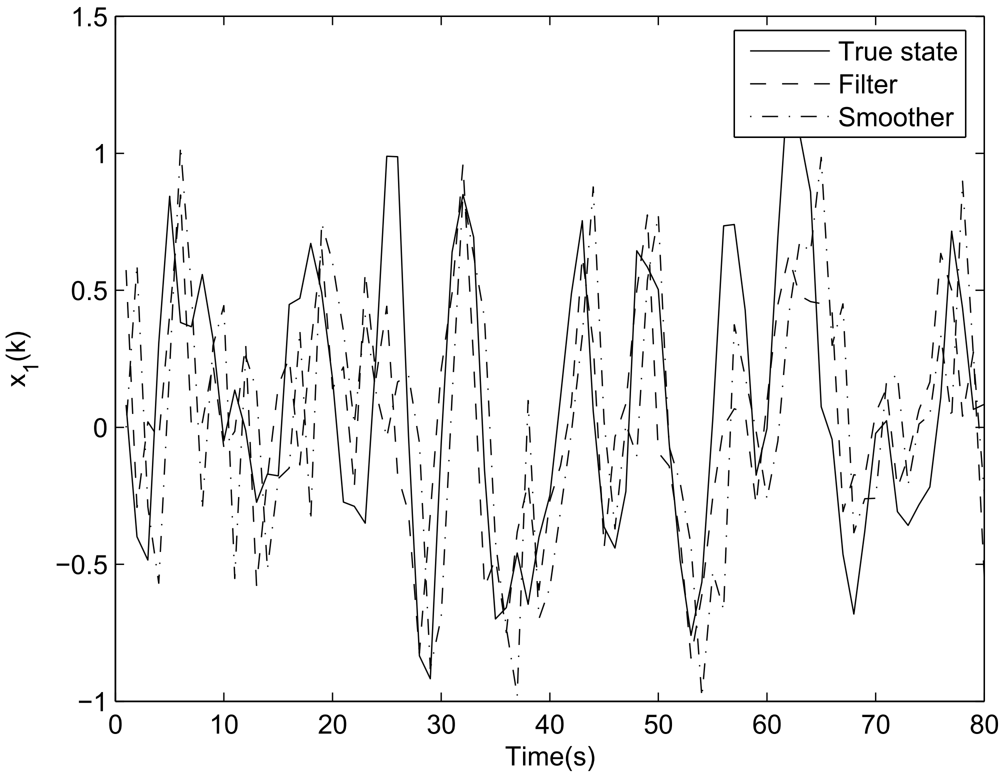

FIGURE 1. True value, filtering result and smoothing result of $x_1(k)$.

Substituting (21) into (16), we have

$$\begin{aligned} z_2(k+1) &= \begin{bmatrix} \bar{H}_{21}(k+1) & \bar{H}_{22}(k+1) \end{bmatrix} \begin{bmatrix} \xi(k+1) \\ \eta(k+1) \end{bmatrix} \\ &\quad + v_2(k+1) \\ &= \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)z_1(k+1) + \Big[\bar{H}_{21}(k+1) \\ &\quad - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)\bar{H}_{11}(k+1)\Big]\xi(k+1) \\ &\quad - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)v_1(k+1) + v_2(k+1) \end{aligned}$$

i.e.

$$\begin{aligned} &z_2(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)z_1(k+1) \\ &= \Big[\bar{H}_{21}(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)\bar{H}_{11}(k+1)\Big] \\ &\quad \times \xi(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)v_1(k+1) \\ &\quad + v_2(k+1) \end{aligned} \tag{22}$$

Substituting (19) into (22), we have

$$\begin{aligned} &z_2(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)z_1(k+1) \\ &= \Big[\bar{H}_{21}(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) \\ &\quad \times \bar{H}_{11}(k+1)\Big] R_1^{-T}(k+1)\bar{A}_1(k)x(k) \\ &\quad + \Big[\bar{H}_{21}(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) \\ &\quad \times \bar{H}_{11}(k+1)\Big] R_1^{-T}(k+1)D_1(k+1)w(k) \\ &\quad - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1)v_1(k+1) + v_2(k+1) \end{aligned} \tag{23}$$

