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New Operations on Interval-Valued Picture Fuzzy Set, Interval-Valued Picture Fuzzy Soft Set and Their Applications

AHMED MOSTAFA KHALIL[1,2], SHENG-GANG LI[1], HARISH GARG[3], HONGXIA LI[4], AND SHENGQUAN MA[5]

[1]College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China
[2]Department of Mathematics, Faculty of Science, Al-Azhar University, Asyut 71524, Egypt
[3]School of Mathematics, Thapar Institute of Engineering & Technology, Patiala 147004, India
[4]School of Mathematics and Statistics, Longdong University, Qingyang 745000, China
[5]College of Information Science and Technology, Hainan Normal University, Haikou 571158, China

Corresponding authors: Sheng-Gang Li (shengganglinew@126.com) and Harish Garg (harishg58iitr@gmail.com)

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ABSTRACT An interval-valued picture fuzzy set (IVPFS) is one of the generalizations of an interval-valued fuzzy set to handle the uncertainties in the data during analysis. The aim of this paper is to introduce and study new operations of IVPFS, along with its properties, and examples. In addition, we present the notion of the interval-valued picture fuzzy soft set theory and investigated their properties. Several operations on such as subset, equal, complement, inf product, sup product, union, and intersection are defined over the interval-valued picture fuzzy soft set and discussed their basic properties. Furthermore, we construct an algorithm using an interval-valued picture fuzzy soft set to solve the decision-making problems and illustrate its applicability through a numerical example. From the study, we conclude that the proposed approach is viable in order to handle the uncertainties during the decision-making problems.

INDEX TERMS Picture fuzzy set, interval-valued picture fuzzy set, interval-valued picture fuzzy soft set, decision-making; soft set.

I. INTRODUCTION

Many complicated problems such as economics, engineering, management science, medical science involve uncertain data. These problems, which one comes face to face in our day-to-day life, cannot be solved using classical mathematical methods due to a large number of uncertainties exists. To cope such uncertainties, a theory of fuzzy set (FS) [1] came into existence in which a membership degree (MD) is assigned to each element. An intuitionistic fuzzy set (IFS) [2] is an extension of FS by adding non-membership degree (NMD) into the analysis such that sum of MD and NMD is not greater than one. Apart from these, a new theory to deal with uncertainties is proposed by Molodtsov [3] and named as a soft set (SS). Since its appearance, several researchers are working on it and developed several operations and method to enrich the environment to handle the uncertainties. For example, Maji *et al.* [4] defined some operations on soft sets. Further, Maji *et al.* [5] presented an application of the soft set in the decision-making problems. A survey on parametric soft sets is presented by Zhan and Alcantud [6]. Maji *et al.* [7] extended SS theory by joining it with existing FS approach and developed the idea of fuzzy soft set (FSS). However, apart from them, some extension models of SSs are rapidly developed such as interval-valued fuzzy soft sets [8], generalized interval-valued fuzzy soft sets [9], interval-valued intuitionistic fuzzy soft sets [10], [11], interval-valued hesitant fuzzy soft sets [12], [13], dual hesitant fuzzy soft set [14], [15], trapezoidal interval type-2 fuzzy soft sets [16], [17], time-neutrosophic soft sets [18]. Furthermore, Maji *et al.* [19] defined the intuitionistic fuzzy soft sets, followed by studies on picture fuzzy soft set [20], and generalized intuitionistic fuzzy soft sets [21]–[23]. Peng and Garg [24] proposed three algorithms to solve interval-valued fuzzy soft decision

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making problem by weighted distance based approximation, combinative distance-based assessment and similarity measure. Peng and Yang [25] proposed algorithms for interval-valued fuzzy soft sets in stochastic multi-criteria decision-making based on regret theory and prospect theory. Zhu and Zhan [26] proposed the concept of fuzzy parameterized fuzzy soft sets, along with decision making. Zhao *et al.* [27] presented a novel decision making approach based on intuitionistic fuzzy soft sets. Das [28] introduced the notion of weighted fuzzy soft multiset and decision-making. Garg and Arora [29] presented an approach to solve the decision making problems by formulating a nonlinear programming model with interval-valued intuitionistic fuzzy soft set information. Deli [30] introduced the notion of interval-valued neutrosophic soft sets and applications to the decision making process. Garg and Arora [31] presented the t-norm operations based Maclaurin Symmetric mean aggregation operators for solving the decision making problems under the dual hesitant fuzzy soft set environment.

From the above studies, it has been observed that all the above- defined decision-making and operations are under the IFS or soft set theories. Although, these theories have been successfully applied in different areas, but there are situations in real life which cannot be represented by IFSs. In other words, we can say that neither the FS nor IFS theory is able to handle the real-life condition more accurately. For instance, in the case of voting, human opinions involving more answers of the types: ''yes'', ''abstain'', ''no'', ''refusal'', which can't be accurately represented in IFSs. Also, if an expert takes an opinion from a certain person about the certain object, then a person may say that 0.3 is the possibility that ''statement is true'', 0.4 says that the ''statement is false'' and 0.2 says that ''he or she is not sure'' of it. This issue is also not handled by the FSs or IFSs. Thus, for handling this situation, Cuong [32], [33] proposed a new notion named picture fuzzy sets (PFSs), which is an extension of FS and IFSs. PFSs consist of three MDs of an element named as the positive membership degree, the neutral membership degree, and the negative membership degree, respectively such that their sum is not greater than unity. In the literature, researchers are working on the PFSs and successfully developed several algorithm to solve different problems such as clustering [34], fuzzy inference [35], and decision-making [36]–[39]. To the best of authors knowledge, a very less work is being done on the interval-valued picture fuzzy set (IVPFS), so the present study define the operations on the IVPFS and later on extend it to soft set environment.

Keeping in view of the PFS advantages, this paper presents a study on the interval-valued PFSs (IVPFS) and present some new operations on interval-valued PFSs. Also, a new concept of interval-valued picture fuzzy soft sets which combines the advantages of the soft sets with the IVPFS. The various desirable relations and properties of the presented set are investigated in details. The present sets are the generalization of the several existing sets such as soft sets, fuzzy soft sets, interval-valued fuzzy soft sets, intuitionistic fuzzy soft sets,

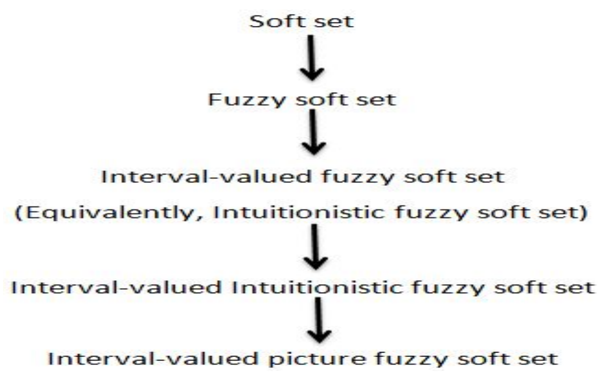

FIGURE 1. The relationship among interval-valued picture fuzzy soft set and other soft sets.

and interval-valued intuitionistic fuzzy soft sets. Based on interval-valued picture fuzzy soft sets matrix, an algorithm for solving the decision-making problem is presented and explain with the help of a numerical example. The relationship among interval-valued picture fuzzy soft set and other soft sets is shown in Figure 1 (which an interval-valued picture fuzzy soft set is a generalization other each the soft sets).

The rest of this paper is organized as follows. In Section II, the basic concepts for a PFS, an interval-valued PFS, new operations of interval-valued PFSs, a soft set, a fuzzy soft set and a picture fuzzy soft set are presented. In Section III, the concept of interval-valued picture fuzzy soft set, some of its operations are defined, and some of its properties are studied. An algorithm is constructed and an application of interval-valued picture fuzzy soft set in the decision-making problem is shown in Section IV. Some applications of the proposed operations are described in Section V. Finally, conclusions are given in Section VI.

II. PRELIMINARIES

A. PICTURE FUZZY SETS

Definition 1 (cf. [32], [33]): Let $\mathbb{I}$ be the subset of R^3 (with point-wise order $\leq$ induced by the ordinary linear order $\leq$ on R, the set of all real numbers) consisting of all element $a = (a_1, a_2, a_3) \in [0, 1]^3$ satisfying $a_1 + a_2 + a_3 \leq 1$. Then a mapping[1] $\Phi : X \longrightarrow \mathbb{I}$ is also called a picture fuzzy set (PFS for short) on X. The set of all PFSs on X is denoted by $\mathbb{I}^X$.

Definition 2 (cf. [32], [33]): Let $\mathbb{I}^{\Delta} = \{([a_1, b_1], [a_2, b_2], [a_3, b_3]) \mid a = (a_1, a_2, a_3) \in \mathbb{I}, b = (b_1, b_2, b_3) \in \mathbb{I}\}$[2] with an order $\leq$ defined by $(a, b) \leq (c, d) \iff [a_1, b_1] \supseteq [c_1, d_1]$, $[a_2, b_2] \supseteq [c_2, d_2]$, and $[a_3, b_3] \supseteq [c_3, d_3]$ $(c =$

[1]Sometimes it is also written as $\Phi = \left\{ \frac{a_x}{x} \mid x \in X \right\}$ or $\Phi = \left\{ \frac{(a_{1,x}, a_{2,x}, a_{3,x})}{x} \mid x \in X \right\}$, where $a_{1,x}$ is called the degree of positive membership of x in Φ, $a_{2,x}$ the degree of neutral membership of x in Φ, and $a_{3,x}$ the degree of negative membership of x in Φ.

[2]It can also be written as $\mathbb{I}^{\Delta} = \{(a, b) \mid a = (a_1, a_2, a_3) \in \mathbb{I}, b = (b_1, b_2, b_3) \in \mathbb{I},\ a_k \leq b_k\ (k = 1, 2, 3)\}$, where $[a, b]$ denotes the ordinary closed interval of R, called interval number. The sum of two interval numbers $[a, b]$ and $[c, d]$ is defined as $[a, b] \oplus [c, d] = [a + c, b + d]$.

$(c_1, c_2, c_3), d = (d_1, d_2, d_3))$. Then a mapping[3] $\Phi : X \longrightarrow \mathbb{I}^{\Delta}$ is also called an interval-valued picture fuzzy set (shortly, IVPFS) on X. The set of all IVPFSs on X is denoted by $\mathbb{I}^{\Delta X}$.

Example 3: Suppose that $X = \{x, y, z\}$ be a set of three universes of discourse, where x represent characterizes of the quality, y indicates the prices of the objects, and z indicates the different of the usages. It may be further assumed that the values of x, y, z are in $\mathbb{I}$ and they are obtained from a expert person. This situation can be characterized by the following IVPFS $\Phi \in \mathbb{I}^{\Delta X}$:

$$\Phi = \left\{ \frac{([0.2, 0.3], [0.1, 0.4], [0.1, 0.3])}{x}, \frac{([0.1, 0.4], [0.3, 0.3], [0.2, 0.2])}{y}, \frac{([0.2, 0.3], [0.1, 0.5], [0.1, 0.2])}{z} \right\}$$

Remark 4: The followings can be looked to be some special cases of the above definition 2.

(1) If $k = 1$, then $\Phi = \left\{ \frac{([a_{1,x}, b_{1,x}])}{x} \,\middle|\, x \in X \right\}$ is called an interval-valued fuzzy set ($0 \le a_{1,x} \le b_{1,x} \le 1$) where $a_{1,x} \in [0, 1]$ is the lower degree of membership of x in Φ, and $b_{1,x} \in [0, 1]$ is the upper degree of membership of x in Φ (cf. [40]).

(2) If $k = 1, 2$, then $\Phi = \left\{ \frac{([a_{1,x}, b_{1,x}], [a_{2,x}, b_{2,x}])}{x} \,\middle|\, x \in X \right\}$ is called an interval-valued intuitionistic fuzzy set ($b_{1,x} + b_{2,x} \le 1$) where $[a_{1,x}, b_{1,x}] \subseteq [0, 1]$ is the degree of membership of x in Φ, and $[a_{2,x}, b_{2,x}] \subseteq [0, 1]$ is the degree of non membership of x in Φ (cf. [41]).

(3) If $k = 1, 2, 3$, then

$$\Phi = \left\{ \frac{([a_{1,x}, b_{1,x}], [a_{2,x}, b_{2,x}], [a_{3,x}, b_{3,x}])}{x} \,\middle|\, x \in X \right\}$$

is called an interval neutrosophic set ($b_{1,x} + b_{2,x} + b_{3,x} \le 3$), where $[a_{1,x}, b_{1,x}] \subseteq [0, 1]$ is the degree of truth membership of x in Φ, $[a_{2,x}, b_{2,x}] \subseteq [0, 1]$ is the degree of indeterminacy membership of x in Φ, and $[a_{3,x}, b_{3,x}] \subseteq [0, 1]$ is the degree of falsity membership of x in Φ (cf. [42]).

Definition 5 (cf. [32], [33]): Let $\Phi^{(s)} = \left\{ \frac{[a_x^{(s)}, b_x^{(s)}]}{x} \mid x \in X \right\} = \left\{ \frac{([a_{1,x}^{(s)}, b_{1,x}^{(s)}],\ [a_{2,x}^{(s)}, b_{2,x}^{(s)}],\ [a_{3,x}^{(s)}, b_{3,x}^{(s)}])}{x} \mid x \in X \right\} \in \mathbb{I}^{\Delta X} (s \in S)$. Then some lattice theory-like notions (such as inclusion, equal, union and intersection) are defined as follows:

(1) $\Phi^{(s)} \subseteq \Phi^{(t)}$ iff $\left(a_x^{(s)}, b_x^{(s)}\right) \le \left(c_x^{(t)}, d_x^{(t)}\right)$ $(\forall x \in X)$, equivalently, $\left[a_{1,x}^{(s)}, b_{1,x}^{(s)}\right] \supseteq \left[c_{1,x}^{(t)}, d_{1,x}^{(t)}\right]$, $\left[a_{2,x}^{(s)}, b_{2,x}^{(s)}\right] \supseteq \left[c_{2,x}^{(t)}, d_{2,x}^{(t)}\right]$, and $\left[a_{3,x}^{(s)}, b_{3,x}^{(s)}\right] \supseteq \left[c_{3,x}^{(t)}, d_{3,x}^{(t)}\right]$ for each $x \in X$.

(2) $\Phi^{(s)} = \Phi^{(t)}$ iff $\Phi^{(s)} \subseteq \Psi^{(t)}$ and $\Phi^{(s)} \supseteq \Psi^{(s)}$.

(3) $$\bigcup_{s \in S} \Phi^{(s)} = \left\{ \frac{\left(\left[\sup\limits_{s\in S} a_{1,x}^{(s)}, \sup\limits_{s\in S} b_{1,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{2,x}^{(s)}, \inf\limits_{s\in S} b_{2,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{3,x}^{(s)}, \inf\limits_{s\in S} b_{3,x}^{(s)}\right] \right)}{x} \,\middle|\, x \in X \right\}$$

(4) $$\bigcap_{s \in S} \Phi^{(s)} = \left\{ \frac{\left(\left[\inf\limits_{s\in S} a_{1,x}^{(s)}, \inf\limits_{s\in S} b_{1,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{2,x}^{(s)}, \inf\limits_{s\in S} b_{2,x}^{(s)}\right], \left[\sup\limits_{s\in S} a_{3,x}^{(s)}, \sup\limits_{s\in S} b_{3,x}^{(s)}\right] \right)}{x} \,\middle|\, x \in X \right\}.$$

Theorem 6 (cf. [32], [33]): Let $\Phi^{(s)} = \left\{ \frac{[a_x^{(s)}, b_x^{(s)}]}{x} \mid x \in X^{(s)} \right\} = \left\{ \frac{([a_{1,x}^{(s)}, b_{1,x}^{(s)}],\ [a_{2,x}^{(s)}, b_{2,x}^{(s)}],\ [a_{3,x}^{(s)}, b_{3,x}^{(s)}])}{x} \mid x \in X^{(s)} \right\} \in \mathbb{I}^{\Delta X^{(s)}}$ $(s \in S)$. Then $\dot{\Phi}$ (called 1-product and written as also $\dot{\Phi} = \Phi^{(1)} \times_1 \Phi^{(2)}$ if $S = \{1, 2\}$), $\hat{\Phi}$ (called 2-product and written as also $\hat{\Phi} = \Phi^{(1)} \times_2 \Phi^{(2)}$ if $S = \{1, 2\}$), and $\check{\Phi}$ (called 3-product and written as also $\check{\Phi} = \Phi^{(1)} \times_3 \Phi^{(2)}$ if $S = \{1, 2\}$) belong to $\mathbb{I}^{\Delta X}$, where

$$\dot{\Phi} = \prod\!\!\prod_{s\in S} \Phi^{(s)} = \left\{ \frac{\left(\left[\prod\limits_{s\in S} a_{1,x}^{(s)}, \prod\limits_{s\in S} b_{1,x}^{(s)}\right], \left[\prod\limits_{s\in S} a_{2,x}^{(s)}, \prod\limits_{s\in S} b_{2,x}^{(s)}\right], \left[\prod\limits_{s\in S} a_{3,x}^{(s)}, \prod\limits_{s\in S} b_{3,x}^{(s)}\right] \right)}{x} \,\middle|\, x \in X \right\}$$

$(|S| \le \aleph_0)$

and $$\hat{\Phi} = \prod\!\!\prod_{s\in S} \Phi^{(s)} = \left\{ \frac{\left(\left[\inf\limits_{s\in S} a_{1,x}^{(s)}, \inf\limits_{s\in S} b_{1,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{2,x}^{(s)}, \inf\limits_{s\in S} b_{2,x}^{(s)}\right], \left[\sup\limits_{s\in S} a_{3,x}^{(s)}, \sup\limits_{s\in S} b_{3,x}^{(s)}\right] \right)}{x} \,\middle|\, x \in X \right\}$$

$$\check{\Phi} = \prod\!\!\prod_{s\in S} \Phi^{(s)} = \left\{ \frac{\left(\left[\sup\limits_{s\in S} a_{1,x}^{(s)}, \sup\limits_{s\in S} b_{1,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{2,x}^{(s)}, \inf\limits_{s\in S} b_{2,x}^{(s)}\right], \left[\inf\limits_{s\in S} a_{3,x}^{(s)}, \inf\limits_{s\in S} b_{3,x}^{(s)}\right] \right)}{x} \,\middle|\, x \in X \right\}$$

B. SOFT SETS, FUZZY SOFT SETS AND PICTURE FUZZY SOFT SETS

Definition 7 (cf. [3]): An alternative name of a mapping[4] $\Phi : I \longrightarrow P(X)$ (the set of all subsets of X) is a soft set over X $(I \subseteq E, E$ is parameter set).

