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A Fuzzy Rough Set Model Based on Reliability Over Dual-Universe and Its Applications

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ABSTRACT In this paper, using the weak fuzzy similarity relation, we propose a new similarity degree which describes the relationships between the 2-tuples in $U \times V$. In addition, we propose the reliability of each 2-tuples and use this reliability to give the calculation method of the similarity degree between the 2-tuples and the approximation set X. Using this similarity degree, a new fuzzy rough set model is proposed. Furthermore, a new fuzzy rough entropy which describing the knowledge information in the dual-universe $U \times V$ is defined. Then, we present a decision-making model to solve the problems, in which the elements in two different universes have different attributes. Finally, we give a significant algorithm which is applied to a decision-making problem in the dual-universe, and the effectiveness of this algorithm is verified by a numerical example.

INDEX TERMS Decision-making method, fuzzy rough entropy, reliability, rough set, similarity degree.

I. INTRODUCTION

Rough set theory [1], [2], a mathematical tool for data analyzing and processing, was first proposed by Pawlak in 1982. As a new soft computing tool for dealing with incomplete knowledge, it has become a hot mathematical framework for data mining, decision making, decision support, pattern recognizing, feature selection, machine learning, intelligent controlling and knowledge discovery [3]–[9].

In rough set theory, there are two representations of rough set definitions [10]. One is a pair of lower and upper approximations, another is three pair-wise disjoint positive, boundary and negative regions. The basic framework of the classical rough set theory is the approximation space composed of the universe and the binary relation. Although the rough set is composed of the upper approximation operator and lower approximation operator, the classification basis of classical rough set theory is the equivalence classes. This makes the rough set model with high misclassification rate and limits its application in a way. By using the probability theory and the variable precision method, some scholars have proposed the probability rough set model and variable precision rough set model [11]–[15]. These two models greatly reduce the misclassification rate of the elements, and make the rough set theory widely applied and popularized.

In recent years, in order to study uncertainty information more precisely, a large number of scholars began to study rough set theory from the following three aspects. Firstly, replace the approximation object X with a non-crisp set, such as fuzzy set. Secondly, replace equivalence relation with similarity relation, reflexive relation, fuzzy relation or fuzzy similarity relation. Dubois and Prade [16] were the first researches to propose the concepts of rough fuzzy set and fuzzy rough set by using the above two ideas. Cock et al. [17] then proposed two new upper and lower approximation operator definitions to solve the problem that an element in the fuzzy rough set may belong to multiple 'soft similar classes' at the same time. In 2016, Aggarwal [18] proposed the probabilistic variable precision fuzzy rough set model. These models were widely used in many fields, such as attribute reduction [19], attribute subset selection [20] and decision support system [21]. Thirdly, change one universe U into two universes U and V. In 2004, Pei and Xu proposed a rough set model based on two universes [22]. Based on this model, a large number of scholars began to study the uncertainty problems in two different universes, and proposed different models to deal with the corresponding questions [23]–[27]. Such as, Shen and Wang [25] gave a variable precision rough

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set model over two different universes and introduced its properties. Sun and Ma [26] proposed the fuzzy rough set model on two different universes and discussed two extended models of the fuzzy rough set model that named the degree fuzzy rough set and the variable precision fuzzy rough set. But these models are based on using one universe to judge the classification of elements in another universe, so there is a certain classification error. Abd El-Monsef et al [28] studied this issue and proposed some models that can make the approximating sets and the approximated sets are on the same universes of discourse.

As we all know, entropy is an important measure to study the uncertainty of information. In 1948, Shannon [29] proposed a system entropy to measure the uncertainty of the actual structure of the system. Liang and Shi [30] introduced the concepts of information entropy, rough entropy and knowledge granulation in rough set theory. In 2009, Bianucci and Cattaneo [31] made a summary for the concepts of information entropy and granulation co-entropy of partitions and coverings. Ma and Sun [32] gave the definition of the rough entropy for probabilistic rough set over two universes. In order to explain the similarity between the quantities mentioned in kinds of frameworks to evaluate a various entropy, Bouchon and Marsala [33] defined the concept of entropy measure. The rough entropy can be used to describe the uncertainty of the approximation set, but it is based on the equivalence class generated by binary relation. This ignores the indistinguishability of the elements in the same equivalence class.

In order to solve above difficulty, in this paper, we mainly discuss a new fuzzy rough set model on dual-universe combined with some of the binary relationships proposed in [34], [35]. We introduce the weak fuzzy similarity relation over two universes, and propose a calculation method for the similarity degree between the 2-tuples in $U \times V$ based on this relation. At the same time, we also give the reliability ξ_i of each 2-tuples in the approximation set X, and define the similarity degree between the 2-tuples and the approximation sets. Then we define the lower and upper approximation operators of the new fuzzy rough set in combination with the similarity degree. This model solves the problem that the relationship between elements in two universes with different attributes is difficult to describe. To explain this advantage, we give a numerical example of clinical diagnosis.

The structure of this paper is as follows. In Section 2, we introduce some basic definitions of rough set theory. In Section 3, we give the calculation methods for the similarity degree of (u_1, v_1) and (u_2, v_2) and the similarity degree between (u_i, v_j) and X based on the weak fuzzy similarity relation firstly. Then, we construct a new fuzzy rough set model based on reliability over dual-universe. This new concept of the fuzzy rough set can classify the elements in $U \times V$. We can solve the problems by giving corresponding decision-making method to different regions. In Section 4, we first define the approximation precision and rough degree of this fuzzy rough set based on reliability over dual-universe. Then we give the concept of the entropy of dual-universe information system. In Section 5, we present a numerical example of clinical diagnosis to verify the advantages of the model presented in this paper. Finally, in Section 6, we give some concluding remarks and future research.

II. PRELIMINARY

In this section, we introduce some basic concepts of rough set, fuzzy rough set and fuzzy similarity relations from [7], [8], [16], [22], [34] and [35].

Generally speaking, the theory begins with the notion of an approximation space, which is a pair (U, R), where U is a non-empty set (the universe of discourse) and R an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive. The relation R decomposes the set U into disjoint classes in such a way that two elements x, y are in the same class iff $(x, y) \in R$. Let U/R denote the quotient set of U by the relation R, and $U/R = \{X_1, X_2, \ldots, X_m\}$ where X_i is an equivalence class of R, $i = 1, 2, \ldots, m$.

If any two elements x, y in U belong to the same equivalence class $X_i \in U/R$, we say that x and y are indiscernible. The equivalence class of R and the empty set $\emptyset$ are the elements in the approximation space (U, R).

Definition 1 (See [7], [8]): Let (U, R) be an arbitrary approximation space. Given any set $X \in 2^U$, in general, it may not describe X in (U, R) precisely. One may characterize X by a pair of lower and upper approximations defined as follows,

$$\underline{apr}(X) = \cup\{[x]_R | [x]_R \subseteq X\} = \{x \in U | [x]_R \subseteq X\},$$
$$\overline{apr}(X) = \cup\{[x]_R | [x]_R \cap X \neq \emptyset\} = \{x \in U | [x]_R \cap X \neq \emptyset\}, \tag{1}$$

The lower approximation $\underline{apr}(X)$ is the union of all the elementary sets which are the subsets of X, and the upper approximation $\overline{apr}(X)$ is the union of all the elementary sets which have a non-empty intersection with X. The interval $[\underline{apr}(X), \overline{apr}(X)]$ is the representation of an ordinary set X in the approximation space (U, R) or simply, the rough set of X.

Furthermore, the positive region, the negative region and boundary region of X about the approximation space (U, R) are defined as follows, respectively,

$$pos(X) = \underline{apr}(X),$$
$$neg(X) = \sim \overline{apr}(X),$$
$$bn(X) = \overline{apr}(X) - \underline{apr}(X);$$

where the symbol $\sim$ stands for the complementarity of sets.

Definition 2 (See [16]): Let X be a set. ϕ be a fuzzy partition on X and F be a fuzzy set in X. The upper and lower approximations $\phi^*(F)$ and $\phi_*(F)$ of a fuzzy set F are defined by

$$M_i \triangleq \mu_{\phi^*(F)}(F_i) = \sup_x \min(\mu_{F_i}(x), \mu_F(x)),$$
$$m_i \triangleq \mu_{\phi_*(F)}(F_i) = \inf_x \max(1 - \mu_{F_i}(x), \mu_F(x)). \tag{2}$$

where M_i being the degree of possible membership of F_i in F, and m_i being the corresponding degree of certain membership. The pair $(\phi_*(F), \phi^*(F))$ can be called a fuzzy rough set.

Definition 3 (See [22]): Let U, V be two non-empty finite universes, and the elements between U with V have the compatible relation C, that is, for any $u \in U$, there must be existed an element $v \in V$ such that the compatible relation between u and v can be defined by a multi-value mapping $r : U \to 2^V$. i.e., $r(u) = \{v \in V|uCv, \text{ for any } u \in 2^U\}$. Where $r(u)$ denotes the set of all elements in V that related with u, based on the compatible relation r, for any $X(X \subseteq V)$, it can be described by the elements in U that is related with the elements of X. That is,

$$\underline{apr}_C(X) = \{u \in U|r(u) \subseteq X\},$$
$$\overline{apr}_C(X) = \{u \in U|r(u) \cap X \neq \emptyset\}. \tag{3}$$

Meanwhile, the positive region, the negative region and boundary region of X are defined as follows, respectively,

$$pos_C(X) = \underline{apr}_C(X),$$
$$neg_C(X) = \sim \overline{apr}_C(X),$$
$$bn_C(X) = \overline{apr}_C(X) - \underline{apr}_C(X),$$

where the symbol $\sim$ stands for the complementarity of sets. If $\overline{apr}_C(X) = \underline{apr}_C(X)$, then X is called definable on two different universes. If $\overline{apr}_C(X) \neq \underline{apr}_C(X)$, then X is called undefinable or rough set on two different universes.