Combing (20) and (23) together, we have a new measurement equation

$$y(k+1) = \Psi(k+1)x(k) + \Delta(k) \begin{bmatrix} w(k) \\ v(k) \end{bmatrix} \tag{24}$$

where

$$\Delta(k) = \begin{bmatrix} D_2(k+1) & 0 \\ \Lambda(k+1) & \Xi(k+1) \end{bmatrix}$$

$$\Xi(k+1) = \begin{bmatrix} -\bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) & I(k+1) \end{bmatrix}$$

$$y(k+1) = \begin{bmatrix} 0 \\ z_2(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) \\ \times z_1(k+1) \end{bmatrix}$$

$$\begin{aligned} \Lambda(k+1) &= \Big[\bar{H}_{21}(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) \\ &\quad \times \bar{H}_{11}(k+1)\Big] R_1^{-T}(k+1)D_1(k+1) \end{aligned}$$

$$\Psi(k+1) = \begin{bmatrix} \bar{A}_2(k) \\ \Big[\bar{H}_{21}(k+1) - \bar{H}_{22}(k+1)\bar{H}_{12}^{-1}(k+1) \\ \times \bar{H}_{11}(k+1)\Big] R_1^{-T}(k+1)\bar{A}_1(k) \end{bmatrix}$$

Using well known projection theory, equation (24) is used to update $\hat{x}(k|k)$ to obtain the one step smoothing result.

$$\hat{x}(k|k+1) = \hat{x}(k|k) + K(k|k+1)\left[y(k+1) - \Psi(k+1)\hat{x}(k|k)\right]$$

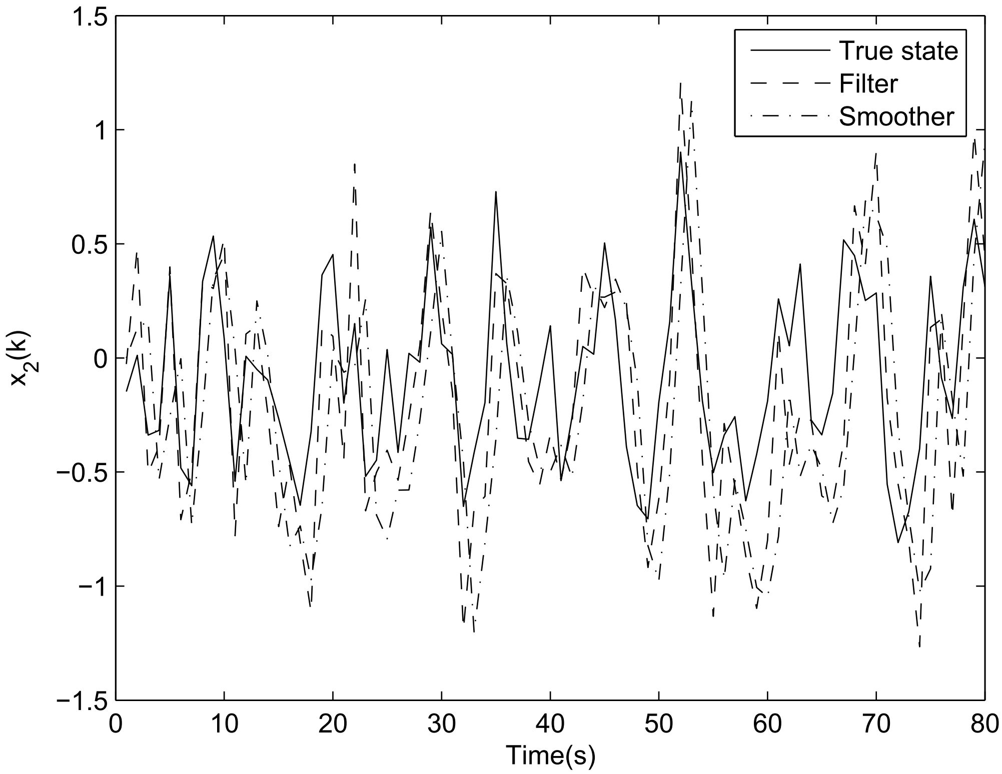

FIGURE 2. True value, filtering result and smoothing result of $x_2(k)$.

where

$$
\begin{aligned}
&K(k|k+1)\\
&\quad = P(k|k)\Psi^T(k+1)\Big\{\Psi(k+1)P(k|k)\Psi^T(k+1)\\
&\qquad + \Delta(k)\begin{bmatrix} W(k+1) & 0 \\ 0 & V(k+1)\end{bmatrix}\Delta^T(k)\Big\}^{-1}
\end{aligned}
$$