[3] Sometimes it is also written as $\Phi = \left\{ \frac{(a_x, b_x)}{x} \,\middle|\, x \in X \right\}$ or $\Phi = \left\{ \frac{([a_{1,x}, b_{1,x}], [a_{2,x}, b_{2,x}], [a_{3,x}, b_{3,x}])}{x} \,\middle|\, x \in X \right\}$, where $[a_{1,x}, b_{1,x}]$ is called the degree of positive membership of x in Φ, $[a_{2,x}, b_{2,x}]$ the degree of neutral membership of x in Φ, and $[a_{3,x}, b_{3,x}]$ the degree of negative membership of x in Φ.

[4] Sometimes it is also written as $\Phi = \{\Phi(i)\}_{i \in I}$ or $\Phi = \{\Phi_i\}_{i \in I}$.

Example 8: Assume that a fund manager in a wealth management firm is assessing four potential investment opportunities $X = \{x_1, x_2, x_3, x_4\}$. The firm mandates that the fund manager has to evaluate the following four parameters of set $I = \{i_1, i_2, i_3, i_4\}$. Each parameter is a word or a sentence, where

i_1 stands for the parameter 'risk',
i_2 stands for the parameter 'growth',
i_3 stands for the parameter 'socio-political issues',
i_4 stands for the parameter 'environmental impacts'.

Suppose that $\Phi(i_1) = \{x_1, x_3\}$, $\Phi(i_2) = \{x_2, x_3\}$, $\Phi(i_3) = X$, $\Phi(i_4) = \{x_2, x_3\}$. Then, we can view the soft set $\Phi : I \longrightarrow P(X)$ describing the 'opportunities of the potential investment'as below:

risk potential investment $= \{x_1, x_3\}$, growth potential investment $= \{x_2, x_3\}$, socio-political issues potential investment $= X$, environmental impacts potential investment $= \{x_2, x_3\}$.

See the following Table 1 for the tabular representation of Φ.

TABLE 1. The tabular representation of soft set (Φ, I).

X	risk	growth	socio-political issues	environmental impacts
x_1	1	0	1	0
x_2	0	1	1	1
x_3	1	1	1	1
x_4	0	0	1	0

Definition 9 (cf. [4]): An alternative name of a mapping $\Phi : I \longrightarrow [0, 1]^X$ (the set of all fuzzy sets of X) is a fuzzy soft set over X $(I \subseteq E)$.

Example 10: We can characterize the fuzzy information by a membership instead of the crisp numbers 0 or 1. The fuzzy soft set Φ in Example 8 can then describe the 'opportunities of the potential investment'under the fuzzy information as indicated below.

$$\Phi(i_1) = \left\{\frac{0.4}{x_1}, \frac{0.6}{x_2}, \frac{0.5}{x_3}, \frac{0.7}{x_4}\right\},$$
$$\Phi(i_2) = \left\{\frac{0.3}{x_1}, \frac{0.5}{x_2}, \frac{0.9}{x_3}, \frac{0.1}{x_4}\right\},$$
$$\Phi(i_3) = \left\{\frac{0.1}{x_1}, \frac{0.4}{x_2}, \frac{0.7}{x_3}, \frac{0.9}{x_4}\right\},$$
$$\Phi(i_4) = \left\{\frac{0.2}{x_1}, \frac{0.3}{x_2}, \frac{0.8}{x_3}, \frac{0.7}{x_4}\right\}.$$

Definition 11 (cf. [20]): A mapping[5] $\Phi : I \longrightarrow \mathbb{I}^X$ is also called a picture fuzzy soft set over X. The set of all picture fuzzy soft sets on X, denoted by $(\mathbb{I}^X)^I$ or $\mathbb{I}^{XI}$.

[5]Sometimes it is also written as $\Phi = \left\{\frac{a_{i,x}}{x} \;\middle|\; x \in X\right\}_{i\in I}$ or $\Phi = \left\{\frac{(a_{i,1,x}, a_{i,2,x}, a_{i,3,x})}{x} \;\middle|\; x \in X\right\}_{i\in I}$, where $a_{i,1,x}$ is the degree of positive membership of x on parameter i in Φ, $a_{i,2,x}$ is the degree of neutral membership of x on parameter i in Φ, and $a_{i,1,x}$ is the degree of negative membership of x on parameter i in Φ.

Example 12: The 'opportunities of the potential investment'in Example 8 can be described by the a picture fuzzy soft set defined by

$$\Phi(i_1) = \left\{\frac{(0.2, 0.1, 0.4)}{x_1}, \frac{(0.1, 0.4, 0.3)}{x_2}, \frac{(0.2, 0.3, 0.5)}{x_3}, \frac{(0.2, 0.3, 0.5)}{x_4}\right\},$$
$$\Phi(i_2) = \left\{\frac{(0.3, 0.4, 0)}{x_1}, \frac{(0.1, 0.2, 0.3)}{x_2}, \frac{(0.7, 0.2, 0.1)}{x_3}, \frac{(0.8, 0.2, 0)}{x_4}\right\},$$
$$\Phi(i_3) = \left\{\frac{(0.5, 0.1, 0.1)}{x_1}, \frac{(0.6, 0.2, 0.2)}{x_2}, \frac{(0.3, 0.3, 0.4)}{x_3}, \frac{(0.2, 0.1, 0.7)}{x_4}\right\},$$
$$\Phi(i_4) = \left\{\frac{(0, 0.1, 0.5)}{x_1}, \frac{(0.5, 0.1, 0.2)}{x_2}, \frac{(0.1, 0.3, 0.4)}{x_3}, \frac{(0.2, 0.3, 0.5)}{x_4}\right\}.$$

III. NEW OPERATIONS OF IVPFSS

In the section, we introduce some new operations of IVPFSs and investigate the basic properties. The usefulness of this section appears in the next section.

Remark 13: For an IVPFS $\Phi = \left\{\frac{[a_x, b_x]}{x} \mid x \in X\right\} = \left\{\frac{([a_{1,x}, b_{1,x}],\ [a_{2,x}, b_{2,x}],\ [a_{3,x}, b_{3,x}])}{x} \mid x \in X\right\} \in \mathbb{I}^{\Delta X}$, let

$$\neg\Phi = \left\{\frac{([a_{3,x}, b_{3,x}], [a_{2,x}, b_{2,x}], [a_{1,x}, b_{1,x}])}{x} \mid x \in X\right\} \tag{1}$$

$$\Box\Phi = \left\{\frac{\left(\frac{1}{3}[1-b_{1,x}, 1-a_{1,x}],\ \frac{1}{3}[1-b_{2,x}, 1-a_{2,x}],\ \frac{1}{3}[1-b_{3,x}, 1-a_{3,x}]\right)}{x} \mid x \in X\right\} \tag{2}$$

Then $1 - b_{1,x} \le 1 - a_{1,x}$, $1 - b_{2,x} \le 1 - a_{2,x}$, $1 - b_{3,x} \le 1 - a_{3,x}$, and $\frac{1}{3}(1-a_{1,x}) + \frac{1}{3}(1-a_{2,x}) + \frac{1}{3}(1-a_{3,x}) \le 1$, which means $\Box\Phi \in \mathbb{I}^{\Delta X}$. Thus $\neg$ and $\Box$ are operations on $\mathbb{I}^{\Delta X}$.

Next, we investigate properties of $\Box$ and $\neg$.

Proposition 14: Let $\Phi = \left\{\frac{[a_x, b_x]}{x} \mid x \in X\right\} = \left\{\frac{([a_{1,x}, b_{1,x}],\ [a_{2,x}, b_{2,x}],\ [a_{3,x}, b_{3,x}])}{x} \mid x \in X\right\} \in \mathbb{I}^{\Delta X}$. Then the followings hold:

(1) $\neg\Box\,\Phi = \Box\neg\Phi$,
(2) $\neg\neg\Phi = \Phi$.

Proof: (1) $\Box\,\neg\Phi = \Box\left\{\frac{([a_{3,x}, b_{3,x}],\ [a_{2,x}, b_{2,x}],\ [a_{1,x}, b_{1,x}])}{x} \mid x \in X\right\}$

$$= \left\{\frac{\left(\frac{1}{3}[1-b_{3,x}, 1-a_{3,x}],\ \frac{1}{3}[1-b_{2,x}, 1-a_{2,x}],\ \frac{1}{3}[1-b_{1,x}, 1-a_{1,x}]\right)}{x} \mid x \in X\right\}$$
$$= \neg\left\{\frac{\left(\frac{1}{3}[1-b_{1,x}, 1-a_{1,x}],\ \frac{1}{3}[1-b_{2,x}, 1-a_{2,x}],\ \frac{1}{3}[1-b_{3,x}, 1-a_{3,x}]\right)}{x} \mid x \in X\right\} = \neg\Box\Phi.$$

(2) is clear. ■

Proposition 15: Let $\Phi^{(s)} = \left\{\frac{[a_x^{(s)}, b_x^{(s)}]}{x} \mid x \in X\right\} = \left\{\frac{([a_{1,x}^{(s)}, b_{1,x}^{(s)}],\ [a_{2,x}^{(s)}, b_{2,x}^{(s)}],\ [a_{3,x}^{(s)}, b_{3,x}^{(s)}])}{x} \mid x \in X\right\} \in \mathbb{I}^{\Delta X}$ $(s \in S)$. Then the followings hold:

(1) $\Box\left(\bigcup_{s\in S} \Phi^{(s)}\right) = \bigcap_{s\in S}(\Box\Phi^{(s)})$.

(2) $\Box\left(\bigcap_{s\in S} \Phi^{(s)}\right) = \bigcup_{s\in S}(\Box\Phi^{(s)})$.

(3) $\neg\left(\bigcup_{s\in S} \Phi^{(s)}\right) = \bigcap_{s\in S}(\neg\Phi^{(s)})$.

(4) $\neg\left(\bigcap_{s\in S} \Phi^{(s)}\right) = \bigcup_{s\in S}(\neg\Phi^{(s)})$.

Proof: (1) By Remark 13 and Definition 5, we have (where p_i is the $i-th$ projection $(i=1,2,3)$)

$$p_1\left(\left(\bigcap_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\inf_{s\in S}\left(1-a_{1,x}^{(s)}\right),\inf_{s\in S}\left(1-b_{1,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\sup_{s\in S}a_{1,x}^{(s)}\right),1-\left(\sup_{s\in S}b_{1,x}^{(s)}\right)\right]=p_1\left(\left(\Box\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_2\left(\left(\bigcap_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\inf_{s\in S}\left(1-a_{2,x}^{(s)}\right),\inf_{s\in S}\left(1-b_{2,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\sup_{s\in S}a_{2,x}^{(s)}\right),1-\left(\sup_{s\in S}b_{2,x}^{(s)}\right)\right]=p_2\left(\left(\Box\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_3\left(\left(\bigcap_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\sup_{s\in S}\left(1-a_{3,x}^{(s)}\right),\sup_{s\in S}\left(1-b_{3,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\inf_{s\in S}a_{3,x}^{(s)}\right),1-\left(\inf_{s\in S}b_{3,x}^{(s)}\right)\right]=p_3\left(\left(\Box\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right).$$

Thus $\Box\left(\bigcup_{s\in S}\Phi^{(s)}\right)=\bigcap_{s\in S}(\Box\Phi^{(s)})$.

(2) By Remark 13 and Definition 5, we have

$$p_1\left(\left(\bigcup_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\sup_{s\in S}\left(1-a_{1,x}^{(s)}\right),\sup_{s\in S}\left(1-b_{1,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\inf_{s\in S}a_{1,x}^{(s)}\right),1-\left(\inf_{s\in S}b_{1,x}^{(s)}\right)\right]=p_1\left(\left(\Box\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_2\left(\left(\bigcup_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\inf_{s\in S}\left(1-a_{2,x}^{(s)}\right),\inf_{s\in S}\left(1-b_{2,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\sup_{s\in S}a_{2,x}^{(s)}\right),1-\left(\sup_{s\in S}b_{2,x}^{(s)}\right)\right]=p_2\left(\left(\Box\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_3\left(\left(\bigcup_{s\in S}(\Box\Phi^{(s)})\right)(x)\right)=\frac{1}{3}\left[\inf_{s\in S}\left(1-a_{3,x}^{(s)}\right),\inf_{s\in S}\left(1-b_{3,x}^{(s)}\right)\right]=\frac{1}{3}\left[1-\left(\sup_{s\in S}a_{3,x}^{(s)}\right),1-\left(\sup_{s\in S}b_{3,x}^{(s)}\right)\right]=p_3\left(\left(\Box\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right).$$

Thus $\Box\left(\bigcap_{s\in S}\Phi^{(s)}\right)=\bigcup_{s\in S}(\Box\Phi^{(s)})$.

(3) By Remark 13 and Definition 5, we have

$$p_1\left(\left(\bigcap_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\inf_{s\in S}a_{3,x}^{(s)},\inf_{s\in S}b_{3,x}^{(s)}\right]=p_1\left(\left(\neg\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_2\left(\left(\bigcap_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\inf_{s\in S}a_{2,x}^{(s)},\inf_{s\in S}b_{2,x}^{(s)}\right]=p_2\left(\left(\neg\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_3\left(\left(\bigcap_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\sup_{s\in S}a_{1,x}^{(s)},\sup_{s\in S}b_{1,x}^{(s)}\right]=p_3\left(\left(\neg\left(\bigcup_{s\in S}\Phi^{(s)}\right)\right)(x)\right).$$

Thus $\neg\left(\bigcup_{s\in S}\Phi^{(s)}\right)=\bigcap_{s\in S}(\neg\Phi^{(s)})$.

(4) By Remark 13 and Definition 5, we have

$$p_1\left(\left(\bigcup_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\sup_{s\in S}a_{3,x}^{(s)},\sup_{s\in S}b_{3,x}^{(s)}\right]=p_1\left(\left(\neg\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_2\left(\left(\bigcup_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\inf_{s\in S}a_{2,x}^{(s)},\inf_{s\in S}b_{2,x}^{(s)}\right]=p_2\left(\left(\neg\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right),$$

$$p_3\left(\left(\bigcup_{s\in S}(\neg\Phi^{(s)})\right)(x)\right)=\left[\inf_{s\in S}a_{1,x}^{(s)},\inf_{s\in S}b_{1,x}^{(s)}\right]=p_3\left(\left(\neg\left(\bigcap_{s\in S}\Phi^{(s)}\right)\right)(x)\right).$$

Thus $\neg\left(\bigcap_{s\in S}\Phi^{(s)}\right)=\bigcup_{s\in S}(\neg\Phi^{(s)})$. ■

Definition 16: For two elements $\Phi^{(s)}=\left\{\frac{(a_x^{(s)},b_x^{(s)})}{x} \mid x\in X\right\} = \left\{\frac{\left([a_{1,x}^{(s)},b_{1,x}^{(s)}],\ [a_{2,x}^{(s)},b_{2,x}^{(s)}],\ [a_{3,x}^{(s)},b_{3,x}^{(s)}]\right)}{x} \mid x \in X\right\}$ and $\Phi^{(t)} = \left\{\frac{(a_x^{(t)},b_x^{(t)})}{x} \mid x \in X\right\} = \left\{\frac{\left([a_{1,x}^{(t)},b_{1,x}^{(t)}],\ [a_{2,x}^{(t)},b_{2,x}^{(t)}],\ [a_{3,x}^{(t)},b_{3,x}^{(t)}]\right)}{x} \mid x \in X\right\}$ of $\mathbb{I}^{\Delta X}$, we define relations $\dot{\subseteq}$ and $\ddot{\subseteq}$ as follows:

(1) $\Phi^{(s)}\dot{\subseteq}\Phi^{(t)}$ iff $\left[a_{1,x}^{(s)}, b_{1,x}^{(s)}\right]\subseteq\left[a_{1,x}^{(t)}, b_{1,x}^{(t)}\right]$, $\left[a_{2,x}^{(s)}, b_{2,x}^{(s)}\right]\subseteq\left[a_{2,x}^{(t)}, b_{2,x}^{(t)}\right]$, and $\left[a_{3,x}^{(s)}, b_{3,x}^{(s)}\right] \subseteq \left[a_{3,x}^{(t)}, b_{3,x}^{(t)}\right]$ for each $x \in X$.

(2) $\Phi^{(s)}\ddot{\subseteq}\Phi^{(t)}$ iff $\left[a_{1,x}^{(s)}, b_{1,x}^{(s)}\right]\subseteq\left[a_{1,x}^{(t)}, b_{1,x}^{(t)}\right]$, $\left[a_{2,x}^{(s)}, b_{2,x}^{(s)}\right]\subseteq\left[a_{2,x}^{(t)}, b_{2,x}^{(t)}\right]$, and $\left[a_{3,x}^{(s)}, b_{3,x}^{(s)}\right] \supseteq \left[a_{3,x}^{(t)}, b_{3,x}^{(t)}\right]$ for each $x \in X$.

It is clear, $\Phi^{(s)} \subseteq \Phi^{(t)}$ and $\Phi^{(s)}\dot{\subseteq}\Phi^{(t)}$ iff $\Phi^{(s)} = \Phi^{(t)}$.

Definition 17: For a $\Phi = \left\{\frac{(a_x,b_x)}{x} \mid x \in X\right\} = \left\{\frac{\left([a_{1,x},b_{1,x}],\ [a_{2,x},b_{2,x}],\ [a_{3,x},b_{3,x}]\right)}{x} \mid x \in X\right\} \in \mathbb{I}^{\Delta X}$, we define $\{\Phi_{(1)}, \Phi_{(2)}, \Phi_{(3)}, \Phi_{(4)}\} \subseteq \mathbb{I}^{\Delta X}$ (i.e. four operations on $\mathbb{I}^{\Delta X}$) as follows:

$$\Phi_{(1)} = \left\{\frac{\left(a_{1,x},\ [a_{2,x}, b_{2,x}],\ a_{3,x}\right)}{x} \mid x \in X\right\} \tag{3}$$

$$\Phi_{(2)} = \left\{\frac{\left(a_{1,x},\ [a_{2,x}, b_{2,x}],\ b_{3,x}\right)}{x} \mid x \in X\right\} \tag{4}$$

$$\Phi_{(3)} = \left\{\frac{\left(b_{1,x},\ [a_{2,x}, b_{2,x}],\ a_{3,x}\right)}{x} \mid x \in X\right\} \tag{5}$$

$$\Phi_{(4)} = \left\{\frac{\left(b_{1,x},\ [a_{2,x}, b_{2,x}],\ b_{3,x}\right)}{x} \mid x \in X\right\} \tag{6}$$

Proposition 18: For a $\Phi = \left\{\frac{(a_x,b_x)}{x} \mid x \in X\right\} = \left\{\frac{\left([a_{1,x},b_{1,x}],\ [a_{2,x},b_{2,x}],\ [a_{3,x},b_{3,x}]\right)}{x} \mid x \in X\right\} \in \mathbb{I}^{\Delta X}$, the followings hold:

(1) $\Phi_{(2)}\ddot{\subseteq}\Phi_{(1)}$.

(2) $\Phi_{(4)}\ddot{\subseteq}\Phi_{(3)}$.

(3) $\Phi_{(1)}\dot{\subseteq}\Phi_{(4)}$.

(4) $\Phi_{(4)}\ddot{\subseteq}\Phi_{(1)}$.