Fuzzy similarity relation and weak fuzzy similarity relation are characterized by the following properties.

Definition 4 (See [35]): A fuzzy similarity relation is a mapping $R : U \times U \to [0, 1]$, such that for $x, y, z \in U$,

(1) Reflexivity: $R(x, x) = 1$,
(2) Symmetry: $R(x, y) = R(y, x)$,
(3) Max-min Transitivity:
$R(x, z) \geq max[min[R(x, y), R(y, z)]], y \in U$.

The weak fuzzy similarity relation has weaker symmetric and transitive properties as follows.

Definition 5 (See [34]): A weak fuzzy similarity relation is a mapping $R : U \times U \to [0, 1]$, such that for $x, y, z \in U$,

(1) Reflexivity: $R(x, x) = 1$,
(2) Conditional Symmetry: if $R(x, y) > 0$ then $R(y, x) > 0$,
(3) Conditional Transitivity: if $R(x, y) \geq R(y, x) > 0$ and $R(y, z) \geq R(z, y) > 0$ then $R(x, z) \geq R(z, x)$.

III. FUZZY ROUGH SET MODEL BASED ON RELIABILITY OVER DUAL-UNIVERSE

In this section, we give a calculation method of the weak fuzzy similarity relation of 2-tuples in the dual-universe. In addition, we combine the reliability of 2-tuples to define a similarity degree between 2-tuples (u_i, v_j) and approximation set X. Using this similarity degree, we define a fuzzy rough set model based on reliability over dual-universe and establish its basic properties.

A. THE SIMILARITY DEGREE BETWEEN 2-TUPLES AND APPROXIMATION SETS BASED ON RELIABILITY

In this section, we mainly introduce the calculation methods for the similarity degrees and give a simple example to illustrate these calculation methods.

Definition 6: Let U, V be two non-empty finite universes, $U \times V$ is the cartesian product of U and V, $U \times V = \{(u_i, v_j)|u_i \in U, v_j \in V, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$. $R : U \times V \to [0, 1]$ is a binary relation. $A = \{a_1, a_2, \ldots, a_p\}$ is the attribute set, $\mu_{a_l}(u_i, v_j)$ is the attribute values of (u_i, v_j) with respect to attribute a_l, $l = 1, 2, \ldots, p$. The pair $\langle U \times V, R, A\rangle$ is called a dual-universe information system. $R_A[(u_i, v_j), (u_s, v_t)]$ is called the similarity degree of (u_i, v_j) and (u_s, v_t) with respect to R, if,

$$R_A[(u_i, v_j), (u_s, v_t)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_i, v_j), \mu_{a_l}(u_s, v_t)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_s, v_t)}. \tag{4}$$

Theorem 1: The similarity degree R_A with regard to R is a weak fuzzy similarity relation.

Proof: For $\forall (u_1, v_1), (u_2, v_2), (u_3, v_3) \in U \times V$,

(1) By (4), we have

$$R_A[(u_1, v_1), (u_1, v_1)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_1, v_1)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1)} = 1.$$

(2) From (4), we have

$$R_A[(u_1, v_1), (u_2, v_2)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_2, v_2)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_2, v_2)} > 0.$$

since $\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_2, v_2) \geq 0$, we have

$$\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_2, v_2)] > 0.$$

Thus, we obtain that

$$\frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_2, v_2), \mu_{a_l}(u_1, v_1)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1)} > 0.$$

(3) From (4), we have

$$\frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_2, v_2)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_2, v_2)} \geq \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_2, v_2), \mu_{a_l}(u_1, v_1)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1)} > 0,$$

we have $\Sigma_{l=1}^{p} \mu_{a_l}(u_2, v_2) \leq \Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1)$. Similarly, $\Sigma_{l=1}^{p} \mu_{a_l}(u_3, v_3) \leq \Sigma_{l=1}^{p} \mu_{a_l}(u_2, v_2)$, so

$$\Sigma_{l=1}^{p} \mu_{a_l}(u_3, v_3) \leq \Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1). \tag{5}$$

By (4), we have

$$R_A[(u_1, v_1), (u_3, v_3)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_3, v_3)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_3, v_3)},$$

and

$$R_A[(u_3, v_3), (u_1, v_1)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_3, v_3), \mu_{a_l}(u_1, v_1)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_1, v_1)},$$

by (5) we get that

$$R_A[(u_1, v_1), (u_3, v_3)] \geq R_A[(u_3, v_3), (u_1, v_1)].$$

Theorem 2: The similarity degree R_A with regard to R is a fuzzy preordering.

Proof: By the Definition (6), for any $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in U \times V$, we have

$$R_A[(u_1, v_1), (u_2, v_2)] = \frac{\Sigma_{l=1}^{p} min[\mu_{a_l}(u_1, v_1), \mu_{a_l}(u_2, v_2)]}{\Sigma_{l=1}^{p} \mu_{a_l}(u_2, v_2)}.$$

It is easy to know that R_A is reflexive. Next, we need to prove that R_A is transitive.

In fact,

$$((u_1, v_1), (u_2, v_2)) \in R_A, \quad ((u_2, v_2), (u_3, v_3)) \in R_A.$$

By the proof of Theorem (3.1), we know that, if

$$R_A[(u_1, v_1), (u_3, v_3)] > 0,$$

and

$$R_A[(u_3, v_3), (u_1, v_1)] > 0.$$

And if

$$R_A[(u_1, v_1), (u_2, v_2)] \geq R_A[(u_2, v_2), (u_1, v_1)] > 0,$$
$$R_A[(u_2, v_2), (u_3, v_3)] \geq R_A[(u_3, v_3), (u_2, v_2)] > 0,$$

then

$$R_A[(u_1, v_1), (u_3, v_3)] \geq R_A[(u_3, v_3), (u_1, v_1)].$$

Thus, we get that

$$((u_1, v_1), (u_3, v_3)) \in R_A,$$

namely R_A is transitive. So the similarity degree R_A with regard to R is a fuzzy preordering.

Theorem 3: The similarity degree R_A with regard to R is a weak ordering.

Proof: We known that a weak ordering w is a fuzzy relation which is transitive and $\mu_w(x, y) > 0$ or $\mu_w(y, x) > 0$ for every $x \neq y$ in [28].

By the Theorem (2), we know R_A is transitive. For every $(u_i, v_j) \neq (u_s, v_t)$, by (4), we have

$$R_A[(u_i, v_j), (u_s, v_t)] > 0, \quad R_A[(u_s, v_t), (u_i, v_j)] > 0.$$

So the similarity degree R_A with regard to R is a weak ordering.

Definition 7: Let $X = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is a fuzzy set. According to the influence of each 2-tuples on the set X, we call $\xi_1, \xi_2, \ldots, \xi_n, (\xi_i \neq 0, i = 1, 2, \ldots, n)$ are the reliability of $(x_i, y_i), (i = 1, 2, \ldots, n)$ to judge X.

Definition 8: Let X, Y are two fuzzy sets, and $(x_1, y_1) \in X \cap Y$. If ξ_1 is the reliability of (x_1, y_1) to judge X, and ξ_2 is the reliability of (x_1, y_1) to judge Y. Then we define the reliability of (x_1, y_1) to judge $X \cap Y$ is $\xi_3 = \frac{\xi_1+\xi_2}{2}$.

Definition 9: Let $\langle U \times V, R, A\rangle$ be a dual-universe information system. $X = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\} \subseteq U \times V$ is a fuzzy set, and ξ_k is the reliability of $(x_k, y_k), (k = 1, 2, \ldots, n)$ to judge X. For $\forall (u_i, v_j) \in U \times V$, $R_A[(u_i, v_j), X]$ is called the similarity degree of (u_i, v_j) and X with respect to R, if,

$$\begin{aligned} R_A[(u_i, v_j), X] &= \xi_1 \cdot R_A[(u_i, v_j), (x_1, y_1)] \\ &\quad + \xi_2 \cdot R_A[(u_i, v_j), (x_2, y_2)] \\ &\quad + \cdots + \xi_n \cdot R_A[(u_i, v_j), (x_n, y_n)]. \end{aligned} \quad (6)$$

Based on the Definition (9), we know that the similarity degree of (u_i, v_j) and X is related to the reliability ξ_i, in which the weak fuzzy similarity relation R over the dual-universe $U \times V$ takes all the values of (0, 1]. As a result, we can get optimal results by giving different reliability ξ_i for different problems.

Example 1: Let $U \times V = \{(u_1, v_1), (u_1, v_2), (u_2, v_1), (u_2, v_2), (x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ be a dual-universe. $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is the attribute set on $U \times V$. Take two approximation object sets $X = \{(x_1, y_1), (x_2, y_2)\}$ and $Y = \{(x_2, y_2), (x_3, y_3)\}$, in which the reliability of (x_1, y_1) to judge X is $\xi_1 = 0.8$, the reliability of (x_2, y_2) to judge X is $\xi_2 = 0.5$, the reliability of (x_2, y_2) to judge Y is $\xi_3 = 0.7$, the reliability of (x_3, y_3) to judge Y is $\xi_4 = 0.4$. The attribute values of 2-tuples in $U \times V$ with respect to attribute set A are as follows, respectively.

