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A Note on Sufficient Conditions of Almost Sure Exponential Stability for Semi-Markovian Jump Stochastic Systems

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ABSTRACT In this paper, we are concerned with the stability analysis of stochastic systems switched by the ergodic semi-Markovian process. The sufficient conditions for almost sure exponential stability are derived by using the method of multiple Lyapunov functions and the ergodic property of the semi-Markov process. Particularly, our results generalize and improve some published results in the literature. An example is presented to illustrate the effectiveness of the obtained results.

INDEX TERMS Almost sure exponential stability, semi-Markov process, ergodic property, stationary distribution, multiple Lyapunov functions.

I. INTRODUCTION

Markovian jump systems are the special cases of hybrid systems, in which the switching signals are modeled as Markov processes. In recent years, the problem of stability analysis for Markovian jump systems has received much attention, and lots of meaningful results have been obtained, we refer the reader to [1]–[17].

In various stability concepts, the almost sure exponential stability heavily relies on the sample path properties of switching signals. In [6], the sufficient conditions of almost sure exponential stability for Markovian jump systems were deduced from the ones of the moment exponential stability. In [8], [15], the sufficient conditions of almost sure exponential stability for Markovian jump stochastic systems were directly obtained from the sample path property of systems.

If it is further assumed that the sojourn time of each visiting state obeys the same exponential distribution, the occurrence number of switching can be regard as Poisson process. In [18], the sufficient conditions of almost sure exponential stability of hybrid systems switched by Poisson process were obtained.

It should be noted that Markov process and Poisson process have the same restriction that the transition rates between two successive switchings are constants. However, many practical systems, such as tolerant control systems (see [19]), the transition rates are time-varying, the switching signals of these systems can not be modeled as Markov process or Poisson process.

The semi-Markov process (see [20], [21]) is the generalization of Markov process and Poisson process. In semi-Markov process, the sojourn times can obey arbitrary distributions, and the transition rates between two successive switchings are time-varying. Many research results on stability analysis of semi-Markovian jump systems (hybrid systems switched by semi-Markov process) have been obtained. In [22]–[24], the sojourn times were assumed to follow the Phase-type distributions and the stochastic stability of semi-Markovian jump systems has been studied. In [25], [26], the sojourn times were assumed to follow the Weibull distributions and the transition rates were assumed to be bounded, then the robust stochastic stability of semi-Markovian jump systems has been studied. [27] studied

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the moment stability in the wide sense of semi-Markovian jump systems in the deterministic case. [28] considered the effect of stochastic noise on the stochastic stability of semi-Markovian jump systems. However, as we know, there are few result on the almost sure exponential stability for semi-Markovian jump stochastic systems.

Motivated by foregoing discussion, in this paper, we shall obtain the sufficient conditions of almost sure exponential stability for semi-Marokvian jump stochastic systems. The main contributions are summarized as follows:

1. The sufficient conditions of almost sure exponential stability for hybrid systems have been provided in some published papers (see [8], [15], [18]), but their sufficient conditions depend on the selected Lyapunov functions, and the expression forms of these sufficient conditions are different. Our results will give the unified form of sufficient conditions of almost sure exponential stability for hybrid systems switched by Markov process or Poisson process.

2. The switching process in this paper is semi-Markov process, which is the generalization of Markov process and Poisson process. By utilizing the ergodic property of semi-Markov process and the stochastic analysis methods, the sufficient conditions of almost sure exponential stability for semi-Markovian jump stochastic systems will be first obtained in this paper.

The remainder of this paper is organized as follows: Section II describes some preliminaries. In Section III, we obtain the sufficient conditions of almost sure exponential stability for semi-Markovian jump stochastic systems via inequalities based on stationary distribution and average sojourn times of semi-Markov process. In Section IV. An example is presented to illustrate the effectiveness of our results. Finally, the paper is concluded in Section V.

Notation. Throughout this paper, $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. $\mathbf{R}^+$ denotes the interval $[0, \infty)$, $|\cdot|$ denotes the absolute value in $\mathbf{R}$ and the Euclidean norm in $\mathbf{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T.

II. PRELIMINARIES

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. In this section, we first give the formal definition of semi-Markov process and some related notions.

Definition 2.1: (see [29]) Let N be a positive integer and $\mathcal{I} = \{1, 2, \cdots, N\}$ be a state space. A stochastic process $\{(r_k, t_k)\}_{k\geq 0}$ taking values in $\mathcal{I} \times \mathbf{R}^+$ and satisfying $0 = t_0 \leq t_1 \leq t_2 \leq \cdots$ is called a Markov renewal process, if

$$\begin{aligned}\mathcal{P}(r_{k+1} &= i, t_{k+1} - t_k \leq t | r_k, \cdots, r_0, t_k, \cdots t_0)\\ &= \mathcal{P}(r_{k+1} = i, t_{k+1} - t_k \leq t | r_k)\\ &= \mathcal{P}(r_1 = i, t_1 - t_0 \leq t | r_0) \qquad (1)\end{aligned}$$

holds for any $i \in \mathcal{I}$, $t \geq 0$ and integer $k \geq 0$.