(5) $\left(\bigcup_{s\in S} \Phi\right)_{(1)} = \bigcup_{s\in S}(\Phi_{(1)})$, $\left(\bigcap_{s\in S} \Phi\right)_{(1)} = \bigcap_{s\in S}(\Phi_{(1)})$.

(6) $\left(\bigcup_{s\in S} \Phi\right)_{(2)} = \bigcup_{s\in S}(\Phi_{(2)})$, $\left(\bigcap_{s\in S} \Phi\right)_{(2)} = \bigcap_{s\in S}(\Phi_{(2)})$.

(7) $\left(\bigcup_{s\in S} \Phi\right)_{(3)} = \bigcup_{s\in S}(\Phi_{(3)})$, $\left(\bigcap_{s\in S} \Phi\right)_{(3)} = \bigcap_{s\in S}(\Phi_{(3)})$.

(8) $\left(\bigcup_{s\in S} \Phi\right)_{(4)} = \bigcup_{s\in S}(\Phi_{(4)})$, $\left(\bigcap_{s\in S} \Phi\right)_{(4)} = \bigcap_{s\in S}(\Phi_{(4)})$.

Proof: It follows from Definitions 5 and 3.3. ∎

Lemma 19: (1) The function $\xi : [0, 1]^2 \longrightarrow R$, defined by

$$\xi(s,t) = \begin{cases} \dfrac{st}{s+t}, & s+t \neq 0, \\ 0, & s+t = 0, \end{cases}$$

is continuous and monotone increase for each variable, and takes the greatest value $\frac{1}{2}$.

(2) The function $\zeta : [0, 1]^2 \longrightarrow R$, defined by

$$\zeta(s,t) = \frac{s+t}{(st+1)}$$

is continuous and monotone increase for each variable, and takes the greatest value 1.

Proof: (1) $\xi(s, t)$ is continuous because $|\xi(s, t)| \leq \frac{1}{2}\frac{s^2+t^2}{s+t} \leq \frac{1}{2}\frac{s^2+t^2}{\sqrt{s^2+t^2}} = \frac{1}{2}\sqrt{s^2+t^2} \longrightarrow 0$ (as $(s, t) \longrightarrow (0, 0)$). As $\frac{\partial\xi(s,t)}{s} = \frac{t^2}{(s+t)^2} \geq 0$ $(s + t \neq 0)$ and $\frac{\partial\xi(s,t)}{t} = \frac{s^2}{(s+t)^2} \geq 0$ $(s + t \neq 0)$, $\xi(s, t)$ is monotone increase for each variable, which implies $\xi(s, t) \leq \xi(1, 1) = \frac{1}{2}$ (i.e. $\xi(s, t)$ takes the greatest value $\frac{1}{2}$ on $[0, 1]^2$).

(2) Clearly, $\zeta(s, t)$ is continuous on $[0, 1]^2$. As $\frac{\partial\zeta(s,t)}{s} = \frac{1-t^2}{(st+1)^2} \geq 0$ and $\frac{\partial\zeta(s,t)}{t} = \frac{1-s^2}{(st+1)^2} \geq 0$, $\zeta(s, t)$ is monotone increase for each variable, which implies $\zeta(s, t) \leq \zeta(1, 1) = 1$ (i.e. $\zeta(s, t)$ takes the greatest value 1 on $[0, 1]^2$). ∎

Proposition 20: If $\Phi^{(s)} = \left\{\frac{(a_x^{(s)},b_x^{(s)})}{x} \mid x \in X\right\} = \left\{\frac{\left([a_{1,x}^{(s)},b_{1,x}^{(s)}],\ [a_{2,x}^{(s)},b_{2,x}^{(s)}],\ [a_{3,x}^{(s)},b_{3,x}^{(s)}]\right)}{x} \mid x \in X\right\}$ and $\Phi^{(t)} = \left\{\frac{(a_x^{(t)},b_x^{(t)})}{x} \mid x \in X\right\} = \left\{\frac{\left([a_{1,x}^{(t)},b_{1,x}^{(t)}],\ [a_{2,x}^{(t)},b_{2,x}^{(t)}],\ [a_{3,x}^{(t)},b_{3,x}^{(t)}]\right)}{x} \mid x \in X\right\}$ belong to $\mathbb{I}^{\Delta X}$, then $\{\Phi^{(s)}\oplus\Phi^{(t)}, \Phi^{(s)}\ominus\Phi^{(t)}, \Phi^{(s)}\boxtimes\Phi^{(t)}, \Phi^{(s)}\boxdot\Phi^{(t)}\} \subseteq \mathbb{I}^{\Delta X}$, where $(x \in X)$

(1) $$\Phi^{(s)}\oplus\Phi^{(t)}(x)=\left(\left[\frac{a_{1,x}^{(s)}+a_{1,x}^{(t)}}{2}, \frac{b_{1,x}^{(s)}+b_{1,x}^{(t)}}{2}\right], \left[\frac{a_{2,x}^{(s)}+a_{2,x}^{(t)}}{2}, \frac{b_{2,x}^{(s)}+b_{2,x}^{(t)}}{2}\right], \left[\frac{a_{3,x}^{(s)}+a_{3,x}^{(t)}}{2}, \frac{b_{3,x}^{(s)}+b_{3,x}^{(t)}}{2}\right]\right)$$

(2) $$\Phi^{(s)}\ominus\Phi^{(t)}(x)=\left(\left[\sqrt{a_{1,x}^{(s)}a_{1,x}^{(t)}}, \sqrt{b_{1,x}^{(s)}b_{1,x}^{(t)}}\right], \left[\sqrt{a_{2,x}^{(s)}a_{2,x}^{(t)}}, \sqrt{b_{2,x}^{(s)}b_{2,x}^{(t)}}\right], \left[\sqrt{a_{3,x}^{(s)}a_{3,x}^{(t)}}, \sqrt{b_{3,x}^{(s)}b_{3,x}^{(t)}}\right]\right)$$

(3) $$\Phi^{(s)}\boxtimes\Phi^{(t)}(x)=\left(\frac{2}{3}\left[\xi\left(a_{1,x}^{(s)}, a_{1,x}^{(t)}\right), \xi\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right)\right], \frac{2}{3}\left[\xi\left(a_{2,x}^{(s)}, a_{2,x}^{(t)}\right), \xi\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right)\right], \frac{2}{3}\left[\xi\left(a_{3,x}^{(s)}, a_{2,x}^{(t)}\right), \xi\left(b_{3,x}^{(s)}, b_{2,x}^{(t)}\right)\right]\right)$$

(4) $$\Phi^{(s)}\boxdot\Phi^{(t)}(x)=\left(\frac{1}{3}\left[\zeta\left(a_{1,x}^{(s)}, a_{1,x}^{(t)}\right), \zeta\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right)\right], \frac{1}{3}\left[\zeta\left(a_{2,x}^{(s)}, a_{2,x}^{(t)}\right), \zeta\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right)\right], \frac{1}{3}\left[\zeta\left(a_{3,x}^{(s)}, a_{2,x}^{(t)}\right), \zeta\left(b_{3,x}^{(s)}, b_{2,x}^{(t)}\right)\right]\right)$$

Proof: Obviously, $\{\Phi^{(s)} \oplus \Phi^{(t)}, \Phi^{(s)} \ominus \Phi^{(t)}\} \subseteq \mathbb{I}^{\Delta X}$. As ξ is monotone increase for each variable, $\xi\left(a_{1,x}^{(s)}, a_{1,x}^{(t)}\right) \leq \xi\left(b_{1,x}^{(s)}, a_{1,x}^{(t)}\right) \leq \xi\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right)$; similarly, $\xi\left(a_{2,x}^{(s)}, a_{2,x}^{(t)}\right) \leq \xi\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right)$ and $\xi\left(a_{3,x}^{(s)}, a_{3,x}^{(t)}\right) \leq \xi\left(b_{3,x}^{(s)}, b_{3,x}^{(t)}\right)$. As ξ takes the greatest value $\frac{1}{2}$ (see Lemma 19(1)), $\frac{2}{3}\xi\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right) + \frac{2}{3}\xi\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right) + \frac{2}{3}\xi\left(b_{3,x}^{(s)}, b_{3,x}^{(t)}\right) \leq 1$. Therefore, $\Phi^{(s)} \boxtimes \Phi^{(t)} \in \mathbb{I}^{\Delta X}$. Similarly, as ζ is monotone increase for each variable, $\zeta\left(a_{1,x}^{(s)}, a_{1,x}^{(t)}\right) \leq \zeta\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right)$; similarly,

$\zeta\left(a_{2,x}^{(s)}, a_{2,x}^{(t)}\right) \leq \zeta\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right)$ and $\zeta\left(a_{3,x}^{(s)}, a_{3,x}^{(t)}\right) \leq \zeta\left(b_{3,x}^{(s)}, b_{3,x}^{(t)}\right)$. As ζ takes the greatest value 1 (see Lemma 19(2)), $\frac{1}{3}\zeta\left(b_{1,x}^{(s)}, b_{1,x}^{(t)}\right) + \frac{1}{3}\zeta\left(b_{2,x}^{(s)}, b_{2,x}^{(t)}\right) + \frac{1}{3}\zeta\left(b_{3,x}^{(s)}, b_{3,x}^{(t)}\right) \leq 1$. Therefore, $\Phi^{(s)} \boxdot \Phi^{(t)} \in \mathbb{I}^{\Delta X}$. ■

Proposition 21: For $\Phi^{(r)} = \left\{\frac{(a_x^{(r)}, b_x^{(r)})}{x} \mid x \in X\right\} = \left\{\frac{([a_{1,x}^{(r)}, b_{1,x}^{(r)}],\ [a_{2,x}^{(r)}, b_{2,x}^{(r)}],\ [a_{3,x}^{(r)}, b_{3,x}^{(r)}])}{x} \mid x \in X\right\}$, $\Phi^{(s)} = \left\{\frac{([a_{1,x}^{(s)}, b_{1,x}^{(s)}],\ [a_{2,x}^{(s)}, b_{2,x}^{(s)}],\ [a_{3,x}^{(s)}, b_{3,x}^{(s)}])}{x} \mid x \in X\right\}$, and $\Phi^{(t)} = \left\{\frac{([a_{1,x}^{(t)}, b_{1,x}^{(t)}],\ [a_{2,x}^{(t)}, b_{2,x}^{(t)}],\ [a_{3,x}^{(t)}, b_{3,x}^{(t)}])}{x} \mid x \in X\right\}$ of $\mathbb{I}^{\Delta X}$, the followings hold:

(1) $\Phi^{(s)} \oplus \Phi^{(t)} = \Phi^{(t)} \oplus \Phi^{(s)}$.
(2) $\Phi^{(s)} \ominus \Phi^{(t)} = \Phi^{(t)} \ominus \Phi^{(s)}$.
(3) $\Phi^{(s)} \boxtimes \Phi^{(t)} = \Phi^{(t)} \boxtimes \Phi^{(s)}$.
(4) $\Phi^{(s)} \boxdot \Phi^{(t)} = \Phi^{(t)} \boxdot \Phi^{(s)}$.
(5) $\neg(\neg\Phi^{(s)} \oplus \neg\Phi^{(t)}) = \Phi^{(s)} \oplus \Phi^{(t)}$.
(6) $\neg(\neg\Phi^{(s)} \ominus \neg\Phi^{(t)}) = \Phi^{(s)} \ominus \Phi^{(t)}$.
(7) $\neg(\neg\Phi^{(s)} \boxtimes \neg\Phi^{(t)}) = \Phi^{(s)} \boxtimes \Phi^{(t)}$.
(8) $\neg(\neg\Phi^{(s)} \boxdot \neg\Phi^{(t)}) = \Phi^{(s)} \boxdot \Phi^{(t)}$.
(9) $\Phi^{(r)} \oplus (\Phi^{(s)} \cap \Phi^{(t)}) = (\Phi^{(r)} \oplus \Phi^{(s)}) \cap (\Phi^{(r)} \oplus \Phi^{(t)})$.
(10) $\Phi^{(r)} \oplus (\Phi^{(s)} \cup \Phi^{(t)}) = (\Phi^{(r)} \oplus \Phi^{(s)}) \cup (\Phi^{(r)} \oplus \Phi^{(t)})$.
(11) $\Phi^{(r)} \boxtimes (\Phi^{(s)} \cap \Phi^{(t)}) = (\Phi^{(r)} \boxtimes \Phi^{(s)}) \cap (\Phi^{(r)} \boxtimes \Phi^{(t)})$.
(12) $\Phi^{(r)} \boxtimes (\Phi^{(s)} \cup \Phi^{(t)}) = (\Phi^{(r)} \boxtimes \Phi^{(s)}) \cup (\Phi^{(r)} \boxtimes \Phi^{(t)})$.

Proof: We only prove (9)-(12).

(9) For each $x \in X$ (note $a + c \leq b + c$ if $a \leq b$),

$$
\begin{aligned}
&(\Phi^{(r)} \oplus (\Phi^{(s)} \cap \Phi^{(t)}))(x)\\
&= \Phi^{(r)}(x) \oplus \left(\begin{array}{l}\left[a_{1,x}^{(s)} \wedge a_{1,x}^{(t)}, b_{1,x}^{(s)} \wedge b_{1,x}^{(t)}\right], \left[a_{2,x}^{(s)} \wedge a_{2,x}^{(t)},\right.\\ \left.b_{2,x}^{(s)} \wedge b_{2,x}^{(t)}\right], \left[a_{3,x}^{(s)} \vee a_{3,x}^{(t)}, b_{3,x}^{(s)} \vee b_{3,x}^{(t)}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\frac{a_{1,x}^{(r)} + (a_{1,x}^{(s)} \wedge a_{1,x}^{(t)})}{2}, \frac{b_{1,x}^{(r)} + (b_{1,x}^{(s)} \wedge b_{1,x}^{(t)})}{2}\right], \left[\frac{a_{2,x}^{(r)} + (a_{2,x}^{(s)} \wedge a_{2,x}^{(t)})}{2},\right.\\ \left.\frac{b_{2,x}^{(r)} + (b_{2,x}^{(s)} \wedge b_{2,x}^{(t)})}{2}\right], \left[\frac{a_{3,x}^{(r)} + (a_{3,x}^{(s)} \vee a_{3,x}^{(t)})}{2}, \frac{b_{3,x}^{(r)} + (b_{3,x}^{(s)} \vee b_{3,x}^{(t)})}{2}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(s)}) \wedge (a_{1,x}^{(r)} + a_{1,x}^{(t)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(s)}) \wedge (b_{1,x}^{(r)} + b_{1,x}^{(t)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(s)}) \wedge (a_{2,x}^{(r)} + a_{2,x}^{(t)})}{2},\right.\\ \left.\frac{(b_{2,x}^{(r)} + b_{2,x}^{(s)}) \wedge (b_{2,x}^{(r)} + b_{2,x}^{(t)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(s)}) \vee (a_{3,x}^{(r)} + a_{3,x}^{(t)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(s)}) \vee (b_{3,x}^{(r)} + b_{3,x}^{(t)})}{2}\right]\end{array}\right)\\
&= \left(\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(s)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(s)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(s)})}{2}, \frac{(b_{2,x}^{(r)} + b_{2,x}^{(s)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(s)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(s)})}{2}\right]\right)\\
&\quad \cap \left(\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(t)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(t)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(t)})}{2}, \frac{(b_{2,x}^{(r)} + b_{2,x}^{(t)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(t)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(t)})}{2}\right]\right)\\
&= (\Phi^{(r)} \oplus \Phi^{(s)}) \cap (\Phi^{(r)} \oplus \Phi^{(t)}).
\end{aligned}
$$

(10) Similar to (9), for each $x \in X$,

$$
\begin{aligned}
&(\Phi^{(r)} \oplus (\Phi^{(s)} \cup \Phi^{(t)}))(x)\\
&= \Phi^{(r)}(x) \oplus \left(\begin{array}{l}\left[a_{1,x}^{(s)} \vee a_{1,x}^{(t)}, b_{1,x}^{(s)} \vee b_{1,x}^{(t)}\right], \left[a_{2,x}^{(s)} \wedge a_{2,x}^{(t)},\right.\\ \left.b_{2,x}^{(s)} \wedge b_{2,x}^{(t)}\right], \left[a_{3,x}^{(s)} \wedge a_{3,x}^{(t)}, b_{3,x}^{(s)} \wedge b_{3,x}^{(t)}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\frac{a_{1,x}^{(r)} + (a_{1,x}^{(s)} \vee a_{1,x}^{(t)})}{2}, \frac{b_{1,x}^{(r)} + (b_{1,x}^{(s)} \vee b_{1,x}^{(t)})}{2}\right], \left[\frac{a_{2,x}^{(r)} + (a_{2,x}^{(s)} \wedge a_{2,x}^{(t)})}{2},\right.\\ \left.\frac{b_{2,x}^{(r)} + (b_{2,x}^{(s)} \wedge b_{2,x}^{(t)})}{2}\right], \left[\frac{a_{3,x}^{(r)} + (a_{3,x}^{(s)} \wedge a_{3,x}^{(t)})}{2}, \frac{b_{3,x}^{(r)} + (b_{3,x}^{(s)} \wedge b_{3,x}^{(t)})}{2}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(s)}) \vee (a_{1,x}^{(r)} + a_{1,x}^{(t)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(s)}) \vee (b_{1,x}^{(r)} + b_{1,x}^{(t)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(s)}) \wedge (a_{2,x}^{(r)} + a_{2,x}^{(t)})}{2},\right.\\ \left.\frac{(b_{2,x}^{(r)} + b_{2,x}^{(s)}) \wedge (b_{2,x}^{(r)} + b_{2,x}^{(t)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(s)}) \wedge (a_{3,x}^{(r)} + a_{3,x}^{(t)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(s)}) \wedge (b_{3,x}^{(r)} + b_{3,x}^{(t)})}{2}\right]\end{array}\right)\\
&= \left(\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(s)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(s)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(s)})}{2}, \frac{(b_{2,x}^{(r)} + b_{2,x}^{(s)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(s)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(s)})}{2}\right]\right)\\
&\quad \cup \left(\left[\frac{(a_{1,x}^{(r)} + a_{1,x}^{(t)})}{2}, \frac{(b_{1,x}^{(r)} + b_{1,x}^{(t)})}{2}\right], \left[\frac{(a_{2,x}^{(r)} + a_{2,x}^{(t)})}{2}, \frac{(b_{2,x}^{(r)} + b_{2,x}^{(t)})}{2}\right], \left[\frac{(a_{3,x}^{(r)} + a_{3,x}^{(t)})}{2}, \frac{(b_{3,x}^{(r)} + b_{3,x}^{(t)})}{2}\right]\right)\\
&= (\Phi^{(r)} \oplus \Phi^{(s)}) \cup (\Phi^{(r)} \oplus \Phi^{(t)}).
\end{aligned}
$$