By the Theorem (1), we can obtain the similarity degrees of (u_i, v_j) and (x_i, y_i) $i, j = 1, 2, 3$ with respect to R in the dual-universe $U \times V$ as follows, respectively,

$$R_A[(u_1, v_1), (x_1, y_1)] = \frac{2}{3}, \quad R_A[(u_1, v_2), (x_1, y_1)] = \frac{11}{27},$$
$$R_A[(u_2, v_1), (x_1, y_1)] = \frac{1}{9}, \quad R_A[(u_2, v_2), (x_1, y_1)] = \frac{14}{27},$$
$$R_A[(u_1, v_1), (x_2, y_2)] = \frac{16}{31}, \quad R_A[(u_1, v_2), (x_2, y_2)] = \frac{6}{31},$$
$$R_A[(u_2, v_1), (x_2, y_2)] = \frac{15}{31}, \quad R_A[(u_2, v_2), (x_2, y_2)] = \frac{10}{31},$$
$$R_A[(u_1, v_1), (x_3, y_3)] = \frac{10}{31}, \quad R_A[(u_1, v_2), (x_3, y_3)] = \frac{11}{31},$$
$$R_A[(u_2, v_1), (x_3, y_3)] = \frac{17}{31}, \quad R_A[(u_2, v_2), (x_3, y_3)] = \frac{15}{31}.$$

By the Definition (9), we can obtain the similarity degrees of (u_1, v_1) and X with respect to R_A in the dual-universe $U \times V$ as follows,

$$\begin{aligned} R_A[(u_1, v_1), X] &= \xi_1 \cdot R_A[(u_1, v_1), (x_1, y_1)] \\ &\quad + \xi_2 \cdot R_A[(u_1, v_1), (x_2, y_2)] \\ &= 0.8 \times \frac{2}{3} + 0.5 \times \frac{16}{31} = 0.79. \end{aligned}$$

By using the same calculation method, we get the following results, respectively,

$$R_A[(u_1, v_2), X] = 0.42, \quad R_A[(u_2, v_1), X] = 0.33,$$

$R_A[(u_2, v_2), X] = 0.58, \quad R_A[(u_1, v_1), Y] = 0.49,$
$R_A[(u_1, v_2), Y] = 0.28, \quad R_A[(u_2, v_1), Y] = 0.56,$
$R_A[(u_2, v_2), Y] = 0.42.$

By Definition (8), we can obtain the reliability of (x_2, y_2) that be used to judge $X \cap Y$ is $\xi_5 = \frac{\xi_2+\xi_3}{2} = 0.6$. So we can calculate the similarity degrees of (u_i, v_j) and $X \cap Y$ with respect to R_A are as follows, respectively,

$$R_A[(u_1, v_1), X \cap Y] = 0.6 \times \frac{16}{31} = 0.31,$$
$$R_A[(u_1, v_2), X \cap Y] = 0.6 \times \frac{6}{31} = 0.12,$$
$$R_A[(u_2, v_1), X \cap Y] = 0.6 \times \frac{15}{31} = 0.29,$$
$$R_A[(u_2, v_2), X \cap Y] = 0.6 \times \frac{10}{31} = 0.19.$$

B. THE FUZZY ROUGH SET MODEL BASED ON RELIABILITY OVER DUAL-UNIVERSE AND ITS PROPERTIES

In this section, we propose a fuzzy rough set model based on reliability over dual-universe, and prove some properties. Then we give a new decision-making method based on this fuzzy rough set model. In other words, according to this fuzzy rough set model, we can divide the 2-tuples in $U \times V$ into three regions, and give corresponding action plans for each region.

Definition 10: Let $\langle U \times V, R, A\rangle$ be a dual-universe information system. $X \subseteq U \times V$ is a fuzzy set, $R_A[(u_i, v_j), X]$ are the similarity degrees of (u_i, v_j) and X with respect to R_A, For any precision control parameters $0 \leq \beta < \alpha \leq 1$, we define a lower approximation of X and an upper approximation of X on the dual-universe $U \times V$ as follows, respectively,

$$\underline{R}_A^{(\xi,\alpha)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] \geq \alpha\},$$
$$\overline{R}_A^{(\xi,\beta)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] > \beta\}. \quad (7)$$

Furthermore, we also define the positive region $POS_A^{(\xi,\alpha)}(X)$, negative region $NEG_A^{(\xi,\beta)}(X)$ and boundary region $BND_A^{(\xi,\alpha,\beta)}(X)$ of X with respect to R_A on the dual-universe $U \times V$ as follows, respectively,

$$POS_A^{(\xi,\alpha)}(X) = \underline{R}_A^{(\xi,\alpha)}(X),$$
$$NEG_A^{(\xi,\beta)}(X) = U \times V - \overline{R}_A^{(\xi,\beta)}(X),$$
$$BND_A^{(\xi,\alpha,\beta)}(X) = \overline{R}_A^{(\xi,\beta)}(X) - \underline{R}_A^{(\xi,\alpha)}(X). \quad (8)$$

Obviously, the relation $\overline{R}_A^{(\xi,\beta)}(X) = POS_A^{(\xi,\alpha)}(X) \bigcup BND_A^{(\xi,\alpha,\beta)}(X)$ is satisfied according to the above definition. And for distinct regions, we can give the corresponding decision-making methods to solve the problems in the dual-universe information systems.

If $\overline{R}_A^{(\xi,\beta)}(X) = \underline{R}_A^{(\xi,\alpha)}(X)$, then X is called the definable set over $U \times V$. To the contrary, X is called undefinable set or the fuzzy rough set based on reliability over $U \times V$ while $\overline{R}_A^{(\xi,\beta)}(X) \neq \underline{R}_A^{(\xi,\alpha)}(X)$.

Example 2: (Continued Example (1)) We taking $\alpha = 0.54$, $\beta = 0.39$. According to the calculation results in Example (1), we can calculate the lower approximation, upper approximation, positive region, negative region and boundary region of X as follows, respectively,

$$\underline{R}_A^{(\xi,0.54)}(X)=\{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] \geq 0.54\} = \{(u_1, v_1), (u_2, v_2)\};$$
$$\overline{R}_A^{(\xi,0.39)}(X)=\{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] > 0.39\} = \{(u_1, v_1), (u_1, v_2), (u_2, v_2)\};$$
$$POS_A^{(\xi,0.54)}(X) = \underline{R}_A^{(\xi,0.54)}(X) = \{(u_1, v_1), (u_2, v_2)\};$$
$$BND_A^{(\xi,0.54,0.39)}(X) = \overline{R}_A^{(\xi,0.39)}(X) - \underline{R}_A^{(\xi,0.54)}(X) = \{(u_1, v_2)\};$$
$$NEG_A^{(\xi,0.39)}(X) = U \times V - \overline{R}_A^{(\xi,0.39)}(X) = \{(u_2, v_1)\}.$$

Similarly, the lower approximation, upper approximation, positive region, negative region and boundary region of Y are calculated as follows, respectively,

$$\underline{R}_A^{(\xi,0.54)}(Y)=\{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), Y] \geq 0.54\} = \{(u_2, v_1)\};$$
$$\overline{R}_A^{(\xi,0.39)}(Y)=\{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), Y] > 0.39\} = \{(u_1, v_1), (u_2, v_1), (u_2, v_2)\};$$
$$POS_A^{(\xi,0.54)}(Y) = \underline{R}_A^{(\xi,0.54)}(Y) = \{(u_2, v_1)\};$$
$$BND_A^{(\xi,0.54,0.39)}(Y) = \overline{R}_A^{(\xi,0.39)}(Y) - \underline{R}_A^{(\xi,0.54)}(Y) = \{(u_1, v_1), (u_2, v_2)\};$$
$$NEG_A^{(\xi,0.39)}(Y) = U \times V - \overline{R}_A^{(\xi,0.39)}(Y) = \{(u_1, v_2)\}.$$

Theorem 4: Let U, V be two non-empty finite universes. $U \times V$ is a dual-universe, $R_A[(u_i, v_j), X]$ is the similarity degree between (u_i, v_j) and X about R_A. For any X, Y, then the lower approximation and the upper approximation have the following properties,

(1) $\underline{R}_A^{(\xi,\alpha)}(X) \subseteq \overline{R}_A^{(\xi,\beta)}(X)$.
(2) $\underline{R}_A^{(\xi,\alpha)}(\emptyset) = \emptyset = \overline{R}_A^{(\xi,\beta)}(\emptyset)$.
(3) $\underline{R}_A^{(\xi,\alpha)}(X \cap Y) \subseteq \underline{R}_A^{(\xi,\alpha)}(X) \cap \underline{R}_A^{(\xi,\alpha)}(Y)$, $\underline{R}_A^{(\xi,\alpha)}(X \cup Y) \supseteq \underline{R}_A^{(\xi,\alpha)}(X) \cup \underline{R}_A^{(\xi,\alpha)}(Y)$.
(4) $\overline{R}_A^{(\xi,\beta)}(X \cap Y) \subseteq \overline{R}_A^{(\xi,\beta)}(X) \cap \overline{R}_A^{(\xi,\beta)}(Y)$, $\overline{R}_A^{(\xi,\beta)}(X \cup Y) \supseteq \overline{R}_A^{(\xi,\beta)}(X) \cup \overline{R}_A^{(\xi,\beta)}(Y)$.
(5) For any $X \subseteq Y$, there are $\underline{R}_A^{(\xi,\alpha)}(X) \subseteq \underline{R}_A^{(\xi,\alpha)}(Y)$ and $\overline{R}_A^{(\xi,\beta)}(X) \subseteq \overline{R}_A^{(\xi,\beta)}(Y)$.
(6) If $0 < \alpha_1 \leq \alpha_2 \leq 1$ and $0 \leq \beta_1 \leq \beta_2 < 1$, then $\underline{R}_A^{(\xi,\alpha_2)}(X) \subseteq \underline{R}_A^{(\xi,\alpha_1)}(X)$ and $\overline{R}_A^{(\xi,\beta_2)}(X) \subseteq \overline{R}_A^{(\xi,\beta_1)}(X)$.

Proof: It is easy to prove by the definitions of the lower and upper approximations. We only prove (5).