Its smooth error covariance is

$$P(k|k+1) = \{I(k|k+1) - K(k|k+1)\Psi(k+1)\}\, p(k|k)$$

Remark 2: When $M(k+1)$ is full rank, similar to the situation discussed in Remark 1, the singular smoother will collapses to the standard Kalman smoother. While if $M(k+1)$ is a full row rank matrix, i.e. $Rank[M(k+1)] = r(k+1) < n$, equation (20) will disappear, which means the singular dynamic equation at time $k+1$ does not affect its former state $x(k)$. In other words, the state equation will provide no help to the smoother. With assumption 2, when the dimension of $z(k+1)$ is equal to $n - s(k+1)$, from the derivation of smoothing process, we can find that the measurement can only be used to obtain the expression of $\eta(k+1)$, and $z_2(k+1)$, which is important to smooth $\hat{x}(k|k)$, will disappear. Furthermore, if both $Rank[M(k+1)] = r(k+1) < n$ and $Dim[z(k+1)] = n - s$ are satisfied, the update of step smoother is 0, i.e. $\hat{x}(k|k+1) = \hat{x}(k|k)$.

V. NUMERICAL EXAMPLE

We consider an example described by equation (1) and (2), with $x(k) = [x_1(k), x_2(k), x_3(k)]^T$. The singular matrix is variant both in value and dimension, and the parameters are given as follows.

$$
M(k) = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2}/2 & 0 \end{bmatrix} & k = 1, 3, \cdots \\ \begin{bmatrix} 1/2 & 0 & 1 \\ 1 & 0 & 1/2 \end{bmatrix} & k = 2, 4, \cdots \end{cases}
$$

$$
A(k) = \begin{cases} \begin{bmatrix} 1/2 & 3/2 & 1/2 \\ 2 & 1 & 3/2 \end{bmatrix} & k = 1, 3, \cdots \\ \begin{bmatrix} 1 & 1 & \sqrt{2} \\ 0 & 1+\sqrt{2} & 1 \\ 3/2 & \sqrt{2}/2 & \sqrt{2} \end{bmatrix} & k = 2, 4, \cdots \end{cases}
$$

$$
W(k) = \begin{cases} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} & k = 1, 3, \cdots \\ \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} & k = 2, 4, \cdots \end{cases}
$$

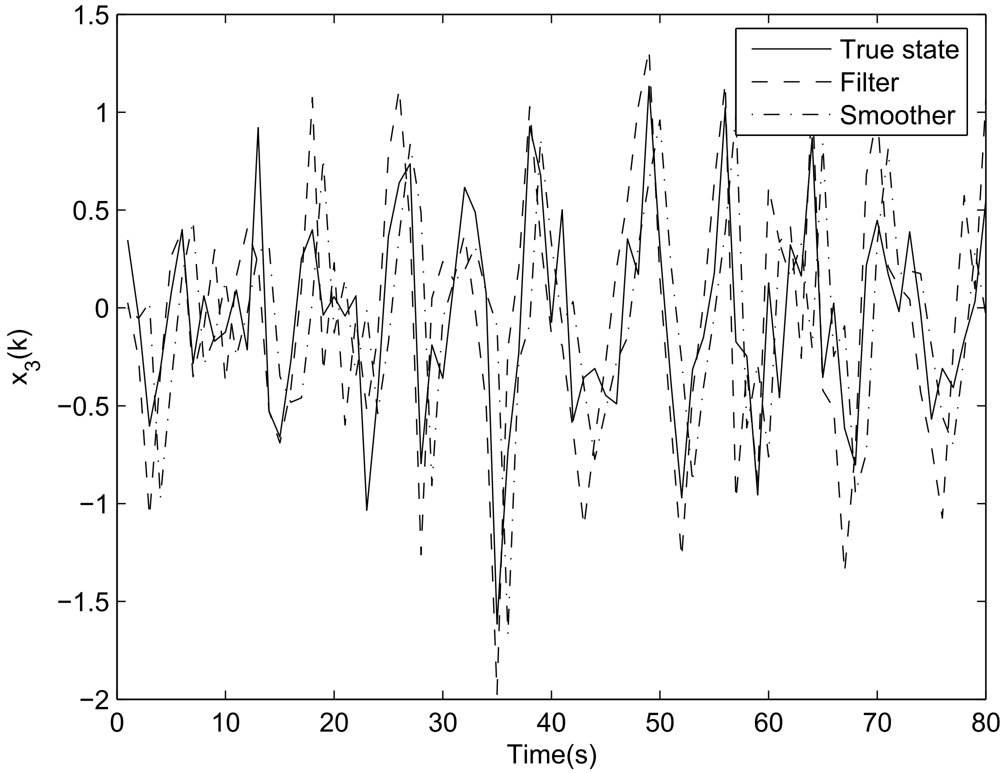

FIGURE 3. True value, filtering result and smoothing result of $x_3(k)$.