(11) For each $x \in X$ (note that $ac \leq bc$ if $a \leq b$ and $c \geq 0$),

$$
\begin{aligned}
&(\Phi^{(r)} \ominus (\Phi^{(s)} \cap \Phi^{(t)}))(x)\\
&= \Phi^{(r)}(x) \ominus \left(\begin{array}{l}\left[a_{1,x}^{(s)} \wedge a_{1,x}^{(t)}, b_{1,x}^{(s)} \wedge b_{1,x}^{(t)}\right], \left[a_{2,x}^{(s)} \wedge a_{2,x}^{(t)},\right.\\ \left.b_{2,x}^{(s)} \wedge b_{2,x}^{(t)}\right], \left[a_{3,x}^{(s)} \vee a_{3,x}^{(t)}, b_{3,x}^{(s)} \vee b_{3,x}^{(t)}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\sqrt{a_{1,x}^{(r)}(a_{1,x}^{(s)} \wedge a_{1,x}^{(t)})}, \sqrt{b_{1,x}^{(r)}(b_{1,x}^{(s)} \wedge b_{1,x}^{(t)})}\right], \left[\sqrt{a_{2,x}^{(r)}(a_{2,x}^{(s)} \wedge a_{2,x}^{(t)})},\right.\\ \left.\sqrt{b_{2,x}^{(r)}(b_{2,x}^{(s)} \wedge b_{2,x}^{(t)})}\right], \left[\sqrt{a_{3,x}^{(r)}(a_{3,x}^{(s)} \vee a_{3,x}^{(t)})}, \sqrt{b_{3,x}^{(r)}(b_{3,x}^{(s)} \vee b_{3,x}^{(t)})}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\sqrt{(a_{1,x}^{(r)}a_{1,x}^{(s)}) \wedge (a_{1,x}^{(r)}a_{1,x}^{(t)})}, \sqrt{(b_{1,x}^{(r)}b_{1,x}^{(s)}) \wedge (b_{1,x}^{(r)}b_{1,x}^{(t)})}\right], \left[\sqrt{(a_{2,x}^{(r)}a_{2,x}^{(s)}) \wedge (a_{2,x}^{(r)}a_{2,x}^{(t)})},\right.\\ \left.\sqrt{(b_{2,x}^{(r)}b_{2,x}^{(s)}) \wedge (b_{2,x}^{(r)}b_{2,x}^{(t)})}\right], \left[\sqrt{(a_{3,x}^{(r)}a_{3,x}^{(s)}) \vee (a_{3,x}^{(r)}a_{3,x}^{(t)})}, \sqrt{(b_{3,x}^{(r)}b_{3,x}^{(s)}) \vee (b_{3,x}^{(r)}b_{3,x}^{(t)})}\right]\end{array}\right)\\
&= \left(\left[\sqrt{a_{1,x}^{(r)}a_{1,x}^{(s)}}, \sqrt{b_{1,x}^{(r)}b_{1,x}^{(s)}}\right], \left[\sqrt{a_{2,x}^{(r)}a_{2,x}^{(s)}}, \sqrt{b_{2,x}^{(r)}b_{2,x}^{(s)}}\right], \left[\sqrt{a_{3,x}^{(r)}a_{3,x}^{(s)}}, \sqrt{b_{3,x}^{(r)}b_{3,x}^{(s)}}\right]\right)\\
&\quad \cap \left(\left[\sqrt{a_{1,x}^{(r)}a_{1,x}^{(t)}}, \sqrt{b_{1,x}^{(r)}b_{1,x}^{(t)}}\right], \left[\sqrt{a_{2,x}^{(r)}a_{2,x}^{(t)}}, \sqrt{b_{2,x}^{(r)}b_{2,x}^{(t)}}\right], \left[\sqrt{a_{3,x}^{(r)}a_{3,x}^{(t)}}, \sqrt{b_{3,x}^{(r)}b_{3,x}^{(t)}}\right]\right)\\
&= (\Phi^{(r)} \ominus \Phi^{(s)})(x) \cap (\Phi^{(r)} \ominus \Phi^{(t)})(x).
\end{aligned}
$$

(12) Similar to (11), for each $x \in X$,

$$
\begin{aligned}
&(\Phi^{(r)} \ominus (\Phi^{(s)} \cup \Phi^{(t)}))(x)\\
&= \Phi^{(r)}(x) \ominus \left(\begin{array}{l}\left[a_{1,x}^{(s)} \vee a_{1,x}^{(t)}, b_{1,x}^{(s)} \vee b_{1,x}^{(t)}\right], \left[a_{2,x}^{(s)} \wedge a_{2,x}^{(t)},\right.\\ \left.b_{2,x}^{(s)} \wedge b_{2,x}^{(t)}\right], \left[a_{3,x}^{(s)} \wedge a_{3,x}^{(t)}, b_{3,x}^{(s)} \wedge b_{3,x}^{(t)}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\sqrt{a_{1,x}^{(r)}(a_{1,x}^{(s)} \vee a_{1,x}^{(t)})}, \sqrt{b_{1,x}^{(r)}(b_{1,x}^{(s)} \vee b_{1,x}^{(t)})}\right], \left[\sqrt{a_{2,x}^{(r)}(a_{2,x}^{(s)} \wedge a_{2,x}^{(t)})},\right.\\ \left.\sqrt{b_{2,x}^{(r)}(b_{2,x}^{(s)} \wedge b_{2,x}^{(t)})}\right], \left[\sqrt{a_{3,x}^{(r)}(a_{3,x}^{(s)} \wedge a_{3,x}^{(t)})}, \sqrt{b_{3,x}^{(r)}(b_{3,x}^{(s)} \wedge b_{3,x}^{(t)})}\right]\end{array}\right)\\
&= \left(\begin{array}{l}\left[\sqrt{(a_{1,x}^{(r)}a_{1,x}^{(s)}) \vee (a_{1,x}^{(r)}a_{1,x}^{(t)})}, \sqrt{(b_{1,x}^{(r)}b_{1,x}^{(s)}) \vee (b_{1,x}^{(r)}b_{1,x}^{(t)})}\right], \left[\sqrt{(a_{2,x}^{(r)}a_{2,x}^{(s)}) \wedge (a_{2,x}^{(r)}a_{2,x}^{(t)})},\right.\\ \left.\sqrt{(b_{2,x}^{(r)}b_{2,x}^{(s)}) \wedge (b_{2,x}^{(r)}b_{2,x}^{(t)})}\right], \left[\sqrt{(a_{3,x}^{(r)}a_{3,x}^{(s)}) \wedge (a_{3,x}^{(r)}a_{3,x}^{(t)})}, \sqrt{(b_{3,x}^{(r)}b_{3,x}^{(s)}) \wedge (b_{3,x}^{(r)}b_{3,x}^{(t)})}\right]\end{array}\right)\\
&= \left(\left[\sqrt{a_{1,x}^{(r)}a_{1,x}^{(s)}}, \sqrt{b_{1,x}^{(r)}b_{1,x}^{(s)}}\right], \left[\sqrt{a_{2,x}^{(r)}a_{2,x}^{(s)}}, \sqrt{b_{2,x}^{(r)}b_{2,x}^{(s)}}\right], \left[\sqrt{a_{3,x}^{(r)}a_{3,x}^{(s)}}, \sqrt{b_{3,x}^{(r)}b_{3,x}^{(s)}}\right]\right)\\
&\quad \cup \left(\left[\sqrt{a_{1,x}^{(r)}a_{1,x}^{(t)}}, \sqrt{b_{1,x}^{(r)}b_{1,x}^{(t)}}\right], \left[\sqrt{a_{2,x}^{(r)}a_{2,x}^{(t)}}, \sqrt{b_{2,x}^{(r)}b_{2,x}^{(t)}}\right], \left[\sqrt{a_{3,x}^{(r)}a_{3,x}^{(t)}}, \sqrt{b_{3,x}^{(r)}b_{3,x}^{(t)}}\right]\right)\\
&= (\Phi^{(r)} \ominus \Phi^{(s)})(x) \cup (\Phi^{(r)} \ominus \Phi^{(t)})(x).
\end{aligned}
$$

■

Proposition 22: Let $\Phi^{(s)} = \left\{\frac{(a_x^{(s)},b_x^{(s)})}{x} \mid x \in X^{(s)}\right\} = \left\{\frac{\left([a_{1,x}^{(s)},b_{1,x}^{(s)}],\ [a_{2,x}^{(s)},b_{2,x}^{(s)}],\ [a_{3,x}^{(s)},b_{3,x}^{(s)}]\right)}{x} \mid x \in X^{(s)}\right\} \in \mathbb{I}^{\Delta X^{(s)}}$ $(s = 1, 2, 3)$, the followings hold:

(1) $\neg(\neg\Phi^{(1)} \times_1 \neg\Phi^{(2)}) = \Phi^{(1)} \times_1 \Phi^{(2)}$.
(2) $\neg(\neg\Phi^{(1)} \times_2 \neg\Phi^{(2)}) = \Phi^{(1)} \times_3 \Phi^{(2)}$.
(3) $\neg(\neg\Phi^{(1)} \times_3 \neg\Phi^{(2)}) = \Phi^{(1)} \times_2 \Phi^{(2)}$.
(4) $(\Phi^{(1)} \times_3 \Phi^{(2)}) \times_3 \Phi^{(3)} = \Phi^{(1)} \times_3 (\Phi^{(2)} \times_3 \Phi^{(3)})$.
(5) $(\Phi^{(1)} \cap \Phi^{(2)}) \times_3 \Phi^{(3)} = (\Phi^{(1)} \times_3 \Phi^{(3)}) \cap (\Phi^{(2)} \times_3 \Phi^{(3)})$.
(6) $(\Phi^{(1)} \cup \Phi^{(2)}) \times_3 \Phi^{(3)} = (\Phi^{(1)} \times_3 \Phi^{(3)}) \cup (\Phi^{(2)} \times_3 \Phi^{(3)})$.

Proof: It is follows from Theorem 6 and Remark 13. ■

Proposition 23: Let $\Phi^{(s)} = \left\{\frac{(a_x^{(s)},b_x^{(s)})}{x} \mid x \in X^{(s)}\right\} = \left\{\frac{\left([a_{1,x}^{(s)},b_{1,x}^{(s)}],\ [a_{2,x}^{(s)},b_{2,x}^{(s)}],\ [a_{3,x}^{(s)},b_{3,x}^{(s)}]\right)}{x} \mid x \in X^{(s)}\right\} \in \mathbb{I}^{\Delta X^{(s)}}$ $(s = 1, 2, 3 \in S)$, the followings hold:

(1) $\left(\Phi^{(1)} \times_1 \Phi^{(2)}\right)_{(1)} = \Phi^{(1)}_{(1)} \times_1 \Phi^{(2)}_{(1)}$.
(2) $\left(\Phi^{(1)} \times_1 \Phi^{(2)}\right)_{(2)} = \Phi^{(1)}_{(2)} \times_1 \Phi^{(2)}_{(2)}$.
(3) $\left(\Phi^{(1)} \times_1 \Phi^{(2)}\right)_{(3)} = \Phi^{(1)}_{(3)} \times_1 \Phi^{(2)}_{(3)}$.
(4) $\left(\Phi^{(1)} \times_1 \Phi^{(2)}\right)_{(4)} = \Phi^{(1)}_{(4)} \times_1 \Phi^{(2)}_{(4)}$.
(5) $\left(\Phi^{(1)} \times_2 \Phi^{(2)}\right)_{(1)} = \Phi^{(1)}_{(1)} \times_2 \Phi^{(2)}_{(1)}$.
(6) $\left(\Phi^{(1)} \times_2 \Phi^{(2)}\right)_{(2)} = \Phi^{(1)}_{(2)} \times_2 \Phi^{(2)}_{(2)}$.
(7) $\left(\Phi^{(1)} \times_2 \Phi^{(2)}\right)_{(3)} = \Phi^{(1)}_{(3)} \times_2 \Phi^{(2)}_{(3)}$.
(8) $\left(\Phi^{(1)} \times_2 \Phi^{(2)}\right)_{(4)} = \Phi^{(1)}_{(4)} \times_2 \Phi^{(2)}_{(4)}$.
(9) $\left(\Phi^{(1)} \times_3 \Phi^{(2)}\right)_{(1)} = \Phi^{(1)}_{(1)} \times_3 \Phi^{(2)}_{(1)}$.
(10) $\left(\Phi^{(1)} \times_3 \Phi^{(2)}\right)_{(2)} = \Phi^{(1)}_{(2)} \times_3 \Phi^{(2)}_{(2)}$.
(11) $\left(\Phi^{(1)} \times_3 \Phi^{(2)}\right)_{(3)} = \Phi^{(1)}_{(3)} \times_3 \Phi^{(2)}_{(3)}$.
(12) $\left(\Phi^{(1)} \times_3 \Phi^{(2)}\right)_{(4)} = \Phi^{(1)}_{(4)} \times_3 \Phi^{(2)}_{(4)}$.

Proof: It is follows from Theorem 6 and Definition 3.3. ■

IV. AN INTERVAL-VALUED PICTURE FUZZY SOFT SETS

In this section, we introduce the concepts of interval-valued picture fuzzy soft set, subset, and equality, and define some operations (i.e. complement, inf product, sup product, union, and intersection).

We begin with the definition of interval-valued picture fuzzy soft set (followed by an example).

Definition 24: An alternative name of a mapping[6] $\Phi : I \longrightarrow \mathbb{I}^{\Delta X}$ is called an interval-valued picture fuzzy soft set over X. The set of all interval-valued picture fuzzy soft sets on X is denoted by $(\mathbb{I}^{\Delta X})^I$ or $\mathbb{I}^{\Delta XI}$.

[6]Sometimes it is also written as $\Phi = \left\{\frac{(a_{i,x},b_{i,x})}{x} \,\middle|\, x \in X\right\}_{i\in I}$ or $\Phi = \left\{\frac{([a_{i,1,x},b_{i,1,x}],[a_{i,2,x},b_{i,2,x}],[a_{i,3,x},b_{i,3,x}])}{x} \,\middle|\, x \in X\right\}_{i\in I}$, where $[a_{i,1,x}, b_{i,1,x}]$ is called the degree of positive membership of x on parameter i in Φ, $[a_{i,2,x}, b_{i,2,x}]$ the degree of neutral membership of x on parameter i in Φ, and $[a_{i,3,x}, b_{i,3,x}]$ the degree of negative membership of x on parameter i in Φ.

Example 25: Assume that $X = \{x_1, x_2, x_3\}$ is a set of three cars under consideration of a decision maker to purchase and $I = \{i_1, i_2, i_3\}$ a set of parameters, where i_1 stands for the parameter 'color', i_2 stands for the parameter 'speed', and i_3 stands for the parameter 'price'. The $\Phi = \{\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}\} \in \mathbb{I}^{\Delta XI}$ describes the "attractiveness of the cars" to this decision maker, where

$$\Phi_{i_1} = \left\{\frac{([0.2, 0.3], [0.1, 0.4], [0.1, 0.3])}{x_1}, \frac{([0.1, 0.4], [0.3, 0.3], [0, 0.1])}{x_2}, \frac{([0.2, 0.3], [0, 0.5], [0.1, 0.2])}{x_3}\right\}$$

$$\Phi_{i_2} = \left\{\frac{([0.15, 0.2], [0.5, 0.6], [0.1, 0.2])}{x_1}, \frac{([0, 0.7], [0.1, 0.2], [0, 0.1])}{x_2}, \frac{([0.13, 0.4], [0.3, 0.3], [0.1, 0.3])}{x_3}\right\}$$

$$\Phi_{i_3} = \left\{\frac{([0.1, 0.2], [0.2, 0.5], [0.1, 0.3])}{x_1}, \frac{([0.2, 0.4], [0.3, 0.5], [0, 0.1])}{x_2}, \frac{([0.1, 0.4], [0.4, 0.4], [0.1, 0.2])}{x_3}\right\}$$

Definition 26: Let $\Phi^{(s)} = \left\{\frac{(a_{i,x}^{(s)},b_{i,x}^{(s)})}{x} \mid x \in X\right\}_{i\in I} = \left\{\frac{\left([a_{i,1,x}^{(s)},b_{i,1,x}^{(s)}],\ [a_{i,2,x}^{(s)},b_{i,2,x}^{(s)}],\ [a_{i,3,x}^{(s)},b_{i,3,x}^{(s)}]\right)}{x} \mid x \in X\right\}_{i\in I} \in \mathbb{I}^{\Delta XI}$ and

$$\Phi^{(t)} = \left\{\frac{(a_{j,x}^{(t)}, b_{j,x}^{(t)})}{x} \mid x \in X\right\}_{j\in J} = \left\{\frac{\left([a_{j,1,x}^{(t)}, b_{j,1,x}^{(t)}],\ [a_{j,2,x}^{(t)}, b_{j,2,x}^{(t)}],\ [a_{j,3,x}^{(t)}, b_{j,3,x}^{(t)}]\right)}{x} \mid x \in X\right\}_{j\in J} \in \mathbb{I}^{\Delta XJ}.$$

Then we define relations $\Subset$, $\dot{\Subset}$, and $\ddot{\Subset}$ as follows:

(1) $\Phi^{(s)} \Subset \Phi^{(t)}$ iff $I \subseteq J$, and $\Phi^{(s)}(i) \subseteq \Phi^{(t)}(j)$, that is, $\left[a_{i,1,x}^{(s)}, b_{i,1,x}^{(s)}\right] \supseteq \left[a_{j,1,x}^{(t)}, b_{j,1,x}^{(t)}\right]$, $\left[a_{i,2,x}^{(s)}, b_{i,2,x}^{(s)}\right] \supseteq \left[a_{j,2,x}^{(t)}, b_{j,2,x}^{(t)}\right]$, and $\left[a_{i,3,x}^{(s)}, b_{i,3,x}^{(s)}\right] \supseteq \left[a_{j,3,x}^{(t)}, b_{j,3,x}^{(t)}\right]$ for each $(x, i) \in X \times I$.

(2) $\Phi^{(s)} \dot{\Subset} \Phi^{(t)}$ iff $I \subseteq J$, and $\Phi^{(s)}(i) \dot{\subseteq} \Phi^{(t)}(j)$, that is, $\left[a_{i,1,x}^{(s)}, b_{i,1,x}^{(s)}\right] \subseteq \left[a_{j,1,x}^{(t)}, b_{j,1,x}^{(t)}\right]$, $\left[a_{i,2,x}^{(s)}, b_{i,2,x}^{(s)}\right] \subseteq \left[a_{j,2,x}^{(t)}, b_{j,2,x}^{(t)}\right]$, and $\left[a_{i,3,x}^{(s)}, b_{i,3,x}^{(s)}\right] \subseteq \left[a_{j,3,x}^{(t)}, b_{j,3,x}^{(t)}\right]$ for each $(x, i) \in X \times I$.