(5) By the Definition (10), we have

$$\underline{R}_A^{(\xi,\alpha)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] \geq \alpha\},$$
$$\overline{R}_A^{(\xi,\beta)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] > \beta\}.$$

Suppose $X = \{(x_1, y_1), (x_2, y_2)\}$, $Y = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$, and the reliability of (x_1, y_1) and (x_2, y_2) to judge X are ξ_1, ξ_2, the reliability of (x_1, y_1), (x_2, y_2) and (x_3, y_3) to judge Y are ξ_3, ξ_4, ξ_5, respectively. By the Definition (9), we have

$$\begin{aligned} R_A[(u_i, v_j), X] &= \xi_1 \cdot R_A[(u_i, v_j), (x_1, y_1)] \\ &\quad + \xi_2 \cdot R_A[(u_i, v_j), (x_2, y_2)], \\ R_A[(u_i, v_j), Y] &= \xi_3 \cdot R_A[(u_i, v_j), (x_1, y_1)] \\ &\quad + \xi_4 \cdot R_A[(u_i, v_j), (x_2, y_2)] \\ &\quad + \xi_5 \cdot R_A[(u_i, v_j), (x_3, y_3)]. \end{aligned}$$

Since $0 \leq \xi_m \leq 1, m = 1, 2, \ldots, 5$, and $R_A[u_i, v_j), (x_n, y_n)] \in [0, 1], n = 1, 2, 3$.

There is

$$[R_A[(u_i, v_j), X] \leq [R_A[(u_i, v_j), Y].$$

Thus, we obtain that

$$\underline{R}_A^{(\xi,\alpha)}(X) \subseteq \underline{R}_A^{(\xi,\alpha)}(Y), \quad \overline{R}_A^{(\xi,\beta)}(X) \subseteq \overline{R}_A^{(\xi,\beta)}(Y).$$

The proof is complete.

According to the Theorem (4) (6), we know that the lower approximation will increase with parameter α decrease, the upper approximation will decrease with parameter β increase, so that positive region will increase with parameter α decrease, negative region will increase with parameter β increase and boundary region will decrease with α decrease and β increase.

Theorem 5: If $\alpha = 1, \beta = 0$, then,

$$\underline{R}_A^{(\xi,\alpha)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] = 1\},$$
$$\overline{R}_A^{(\xi,\beta)}(X) = \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] > 0\}.$$

and

$$\begin{aligned} POS_A^{(\xi,\alpha)}(X) &= \underline{R}_A^{(\xi,\alpha)}(X) \\ &= \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] = 1\}, \\ BND_A^{(\xi,\alpha,\beta)}(X) &= \overline{R}_A^{(\xi,\beta)}(X) - \underline{R}_A^{(\xi,\alpha)}(X) \\ &= \{(u_i, v_j) \in U \times V | 0 < R_A[(u_i, v_j), X] \\ &< 1\}, \\ NEG_A^{(\xi,\beta)}(X) &= U \times V - \overline{R}_A^{(\xi,\beta)}(X) \\ &= \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] = 0\}. \end{aligned}$$

Theorem 6: Let $\langle U \times V, R, A \rangle$ be a dual-universe information system. For any $0 < \gamma < 1, X \subseteq U \times V$, we have,

(1) $\lim\limits_{\alpha \to \gamma^-} \underline{R}_A^{(\xi,\alpha)}(X) = \bigcap\limits_{0<\alpha<\gamma} \underline{R}_A^{(\xi,\alpha)}(X) = \underline{R}_A^{(\xi,\gamma)}(X),$

(2) $\lim\limits_{\beta \to \gamma^+} \overline{R}_A^{(\xi,\beta)}(X) = \bigcup\limits_{\gamma<\beta<1} \overline{R}_A^{(\xi,\beta)}(X) = \overline{R}_A^{(\xi,\gamma)}(X).$

Proof: (1) By the Theorem (4), we have

$$\lim_{\alpha \to \gamma^-} \underline{R}_A^{(\xi,\alpha)}(X) = \bigcap_{0<\alpha<\gamma} \underline{R}_A^{(\xi,\alpha)}(X).$$

If $\alpha < \gamma$, then $\underline{R}_A^{(\xi,\alpha)}(X) \supseteq \underline{R}_A^{(\xi,\gamma)}(X)$. Thus $\lim\limits_{\alpha \to \gamma^-} \underline{R}_A^{(\xi,\alpha)}(X) = \bigcap\limits_{0<\alpha<\gamma} \underline{R}_A^{(\xi,\alpha)}(X) \supseteq \underline{R}_A^{(\xi,\gamma)}(X)$. Conversely, $\forall (u.v) \in \bigcap\limits_{0<\alpha<\gamma} \underline{R}_A^{\xi,\alpha}(X)$, for any $0 < \alpha < \gamma$, we have $R_A[(u, v), X] \geq \alpha$.

Then $R_A[(u, v), X] \geq \gamma$ which implies that $(u, v) \in \underline{R}_A^{(\xi,\gamma)}(X)$.

Otherwise, if $R_A[(u, v), X] < \gamma$, take $\alpha = \frac{R_A[(u,v),X]+1}{2}$, then $0 < \alpha < \gamma$, but $R_A[(u, v), X] < \alpha$. This is in contradiction with the previous conclusions.

Therefore, we can get the following conclusions,

$$\lim_{\alpha \to \gamma^-} \underline{R}_A^{(\xi,\alpha)}(X) = \bigcap_{0<\alpha<\gamma} \underline{R}_A^{(\xi,\alpha)}(X) = \underline{R}_A^{(\xi,\gamma)}(X).$$

(2) By the Theorem (4), we have

$$\lim_{\beta \to \gamma^+} \overline{R}_A^{(\xi,\beta)}(X) = \bigcup_{\gamma<\beta<1} \overline{R}_A^{(\xi,\beta)}(X).$$

If $\beta > \gamma$, then $\overline{R}_A^{(\xi,\beta)}(X) \subseteq \overline{R}_A^{(\xi,\gamma)}(X)$. Thus $\lim\limits_{\beta \to \gamma^+} \overline{R}_A^{(\xi,\beta)}(X) = \bigcup\limits_{\gamma<\beta<1} \overline{R}_A^{(\xi,\beta)}(X) \subseteq \overline{R}_A^{(\xi,\gamma)}(X)$. Conversely, $\forall (u, v) \in \overline{R}_A^{(\xi,\beta)}(X)$, for any $\gamma < \beta < 1$, we have $R_A[(u, v), X] > \beta$.

We get that $R_A[(u, v), X] > \gamma$, and $(u, v) \in \overline{R}_A^{(\xi,\gamma)}(X)$.

Therefore, we can get the following conclusions,

$$\lim_{\beta \to \gamma^+} \overline{R}_A^{(\xi,\beta)}(X) = \bigcup_{\gamma<\beta<1} \overline{R}_A^{(\xi,\beta)}(X) = \overline{R}_A^{(\xi,\gamma)}(X).$$

The proof is complete.

Theorem (6) shows that the lower approximation $\underline{R}_A^{(\xi,\alpha)}(X)$ is left continuous with α, and the upper approximation $\overline{R}_A^{(\xi,\beta)}(X)$ is right continuous with β.

Theorem 7: Let $\langle U \times V, R, A \rangle$ be a dual-universe information system. For any $0 < \gamma < 1, X \subseteq U \times V$, we have,

(1)

$$\begin{aligned} \lim_{\alpha \to \gamma^+} \underline{R}_A^{(\xi,\alpha)}(X) &= \bigcup_{\gamma<\alpha\leq 1} \underline{R}_A^{(\xi,\alpha)}(X) \\ &\subseteq \underline{R}_A^{(\xi,\gamma)}(X), \end{aligned}$$

(2)

$$\begin{aligned} \lim_{\beta \to \gamma^-} \overline{R}_A^{(\xi,\beta)}(X) &= \bigcap_{0\leq\beta<\gamma} \overline{R}_A^{(\xi,\beta)}(X) \\ &\supseteq \overline{R}_A^{(\xi,\gamma)}(X), \end{aligned}$$

(3)

$$\begin{aligned} &\lim_{\alpha \to \gamma^+, \beta \to \gamma^-} BND^{(\xi,\alpha,\beta)}(X) \\ &= \bigcap_{0\leq\beta<\gamma<\alpha\leq 1} (\overline{R}_A^{(\xi,\beta)}(X) - \underline{R}_A^{(\xi,\alpha)}(X)) \\ &\supseteq \underline{R}_A^{(\xi,\gamma)}(X) - \overline{R}_A^{(\xi,\gamma)}(X) \\ &= \{(u_i, v_j) \in U \times V | R_A[(u_i, v_j), X] = \gamma\}. \end{aligned}$$

Proof: (1) By the Theorem (4), we have

$$\lim_{\alpha\to\gamma^+}\underline{R}_A^{(\xi,\alpha)}(X)=\bigcup_{\gamma<\alpha\le 1}\underline{R}_A^{(\xi,\alpha)}(X).$$

By the lower and upper approximations, if $\alpha>\gamma$, we have

$$\begin{aligned}\underline{R}_A^{(\xi,\alpha)}(X)&=\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]\ge\alpha\}\\&\subseteq\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]>\gamma\}\\&=\overline{R}_A^{(\xi,\gamma)}(X).\end{aligned}$$

Then there is $\bigcup_{\gamma<\alpha\le 1}\underline{R}_A^{(\xi,\alpha)}(X)\subseteq\overline{R}_A^{(\xi,\gamma)}(X)$, we have

$$\lim_{\alpha\to\gamma^+}\underline{R}_A^{(\xi,\alpha)}(X)=\bigcup_{\gamma<\alpha\le 1}\underline{R}_A^{(\xi,\alpha)}(X)\subseteq\overline{R}_A^{(\xi,\gamma)}(X).$$

(2) By the Theorem (4), we have

$$\lim_{\beta\to\gamma^-}\overline{R}_A^{(\xi,\beta)}(X)=\bigcap_{0\le\beta<\gamma}\overline{R}_A^{(\xi,\beta)}(X).$$

By the lower and upper approximations, if $\beta<\gamma$, we have

$$\begin{aligned}\overline{R}_A^{(\xi,\beta)}(X)&=\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]>\beta\}\\&\supseteq\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]\ge\gamma\}\\&=\underline{R}_A^{(\xi,\gamma)}(X).\end{aligned}$$

Then there is $\bigcap_{0\le\beta<\gamma}\overline{R}_A^{(\xi,\beta)}(X)\supseteq\underline{R}_A^{(\xi,\gamma)}(X)$, we have $\lim_{\beta\to\gamma^-}\overline{R}_A^{(\xi,\beta)}(X)=\bigcap_{0\le\beta<\gamma}\overline{R}_A^{(\xi,\beta)}(X)\supseteq\underline{R}_A^{(\xi,\gamma)}(X)$.