Thus, $\{r_k, k \geq 0\}$ is a homogeneous Markov chain and its transition probabilities can be described as

$$p_{ij} := \mathcal{P}(r_{k+1} = j | r_k = i), \quad \forall k \geq 0, \qquad (2)$$

for any $i, j \in \mathcal{I}$. Then the transition probability matrix of $\{r_k, k \geq 0\}$ is given by

$$P = (p_{ij})_{N\times N}. \qquad (3)$$

For each $i \in \mathcal{I}$, let τ_i be the sojourn time of each visiting state $i \in \mathcal{I}$, and the distribution function of τ_i is given by

$$F_i(t) = \mathcal{P}(\tau_i \leq t) = \mathcal{P}(t_{k+1} - t_k \leq t | r_k = i), \quad \forall k \geq 0. \qquad (4)$$

A continuous-time stochastic process $\{r(t), t \geq 0\}$, which is called semi-Markov process (see Eq.16.3 of [21]), can be defined as:

$$r(t) = r_k, \ t_k \leq t < t_{k+1}, \quad k = 0, 1, 2, \cdots, \qquad (5)$$

in which $\{r_k, k \geq 0\}$ is called the embedded Markov chain of $\{r(t), t \geq 0\}$.

Remark 2.2: By the above statement, the probability structure of semi-Markov process $\{r(t), t \geq 0\}$ can be characterized by two notions: the transition probability matrix P of its embedded Markov chain and the distribution functions $F_i(t)$ of sojourn time of each visiting state $i \in \mathcal{I}$.

In this paper, we assume that the embedded Markov chain $\{r_k, k \geq 0\}$ is ergodic and its stationary distribution is denoted by $\widetilde{\pi} = (\widetilde{\pi}_1, \widetilde{\pi}_2, \cdots, \widetilde{\pi}_N)$. The stationary distribution of semi-Markov process $\{r(t), t \geq 0\}$, $\pi = (\pi_1, \pi_2, \cdots, \pi_N)$, is related to the stationary distribution of $\{r_k, k \geq 0\}$, $\widetilde{\pi}$, by (see Eq. 16.10 of [21])

$$\pi_i = \frac{\widetilde{\pi}_i \mathbf{E}(\tau_i)}{\sum_{j\in\mathcal{I}} \widetilde{\pi}_j \mathbf{E}(\tau_j)}, \quad i \in \mathcal{I}. \qquad (6)$$

By the strong law of large numbers (see Eq. 16.13 and Eq. 16.14 of [21]), we have

$$\lim_{t\to\infty} \frac{\Pi_i(t)}{N_i(t)} = \mathbf{E}(\tau_i), \ a.s., \qquad (7)$$

and

$$\lim_{t\to\infty} \frac{N_i(t)}{N(t)} = \widetilde{\pi}_i, \ a.s.. \qquad (8)$$

This paper is concerned with the following hybrid system:

$$\begin{aligned}&\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}B(t), \qquad (9)\\ &x(0) = x_0,\end{aligned}$$

in which, $\{r(t), t \geq 0\}$ is a switching process. We assume that almost every sample path of $r(t)$ is right continuous step function with a finite number of simple jumps on a finite time interval. $B(t)$ is a d-dimensional Brownian motion. $f : \mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I} \to \mathbf{R}^n$ and $g : \mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I} \to \mathbf{R}^{n\times d}$. Both f and g satisfy the Lipschitz condition and the linear growth condition. Obviously, these conditions can ensure that system (9) has a unique solution (the proof is similar to Theorem 3.13 of [7], so we omit it), we denote this solution by $x(t; x_0)$ on $t \geq 0$, or $x(t)$ for simplicity. We assume that $f(0, t, i) \equiv 0$, $g(0, t, i) \equiv 0$ for all $i \in \mathcal{I}$, which implies that system (9) admits a trivial solution $x(t; 0) \equiv 0$.

The purpose of this paper is to give the unified form of sufficient conditions for almost sure exponential stability of hybrid system (9). We first give the formal definition of such stability as follow:

Definition 2.3: The trivial solution of system (9), or simply, system (9) is said to be almost surely exponentially stable, if for any $x_0 \in \mathbf{R}^n$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln |x(t; x_0)| < 0, \ a.s.. \tag{10}$$

Let $C^{2,1}(\mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I}; \mathbf{R}^+)$ be the set of nonnegative functions $V(x, t, i) : \mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I} \to \mathbf{R}^+$, which are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I}; \mathbf{R}^+)$, for each $i \in \mathcal{I}$, we define the operators $\mathcal{L}V : \mathbf{R}^n \times \mathbf{R}^+ \times \{i\} \to \mathbf{R}$ and $\mathcal{H}V : \mathbf{R}^n \times \mathbf{R}^+ \times \{i\} \to \mathbf{R}$ as

$$\mathcal{L}V(x, t, i) = V_t(x, t, i) + V_x(x, t, i)f(x, t, i) + \frac{1}{2}trace[g^T(x, t, i)V_{xx}(x, t, i)g(x, t, i)] \tag{11}$$

and

$$\mathcal{H}V(x, t, i) = V_x(x, t, i)g(x, t, i), \tag{12}$$

where

$$V_t(x, t, i) = \frac{\partial V(x, t, i)}{\partial t},$$
$$V_x(x, t, i) = (\frac{\partial V(x, t, i)}{\partial x_1}, \cdots, \frac{\partial V(x, t, i)}{\partial x_n}),$$

and

$$V_{xx}(x, t, i) = (\frac{\partial^2 V(x, t, i)}{\partial x_k \partial x_l})_{n\times n}. \tag{13}$$