(3) $\Phi^{(s)} \ddot{\Subset} \Phi^{(t)}$ iff $I \subseteq J$, and $\Phi^{(s)}(i) \ddot{\subseteq} \Phi^{(t)}(j)$, that is, $\left[a_{i,1,x}^{(s)}, b_{i,1,x}^{(s)}\right] \subseteq \left[a_{j,1,x}^{(t)}, b_{j,1,x}^{(t)}\right]$, $\left[a_{i,2,x}^{(s)}, b_{i,2,x}^{(s)}\right] \subseteq \left[a_{j,2,x}^{(t)}, b_{j,2,x}^{(t)}\right]$, and $\left[a_{i,3,x}^{(s)}, b_{i,3,x}^{(s)}\right] \supseteq \left[a_{j,3,x}^{(t)}, b_{j,3,x}^{(t)}\right]$ for each $(x, i) \in X \times I$.

Example 27: Assume that $X = \{x_1, x_2, x_3, x_4\}$ is a universe set, $E = \{i_1, i_2, i_3, i_4\}$ is a set of parameters, $I = \{i_1, i_2\}$ and $J = \{i_1, i_2, i_3\}$ are subset of E, and $\Phi^{(s)} = \left\{\Phi_{i_1}^{(s)}, \Phi_{i_2}^{(s)}\right\} \in \mathbb{I}^{\Delta XI}$ and $\Phi^{(t)} = \left\{\Phi_{i_1}^{(t)}, \Phi_{i_2}^{(t)}, \Phi_{i_3}^{(t)}\right\} \in \mathbb{I}^{\Delta XJ}$ are defined by

$$\Phi^{(s)}_{i_1}=\left\{\frac{([0.1,0.1],[0.3,0.4],[0,0.3])}{x_1},\frac{([0,0.25],[0.1,0.3],[0.2,0.2])}{x_2},\frac{([0.2,0.4],[0.2,0.3],[0.1,0.3])}{x_3}\right\}$$

$$\Phi^{(s)}_{i_2}=\left\{\frac{([0.18,0.2],[0.1,0.4],[0.3,0.35])}{x_1},\frac{([0.14,0.22],[0.2,0.3],[0.4,0.42])}{x_2},\frac{([0.25,0.3],[0,0.2],[0.3,0.5])}{x_3}\right\}$$

$$\Phi^{(t)}_{i_1}=\left\{\frac{([0.2,0.2],[0.4,0.5],[0,0.3])}{x_1},\frac{([0.1,0.5],[0.2,0.3],[0.1,0.2])}{x_2},\frac{([0.3,0.4],[0.3,0.3],[0,0.2])}{x_3}\right\}$$

$$\Phi^{(t)}_{i_2}=\left\{\frac{([0.19,0.25],[0.2,0.4],[0.26,0.3])}{x_1},\frac{([0.15,0.3],[0.2,0.4],[0.3,0.3])}{x_2},\frac{([0.25,0.35],[0.1,0.25],[0.2,0.4])}{x_3}\right\}$$

$$\Phi^{(t)}_{i_3}=\left\{\frac{([0.15,0.2],[0.3,0.3],[0.1,0.2])}{x_1},\frac{([0,0.35],[0.1,0.2],[0,0.1])}{x_2},\frac{([0.13,0.4],[0.3,0.3],[0.1,0.3])}{x_3}\right\}.$$

From Definition 26, we can see $\Phi^{(s)}\ddot{\Subset}\Phi^{(t)}$, but $\Phi^{(s)}\not\Subset\Phi^{(t)}$ and $\Phi^{(s)}\dot{\not\Subset}\Phi^{(t)}$.

Definition 28: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}$.
Then we define relations $\boldsymbol{=}$, $\dot{=}$, and $\ddot{=}$ as follows:

(1) $\Phi^{(s)}\boldsymbol{=}\Phi^{(t)}$ iff $\Phi^{(s)}\Subset\Phi^{(t)}$ and $\Phi^{(t)}\Subset\Phi^{(s)}$.
(2) $\Phi^{(s)}\dot{=}\Phi^{(t)}$ iff $\Phi^{(s)}\dot{\Subset}\Phi^{(t)}$ and $\Phi^{(t)}\dot{\Subset}\Phi^{(s)}$.
(3) $\Phi^{(s)}\ddot{=}\Phi^{(t)}$ iff $\Phi^{(s)}\ddot{\Subset}\Phi^{(t)}$ and $\Phi^{(t)}\ddot{\Subset}\Phi^{(s)}$.

In order to give a deeper insight into this issue, we propose a marched complement of an interval-valued picture fuzzy soft set.

Definition 29: Let $\Phi=\left\{\frac{(a_{i,x},b_{i,x})}{x}\ \middle|\ x\in X\right\}_{i\in I}=\left\{\frac{\left([a_{i,1,x},b_{i,1,x}],[a_{i,2,x},b_{i,2,x}],[a_{i,3,x},b_{i,3,x}]\right)}{x}\ \middle|\ x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$.
The complement of Φ, denoted by $\neg\Phi$, is a mapping $\neg\Phi : I\longrightarrow\mathbb{I}^{\Delta X}$, given by

$$\neg\Phi=\{\neg\Phi(i)\}_{i\in I}=\left\{\frac{\left([a_{i,3,x},b_{i,3,x}],[a_{i,2,x},b_{i,2,x}],[a_{i,1,x},b_{i,1,x}]\right)}{x}\ \middle|\ x\in X\right\}_{i\in I}.$$

It is worth noting that in the above definition, the parameter set of the complement $\neg\Phi$ is still the original parameter set I (instead of $\neg I$). Clearly, $\neg\neg\Phi=\Phi$.

Example 30: The complement $\neg\Phi=\{\neg\Phi_{i_1},\neg\Phi_{i_2},\neg\Phi_{i_3}\}$ of Φ in Example 25 is given by

$$\neg\Phi_{i_1}=\left\{\frac{([0.1,0.3],[0.1,0.4],[0.2,0.3])}{x_1},\frac{([0,0.1],[0.3,0.3],[0.1,0.4])}{x_2},\frac{([0.1,0.2],[0,0.5],[0.2,0.3])}{x_3}\right\}$$

$$\neg\Phi_{i_2}=\left\{\frac{([0.1,0.2],[0.5,0.6],[0.15,0.2])}{x_1},\frac{([0,0.1],[0.1,0.2],[0,0.7])}{x_2},\frac{([0.1,0.3],[0.3,0.3],[0.13,0.4])}{x_3}\right\}$$

$$\neg\Phi_{i_3}=\left\{\frac{([0.1,0.3],[0.2,0.5],[0.1,0.2])}{x_1},\frac{([0,0.1],[0.3,0.5],[0.2,0.4])}{x_2},\frac{([0.1,0.2],[0.4,0.4],[0.1,0.4])}{x_3}\right\}.$$

Remark 31: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$
and

$$\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}.$$

No one of the followings is true:

(1) $\Phi^{(s)}\Subset\Phi^{(t)}\Longleftrightarrow\neg\Phi^{(t)}\Subset\neg\Phi^{(s)}$.
(2) $\Phi^{(s)}\dot{\Subset}\Phi^{(t)}\Longleftrightarrow\neg\Phi^{(t)}\dot{\Subset}\neg\Phi^{(s)}$.
(3) $\Phi^{(s)}\ddot{\Subset}\Phi^{(t)}\Longleftrightarrow\neg\Phi^{(t)}\ddot{\Subset}\neg\Phi^{(s)}$.

Example 32: Suppose that $X=\{x_1,x_2\}$, $I=\{i\}$, $J=\{j\}$, and $\Phi^{(s)}=\{\Phi^{(s)}_i\}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\{\Phi^{(t)}_j\}\in\mathbb{I}^{\Delta XJ}$ are defined by

$$\Phi^{(s)}_i=\left\{\frac{([0.2,0.4],[0.3,0.5],[0.1,0.1])}{x_1},\frac{([0.3,0.3],[0.1,0.4],[0.2,0.3])}{x_2}\right\}$$

and

$$\Phi^{(t)}_j=\left\{\frac{([0.1,0.3],[0.2,0.4],[0,0.1])}{x_1},\frac{([0.2,0.3],[0,0.3],[0.1,0.2])}{x_2}\right\}.$$

It follows that

$$\neg\Phi^{(s)}_i=\left\{\frac{([0.1,0.1],[0.3,0.5],[0.2,0.4])}{x_1},\frac{([0.2,0.3],[0.1,0.4],[0.3,0.3])}{x_2}\right\}$$

and

$$\neg\Phi^{(t)}_j=\left\{\frac{([0,0.1],[0.2,0.4],[0.1,0.3])}{x_1},\frac{([0.1,0.2],[0,0.3],[0.2,0.3])}{x_2}\right\}.$$

Thus $\Phi^{(s)}\Subset\Phi^{(t)}\Longleftrightarrow\neg\Phi^{(t)}\Subset\neg\Phi^{(s)}$ is not true.

Example 33: Assume that $X=\{x_1,x_2\}$, $I=\{i\}$, $J=\{j\}$, and $\Phi^{(s)}=\{\Phi^{(s)}_i\}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\{\Phi^{(t)}_j\}\in\mathbb{I}^{\Delta XJ}$ are defined by

$$\Phi^{(s)}_i=\left\{\frac{([0.1,0.3],[0.2,0.4],[0,0.1])}{x_1},\frac{([0.2,0.3],[0,0.3],[0.1,0.2])}{x_2}\right\}$$

and

$$\Phi^{(t)}_j=\left\{\frac{([0.2,0.4],[0.3,0.5],[0.1,0.1])}{x_1},\frac{([0.3,0.3],[0.1,0.4],[0.2,0.3])}{x_2}\right\}.$$

It follows that

$$\neg\Phi^{(s)}_i=\left\{\frac{([0,0.1],[0.2,0.4],[0.1,0.3])}{x_1},\frac{([0.1,0.2],[0,0.3],[0.2,0.3])}{x_2}\right\}$$

and

$$\neg\Phi^{(t)}_j=\left\{\frac{([0.1,0.1],[0.3,0.5],[0.2,0.4])}{x_1},\frac{([0.2,0.3],[0.1,0.4],[0.3,0.3])}{x_2}\right\}.$$

Thus $\Phi^{(s)}\dot{\Subset}\Phi^{(t)}\Longleftrightarrow\neg\Phi^{(t)}\dot{\Subset}\neg\Phi^{(s)}$ is not true.

Example 34: Suppose that $X=\{x_1,x_2\}$, $I=\{i\}$, $J=\{j\}$, and $\Phi^{(s)}=\{\Phi^{(s)}_i\}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\{\Phi^{(t)}_j\}\in\mathbb{I}^{\Delta XJ}$ are defined by

$$\Phi^{(s)}_i=\left\{\frac{([0.1,0.3],[0.2,0.4],[0.1,0.1])}{x_1},\frac{([0.2,0.3],[0,0.3],[0.3,0.4])}{x_2}\right\}$$

and

$$\Phi_j^{(t)} = \left\{ \frac{([0.2,0.4],[0.3,0.5],[0,0.1])}{x_1}, \frac{([0.3,0.3],[0.1,0.4],[0.1,0.2])}{x_2} \right\}.$$

It follows that

$$\neg\Phi_i^{(s)} = \left\{ \frac{([0.1,0.1],[0.2,0.4],[0.1,0.3])}{x_1}, \frac{([0.3,0.4],[0,0.3],[0.2,0.3])}{x_2} \right\}$$

and

$$\neg\Phi_j^{(t)} = \left\{ \frac{([0,0.1],[0.3,0.5],[0.2,0.4])}{x_1}, \frac{([0.1,0.2],[0.1,0.4],[0.3,0.3])}{x_2} \right\}.$$

Thus $\Phi^{(s)} \ddot{\Subset} \Phi^{(t)} \Longleftrightarrow \neg\Phi^{(t)} \ddot{\Subset} \neg\Phi^{(s)}$ is not true.

In the following we introduce the definitions of inf product and sup product for interval-valued picture fuzzy soft sets, give some examples, and derive their properties.

Definition 35: Let $\Phi^{(s)} = \left\{ \frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X \right\}_{i\in I} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i\in I} \in \mathbb{I}^{\Delta XI}$ and $\Phi^{(t)} = \left\{ \frac{(a^{(t)}_{j,x}, b^{(t)}_{j,x})}{x} \mid x \in X \right\}_{j\in J} = \left\{ \frac{\left([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]\right)}{x} \mid x \in X \right\}_{j\in J} \in \mathbb{I}^{\Delta XJ}$. Then we define the inf product $\Phi^{(s)} \hat{\otimes} \Phi^{(t)} \in \mathbb{I}^{\Delta X(I\times J)}$ of $\Phi^{(s)}$ and $\Phi^{(t)}$ as[7] $\Phi^{(s)} \hat{\otimes} \Phi^{(t)} = \left\{ (\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j) \right\}_{(i,j)\in I\times J}$, where

$$(\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j) = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}\right]\right)}{x} \,\middle|\, x \in X \right\}.$$

Next, we present an application of inf product of interval-valued picture fuzzy soft theory in decision-making problems.

Let $X = \{x_1, x_2, \cdots, x_p\}$ (a p-element set) be the set of option schemes, I and J be the sets of scheme parameters, $\Phi^{(s)} \in \mathbb{I}^{\Delta XI}$ and $\Phi^{(t)} \in \mathbb{I}^{\Delta XJ}$ are the evaluation interval-valued picture fuzzy soft sets. For each $i \in I$, the interval-valued $[a_{i,k,x}, b_{i,k,x}]$ $(k = 1, 2, 3)$ in

$$\Phi^{(s)} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}], [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}], [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \,\middle|\, x \in X \right\}_{i\in I}$$

is the evaluation of expert group k to scheme x on parameter i, where $[a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}]$ is the degree of positive membership of scheme x on parameter i in $\Phi^{(s)}$, $[a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}]$ is the degree of neutral membership of scheme x on parameter i in $\Phi^{(s)}$, and $[a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]$ is the degree of negative membership of scheme x on parameter i in $\Phi^{(s)}$ (clearly, $b^{(s)}_{i,1,x} + b^{(s)}_{i,2,x} + b^{(s)}_{i,3,x} \leq 1$). Also, for each $j \in J$, the interval-valued $[a^{(s)}_{j,k,x}, b^{(s)}_{j,k,x}]$ $(k = 1, 2, 3)$ in

$$\Phi^{(t)} = \left\{ \frac{\left([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}], [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}], [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]\right)}{x} \,\middle|\, x \in X \right\}_{j\in J}$$

is the evaluation of expert group k to scheme x on parameter j, where $[a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}]$ is the degree of positive membership of scheme x on parameter j in $\Phi^{(t)}$, $[a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}]$ is the degree of neutral membership of scheme x on parameter j in $\Phi^{(t)}$, and $[a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]$ is the degree of negative membership of scheme x on parameter j in $\Phi^{(t)}$ (clearly, $b^{(t)}_{j,1,x} + b^{(t)}_{j,2,x} + b^{(t)}_{j,3,x} \leq 1$). Thus we get the inf product $\Phi^{(s)} \hat{\otimes} \Phi^{(t)}$ which satisfies ($(i,j) \in I \times J$)

$$(\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j) = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}\right]\right)}{x} \,\middle|\, x \in X \right\}.$$

where the interval-valued vector $\left([a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}], [a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}], [a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}]\right)$ can be looked to be an evaluation of expert group k to scheme x on parameter pair (i, j), $[a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}]$ can be looked to be the degree of positive membership, $[a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}]$ can be looked to be the degree of neutral membership and $[a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}]$ can be looked to be the degree of negative membership (clearly, $(b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}) + (b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}) + (b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}) \leq 1$).

Define the choice value $C(x, i, j)$ (an interval number) and the score $S(x) = \sum\limits_{(i,j)\in I\times J} S(x, i, j)$ (a real number) of scheme x, respectively, as follows ($x \in X,\ (i, j) \in I \times J$):

$$\begin{aligned} C(x, i, j) = p_1&\left[(\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j)(x)\right] \\ &\oplus p_2\left[(\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j)(x)\right] \\ &\oplus p_3\left[(\Phi^{(s)} \hat{\otimes} \Phi^{(t)})(i,j)(x)\right] \end{aligned}$$

[7] It can also be written as

$$\Phi^{(s)} \hat{\otimes} \Phi^{(t)} = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}\right]\right)}{x} \,\middle|\, x \in X \right\}_{(i,j)\in I\times J}.$$

or

$$\Phi^{(s)} \hat{\otimes} \Phi^{(t)} = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x} \wedge a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x} \wedge b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x} \wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x} \wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x} \vee a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x} \vee b^{(t)}_{j,3,x}\right]\right)}{x} \,\middle|\, x \in X, (i,j) \in I \times J \right\}.$$

$S(x,i,j)=\sum_{y\in X}\Big[p_1(C(x,i,j))-p_1(C(y,i,j))+p_2(C(x,i,j))-p_2(C(y,i,j))\Big]$.

Then $\hat{x}=\arg\max\{S(x_1),S(x_2),\cdots,S(x_q)\}$ will be the best choice.

The corresponding algorithm is as follow:

Algorithm I

Step 1: Input an interval-valued picture fuzzy soft sets $\Phi^{(s)}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}\in\mathbb{I}^{\Delta XJ}$.