(3) It is easy to know that if $0\le\beta<\gamma<\alpha\le 1$, then $\underline{R}_A^{(\xi,\alpha)}(X)\subseteq\overline{R}_A^{(\xi,\gamma)}(X)$ and $\overline{R}_A^{(\xi,\beta)}(X)\supseteq\underline{R}_A^{(\xi,\gamma)}(X)$. So there is

$$\overline{R}_A^{(\xi,\beta)}(X)-\underline{R}_A^{(\xi,\alpha)}(X)\supseteq\underline{R}_A^{(\xi,\gamma)}(X)-\overline{R}_A^{(\xi,\gamma)}(X),$$

that is to say,

$$\begin{aligned}&\bigcap_{0\le\beta<\gamma<\alpha\le 1}(\overline{R}_A^{(\xi,\beta)}(X)-\underline{R}_A^{(\xi,\alpha)}(X))\\&\supseteq\underline{R}_A^{(\xi,\gamma)}(X)-\overline{R}_A^{(\xi,\gamma)}(X)\\&=\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]=\gamma\}.\end{aligned}$$

Since $BND^{(\xi,\alpha,\beta)}(X)$ will decrease as β increase and α decrease, then $\lim_{\alpha\to\gamma^+,\beta\to\gamma^-}BND^{(\xi,\alpha,\beta)}(X)=\bigcap_{0\le\beta<\gamma<\alpha\le 1}(\overline{R}_A^{(\xi,\beta)}(X)-\underline{R}_A^{(\xi,\alpha)}(X))$.

Thus, we get that

$$\begin{aligned}&\lim_{\alpha\to\gamma^+,\beta\to\gamma^-}BND^{(\xi,\alpha,\beta)}(X)\\&=\bigcap_{0\le\beta<\gamma<\alpha\le 1}(\overline{R}_A^{(\xi,\beta)}(X)-\underline{R}_A^{(\xi,\alpha)}(X))\\&\supseteq\underline{R}_A^{(\xi,\gamma)}(X)-\overline{R}_A^{(\xi,\gamma)}(X)\\&=\{(u_i,v_j)\in U\times V|R_A[(u_i,v_j),X]=\gamma\}.\end{aligned}$$

Theorem 8: Let $\langle U\times V,R,A\rangle$ be a dual-universe information system, where $U\times V=\{(u_1,v_1),(u_1,v_2),\ldots,(u_1,v_m),(u_2,v_1),\ldots,(u_n,v_m)\}$ and R_A be the similarity degree of $U\times V$. For any $0\le\beta<\alpha\le 1$ and $X\subseteq U\times V$, if $R_A[(u_1,v_1),X]\le R_A[(u_1,v_2),X]\le\ldots\le R_A[(u_1,v_m),X]\le R_A[(u_2,v_1),X]\le\ldots\le R_A[(u_n,v_m),X]$, then,

(1) If $(u_i,v_j)\in\underline{R}_A^{(\xi,\alpha)}(X)$, then

$$(u_{i+1},v_{j+1}),(u_{i+1},v_{j+2}),\ldots,(u_n,v_m)\in\underline{R}_A^{(\xi,\alpha)}(X).$$

(2) If $(u_i,v_j)\in\overline{R}_A^{(\xi,\beta)}(X)$, then

$$(u_{i+1},v_{j+1}),(u_{i+1},v_{j+2}),\ldots,(u_n,v_m)\in\overline{R}_A^{(\xi,\beta)}(X).$$

Proof: Without losing generality, we only prove the case of $U\times V=\{(u_1,v_1),(u_1,v_2),(u_1,v_3),(u_2,v_1),(u_2,v_2),(u_2,v_3)\}$.

(1) Since $(u_2,v_1)\in\underline{R}_A^{(\xi,\alpha)}(X)$, we know that,

$$R_A[(u_2,v_1),X]\ge\alpha,$$

and

$$R_A[(u_2,v_1),X]\le R_A[(u_2,v_2),X]\le R_A[(u_2,v_3),X]$$

are satisfied.

Therefore, $(u_2,v_2),(u_2,v_3)\in\underline{R}_A^{(\xi,\alpha)}(X)$.

(2) Since $(u_2,v_1)\in\overline{R}_A^{(\xi,\beta)}(X)$, we know that,

$$R_A[(u_2,v_1),X]>\beta,$$

and

$$R_A[(u_2,v_1),X]\le R_A[(u_2,v_2),X]\le R_A[(u_2,v_3),X]$$

are satisfied.

Therefore, $(u_2,v_2),(u_2,v_3)\in\overline{R}_A^{(\xi,\beta)}(X)$.

IV. APPROXIMATION PRECISION, ROUGH DEGREE AND FUZZY ROUGH ENTROPY

A. THE APPROXIMATION PRECISION AND ROUGH DEGREE OF THE NEW FUZZY ROUGH SET MODEL

In this section, we give the notion of the approximation precision and rough degree of this fuzzy rough set based on reliability over dual-universe.

Definition 11: Let U, V be two non-empty finite universes, $X\subseteq U\times V$ is a fuzzy set, $R_A[(u_i,v_j),X]$ is the similarity degree between (u_i,v_j) and X with respect to R_A. The approximation precision $\rho^{(\xi,\alpha,\beta)}(X)$ of X about R_A is defined as follows,

$$\rho^{(\xi,\alpha,\beta)}(X)=\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|},\tag{9}$$

Let $\varphi^{(\xi,\alpha,\beta)}(X)=1-\rho^{(\xi,\alpha,\beta)}(X)$ is the rough degree of X about R_A.

Where $X\ne\emptyset$, $|X|$ denotes the cardinality of the set X, and $0\le\rho^{(\xi,\alpha,\beta)}(X)\le 1$, $0\le\varphi^{(\xi,\alpha\beta)}(X)\le 1$.

Example 3: (Continued Example (2)) By the Definition (11), we have

$$\rho^{(\xi,0.54,0.39)}(X)=\frac{|\underline{R}_A^{(\xi,0.54)}(X)|}{|\overline{R}_A^{(\xi,0.39)}(X)|}=\frac{2}{3};$$
$$\varphi^{(\xi,0.54,0.39)}(X)=1-\rho^{(\xi,0.54,0.39)}(X)=\frac{1}{3};$$
$$\rho^{(\xi,0.54,0.39)}(Y)=\frac{|\underline{R}_A^{(\xi,0.54)}(Y)|}{|\overline{R}_A^{(\xi,0.39)}(Y)|}=\frac{1}{3};$$
$$\varphi^{(\xi,0.54,0.39)}(Y)=1-\rho^{(\xi,0.54,0.39)}(Y)=\frac{2}{3}.$$

Theorem 9: Let $\langle U\times V,\ R,A\rangle$ be a dual-universe information system. For any $0\le\beta<\alpha\le 1$, and $\emptyset\neq X\subseteq Y\subseteq U\times V$,

(1) If $\underline{R}_A^{(\xi,\alpha)}(X)=\underline{R}_A^{(\xi,\alpha)}(Y)$, then $\rho^{(\xi,\alpha,\beta)}(X)\ge\rho^{(\xi,\alpha,\beta)}(Y)$, $\varphi^{(\xi,\alpha,\beta)}(X)\le\varphi^{(\xi,\alpha,\beta)}(Y)$.

(2) If $\overline{R}_A^{(\xi,\beta)}(X)=\overline{R}_A^{(\xi,\beta)}(Y)$, then $\rho^{(\xi,\alpha,\beta)}(X)\le\rho^{(\xi,\alpha,\beta)}(Y)$, $\varphi^{(\xi,\alpha,\beta)}(X)\ge\varphi^{(\xi,\alpha,\beta)}(Y)$.

(3) If $\underline{R}_A^{(\xi,\alpha)}(X)=\underline{R}_A^{(\xi,\alpha)}(Y)$, $\overline{R}_A^{(\xi,\beta)}(X)=\overline{R}_A^{(\xi,\beta)}(Y)$, then $\rho^{(\xi,\alpha,\beta)}(X)=\rho^{(\xi,\alpha,\beta)}(Y)$, $\varphi^{(\xi,\alpha,\beta)}(X)=\varphi^{(\xi,\alpha,\beta)}(Y)$.