The measurement matrix and measurement noises covariance are time-invariant, which are $H(k) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $R(k) = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 1 & 0.2 \end{bmatrix}$.

Figs.1-3 display the simulation results of true state, filtering result, and the smoothing result.

VI. CONCLUSION

In this paper, we have developed a predictor of part of the state and a filter of full state for singular systems. Differently from previous methods, the key to our approach is to decompose the original state space into two parts by analyzing the singular matrix. The parameters in these two subspaces are estimated separately. One component, i.e. the prediction of part of the state, is given based on the state equation and measurements $z(1), z(2), \cdots, z(k-1)$ and the other component is estimated by the former component and current measurement $z(k)$. Furthermore, the decomposition allows for singular estimates to be reconstructed from the reduced state space to the original state space. For the same dynamic system, by establishing the relationship between measurement $z(k+1)$ and the state $x(k)$, we also presented a one-step smoother. Lastly, a numerical example is presented to show the effectiveness of state space decomposition method.

APPENDIX

The following gives a proof of $Rank\left[\bar{H}_{12}(k)\right] = n - s(k)$ Let us first introduce a lemma.

Lemma 2: Let $A \in R^{m\times n}$ $(m > n)$ be full column rank matrix, and $B \in R^{n\times r}$ $(n > r)$, We have the conclusion that $Rank[AB] = Rank[B]$.

Proof: Since $Rank[A] = n$, there exist n rows of A, denoted as A_1, such that $Rank[A_1] = n$. Without loss of generality, we may assume that A_1 is composed of first n rows of A and the last $m-n$ rows are denoted as A_2. then

$$Rank[AB] = Rank\begin{bmatrix} A_1B \\ A_2B \end{bmatrix} \geq Rank[A_1B] = Rank[B]$$

also because

$$Rank[AB] \leq Rank[B]$$

We have that

$$Rank[AB] = Rank[B]$$

From assumption 2 and the definition of $Q_2(k)$, we have

$$Rank\begin{bmatrix} M(k) \\ H_1(k) \end{bmatrix} = n$$

and

$$Rank[Q_2(k)] = n - s(k)$$

From lemma 2, we have

$$Rank\left\{\begin{bmatrix} M(k) \\ H_1(k) \end{bmatrix} Q_2(k)\right\} = n - s(k)$$

also because

$$M(k) = D^{-1}(k)\begin{bmatrix} M_1(k) \\ 0 \end{bmatrix} = D^{-1}(k)\begin{bmatrix} R_1^T(k)Q_1^T(k) \\ 0 \end{bmatrix}$$

Combining above two equations, we obtain

$$\begin{aligned} & Rank\left\{\begin{bmatrix} M(k) \\ H_1(k) \end{bmatrix} Q_2(k)\right\} \\ &= Rank\left\{\begin{bmatrix} D^{-1}(k)\begin{pmatrix} M_1(k) \\ 0 \end{pmatrix} \\ H_1(k) \end{bmatrix} Q_2(k)\right\} \\ &= Rank\left\{\begin{bmatrix} D^{-1}(k)\begin{pmatrix} M_1(k) \\ 0 \end{pmatrix} \\ H_1(k) \end{bmatrix} Q_2(k)\right\} \\ &= Rank\left\{\begin{bmatrix} D^{-1}(k)\begin{pmatrix} R_1^T(k)Q_1^T(k) \\ 0 \end{pmatrix} \\ H_1(k) \end{bmatrix} Q_2(k)\right\} \\ &= Rank\begin{bmatrix} D^{-1}(k)\begin{pmatrix} R_1^T(k)Q_1^T(k) \\ 0 \end{pmatrix} Q_2(k) \\ H_1(k)Q_2(k) \end{bmatrix} \\ &= Rank\begin{bmatrix} 0 \\ \bar{H}_{12}(k) \end{bmatrix} \\ &= n - s(k) \end{aligned}$$

which means

$$Rank[\bar{H}_{12}(k)] = n - s(k)$$

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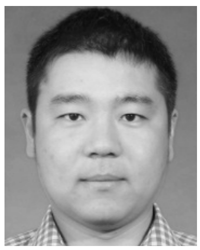

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