Step 2: Compute and write the inf product $\Phi^{(s)}\hat{\otimes}\Phi^{(t)}$ which satisfies $((i,j)\in I\times J)$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i,j)=\left\{\frac{\left(\left[a^{(s)}_{i,1,x}\wedge a^{(t)}_{j,1,x},b^{(s)}_{i,1,x}\wedge b^{(t)}_{j,1,x}\right],\left[a^{(s)}_{i,2,x}\wedge a^{(t)}_{j,2,x},b^{(s)}_{i,2,x}\wedge b^{(t)}_{j,2,x}\right],\left[a^{(s)}_{i,3,x}\vee a^{(t)}_{j,3,x},b^{(s)}_{i,3,x}\vee b^{(t)}_{j,3,x}\right]\right)}{x}\,\middle|\,x\in X\right\}.$$

Step 3: Compute the choice value $C(x,i,j)$ (an interval number) and the score $S(x)=\sum_{(i,j)\in I\times J}S(x,i,j)$ (a real number) of scheme x, respectively, as follows ($x\in X$, $(i,j)\in I\times J$):

$$\begin{aligned}C(x,i,j)&=p_1\Big[(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i,j)(x)\Big]\\&\quad\oplus p_2\Big[(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i,j)(x)\Big]\\&\quad\oplus p_3\Big[(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i,j)(x)\Big],\\S(x,i,j)&=\sum_{y\in X}\Big[p_1(C(x,i,j))-p_1(C(y,i,j))\\&\quad+p_2(C(x,i,j))-p_2(C(y,i,j))\Big].\end{aligned}$$

Step 4: Determine the best choice

$$\hat{x}=\arg\max\{S(x_1),S(x_2),\cdots,S(x_q)\}.$$

The working of Algorithm I is demonstrated through a numerical example as below:

Example 36: Assume that a general manager Mr. Z in Customer Care Center of a company want one candidate out of five x_1,x_2,x_3,x_4,x_5 (write $X=\{x_1,x_2,x_3,x_4,x_5\}$) who have applied for a job position of Office Representative in Customer Care Center of a company. The general manager has to evaluate the following three parameters i_1,i_2,i_3 (write $I=J=\{i_1,i_2,i_3\}$), where i_1 stands for the 'Hard Working ', i_2 stands for the 'Optimism ', and i_3 stands for 'Individualism '. The evaluation result can be described by interval-valued picture fuzzy soft sets $\Phi^{(s)}=\left\{\Phi^{(s)}_{i_1},\Phi^{(s)}_{i_2},\Phi^{(s)}_{i_3}\right\}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\left\{\Phi^{(t)}_{i_1},\Phi^{(t)}_{i_2},\Phi^{(t)}_{i_3}\right\}\in\mathbb{I}^{\Delta XJ}$ which are given as

$$\Phi^{(s)}_{i_1}=\left\{\frac{([0.2,0.3],[0.1,0.2],[0.25,0.3])}{x_1},\frac{([0.12,0.4],[0,0.3],[0,0.25])}{x_2},\frac{([0.2,0.3],[0,0.41],[0.1,0.2])}{x_3},\frac{([0.1,0.4],[0.1,0.2],[0,0.3])}{x_4},\frac{([0.1,0.3],[0.2,0.3],[0,0.2])}{x_5}\right\}$$

$$\Phi^{(s)}_{i_2}=\left\{\frac{([0.15,0.3],[0.18,0.2],[0.17,0.4])}{x_1},\frac{([0.2,0.3],[0.17,0.2],[0,0.3])}{x_2},\frac{([0.13,0.4],[0.2,0.3],[0.1,0.3])}{x_3},\frac{([0.1,0.4],[0,0.4],[0.1,0.1])}{x_4},\frac{([0.19,0.4],[0.15,0.2],[0,0.3])}{x_5}\right\}$$

$$\Phi^{(s)}_{i_3}=\left\{\frac{([0.1,0.29],[0.2,0.4],[0.1,0.25])}{x_1},\frac{([0.2,0.4],[0.3,0.5],[0,0.1])}{x_2},\frac{([0.1,0.4],[0.3,0.4],[0.1,0.2])}{x_3},\frac{([0.2,0.3],[0.2,0.6],[0,0.1])}{x_4},\frac{([0.1,0.25],[0.19,0.2],[0,0.3])}{x_5}\right\}$$

$$\Phi^{(t)}_{i_1}=\left\{\frac{([0.22,0.3],[0.1,0.25],[0.11,0.3])}{x_1},\frac{([0,0.4],[0,0.25],[0.15,0.3])}{x_2},\frac{([0.2,0.4],[0.2,0.35],[0.1,0.2])}{x_3},\frac{([0.1,0.2],[0.14,0.25],[0,0.3])}{x_4},\frac{([0.1,0.32],[0.25,0.3],[0.1,0.2])}{x_5}\right\}$$

$$\Phi^{(t)}_{i_2}=\left\{\frac{([0.1,0.2],[0.18,0.3],[0.19,0.4])}{x_1},\frac{([0.19,0.3],[0.17,0.2],[0.16,0.4])}{x_2},\frac{([0.13,0.4],[0.12,0.3],[0.1,0.22])}{x_3},\frac{([0.1,0.33],[0,0.4],[0.19,0.2])}{x_4},\frac{([0.19,0.4],[0.15,0.22],[0,0.3])}{x_5}\right\}$$

$$\Phi^{(t)}_{i_3}=\left\{\frac{([0.1,0.29],[0.2,0.4],[0,0.25])}{x_1},\frac{([0.2,0.4],[0.3,0.4],[0,0.1])}{x_2},\frac{([0.1,0.22],[0.3,0.4],[0.1,0.25])}{x_3},\frac{([0.12,0.3],[0.2,0.5],[0,0.1])}{x_4},\frac{([0.1,0.25],[0.19,0.2],[0.13,0.3])}{x_5}\right\}.$$

Thus we can write the inf product $\Phi^{(s)}\hat{\otimes}\Phi^{(t)}$ in Step 2 of Algorithm I:

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_1,i_1)=\left\{\frac{([0.2,0.3],[0.1,0.2],[0.25,0.3])}{x_1},\frac{([0,0.4],[0,0.25],[0.15,0.3])}{x_2},\frac{([0.2,0.3],[0,0.35],[0.1,0.2])}{x_3},\frac{([0.1,0.2],[0.1,0.2],[0,0.3])}{x_4},\frac{([0.1,0.3],[0.2,0.3],[0.1,0.2])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_1,i_2)=\left\{\frac{([0.1,0.2],[0.1,0.2],[0.25,0.4])}{x_1},\frac{([0.12,0.3],[0,0.2],[0.16,0.4])}{x_2},\frac{([0.13,0.3],[0,0.3],[0.1,0.22])}{x_3},\frac{([0.1,0.33],[0,0.2],[0.19,0.3])}{x_4},\frac{([0.1,0.3],[0.15,0.22],[0,0.3])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_1,i_3)=\left\{\frac{([0.1,0.29],[0.1,0.2],[0.25,0.3])}{x_1},\frac{([0.12,0.4],[0,0.3],[0,0.25])}{x_2},\frac{([0.1,0.22],[0,0.4],[0.1,0.25])}{x_3},\frac{([0.1,0.3],[0.1,0.2],[0,0.3])}{x_4},\frac{([0.1,0.25],[0.19,0.2],[0.13,0.3])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_2,i_1)=\left\{\frac{([0.15,0.3],[0.1,0.2],[0.17,0.4])}{x_1},\frac{([0,0.3],[0,0.2],[0.15,0.3])}{x_2},\frac{([0.13,0.4],[0.2,0.3],[0.1,0.3])}{x_3},\frac{([0.1,0.2],[0,0.25],[0.1,0.3])}{x_4},\frac{([0.1,0.32],[0.15,0.2],[0.1,0.3])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_2,i_2)=\left\{\frac{([0.1,0.2],[0.18,0.2],[0.19,0.4])}{x_1},\frac{([0.19,0.3],[0.17,0.2],[0.16,0.4])}{x_2},\frac{([0.13,0.4],[0,0.3],[0.1,0.22])}{x_3},\frac{([0.1,0.33],[0,0.4],[0.19,0.2])}{x_4},\frac{([0.19,0.4],[0.15,0.2],[0,0.3])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_2,i_3)=\left\{\frac{([0.1,0.29],[0.18,0.2],[0.17,0.4])}{x_1},\frac{([0.2,0.3],[0.17,0.2],[0,0.3])}{x_2},\frac{([0.1,0.22],[0.2,0.3],[0.1,0.3])}{x_3},\frac{([0.1,0.3],[0,0.4],[0.1,0.1])}{x_4},\frac{([0.1,0.25],[0.15,0.2],[0.13,0.3])}{x_5}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_3,i_1)=\left\{\begin{array}{l}\frac{([0.1,0.29],[0.1,0.25],[0.11,0.3])}{x_1},\frac{([0,0.4],[0,0.25],[0.15,0.3])}{x_2},\\ \frac{([0.1,0.4],[0.2,0.35],[0.1,0.2])}{x_3},\frac{([0.1,0.2],[0.14,0.25],[0,0.3])}{x_4},\\ \frac{([0.1,0.25],[0.19,0.2],[0.1,0.3])}{x_5}\end{array}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_3,i_2)=\left\{\begin{array}{l}\frac{([0.1,0.2],[0.18,0.3],[0.19,0.4])}{x_1},\frac{([0.19,0.3],[0.17,0.2],[0.16,0.4])}{x_2},\\ \frac{([0.13,0.4],[0.12,0.3],[0.1,0.22])}{x_3},\frac{([0.1,0.3],[0,0.4],[0.19,0.2])}{x_4},\\ \frac{([0.1,0.25],[0.15,0.2],[0,0.3])}{x_5}\end{array}\right\}$$

$$(\Phi^{(s)}\hat{\otimes}\Phi^{(t)})(i_3,i_3)=\left\{\begin{array}{l}\frac{([0.1,0.29],[0.2,0.4],[0.1,0.25])}{x_1},\frac{([0.2,0.4],[0.3,0.4],[0,0.1])}{x_2},\\ \frac{([0.1,0.22],[0.3,0.4],[0.1,0.25])}{x_3},\frac{([0.12,0.3],[0.2,0.5],[0,0.1])}{x_4},\\ \frac{([0.1,0.25],[0.19,0.2],[0.13,0.3])}{x_5}\end{array}\right\}.$$

By computing we obtain $C(x,i,j)$, $S(x,i,j)$ and the score $S(x)$ of schemes x, respectively, as follows ($x_1,x_2,x_3,x_4,x_5 \in X$, $(i,j)\in I\times J$):

$C(x_1,i_1,i_1) = [0.2+0.1+0.25, 0.3+0.2+03] = [0.55,0.8]$. Similarity, $C(x_1,i_1,i_2) = [0.45,0.8]$, $C(x_1,i_1,i_3) = [0.45,0.79]$, $C(x_1,i_2,i_1) = [0.42,0.9]$, $C(x_1,i_2,i_2) = [0.47,0.8]$, $C(x_1,i_2,i_3) = [0.45,0.89]$, $C(x_1,i_3,i_1) = [0.31,0.84]$, $C(x_1,i_3,i_2) = [0.47,0.9]$, $C(x_1,i_3,i_3) = [0.4,0.94]$; $C(x_2,i_1,i_1) = [0.15,0.95]$, $C(x_2,i_1,i_2) = [0.28,0.9]$, $C(x_2,i_1,i_3) = [0.12,0.95]$, $C(x_2,i_2,i_1) = [0.15,0.8]$, $C(x_2,i_2,i_2) = [0.52,0.9]$, $C(x_2,i_2,i_3) = [0.37,0.8]$, $C(x_2,i_3,i_1) = [0.15,0.95]$, $C(x_2,i_3,i_2) = [0.52,0.9]$, $C(x_2,i_3,i_3) = [0.5,0.9]$; $C(x_3,i_1,i_1) = [0.3,0.85]$, $C(x_3,i_1,i_2) = [0.23,0.82]$, $C(x_3,i_1,i_3) = [0.2,0.87]$, $C(x_3,i_2,i_1) = [0.43,1]$, $C(x_3,i_2,i_2) = [0.23,0.92]$, $C(x_3,i_2,i_3) = [0.4,0.82]$, $C(x_3,i_3,i_1) = [0.4,0.95]$, $C(x_3,i_3,i_2) = [0.35,0.92]$, $C(x_3,i_3,i_3) = [0.5,0.87]$; $C(x_4,i_1,i_1) = [0.2,0.7]$, $C(x_4,i_1,i_2) = [0.29,0.83]$, $C(x_4,i_1,i_3) = [0.2,0.8]$, $C(x_4,i_2,i_1) = [0.2,0.75]$, $C(x_4,i_2,i_2) = [0.29,0.93]$, $C(x_4,i_2,i_3) = [0.2,0.8]$, $C(x_4,i_3,i_1) = [0.24,0.75]$, $C(x_4,i_3,i_2) = [0.29,0.9]$, $C(x_4,i_3,i_3) = [0.32,0.9]$; $C(x_5,i_1,i_1) = [0.4,0.8]$, $C(x_5,i_1,i_2) = [0.25,0.82]$, $C(x_5,i_1,i_3) = [0.42,0.75]$, $C(x_5,i_2,i_1) = [0.35,0.82]$, $C(x_5,i_2,i_2) = [0.34,0.9]$, $C(x_5,i_2,i_3) = [0.38,0.75]$, $C(x_5,i_3,i_1) = [0.39,0.75]$, $C(x_5,i_3,i_2) = [0.25,0.75]$, $C(x_5,i_3,i_3) = [0.42,0.75]$.

And $S(x_1,i_1,i_1) = (((0.55-0.15)+(0.8-0.95))+((0.55-0.3)+(0.8-0.85))+((0.55-0.2)+(0.8-0.7))+((0.55-0.4)+(0.8-0.8))) = 1.05$. Similarity, $S(x_1,i_1,i_2) = 0.58$, $S(x_1,i_1,i_3) = 0.65$, $S(x_1,i_2,i_1) = 0.78$, $S(x_1,i_2,i_2) = 0.05$, $S(x_1,i_2,i_3) = 0.84$, $S(x_1,i_3,i_1) = 0.02$, $S(x_1,i_3,i_2) = 0.6$, $S(x_1,i_3,i_3) = 0.2$; $S(x_2,i_1,i_1) = -0.2$, $S(x_2,i_1,i_2) = 0.24$, $S(x_2,i_1,i_3) = -0.2$, $S(x_2,i_2,i_1) = -0.98$, $S(x_2,i_2,i_2) = 0.8$, $S(x_2,i_2,i_3) = -0.01$, $S(x_2,i_3,i_1) = -0.23$, $S(x_2,i_3,i_2) = 0.85$, $S(x_2,i_3,i_3) = 0.5$; $S(x_3,i_1,i_1) = 0.05$, $S(x_3,i_1,i_2) = -0.42$, $S(x_3,i_1,i_3) = -0.2$, $S(x_3,i_2,i_1) = 1.33$, $S(x_3,i_2,i_2) = -0.65$, $S(x_3,i_2,i_3) = 0.24$, $S(x_3,i_3,i_1) = -0.88$, $S(x_3,i_3,i_2) = 0.1$, $S(x_3,i_3,i_3) = -1.39$; $S(x_4,i_1,i_1) = -1.2$, $S(x_4,i_1,i_2) = -0.07$, $S(x_4,i_1,i_3) = -0.55$, $S(x_4,i_2,i_1) = -1.07$, $S(x_4,i_2,i_2) = -0.18$, $S(x_4,i_2,i_3) = -0.86$, $S(x_4,i_3,i_1) = -0.78$, $S(x_4,i_3,i_2) = -0.3$, $S(x_4,i_3,i_3) = -0.4$; $S(x_5,i_1,i_1) = -0.5$, $S(x_5,i_1,i_2) = -0.32$, $S(x_5,i_1,i_3) = 0.3$, $S(x_5,i_2,i_1) = 0.03$, $S(x_5,i_2,i_2) = -0.1$, $S(x_5,i_2,i_3) = -0.21$, $S(x_5,i_3,i_1) = -0.03$, $S(x_5,i_3,i_2) = -0.1.25$, $S(x_5,i_3,i_3) = -0.65$.

It follows $S(x_1) = 4.77$, $S(x_2) = 0.77$, $S(x_3) = -1.85$, $S(x_4) = -5.41$, $S(x_5) = -2.73$. Thus, based on the values $S(x)$, we get x_1 is the best choice.

Now we introduce the sup product for interval-valued picture fuzzy soft sets.

Definition 37: Let $\Phi^{(s)} = \left\{\frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X\right\}_{i\in I} = \left\{\frac{([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}])}{x} \mid x \in X\right\}_{i\in I} \in \mathbb{I}^{\Delta XI}$ and $\Phi^{(t)} = \left\{\frac{(a^{(t)}_{j,x}, b^{(t)}_{j,x})}{x} \mid x \in X\right\}_{j\in J} = \left\{\frac{([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}])}{x} \mid x \in X\right\}_{j\in J} \in \mathbb{I}^{\Delta XJ}$. Then we define the sup product $\Phi^{(s)}\check{\otimes}\Phi^{(t)} \in \mathbb{I}^{\Delta X(I\times J)}$ of $\Phi^{(s)}$ and $\Phi^{(t)}$ as[8] $\Phi^{(s)}\check{\otimes}\Phi^{(t)} = \{(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i,j)\}_{(i,j)\in I\times J}$, where

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i,j) = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x}\vee a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x}\vee b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x}\wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x}\wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x}\wedge a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x}\wedge b^{(t)}_{j,3,x}\right]\right)}{x} \;\middle|\; x \in X \right\}.$$

Example 38: The sup product $\Phi^{(s)}\check{\otimes}\Phi^{(t)}$ in Example 27 is given by

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_1,i_1)=\left\{\begin{array}{l}\frac{([0.2,0.2],[0.3,0.4],[0,0.3])}{x_1},\frac{([0.1,0.5],[0.1,0.3],[0.1,0.2])}{x_2},\\ \frac{([0.3,0.4],[0.2,0.3],[0,0.2])}{x_3}\end{array}\right\}$$

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_1,i_2)=\left\{\begin{array}{l}\frac{([0.19,0.25],[0.2,0.4],[0,0.3])}{x_1},\frac{([0.15,0.3],[0.1,0.3],[0.2,0.2])}{x_2},\\ \frac{([0.25,0.4],[0.1,0.25],[0.1,0.3])}{x_3}\end{array}\right\}$$

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_1,i_3)=\left\{\begin{array}{l}\frac{([0.15,0.2],[0.3,0.3],[0,0.2])}{x_1},\frac{([0,0.35],[0.1,0.2],[0,0.1])}{x_2},\\ \frac{([0.2,0.4],[0.2,0.3],[0.1,0.3])}{x_3}\end{array}\right\}$$

[8] It can also be written as

$$\Phi^{(s)}\check{\otimes}\Phi^{(t)} = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x}\vee a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x}\vee b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x}\wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x}\wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x}\wedge a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x}\wedge b^{(t)}_{j,3,x}\right]\right)}{x} \;\middle|\; x \in X \right\}_{(i,j)\in I\times J}.$$

or

$$\Phi^{(s)}\check{\otimes}\Phi^{(t)} = \left\{ \frac{\left(\left[a^{(s)}_{i,1,x}\vee a^{(t)}_{j,1,x}, b^{(s)}_{i,1,x}\vee b^{(t)}_{j,1,x}\right], \left[a^{(s)}_{i,2,x}\wedge a^{(t)}_{j,2,x}, b^{(s)}_{i,2,x}\wedge b^{(t)}_{j,2,x}\right], \left[a^{(s)}_{i,3,x}\wedge a^{(t)}_{j,3,x}, b^{(s)}_{i,3,x}\wedge b^{(t)}_{j,3,x}\right]\right)}{x} \;\middle|\; x \in X, (i,j)\in I\times J \right\}.$$

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_2,i_1)=\left\{\frac{([0.2,0.2],[0.1,0.4],[0,0.3])}{x_1},\frac{([0.14,0.5],[0.2,0.3],[0.1,0.2])}{x_2},\frac{([0.3,0.4],[0,0.2],[0,0.2])}{x_3}\right\}$$

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_2,i_2)=\left\{\frac{([0.19,0.25],[0.1,0.4],[0.26,0.3])}{x_1},\frac{([0.15,0.3],[0.2,0.3],[0.3,0.3])}{x_2},\frac{([0.25,0.35],[0,0.2],[0.2,0.4])}{x_3}\right\}$$

$$(\Phi^{(s)}\check{\otimes}\Phi^{(t)})(i_2,i_3)=\left\{\frac{([0.18,0.2],[0.1,0.3],[0.1,0.2])}{x_1},\frac{([0.14,0.35],[0.1,0.2],[0,0.1])}{x_2},\frac{([0.25,0.4],[0,0.2],[0.1,0.3])}{x_3}\right\}.$$

Proposition 39: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}$.
Then the followings hold:

(1) $\neg\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)=\left(\neg\Phi^{(s)}\right)\check{\otimes}\left(\neg\Phi^{(t)}\right)$.
(2) $\neg\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)=\left(\neg\Phi^{(s)}\right)\hat{\otimes}\left(\neg\Phi^{(t)}\right)$.
(3) $\neg\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\doteq\left(\neg\Phi^{(s)}\right)\check{\otimes}\left(\neg\Phi^{(t)}\right)$.
(4) $\neg\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\doteq\left(\neg\Phi^{(s)}\right)\hat{\otimes}\left(\neg\Phi^{(t)}\right)$.
(5) $\neg\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\ddot{=}\left(\neg\Phi^{(s)}\right)\check{\otimes}\left(\neg\Phi^{(t)}\right)$.
(6) $\neg\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\ddot{=}\left(\neg\Phi^{(s)}\right)\hat{\otimes}\left(\neg\Phi^{(t)}\right)$.