Proof: By the Theorem (4), if $X\subseteq Y$, we have

$$\underline{R}_A^{(\xi,\alpha)}(X)\subseteq\underline{R}_A^{(\xi,\alpha)}(Y),\quad \overline{R}_A^{(\xi,\beta)}(X)\subseteq\overline{R}_A^{(\xi,\beta)}(Y).$$

That is to say,

$$|\underline{R}_A^{(\xi,\alpha)}(X)|\le|\underline{R}_A^{(\xi,\alpha)}(Y)|,\quad |\overline{R}_A^{(\xi,\beta)}(X)|\le|\overline{R}_A^{(\xi,\beta)}(Y)|.$$

By the Definition (11), we know

$$\rho^{(\xi,\alpha,\beta)}(X)=\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|},\quad \varphi^{(\xi,\alpha,\beta)}(X)=1-\rho^{(\xi,\alpha,\beta)}(X).$$

(1) If $\underline{R}_A^{(\xi,\alpha)}(X)=\underline{R}_A^{(\xi,\alpha)}(Y)$, there is

$$|\underline{R}_A^{(\xi,\alpha)}(X)|=|\underline{R}_A^{(\xi,\alpha)}(Y)|.$$

Thus, we obtain

$$\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|}\ge\frac{|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(Y)|}.$$

That is to say,

$$\rho^{(\xi,\alpha,\beta)}(X)\ge\rho^{(\xi,\alpha,\beta)}(Y),\quad \varphi^{(\xi,\alpha,\beta)}(X)\le\varphi^{(\xi,\alpha,\beta)}(Y).$$

(2) If $\overline{R}_A^{(\xi,\beta)}(X)=\overline{R}_A^{(\xi,\beta)}(Y)$, there is

$$|\overline{R}_A^{(\xi,\beta)}(X)|\le|\overline{R}_A^{(\xi,\beta)}(Y)|.$$

Thus, we obtain,

$$\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|}\le\frac{|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(Y)|}.$$

That is to say,

$$\rho^{(\xi,\alpha,\beta)}(X)\le\rho^{(\xi,\alpha,\beta)}(Y),\quad \varphi^{(\xi,\alpha,\beta)}(X)\ge\varphi^{(\xi,\alpha,\beta)}(Y).$$

(3) If $\underline{R}_A^{(\xi,\alpha)}(X)=\underline{R}_A^{(\xi,\alpha)}(Y)$, and $\overline{R}_A^{(\xi,\beta)}(X)=\overline{R}_A^{(\xi,\beta)}(Y)$; there are

$$|\underline{R}_A^{(\xi,\alpha)}(X)|=|\underline{R}_A^{(\xi,\alpha)}(Y)|,\quad |\overline{R}_A^{(\xi,\beta)}(X)|=|\overline{R}_A^{(\xi,\beta)}(Y)|.$$

Thus, we obtain,

$$\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|}=\frac{|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(Y)|}.$$

That is to say,

$$\rho^{(\xi,\alpha,\beta)}(X)=\rho^{(\xi,\alpha,\beta)}(Y),\quad \varphi^{(\xi,\alpha,\beta)}(X)=\varphi^{(\xi,\alpha,\beta)}(Y).$$

Theorem 10: Let $\langle U\times V,\ R,A\rangle$ be a dual-universe information system. For any $0\le\beta_1\le\beta_2<\alpha_1\le\alpha_2\le 1$, then,

(1) $\underline{R}_A^{(\xi,\alpha_2)}(X)\subseteq\underline{R}_A^{(\xi,\alpha_1)}(X)$, $\overline{R}_A^{(\xi,\beta_2)}(X)\subseteq\overline{R}_A^{(\xi,\beta_1)}(X)$;

(2) $\rho^{(\xi,\alpha_1,\beta_2)}(X)\le\rho^{(\xi,\alpha_2,\beta_1)}(X)$.

Proof: By the Definition (10),

(1) Assume

$$0\le\beta_1\le\beta_2<\alpha_1\le\alpha_2\le 1,$$

we know

$$\underline{R}_A^{(\xi,\alpha_2)}(X)\subseteq\underline{R}_A^{(\xi,\alpha_1)}(X),$$

and

$$\overline{R}_A^{(\xi,\beta_2)}(X)\subseteq\overline{R}_A^{(\xi,\beta_1)}(X).$$

(2) It is easy to know

$$|\underline{R}_A^{(\xi,\alpha_2)}(X)|\le|\underline{R}_A^{(\xi,\alpha_1)}(X)|,$$

and

$$|\overline{R}_A^{(\xi,\beta_2)}(X)|\le|\overline{R}_A^{(\xi,\beta_1)}(X)|.$$

By the Definition (11), we have

$$\rho^{(\xi,\alpha_1,\beta_2)}(X)=\frac{|\underline{R}_A^{(\xi,\alpha_1)}(X)|}{|\overline{R}_A^{(\xi,\beta_2)}(X)|},$$

and

$$\rho^{(\xi,\alpha_2,\beta_1)}(X)=\frac{|\underline{R}_A^{(\xi,\alpha_2)}(X)|}{|\overline{R}_A^{(\xi,\beta_1)}(X)|}.$$

Thus, we obtain

$$\begin{aligned}\frac{\rho^{(\xi,\alpha_1,\beta_2)}(X)}{\rho^{(\xi,\alpha_2,\beta_1)}(X)}&=\frac{|\underline{R}_A^{(\xi,\alpha_1)}(X)|}{|\overline{R}_A^{(\xi,\beta_2)}(X)|}\Big/\frac{|\underline{R}_A^{(\xi,\alpha_2)}(X)|}{|\overline{R}_A^{(\xi,\beta_1)}(X)|}\\&=\frac{|\underline{R}_A^{(\xi,\alpha_1)}(X)|\cdot|\overline{R}_A^{(\xi,\beta_1)}(X)|}{|\overline{R}_A^{(\xi,\beta_2)}(X)|\cdot|\underline{R}_A^{(\xi,\alpha_2)}(X)|}\\&\le 1.\end{aligned}$$

That is

$$\rho^{(\xi,\alpha_1,\beta_2)}(X)\le\rho^{(\xi,\alpha_2,\beta_1)}(X).$$

Theorem 11: Let $\langle U\times V,\ R,A\rangle$ be a dual-universe information system. For any $0\le\beta<\alpha\le 1, X,Y\subseteq U\times V$. Then the following relations fold for approximation precision and rough degree of X, Y, $X\cup Y$ and $X\cap Y$.

(1)

$$\begin{aligned}&\varphi^{(\xi,\alpha,\beta)}(X\cup Y)|\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)|\\&\quad\le\varphi^{(\xi,\alpha,\beta)}(X)|\overline{R}_A^{(\xi,\beta)}(X)|+\varphi^{(\xi,\alpha,\beta)}(Y)|\overline{R}_A^{(\xi,\beta)}(Y)|\\&\quad-\varphi^{(\xi,\alpha,\beta)}(X\cap Y)|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|.\end{aligned}$$

(2)

$$\begin{aligned}\rho^{(\xi,\alpha,\beta)}(X\cup Y)&|\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\geq \rho^{(\xi,\alpha,\beta)}(X)|\overline{R}_A^{(\xi,\beta)}(X)| + \rho^{(\xi,\alpha,\beta)}(Y)|\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\quad - \rho^{(\xi,\alpha,\beta)}(X\cap Y)|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|.\end{aligned}$$

Proof: By the Definition (11), we have

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cup Y) &= 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(X\cup Y)|}{|\overline{R}_A^{(\xi,\beta)}(X\cup Y)|}\\ &\leq 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|\cup|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(X)|\cup|\overline{R}_A^{(\xi,\beta)}(Y)|}.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cup Y)&|\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\leq |\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)\cup\underline{R}_A^{(\xi,\alpha)}(Y)|.\end{aligned}$$

Similarly,

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cap Y) &= 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(X\cap Y)|}{|\overline{R}_A^{(\xi,\beta)}(X\cap Y)|}\\ &\leq 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|\cap|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(X)|\cap|\overline{R}_A^{(\xi,\beta)}(Y)|}.\end{aligned}$$

We have

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cap Y)&|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\leq |\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)\cap\underline{R}_A^{(\xi,\alpha)}(Y)|.\end{aligned}$$

As known, for any finite sets X and Y, the relation $|A\cup B| = |A|+|B|-|A\cap B|$ holds.Then,

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cup Y)&|\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\leq |\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)\cup\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &= |\overline{R}_A^{(\xi,\beta)}(X)| + |\overline{R}_A^{(\xi,\beta)}(Y)| - |\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\quad - |\underline{R}_A^{(\xi,\alpha)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(Y)| + |\underline{R}_A^{(\xi,\alpha)}(X)\cap\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &= |\overline{R}_A^{(\xi,\beta)}(X)| + |\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &\quad - \{|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)\cap\underline{R}_A^{(\xi,\alpha)}(Y)|\}\\ &= |\overline{R}_A^{(\xi,\beta)}(X)| + |\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &\quad + \{|\underline{R}_A^{(\xi,\alpha)}(X)\cap\underline{R}_A^{(\xi,\alpha)}(Y)| - |\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|\}\\ &\leq |\overline{R}_A^{(\xi,\beta)}(X)| + |\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &\quad - \varphi^{(\xi,\alpha,\beta)}(X\cap Y)|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|.\end{aligned}$$

Furthermore, by the relation of $\varphi^{(\xi,\alpha,\beta)}(X) = 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(X)|}{|\overline{R}_A^{(\xi,\beta)}(X)|}$ and $\varphi^{(\xi,\alpha,\beta)}(Y) = 1-\frac{|\underline{R}_A^{(\xi,\alpha)}(Y)|}{|\overline{R}_A^{(\xi,\beta)}(Y)|}$, we have

$$|\overline{R}_A^{(\xi,\beta)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(X)| = \varphi^{(\xi,\alpha,\beta)}(X)|\overline{R}_A^{(\xi,\beta)}(X)|,$$

and

$$|\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(Y)| = \varphi^{(\xi,\alpha,\beta)}(Y)|\overline{R}_A^{(\xi,\beta)}(Y)|.$$

Thus, we have

$$\begin{aligned}\varphi^{(\xi,\alpha,\beta)}(X\cup Y)&|\overline{R}_A^{(\xi,\beta)}(X)\cup\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\leq |\overline{R}_A^{(\xi,\beta)}(X)| + |\overline{R}_A^{(\xi,\beta)}(Y)| - |\underline{R}_A^{(\xi,\alpha)}(X)| - |\underline{R}_A^{(\xi,\alpha)}(Y)|\\ &\quad - \varphi^{(\xi,\alpha,\beta)}(X\cap Y)|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|\\ &= \varphi^{(\xi,\alpha,\beta)}(X)|\overline{R}_A^{(\xi,\beta)}(X)| + \varphi^{(\xi,\alpha,\beta)}(Y)|\overline{R}_A^{(\xi,\beta)}(Y)|\\ &\quad - \varphi^{(\xi,\alpha,\beta)}(X\cap Y)|\overline{R}_A^{(\xi,\beta)}(X)\cap\overline{R}_A^{(\xi,\beta)}(Y)|.\end{aligned}$$

So we prove the assertion (1), and the assertion (2) is easy to prove by the relation $\varphi^{(\xi,\alpha,\beta)}(X) = 1-\rho^{(\xi,\alpha,\beta)}(X)$.

B. A NEW FUZZY ROUGH ENTROPY OVER THE DUAL-UNIVERSE

In [32], Ma and Sun mentioned that there are some limitations in describing the uncertainty of probabilistic rough set X by using the precision and rough degree. Firstly, only the elements contained in the upper and lower approximation operators are used, while the other elements in the two universes are not used. Secondly, compared with different binary relations R and Q, the precision and rough degree of probabilistic rough set X may be the same, that is, it can not explain the difference of the binary relations. To solve the above problems, Ma and Sun [32] proposed the concept of rough entropy of probability rough sets over two universes. However, this kind of rough entropy is defined by equivalence class, and ignores the indistinguishability of elements in equivalence classes. In order to solve this difficulty, the entropy of the dual-universe is proposed by using the granulation information in the dual-universe.

Firstly, we give the definition of Shannon entropy [29], [31].

Let U be a non-empty finite universe, R be a binary equivalence relation on universe U. $U/R = \{X_1, X_2, \ldots, X_n\}$ is a partition determined by the equivalence relation R. $p_i = P(X_i) = \frac{|X_i|}{|U|}$ is the probability distribution on U/R. We call

$$H(U/R) = -\Sigma_{i=1}^{n} p_i \log_2 p_i$$

is the information entropy of U/R.

If there is $p_i = 0$ or $X_i = \emptyset$, then define $0\log_2 0 = 0$.

Based on the definition of Shannon entropy, we give the general entropy of dual-universe $U\times V$ and the fuzzy rough entropy of the object set X as follows,

Let $\langle U \times V, R, A\rangle$ be a dual-universe information system. For convenience, this section unify the 2-tuples in dual-universe $U\times V$ as (u_i, v_i). For example, if $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2\}$, then $U \times V = \{(u_1, v_1), (u_2, v_1), (u_3, v_1), (u_1, v_2), (u_2, v_2), (u_3, v_2)\} = \{(u_1, v_1), (u_2, v_2), (u_3, v_3), (u_4, v_4), (u_5, v_5), (u_6, v_6)\}$. The attribute set $A = \{a_1, a_2, \ldots, a_m\}$, and the attribute values of $(u_i, v_i), (i = 1, 2, \ldots, 6)$ are represented as $\mu_{a_1}(u_i, v_i), \mu_{a_2}(u_i, v_i), \ldots, \mu_{a_m}(u_i, v_i)$. Because the effects of attribute a_m on 2-tuple (u_i, v_i) are different, we assign corresponding weights to different attributes such as, $\omega_1, \omega_2, \ldots, \omega_m$, where $0 < \omega_k < \frac{1}{2}$.

TABLE 1. Dual-universe information system.

$U\times V$	a_1	a_2	a_3	a_4	a_5	a_6
(u_1,v_1)	1.0	0.8	0.2	0.0	0.0	0.1
(u_1,v_2)	0.0	0.2	0.8	0.2	0.0	0.0
(u_2,v_1)	0.0	0.0	0.0	0.8	1.0	1.0
(u_2,v_2)	0.0	0.4	0.8	0.2	0.1	0.1
(x_1,y_1)	0.8	0.9	0.7	0.2	0.1	0.0
(x_2,y_2)	0.7	0.6	0.2	0.9	0.3	0.4
(x_3,y_3)	0.2	0.5	0.7	0.6	0.3	0.8

Definition 12: Let $\langle U\times V,\ R, A\rangle$ be a dual-universe information system. An index value is a mapping $IV : U\times V \to (0, 1)$ as follows,

$$IV(u_i, v_i) = \Sigma_{k=1}^{m}\omega_k \cdot \mu_{a_k}(u_i, v_i). \tag{10}$$

where $\mu_{a_k}(u_i, v_i)$ is the attribute value of the 2-tuples (u_i, v_i), and ω_k is the weight of the attribution a_k.

Definition 13: Let $\langle U\times V,\ R, A\rangle$ be a dual-universe information system. IV be an index value function of dual-universe $U\times V$. Then, the entropy of dual-universe $U\times V$ is defined as follows,

$$E_{IV}(U\times V) = -\frac{1}{|U\times V|}\Sigma_{i=1}^{|U\times V|}\log_2 IV(u_i, v_i). \tag{11}$$

Theorem 12: Let $\langle U\times V,\ R, A\rangle$ be a dual-universe information system. $E_{IV}(U\times V)$ be the entropy of dual-universe $U\times V$. The following assertions hold,

(1) $0 \le E_{IV}(U\times V) \le 2\log_2 10$,
(2) If $IV(u_i, v_i) = 1, i = 1, 2, \ldots, |U\times V|$, the entropy of dual-universe $E_{IV}(U\times V)$ takes minimum value 0,
(2) If $IV(u_i, v_i) = \frac{1}{100}, i = 1, 2, \ldots, |U\times V|$, the entropy of dual-universe $E_{IV}(U\times V)$ takes maximum value $2\log_2 10$.

Proof: It can be easily proved by the Definition (13).

Definition 14: Let $\langle U\times V,\ R, A\rangle$ be a dual-universe information system. For any fuzzy set $X\subseteq U\times V$, $\varphi^{(\xi,\alpha,\beta)}(X)$ is the rough degree of X in $U\times V$. Then, the fuzzy rough entropy of the object set X in $U\times V$ is defined as follows,

$$E_{IV}(X) = -\varphi^{(\xi,\alpha,\beta)}(X)\frac{1}{|U\times V|}\Sigma_{i=1}^{|U\times V|}\log_2 IV(u_i, v_i). \tag{12}$$

Clearly, the fuzzy rough entropy of the object set X can also be described as follows,

$$E_{IV}(X) = \varphi^{(\xi,\alpha,\beta)}(X)E_{IV}(U\times V). \tag{13}$$

Example 4: (Continued the Example (3)) For convenience, we change the representations of the 2-tuples in Table 1, even if $(u_1, v_1) = (u_1, v_1), (u_2, v_2) = (u_2, v_2), (u_1, v_2) = (u_3, v_3), (u_2, v_1) = (u_4, v_4), (x_1, y_1) = (u_5, v_5), (x_2, y_2) = (u_6, v_6), (x_3, y_3) = (u_7, v_7)$. So, $U\times V = \{(u_i, v_i), i = 1, 2, \ldots, 7\}$, $X = \{(u_5, v_5), (u_6, v_6)\}$ and $Y = \{(u_6, v_6), (u_7, v_7)\}$. Let the weights of the attributions in $U\times V$ are $\omega_1 = 0.1$, $\omega_2 = 0.3$, $\omega_3 = 0.4$, $\omega_4 = 0.2$, $\omega_5 = 0.3$, $\omega_6 = 0.2$. Thus, the attribute values of 2-tuples in $U\times V$ with respect to attribute set A are as follows, respectively.

TABLE 2. Dual-universe information system.

$U\times V$	a_1	a_2	a_3	a_4	a_5	a_6
ω_i	0.1	0.3	0.4	0.2	0.3	0.2
(u_1,v_1)	1.0	0.8	0.2	0.0	0.0	0.1
(u_2,v_2)	0.0	0.4	0.8	0.2	0.1	0.1
(u_3,v_3)	0.0	0.2	0.8	0.2	0.0	0.0
(u_4,v_4)	0.0	0.0	0.0	0.8	1.0	1.0
(u_5,v_5)	0.8	0.9	0.7	0.2	0.1	0.0
(u_6,v_6)	0.7	0.6	0.2	0.9	0.3	0.4
(u_7,v_7)	0.2	0.5	0.7	0.6	0.3	0.8

By the Definition (12), we can calculate the following results,

$$IV(u_1, v_1) = 0.44,\ IV(u_2, v_2) = 0.53,$$
$$IV(u_3, v_3) = 0.42,\ IV(u_4, v_4) = 0.66,$$
$$IV(u_5, v_5) = 0.70,\ IV(u_6, v_6) = 0.68,$$
$$IV(u_7, v_7) = 0.82.$$

So, the entropy of dual-universe $U\times V$ is follows,

$$\begin{aligned}E_{IV}(U\times V) &= -\frac{1}{|U\times V|}\Sigma_{i=1}^{|U\times V|}\log_2 IV(u_i, v_i)\\ &= -\frac{1}{7}\Sigma_{i=1}^{7}\log_2 IV(u_i, v_i)\\ &= -\frac{1}{7}\cdot(\log_2\frac{44}{100} + \log_2\frac{53}{100} + \log_2\frac{42}{100}\\ &\quad + \log_2\frac{66}{100} + \log_2\frac{70}{100} + \log_2\frac{68}{100}\\ &\quad + \log_2\frac{82}{100})\\ &\approx 0.758.\end{aligned}$$

By the Example (3), we know that $\varphi^{(\xi,0.54,0.39)}(X) = \frac{1}{3}$, $\varphi^{(\xi,0.54,0.39)}(Y) = \frac{2}{3}$. Then the fuzzy rough entropy of object set X and Y in $U\times V$ as follows, respectively,

$$\begin{aligned}E_{IV}(X) &= -\varphi^{(\xi,0.54,0.39)}(X)\frac{1}{|U\times V|}\Sigma_{i=1}^{|U\times V|}\log_2 IV(u_i, v_i)\\ &= \varphi^{(\xi,0.54,0.39)}(X)E_{IV}(U\times V)\\ &= \frac{1}{3}E_{IV}(U\times V)\\ &\approx 0.253.\\ E_{IV}(Y) &= -\varphi^{(\xi,0.54,0.39)}(Y)\frac{1}{|U\times V|}\Sigma_{i=1}^{|U\times V|}\log_2 IV(u_i, v_i)\\ &= \varphi^{(\xi,0.54,0.39)}(Y)E_{IV}(U\times V)\\ &= \frac{2}{3}E_{IV}(U\times V)\\ &\approx 0.506.\end{aligned}$$

V. A NUMERICAL EXAMPLE ABOUT THE FUZZY ROUGH SET MODEL BASED ON RELIABILITY OVER THE DUAL-UNIVERSE

In this section, we give an algorithm for the decision-making model and provide a numerical example about clinical diagnosis to verify the advantages of the decision-making method presented in this paper.