Proof: It is follows from Definitions 29, 35, and 37. ■

Proposition 40: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$, $\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}$, and $\Phi^{(u)}=\left\{\frac{(a^{(u)}_{k,x},b^{(u)}_{k,x})}{x}\mid x\in X\right\}_{k\in K}=\left\{\frac{\left([a^{(u)}_{k,1,x},b^{(u)}_{k,1,x}],\ [a^{(u)}_{k,2,x},b^{(u)}_{k,2,x}],\ [a^{(u)}_{k,3,x},b^{(u)}_{k,3,x}]\right)}{x}\mid x\in X\right\}_{k\in K}\in\mathbb{I}^{\Delta XK}$. Then the followings hold:

(1) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)=\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\hat{\otimes}\Phi^{(u)}$.
(2) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)=\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\check{\otimes}\Phi^{(u)}$.
(3) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)\doteq\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\hat{\otimes}\Phi^{(u)}$.
(4) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)\doteq\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\check{\otimes}\Phi^{(u)}$.
(5) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)\ddot{=}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\hat{\otimes}\Phi^{(u)}$.
(6) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)\ddot{=}\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\check{\otimes}\Phi^{(u)}$.
(7) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)\Subset\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\check{\otimes}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(u)}\right)$.
(8) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)\Subset\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\hat{\otimes}\left(\Phi^{(s)}\check{\otimes}\Phi^{(u)}\right)$.
(9) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)\dot{\Subset}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\check{\otimes}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(u)}\right)$.
(10) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)\dot{\Subset}\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\hat{\otimes}\left(\Phi^{(s)}\check{\otimes}\Phi^{(u)}\right)$.
(11) $\Phi^{(s)}\hat{\otimes}\left(\Phi^{(t)}\check{\otimes}\Phi^{(u)}\right)\ddot{\Subset}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(t)}\right)\check{\otimes}\left(\Phi^{(s)}\hat{\otimes}\Phi^{(u)}\right)$.
(12) $\Phi^{(s)}\check{\otimes}\left(\Phi^{(t)}\hat{\otimes}\Phi^{(u)}\right)\ddot{\Subset}\left(\Phi^{(s)}\check{\otimes}\Phi^{(t)}\right)\hat{\otimes}\left(\Phi^{(s)}\check{\otimes}\Phi^{(u)}\right)$.

Proof: It is follows from Definitions 26, 28, 35 and 37. ■

In the following we introduce the definitions of union and intersection of interval-valued picture fuzzy soft sets, give some examples, and derive their properties.

Definition 41: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$ and $\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}$.
Then we define union $\Phi^{(s)}\Cup\Phi^{(t)}$ and intersection $\Phi^{(s)}\Cap\Phi^{(t)}$ of $\Phi^{(s)}$ and $\Phi^{(t)}$ as follows:

(1) $$\Phi^{(s)}\Cup\Phi^{(t)}=\left\{\frac{\left(\left[a^{(s)}_{k,1,x}\vee a^{(t)}_{k,1,x},b^{(s)}_{k,1,x}\vee b^{(t)}_{k,1,x}\right],\left[a^{(s)}_{k,2,x}\wedge a^{(t)}_{k,2,x},b^{(s)}_{k,2,x}\wedge b^{(t)}_{k,2,x}\right],\left[a^{(s)}_{k,3,x}\wedge a^{(t)}_{k,3,x},b^{(s)}_{k,3,x}\wedge b^{(t)}_{k,3,x}\right]\right)}{x}\,\middle|\,x\in X\right\}_{k\in I\cap J}$$

(2) $$\Phi^{(s)}\Cap\Phi^{(t)}=\left\{\frac{\left(\left[a^{(s)}_{k,1,x}\wedge a^{(t)}_{k,1,x},b^{(s)}_{k,1,x}\wedge b^{(t)}_{k,1,x}\right],\left[a^{(s)}_{k,2,x}\wedge a^{(t)}_{k,2,x},b^{(s)}_{k,2,x}\wedge b^{(t)}_{k,2,x}\right],\left[a^{(s)}_{k,3,x}\vee a^{(t)}_{k,3,x},b^{(s)}_{k,3,x}\vee b^{(t)}_{k,3,x}\right]\right)}{x}\,\middle|\,x\in X\right\}_{k\in I\cap J}.$$

Example 42: Suppose $\Phi^{(s)}$ and $\Phi^{(t)}$ are as in Example 27, then $\Phi^{(s)}\Cup\Phi^{(t)}$ and $\Phi^{(s)}\Cap\Phi^{(t)}$ are given as follows:

$$(\Phi^{(s)}\Cup\Phi^{(t)})(i_1,i_1)=\left\{\frac{([0.2,0.2],[0.3,0.4],[0,0.3])}{x_1},\frac{([0.1,0.5],[0.1,0.3],[0.1,0.2])}{x_2},\frac{([0.3,0.4],[0.2,0.3],[0,0.2])}{x_3}\right\},$$

$$(\Phi^{(s)}\Cup\Phi^{(t)})(i_2,i_2)=\left\{\frac{([0.19,0.25],[0.1,0.4],[0.26,0.3])}{x_1},\frac{([0.15,0.3],[0.2,0.3],[0.3,0.3])}{x_2},\frac{([0.25,0.35],[0,0.2],[0.2,0.4])}{x_3}\right\},$$

$$(\Phi^{(s)}\Cap\Phi^{(t)})(i_1,i_1)=\left\{\frac{([0.1,0.1],[0.3,0.4],[0,0.3])}{x_1},\frac{([0,0.25],[0.1,0.3],[0.2,0.2])}{x_2},\frac{([0.2,0.4],[0.2,0.3],[0.1,0.3])}{x_3}\right\},$$

$$(\Phi^{(s)}\Cap\Phi^{(t)})(i_2,i_2)=\left\{\frac{([0.18,0.2],[0.1,0.4],[0.3,0.35])}{x_1},\frac{([0.14,0.22],[0.2,0.3],[0.4,0.42])}{x_2},\frac{([0.25,0.3],[0,0.2],[0.3,0.5])}{x_3}\right\}.$$

Proposition 43: Let $\Phi^{(s)}=\left\{\frac{(a^{(s)}_{i,x},b^{(s)}_{i,x})}{x}\mid x\in X\right\}_{i\in I}=\left\{\frac{\left([a^{(s)}_{i,1,x},b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x},b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x},b^{(s)}_{i,3,x}]\right)}{x}\mid x\in X\right\}_{i\in I}\in\mathbb{I}^{\Delta XI}$, $\Phi^{(t)}=\left\{\frac{(a^{(t)}_{j,x},b^{(t)}_{j,x})}{x}\mid x\in X\right\}_{j\in J}=\left\{\frac{\left([a^{(t)}_{j,1,x},b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x},b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x},b^{(t)}_{j,3,x}]\right)}{x}\mid x\in X\right\}_{j\in J}\in\mathbb{I}^{\Delta XJ}$, and $\Phi^{(u)}=\left\{\frac{(a^{(u)}_{k,x},b^{(u)}_{k,x})}{x}\mid x\in X\right\}_{k\in K}=\left\{\frac{\left([a^{(u)}_{k,1,x},b^{(u)}_{k,1,x}],\ [a^{(u)}_{k,2,x},b^{(u)}_{k,2,x}],\ [a^{(u)}_{k,3,x},b^{(u)}_{k,3,x}]\right)}{x}\mid x\in X\right\}_{k\in K}\in\mathbb{I}^{\Delta XK}$. Then the followings hold:

(1) $\Phi^{(s)}\Cup\Phi^{(s)}=\Phi^{(s)}$.
(2) $\Phi^{(s)}\Cup\Phi^{(s)}\doteq\Phi^{(s)}$.
(3) $\Phi^{(s)}\Cup\Phi^{(s)}\ddot{=}\Phi^{(s)}$.
(4) $\Phi^{(s)}\Cup\Phi^{(t)}=\Phi^{(t)}\Cup\Phi^{(s)}$.
(5) $\Phi^{(s)}\Cup\Phi^{(t)}\doteq\Phi^{(t)}\Cup\Phi^{(s)}$.

(6) $\Phi^{(s)} \Cup \Phi^{(t)} \ddot{=} \Phi^{(t)} \Cup \Phi^{(s)}$.
(7) $\left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cup \Phi^{(u)} = \Phi^{(s)} \Cup \left(\Phi^{(t)} \Cup \Phi^{(u)}\right)$.
(8) $\left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cup \Phi^{(u)} \doteq \Phi^{(s)} \Cup \left(\Phi^{(t)} \Cup \Phi^{(u)}\right)$.
(9) $\left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cup \Phi^{(u)} \ddot{=} \Phi^{(s)} \Cup \left(\Phi^{(t)} \Cup \Phi^{(u)}\right)$.

Proof: This proof is easily obtained from Definitions 28 and 41 (1). ■

Proposition 44: Let $\Phi^{(s)} = \left\{ \frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X \right\}_{i \in I} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i \in I} \in \mathbb{I}^{\Delta XI}$, $\Phi^{(t)} = \left\{ \frac{(a^{(t)}_{j,x}, b^{(t)}_{j,x})}{x} \mid x \in X \right\}_{j \in J} = \left\{ \frac{\left([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]\right)}{x} \mid x \in X \right\}_{j \in J} \in \mathbb{I}^{\Delta XJ}$, and $\Phi^{(u)} = \left\{ \frac{(a^{(u)}_{k,x}, b^{(u)}_{k,x})}{x} \mid x \in X \right\}_{k \in K} = \left\{ \frac{\left([a^{(u)}_{k,1,x}, b^{(u)}_{k,1,x}],\ [a^{(u)}_{k,2,x}, b^{(u)}_{k,2,x}],\ [a^{(u)}_{k,3,x}, b^{(u)}_{k,3,x}]\right)}{x} \mid x \in X \right\}_{k \in K} \in \mathbb{I}^{\Delta XK}$. Then the followings hold:

(1) $\Phi^{(s)} \Cap \Phi^{(s)} = \Phi^{(s)}$.
(2) $\Phi^{(s)} \Cap \Phi^{(s)} \doteq \Phi^{(s)}$.
(3) $\Phi^{(s)} \Cap \Phi^{(s)} \ddot{=} \Phi^{(s)}$.
(4) $\Phi^{(s)} \Cap \Phi^{(t)} = \Phi^{(t)} \Cap \Phi^{(s)}$.
(5) $\Phi^{(s)} \Cap \Phi^{(t)} \doteq \Phi^{(t)} \Cap \Phi^{(s)}$.
(6) $\Phi^{(s)} \Cap \Phi^{(t)} \ddot{=} \Phi^{(t)} \Cap \Phi^{(s)}$.
(7) $\left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cap \Phi^{(u)} = \Phi^{(s)} \Cap \left(\Phi^{(t)} \Cap \Phi^{(u)}\right)$.
(8) $\left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cap \Phi^{(u)} \doteq \Phi^{(s)} \Cap \left(\Phi^{(t)} \Cap \Phi^{(u)}\right)$.
(9) $\left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cup \Phi^{(u)} \ddot{=} \Phi^{(s)} \Cap \left(\Phi^{(t)} \Cap \Phi^{(u)}\right)$.

Proof: It is follows from Definitions 28 and 41 (2). ■

Proposition 45: Let $\Phi^{(s)} = \left\{ \frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X \right\}_{i \in I} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i \in I} \in \mathbb{I}^{\Delta XI}$ and $\Phi^{(t)} = \left\{ \frac{(a^{(t)}_{j,x}, b^{(t)}_{j,x})}{x} \mid x \in X \right\}_{j \in J} = \left\{ \frac{\left([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]\right)}{x} \mid x \in X \right\}_{j \in J} \in \mathbb{I}^{\Delta XJ}$.
Then the followings hold:

(1) $\neg \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) = (\neg \Phi^{(s)}) \Cap (\neg \Phi^{(t)})$.
(2) $\neg \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) = (\neg \Phi^{(s)}) \Cup (\neg \Phi^{(t)})$.
(3) $\neg \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \doteq (\neg \Phi^{(s)}) \Cap (\neg \Phi^{(t)})$.
(4) $\neg \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \doteq (\neg \Phi^{(s)}) \Cup (\neg \Phi^{(t)})$.
(5) $\neg \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \ddot{=} (\neg \Phi^{(s)}) \Cap (\neg \Phi^{(t)})$.
(6) $\neg \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \ddot{=} (\neg \Phi^{(s)}) \Cup (\neg \Phi^{(t)})$.

Proof: It is follows from Definitions 28, 29 and 41. ■

Proposition 46: Let $\Phi^{(s)} = \left\{ \frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X \right\}_{i \in I} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i \in I} \in \mathbb{I}^{\Delta XI}$, $\Phi^{(t)} = \left\{ \frac{(a^{(t)}_{j,x}, b^{(t)}_{j,x})}{x} \mid x \in X \right\}_{j \in J} = \left\{ \frac{\left([a^{(t)}_{j,1,x}, b^{(t)}_{j,1,x}],\ [a^{(t)}_{j,2,x}, b^{(t)}_{j,2,x}],\ [a^{(t)}_{j,3,x}, b^{(t)}_{j,3,x}]\right)}{x} \mid x \in X \right\}_{j \in J} \in \mathbb{I}^{\Delta XJ}$, and $\Phi^{(u)} = \left\{ \frac{(a^{(u)}_{k,x}, b^{(u)}_{k,x})}{x} \mid x \in X \right\}_{k \in K} = \left\{ \frac{\left([a^{(u)}_{k,1,x}, b^{(u)}_{k,1,x}],\ [a^{(u)}_{k,2,x}, b^{(u)}_{k,2,x}],\ [a^{(u)}_{k,3,x}, b^{(u)}_{k,3,x}]\right)}{x} \mid x \in X \right\}_{k \in K} \in \mathbb{I}^{\Delta XK}$. Then the followings hold:

(1) $\Phi^{(s)} \Cup \left(\Phi^{(t)} \Cap \Phi^{(u)}\right) = \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cap \left(\Phi^{(s)} \Cup \Phi^{(u)}\right)$.
(2) $\Phi^{(s)} \Cap \left(\Phi^{(t)} \Cup \Phi^{(u)}\right) = \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cup \left(\Phi^{(s)} \Cap \Phi^{(u)}\right)$.
(3) $\Phi^{(s)} \Cup \left(\Phi^{(t)} \Cap \Phi^{(u)}\right) \doteq \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cap \left(\Phi^{(s)} \Cup \Phi^{(u)}\right)$.
(4) $\Phi^{(s)} \Cap \left(\Phi^{(t)} \Cup \Phi^{(u)}\right) \doteq \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cup \left(\Phi^{(s)} \Cap \Phi^{(u)}\right)$.
(5) $\Phi^{(s)} \Cup \left(\Phi^{(t)} \Cap \Phi^{(u)}\right) \ddot{=} \left(\Phi^{(s)} \Cup \Phi^{(t)}\right) \Cap \left(\Phi^{(s)} \Cup \Phi^{(u)}\right)$.
(6) $\Phi^{(s)} \Cap \left(\Phi^{(t)} \Cup \Phi^{(u)}\right) \ddot{=} \left(\Phi^{(s)} \Cap \Phi^{(t)}\right) \Cup \left(\Phi^{(s)} \Cap \Phi^{(u)}\right)$.

Proof: It is follows from Definitions 28 and 41. ■

It is easy to extend the results of Section 3 based on the following:

Definition 47: Let $\Phi^{(s)} = \left\{ \frac{(a^{(s)}_{i,x}, b^{(s)}_{i,x})}{x} \mid x \in X \right\}_{i \in I} = \left\{ \frac{\left([a^{(s)}_{i,1,x}, b^{(s)}_{i,1,x}],\ [a^{(s)}_{i,2,x}, b^{(s)}_{i,2,x}],\ [a^{(s)}_{i,3,x}, b^{(s)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i \in I}$ and $\Phi^{(t)} = \left\{ \frac{(a^{(t)}_{i,x}, b^{(t)}_{i,x})}{x} \mid x \in X \right\}_{i \in I} = \left\{ \frac{\left([a^{(t)}_{i,1,x}, b^{(t)}_{i,1,x}],\ [a^{(t)}_{i,2,x}, b^{(t)}_{i,2,x}],\ [a^{(t)}_{i,3,x}, b^{(t)}_{i,3,x}]\right)}{x} \mid x \in X \right\}_{i \in I}$ belong to $\mathbb{I}^{\Delta XI}$. Then we can define the following nine operations on $\mathbb{I}^{\Delta XI}$:

(1) $\Box \Phi^{(s)} = \left\{ \Box [\Phi^{(s)}(i)] \right\}_{i \in I}$.
(2) $\Phi^{(s)}_{(1)} = \left\{ [\Phi^{(s)}_{(1)}(i)] \right\}_{i \in I}$.
(3) $\Phi^{(s)}_{(2)} = \left\{ [\Phi^{(s)}_{(2)}(i)] \right\}_{i \in I}$.
(4) $\Phi^{(s)}_{(3)} = \left\{ [\Phi^{(s)}_{(3)}(i)] \right\}_{i \in I}$.
(5) $\Phi^{(s)}_{(4)} = \left\{ [\Phi^{(s)}_{(4)}(i)] \right\}_{i \in I}$.
(6) $\Phi^{(s)} \oplus \Phi^{(t)} = \left\{ [\Phi^{(s)}(i)] \oplus [\Phi^{(t)}(i)] \right\}_{i \in I}$.
(7) $\Phi^{(s)} \ominus \Phi^{(t)} = \left\{ [\Phi^{(s)}(i)] \ominus [\Phi^{(t)}(i)] \right\}_{i \in I}$.
(8) $\Phi^{(s)} \boxtimes \Phi^{(t)} = \left\{ [\Phi^{(s)}(i)] \boxtimes [\Phi^{(t)}(i)] \right\}_{i \in I}$.
(9) $\Phi^{(s)} \boxdot \Phi^{(t)} = \left\{ [\Phi^{(s)}(i)] \boxdot [\Phi^{(t)}(i)] \right\}_{i \in I}$.

V. APPLICATION OF INTERVAL-VALUED PICTURE FUZZY SOFT SET IN DECISION-MAKING

In this section, we propose, based on the interval-valued picture fuzzy soft sets, a new approach for decision-making problems.