A. AN ALGORITHM FOR THE DECISION-MAKING MODEL

In this section, we propose an algorithm for the decision-making model based on the new fuzzy rough set. The specific steps are as follows.

Step 1: Calculate the similarity degrees of (u_i, v_j) and (x_n, y_n) according to Definition (6).

Step 2: Calculate the similarity degrees of (u_i, v_j) and X with respect to R_A according to Definition (9).

Step 3: Calculate the lower approximation, upper approximation, positive region, negative region and boundary region of X. And the 2-tuples in the dual-universe will be divided into the positive, negative and boundary regions.

Step 4: Calculate the precision $\rho^{(\xi,\alpha,\beta)}(X)$ and the rough degree $\varphi^{(\xi,\alpha,\beta)}(X)$ of X.

Step 5: The decision results are obtained by analyzing the calculated values and practical examples. In other words, the experts will give different action plans for 2-tuples in different regions.

B. A NUMERICAL EXAMPLE ABOUT CLINICAL DIAGNOSIS

In this section, we give a simple application of the fuzzy rough set model based on reliability over dual-universe. We all know that each person's body index has a certain degree of influence on his illness, and the severity of the same symptom will also affect the disease. Therefore, different treatments should be given in different situations. However, the existing researches on disease diagnosis have ignored this problem. The fuzzy rough set model based on reliability over dual-universe proposed in this paper solves this difficulty well.

Suppose that U and V are the sufferer set and the symptom set, respectively. We get a dual-universe $U \times V$ with some 2-tuples (u_i, v_j) by calculating their cartesian product. $A = \{a_1, a_2, \dots, a_m\}$ is an attribute set that represents the patient's body index. For any disease $X = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ which the 2-tuples in X has the same attributes as the 2-tuples in $U \times V$, and $\xi_1, \xi_2, \dots, \xi_n$ are the reliability of (x_k, y_k), $(k = 1, 2, \dots, n)$ to judge X. The detailed calculation and analysis are as follows.

Example 5: Let $U \times V = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_2, v_1), (u_2, v_2), (u_2, v_3), (x_1, y_1), (x_2, y_2)\}$ is a dual-universe, $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is the attribute set on $U \times V$. Take an approximation object set $X = \{(x_1, y_1), (x_2, y_2)\}$, in which the reliability of (x_1, y_1) to judge X is $\xi_1 = 0.8$, the reliability of (x_2, y_2) to judge X is $\xi_2 = 0.5$. The attribute values of 2-tuples in $U \times V$ and X with respect to attribute set A are as follows, respectively,

TABLE 3. Dual-universe information system.

$U \times V$	a_1	a_2	a_3	a_4	a_5	a_6
(u_1, v_1)	1.0	0.8	0.2	0.0	0.0	0.1
(u_1, v_2)	0.0	0.2	0.8	0.2	0.0	0.0
(u_1, v_3)	0.1	0.2	0.3	0.2	1.0	0.6
(u_2, v_1)	0.0	0.0	0.0	0.8	1.0	1.0
(u_2, v_2)	0.0	0.4	0.8	0.2	0.1	0.1
(u_2, v_3)	0.3	0.2	0.8	0.7	0.2	0.0
(x_1, y_1)	0.8	0.9	0.7	0.2	0.1	0.0
(x_2, y_2)	0.7	0.6	0.2	0.9	0.3	0.4

We taking $\alpha = 0.57, \beta = 0.45$. By the Definition (6), we can obtain the similarity degrees of $(u_i, v_j)(i = 1, 2, j = 1, 2, 3)$ and $(x_k, y_k)(k = 1, 2)$ as follows,

$$R_A[(u_1, v_1), (x_1, y_1)] = \frac{2}{3}, \quad R_A[(u_1, v_1), (x_2, y_2)] = \frac{16}{31},$$
$$R_A[(u_1, v_2), (x_1, y_1)] = \frac{11}{27}, \quad R_A[(u_1, v_2), (x_2, y_2)] = \frac{6}{31},$$
$$R_A[(u_1, v_3), (x_1, y_1)] = \frac{1}{3}, \quad R_A[(u_1, v_3), (x_2, y_2)] = \frac{14}{31},$$
$$R_A[(u_2, v_1), (x_1, y_1)] = \frac{1}{9}, \quad R_A[(u_2, v_1), (x_2, y_2)] = \frac{15}{31},$$
$$R_A[(u_2, v_2), (x_1, y_1)] = \frac{14}{27}, \quad R_A[(u_2, v_2), (x_2, y_2)] = \frac{10}{31},$$
$$R_A[(u_2, v_3), (x_1, y_1)] = \frac{5}{9}, \quad R_A[(u_2, v_3), (x_2, y_2)] = \frac{16}{31}.$$

By the Definition (9), we can obtain the similarity degree of (u_1, v_1) and X with respect to R_A in the dual-universe $U \times V$ as follows,

$$\begin{aligned} R_A[(u_1, v_1), X] &= \xi_1 \cdot R_A[(u_1, v_1), (x_1, y_1)] \\ &\quad + \xi_2 \cdot R_A[(u_1, v_1), (x_2, y_2)] \\ &= 0.8 \times \frac{2}{3} + 0.5 \times \frac{16}{31} = 0.79, \end{aligned}$$

Through the same calculation method, we get the following results, respectively,

$$R_A[(u_1, v_2), X] = 0.42, \quad R_A[(u_1, v_3), X] = 0.49,$$
$$R_A[(u_2, v_1), X] = 0.33, \quad R_A[(u_2, v_2), X] = 0.58,$$
$$R_A[(u_2, v_3), X] = 0.70.$$

Then we can calculate the lower approximation, upper approximation, positive region, negative region and boundary region of the X as follows, respectively,

$$\begin{aligned} \underline{R}_A^{(\xi,0.57)}(X) &= \{(u_1, v_1), (u_2, v_2), (u_2, v_3)\}; \\ \overline{R}_A^{(\xi,0.45)}(X) &= \{(u_1, v_1), (u_1, v_3), (u_2, v_2), (u_2, v_3)\}; \\ POS_A^{(\xi,0.57)}(X) &= \underline{R}_A^{(\xi,0.57)}(X) \\ &= \{(u_1, v_1), (u_2, v_2), (u_2, v_3)\}; \\ BND_A^{(\xi,0.57,0.45)}(X) &= \overline{R}_A^{(\xi,0.45)}(X) - \underline{R}_A^{(\xi,0.57)}(X) \\ &= \{(u_1, v_3)\}; \\ NEG_A^{(\xi,0.45)}(X) &= U \times V - \overline{R}_A^{(\xi,0.45)}(X) \\ &= \{(u_1, v_2), (u_2, v_1)\}. \end{aligned}$$

So the precision $\rho^{(\xi,0.57,0.45)}(X)$ and the rough degree $\varphi^{(\xi,0.57,0.45)}(X)$ of X $(X \in U \times V)$ can also be calculated as follows, respectively,

$$\rho^{(\xi,0.57,0.45)}(X) = \frac{|\underline{R}_A^{(\xi,0.57)}(X)|}{|\overline{R}_A^{(\xi,0.45)}(X)|} = \frac{3}{4};$$
$$\varphi^{(\xi,0.57,0.45)}(X) = 1 - \rho^{(\xi,0.57,0.45)}(X) = \frac{1}{4}.$$

Then, we get the following conclusions:

(1) The sufferers (u_1, v_1), (u_2, v_2) and (u_2, v_3) must ill with the disease X, and the doctor must immediately treat them.

(2) The sufferers (u_1, v_2), (u_2, v_1) are not ill with the disease X, and they do not need treatment of this department doctor.

(3) Since sufferer (u_1, v_3) belong to the boundary region, we can not judge that he (or she) is ill with disease X or not. So he (or she) need the second thoughts of the department doctor.

From this numerical example, we know that physical indicators and the severity of symptoms are the main factors in determining the sufferer is ill with disease or not. For example, the sufferer u_1 with the symptom v_1 must ill with the disease X, but the sufferer u_1 with the symptom v_2 is not ill with the disease X. This is the biggest advantage of using the model presented in this paper to deal with medical problems.

VI. CONCLUSIONS

In this paper, we define the calculation methods for the similarity degree of the 2-tuples in $U \times V$ and the similarity degree between (u_i, v_j) and X. Based on these definitions of the similarity degree, we construct a fuzzy rough set model based on reliability over dual-universe, and give a new decision-making method based on this fuzzy rough set model. This model is to discuss the information on dual-universe, which can reduce classification error to a greater extent. Finally we use a numerical example of clinical diagnosis to verify and interpret the advantages of the model presented in this paper.

The fuzzy rough set defined in this paper based on weak fuzzy similarity degree have the following advantages:

(1) We use the cartesian product of different elements in the different universes to generate a new dual-universe consisting of 2-tuples. And these 2-tuples are represented under the same attribute vector, which overcomes the difficulty of data analysis and processing because of different element attributes in different universes.

(2) The method of calculating the weak fuzzy similarity proposed in this paper is based on the dual-universe information system, which avoids the error of using the elements in one universe to judge the element classification problem in the other universe with the traditional binary relation.

(3) The method of dealing with problems in dual-universe information systems mentioned in this paper is easier to be extended to multi-universe and multi-attribute problems.

Our future study is to extend this model to multiple fields for dealing with the problems in multiple universes.

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