Let $X = \{x_1, x_2, \cdots, x_p\}$ be the set of option schemes, $I = \{1, 2, \cdots, q\}$ the set of scheme parameters, and $\Phi(i) = \left\{ \frac{\left([a_{i,1,x}, b_{i,1,x}], [a_{i,2,x}, b_{i,2,x}], [a_{i,3,x}, b_{i,3,x}]\right)}{x} \middle| x \in X \right\}_{i \in I}$ the evaluation interval-valued picture fuzzy soft set (where p, and q are natural numbers). Then we get the following interval-valued picture fuzzy soft set decision-making matrix (a deformation of Φ), $\mathcal{D}^{(3)}_{q \times p}$, as shown at the top of the next page, where $d_{i,k,x} = [a_{i,k,x}, b_{i,k,x}]$ (the eval-

$$\mathcal{D}^{(3)}_{q\times p}=\left(d_{i,k,x_j}\right)_{q\times 3\times p}=\begin{pmatrix}\left(d_{1,1,x_1},d_{1,2,x_1},d_{1,3,x_1}\right) & \left(d_{1,1,x_2},d_{1,2,x_2},d_{1,3,x_2}\right) & \cdots & \left(d_{1,1,x_p},d_{1,2,x_p},d_{1,3,x_p}\right)\\ \left(d_{2,1,x_1},d_{2,2,x_1},d_{2,3,x_1}\right) & \left(d_{2,1,x_2},d_{2,2,x_2},d_{2,3,x_2}\right) & \cdots & \left(d_{2,1,x_p},d_{2,2,x_p},d_{2,3,x_p}\right)\\ \left(d_{3,1,x_1},d_{3,2,x_1},d_{3,3,x_1}\right) & \left(d_{3,1,x_2},d_{3,2,x_2},d_{3,3,x_2}\right) & \cdots & \left(d_{3,1,x_p},d_{3,2,x_p},d_{3,3,x_p}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \left(d_{q,1,x_1},d_{q,2,x_1},d_{q,3,x_1}\right) & \left(d_{q,1,x_2},d_{q,2,x_2},d_{q,3,x_2}\right) & \cdots & \left(d_{q,1,x_p},d_{q,2,x_p},d_{q,3,x_p}\right)\end{pmatrix}.$$

$$\mathcal{D}^{(3)}_{q\times p}=\begin{pmatrix}\left(d_{1,1,x_1},d_{1,2,x_1},d_{1,3,x_1}\right) & \left(d_{1,1,x_2},d_{1,2,x_2},d_{1,3,x_2}\right) & \cdots & \left(d_{1,1,x_p},d_{1,2,x_p},d_{1,3,x_p}\right)\\ \left(d_{2,1,x_1},d_{2,2,x_1},d_{2,3,x_1}\right) & \left(d_{2,1,x_2},d_{2,2,x_2},d_{2,3,x_2}\right) & \cdots & \left(d_{2,1,x_p},d_{2,2,x_p},d_{2,3,x_p}\right)\\ \left(d_{3,1,x_1},d_{3,2,x_1},d_{3,3,x_1}\right) & \left(d_{3,1,x_2},d_{3,2,x_2},d_{3,3,x_2}\right) & \cdots & \left(d_{3,1,x_p},d_{3,2,x_p},d_{3,3,x_p}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \left(d_{q,1,x_1},d_{q,2,x_1},d_{q,3,x_1}\right) & \left(d_{q,1,x_2},d_{q,2,x_2},d_{q,3,x_2}\right) & \cdots & \left(d_{q,1,x_p},d_{q,2,x_p},d_{q,3,x_p}\right)\end{pmatrix}.$$

uation of expert group k to scheme x on parameter i), $[a_{i,1,x}, b_{i,1,x}]$ is the degree of positive membership of scheme x on parameter i in Φ, $[a_{i,2,x}, b_{i,2,x}]$ is the degree of neutral membership of scheme x on parameter i in Φ, and $[a_{i,3,x}, b_{i,3,x}]$ is the degree of negative membership of scheme x on parameter i in Φ (clearly, $b_{i,1,x} + b_{i,2,x} + b_{i,3,x} \leq 1$).

Replacing elements $\left(d_{i,1,x}, d_{i,2,x}, d_{i,3,x}\right)$ of $\mathcal{D}^{(3)}_{q\times p}$ by $d_{i,x} = \sum_{k=1}^{3}\left(b_{i,k,x} - a_{i,k,x}\right)$, we get the following element-length matrix $D_{q\times p}$ of $\mathcal{D}^{(3)}_{q\times p}$:

$$D_{q\times p}=\begin{pmatrix} d_{1,x_1} & d_{1,x_2} & \cdots, & d_{1,x_p}\\ d_{2,x_1} & d_{2,x_2} & \cdots, & d_{2,x_p}\\ \vdots & \vdots & \ddots & \vdots\\ d_{q,x_1} & d_{q,x_2} & \cdots, & d_{q,x_p}\end{pmatrix}.$$

Define the max-decision $d^1(x_j)$, the min-decision $d^0(x_j)$, and the score $S(x_j)$ of schemes x_j as follows ($j = 1, 2, \cdots, p$):
$d^1(x_j) = \sum_{i=1}^{q}\left(1 - d_{i,x_j}\right)^2$, $d^0(x_j) = \sum_{i=1}^{q} d^2_{i,x_j}$, and $S(x_j) = \frac{d^0(x_j)}{d^1(x_j)+d^0(x_j)}$.

Then $\hat{x} = \arg\max\left\{S(x_1), S(x_2), \cdots, S(x_p)\right\}$ will be the best choice.[9]

The corresponding algorithm is as follow:

Algorithm II

Step 1: Input $\Phi \in \mathbb{I}^{\Delta XI}$.

Step 2: Write the interval-valued picture fuzzy soft set decision-making matrix $\mathcal{D}^{(3)}_{q\times p}$, as shown at the top of this page.

Step 3: Compute and write the element-length matrix (where $d_{i,x} = \sum_{k=1}^{3}\left(b_{i,k,x} - a_{i,k,x}\right)$):

$$D_{q\times p}=\begin{pmatrix} d_{1,x_1} & d_{1,x_2} & \cdots, & d_{1,x_p}\\ d_{2,x_1} & d_{2,x_2} & \cdots, & d_{2,x_p}\\ \vdots & \vdots & \ddots & \vdots\\ d_{q,x_1} & d_{q,x_2} & \cdots, & d_{q,x_p}\end{pmatrix}.$$

Step 4: Compute the max-decision $d^1(x_j)$, the min-decision $d^0(x_j)$, and the score $S(x_j)$ of schemes x_j ($j = 1, 2, \cdots, p$):
$d^1(x_j) = \sum_{i=1}^{q}\left(1 - d_{i,x_j}\right)^2$, $d^0(x_j) = \sum_{i=1}^{q} d^2_{i,x_j}$, and $S(x_j) = \frac{d^0(x_j)}{d^1(x_j)+d^0(x_j)}$.

Step 5: Determine the best choice

$$\hat{x} = \arg\max\left\{S(x_1), S(x_2), \cdots, S(x_p)\right\}.$$

We show that the principal and steps of the approach to decision-making proposed in this paper using the following example.

Example 48: Assume that a fund manager Mr. M in a wealth management firm is assessing five potential investment opportunities x_1, x_2, x_3, x_4, x_5 (let $X = \{x_1, x_2, x_3, x_4, x_5\}$). The firm mandates that the fund manager has to evaluate the following three parameters i_1, i_2, i_3 (let $I = \{i_1, i_2, i_3\}$), where i_1 stands for the 'risk', i_2 stands for the 'growth', and i_3 stands for 'environmental impacts'. For each parameter $i \in I$, Mr. M invite three expert groups to evaluate if x_j the best opportunity of potential investment in view of i ($j = 1, 2, 3, 4, 5$). The evaluation result can be described by interval-valued picture fuzzy soft set $\Phi = \{\Phi_{i_1}, \Phi_{i_2}, \Phi_{i_3}\} \in \mathbb{I}^{\Delta XI}$ which is given as follows:

$$\Phi_{i_1}=\left\{\begin{array}{l}\frac{([0.2,0.3],[0.1,0.2],[0.19,0.3])}{x_1},\frac{([0.1,0.4],[0,0.3],[0,0.1])}{x_2},\\ \frac{([0.2,0.3],[0,0.4],[0.1,0.2])}{x_3},\frac{([0.1,0.4],[0.1,0.2],[0,0.3])}{x_4},\\ \frac{([0.1,0.3],[0.2,0.3],[0,0.2])}{x_5}\end{array}\right\}$$

[9] To understand the motivation behind this method, let ρ be the Euclidean metric on R^q, $\mathbf{0} = (0, \cdots, 0)^T \in R^q$, $\mathbf{1} = (1, \cdots, 1)^T \in R^q$, and $\boldsymbol{d}_j = (d_{1,x_j}, d_{2,x_j}, \cdots, d_{q,x_j})^T \in R^q$. Then $S(x_j) = \frac{[\rho(\boldsymbol{d}_j,\mathbf{0})]^2}{[\rho(\boldsymbol{d}_j,\mathbf{1})]^2+[\rho(\boldsymbol{d}_j,\mathbf{0})]^2}$ ($j = 1, 2, \cdots, p$).

$$\mathcal{D}^{(3)}_{3\times5} = \begin{pmatrix} ([0.2, 0.3], [0.1, 0.2], [0.19, 0.3]) & ([0.15, 0.2], [0.18, 0.2], [0.17, 0.4]) & ([0.1, 0.29], [0.2, 0.4], [0.1, 0.25]) \\ ([0.1, 0.4], [0, 0.3], [0, 0.1]) & ([0.19, 0.3], [0.17, 0.2], [0, 0.1]) & ([0.2, 0.4], [0.3, 0.5], [0, 0.1]) \\ ([0.2, 0.3], [0, 0.4], [0.1, 0.2]) & ([0.13, 0.4], [0.2, 0.3], [0.1, 0.3]) & ([0.1, 0.4], [0.3, 0.4], [0.1, 0.2]) \\ ([0.1, 0.4], [0.1, 0.2], [0, 0.3])) & ([0.1, 0.4], [0, 0.4], [0.1, 0.1]) & ([0.2, 0.3], [0.2, 0.6], [0, 0.1]) \\ ([0.1, 0.3], [0.2, 0.3], [0, 0.2]) & ([0.19, 0.4], [0.15, 0.2], [0, 0.3]) & ([0.1, 0.25], [0.19, 0.2], [0, 0.3]) \end{pmatrix}.$$

$$\Phi_{i_2} = \left\{ \frac{([0.15, 0.2], [0.18, 0.2], [0.17, 0.4])}{x_1}, \frac{([0.19, 0.3], [0.17, 0.2], [0, 0.1])}{x_2}, \frac{([0.13, 0.4], [0.2, 0.3], [0.1, 0.3])}{x_3}, \frac{([0.1, 0.4], [0, 0.4], [0.1, 0.1])}{x_4}, \frac{([0.19, 0.4], [0.15, 0.2], [0, 0.3])}{x_5} \right\}$$

$$\Phi_{i_3} = \left\{ \frac{([0.1, 0.29], [0.2, 0.4], [0.1, 0.25])}{x_1}, \frac{([0.2, 0.4], [0.3, 0.5], [0, 0.1])}{x_2}, \frac{([0.1, 0.4], [0.3, 0.4], [0.1, 0.2])}{x_3}, \frac{([0.2, 0.3], [0.2, 0.6], [0, 0.1])}{x_4}, \frac{([0.1, 0.25], [0.19, 0.2], [0, 0.3])}{x_5} \right\}.$$

Here we explain the practice meaning of Φ by taking the value $\Phi_{i_1}(x_1)$ for example: the interval valued $[0.2, 0.3]$ in $\Phi_{i_1}(x_1)$ is the evaluation of expert group 1 to 'if x_1 the best opportunity of potential investment in view of risk' which indicates that 20% say yes but 30% say no, and the interval valued $[0.1, 0.2]$ in $\Phi_{i_1}(x_1)$ is the evaluation of expert group 2 to 'if x_1 the best opportunity of potential investment in view of risk' which indicates that 10% say yes but 20% say no, and the interval valued $[0.19, 0.3]$ in $\Phi_{i_1}(x_1)$ is the evaluation of expert group 3 to 'if x_1 the best opportunity of potential investment in view of risk' which indicates that 19% say yes but 30% say no. Thus we can write the interval-valued picture fuzzy soft set decision-making matrix $\mathcal{D}^{(3)}_{3\times5}$ in Step 2 of Algorithm II, $\mathcal{D}^{(3)}_{3\times5}$, as shown at the top of this page.

Then we get the following element-length matrix $D_{3\times5}$ of $\mathcal{D}^{(3)}_{3\times5}$ in Step 3 of Algorithm II:

$$D_{3\times5} = \begin{pmatrix} 0.31 & 0.3 & 0.54 \\ 0.7 & 0.24 & 0.5 \\ 0.6 & 0.57 & 0.5 \\ 0.7 & 0.7 & 0.6 \\ 0.5 & 0.56 & 0.46 \end{pmatrix}.$$

By computing we obtain the max-decision $d^1(x_j)$, the min-decision $d^0(x_j)$, and the score $S(x_j)$ of schemes x_j ($j = 1, 2, 3, 4, 5$):

$d^1(x_1) = 1.7777$, $d^1(x_2) = 0.9176$, $d^1(x_3) = 0.5949$, $d^1(x_4) = 0.34$, $d^1(x_5) = 0.7352$; $d^0(x_1) = 0.4777$, $d^0(x_2) = 0.7976$, $d^0(x_3) = 0.9349$, $d^0(x_4) = 1.34$, $d^0(x_5) = 0.7752$; $S(x_1) = 0.2118$, $S(x_2) = 0.4650$, $S(x_3) = 0.6111$, $S(x_4) = 0.7976$, $S(x_5) = 0.5132$.

Finally, we know from Step 5 that x_4 is the best choice.

VI. CONCLUSIONS

In this paper, we study and define new operations of IVPFSs and interval-valued picture fuzzy soft sets. The basic properties of the interval-valued picture fuzzy soft sets are also introduced and discussed. Subsequently, we give a new approach to decision-making problem based on the interval-valued picture fuzzy soft set matrix by analyzing the limitations and advantages in the existing literature. The decision steps of the decision-making method are constructed. The proposed approach will yield an objective decision result based on information from the decision problem only. An illustrative example is used to show the applicability of the interval-valued picture fuzzy soft sets to solve the fuzzy decision-making problems. In the future, the present work can be extended to present some new aggregation and information measures to solve some group decision-making problem under diverse fuzzy environment [43]–[46].

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AHMED MOSTAFA KHALIL received the M.Sc. degree in pure mathematics from Al-Azhar University, Egypt, in 2016. He is currently pursuing the Ph.D. degree with the College of Mathematics and Information Science, Shaanxi Normal University, China.

He is currently an Assistant Lecturer with the Department of Mathematics, Faculty of Science, Al-Azhar University. He has published several technical papers that have been published in refereed International Journals including *Applied Mathematical Modelling*, *Mathematical Problems in Engineering*, *Applied Soft Computing* Journal, the *International Journal of Fuzzy Systems*, the *Hacettepe Journal of Mathematics and Statistics* and the *Journal of Intelligent and Fuzzy Systems*. Recently, he obtained the 2018 Graduate Training Innovation Fund Project "Fuzzy Soft Expert System and Its Medical Application" (No. 2018CBLY001). His research interests include fuzzy sets, soft sets, fuzzy soft sets, and decision-making problems.

SHENG-GANG LI received the M.S. degree in mathematics from Shaanxi Normal University, Xian, China, in 1988, and the Ph.D. degree in mathematics from Sichuan University, Chengdu, China, in 1998. He is currently a Professor with the College of Mathematics and Information Science, Shaanxi Normal University.

He has authored/coauthored over 70 technical papers published in refereed International Journals including *Information Sciences*, the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS: SYSTEMS, the *International Journal of Machine Learning and Cybernetics*, the *International Journal of Fuzzy Systems*, the *Journal of Computational Analysis and Applications*, the *Journal of Intelligent and Fuzzy Systems*, *Soft Computing*, *Advances in Difference Equations*, and many more. His Scopus citations are over 290. His research interests are in the fields of fuzzy topology, fuzzy matroid, fuzzy control theory, and fuzzy decision-making problems.

HARISH GARG was born in 1986. He received the M.Sc. degree in mathematics from Punjabi University, Patiala, Punjab, India, in 2008, and the Ph.D. degree in applied mathematics with speciality in reliability theory and soft computing techniques from IIT Roorkee, India, in 2013. He is currently an Assistant Professor (Senior Grade) with the School of Mathematics, Thapar Institute of Engineering & Technology (Deemed University), Patiala, India.

He has authored over 180 papers published in refereed International Journals including *Information Sciences*, the IEEE Transactions on Fuzzy Systems, *Cognitive Computation*, *Applied Soft Computing*, *Experts Systems with Applications*, the *International Journal of Intelligent Systems*, the IEEE Access, the *Journal of Intelligent and Fuzzy Systems*, *Expert Systems*, the *Journal of Manufacturing Systems*, *Applied Mathematics & Computations*, *ISA Transactions*, the IEEE/CAA Journal of Automatic Sinica, the IEEE Transactions on Emerging Topics in Computational Intelligence, *Applied Intelligence*, *Computer and Industrial Engineering*, *Computer and Operations Research*, the *International Journal of Uncertainty*, *Fuzziness and Knowledge-based Systems*, the *Journal of Industrial and Management Optimization*, the *Journal of Multiple Logic and Soft Computing*, the *Japan Journal of Industrial and Applied Mathematics*, and many more. He has also authored seven book chapters. His research interests are in the fields of computational intelligence, multicriteria decision making, fuzzy decision making, Pythagorean fuzzy sets, and computing with words and soft computing.

Dr. Garg is the Associate Editor for the *Journal of Intelligent & Fuzzy Systems*, *Complex and Intelligent Systems*, *Neutrosophic Sets and Systems*, and so on. In 2016–2019, he was a receipt of the Outstanding Reviewer for the various journals including ASOC, AMM, EAAI, and RESS. His Google citations are over 4010. For more details, visit http://sites.google.com/site/harishg58iitr/

HONGXIA LI received the B.S. degree in mathematics from Northwest Normal University, Lanzhou, China, in 2008. She is currently an Associate Professor with the School of Mathematics and Statistics, Longdong University. Her current research interests include fuzzy convex analysis and fuzzy optimization.

SHENGQUAN MA received the B.S. degree in mathematics from Northwest Minzu University, Lanzhou, China, in 1985, and the Ph.D. degree in mathematics from Shaanxi Normal University, Xi'an, China, in 2015. He is currently a Professor with the School of Information Science and Technology, Hainan Normal University.

His has published more than 50 technical papers published in refereed International Journals and Scopus citations are over 80. His current research interests include fuzzy measures, complex fuzzy set-valued integrals, complex fuzzy analysis, and its application.