With these notions, for each $i \in \mathcal{I}$, the Itô formula (see Chap 4 of [30]) can be written as

$$dV(x, t, i) = \mathcal{L}V(x, t, i)dt + \mathcal{H}V(x, t, i)dB(t). \tag{14}$$

III. MAIN RESULTS

In this section, we first give the sufficient conditions of almost sure exponential stability for semi-Markovian jump system, and then we will give some corollaries to illustrate that our research results are the generalization and improvement of some published results.

Theorem 3.1: Assume that there exist a function $V \in C^{2,1}(\mathbf{R}^n \times \mathbf{R}^+ \times \mathcal{I}; \mathbf{R}^+)$, and constants $c > 0, p > 0, \gamma_i \geq 0, \mu_i > 0, \lambda_i \in \mathbf{R}$, such that for any $i, j \in \mathcal{I}$,

$$c|x|^p \leq V(x, t, i), \tag{15}$$
$$V(x, t, j) \leq \mu_i V(x, t, i), \tag{16}$$
$$\mathcal{L}V(x, t, i) - \gamma_i|\mathcal{H}V(x, t, i)| + \left[\frac{1}{2}\gamma_i^2 + \frac{\ln \mu_i}{\mathbf{E}(\tau_i)}\right]V(x, t, i) \leq \lambda_i V(x, t, i), \tag{17}$$

Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln(|x(t; x_0)|) \leq \frac{1}{p}\sum_{i\in\mathcal{I}} \pi_i\lambda_i, \ a.s. \tag{18}$$

for all $x_0 \in \mathbf{R}^n$. In particular, if

$$\sum_{i\in\mathcal{I}} \pi_i\lambda_i < 0, \tag{19}$$

then system (9) switched by semi-Markov process with stationary distribution π is almost surely exponentially stable.

Proof: It is clear that assertion (10) holds when $x_0 = 0$ since in this case the solution $x(t; 0) \equiv 0$. Fixed any initial value $x_0 \neq 0$, the solution $x(t)$ will never reach zero with probability one (the detailed proof is similar to Lemma 2.1 of [15], so we omit). For each $i \in \mathcal{I}$, applying the Itô formula to $\ln V(x(t), t, i)$ to obtain

$$d[\ln V(x(t), t, i)] = \left[\frac{\mathcal{L}V(x(t), t, i)}{V(x(t), t, i)} - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, i)|^2}{V^2(x(t), t, i)}\right]dt + \frac{\mathcal{H}V(x(t), t, i)}{V(x(t), t, i)}dB(t), \tag{20}$$

thus, for any $t \in [t_k, t_{k+1}), k \geq 0$, we have

$$\begin{aligned}\ln V(x(t), t, r(t)) = &\ln V(x(t_k), t_k, r(t_k)) \\ &+ \int_{t_k}^t \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))} - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]dt \\ &+ \int_{t_k}^t \frac{\mathcal{H}V(x(t), t, r(t))}{V(x(t), t, r(t))}dB(t).\end{aligned} \tag{21}$$

Similarly, for any $k \geq 1$, we have

$$\begin{aligned}\ln V(x(t_k), t_k, r(t_{k-1})) = &\ln V(x(t_{k-1}), t_{k-1}, r(t_{k-1})) \\ &+ \int_{t_{k-1}}^{t_k} \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))} - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]dt \\ &+ \int_{t_{k-1}}^{t_k} \frac{\mathcal{H}V(x(t), t, r(t))}{V(x(t), t, r(t))}dB(t).\end{aligned} \tag{22}$$

The condition (16) implies that for any $k \geq 1$,

$$\ln V(x(t_k), t_k, r(t_k)) \leq \ln \mu_{r(t_{k-1})} + \ln V(x(t_k), t_k, r(t_{k-1})), \tag{23}$$

which together with (22) implies that for any $t \in [t_k, t_{k+1})$, $k \geq 1$,

$$\begin{aligned}\ln V(x(t), t, r(t)) \leq &\ln \mu_{r(t_{k-1})} + \ln V(x(t_{k-1}), t_{k-1}, r(t_{k-1})) \\ &+ \int_{t_{k-1}}^t \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))} - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]dt \\ &+ \int_{t_{k-1}}^t \frac{\mathcal{H}V(x(t), t, r(t))}{V(x(t), t, r(t))}dB(t).\end{aligned} \tag{24}$$

Iterating the above procedure, we have that for any $t \in [t_k, t_{k+1})$, $k \geq 1$,

$$\begin{aligned}\ln V(x(t), t, r(t)) \leq \ln V(x_0, 0, r_0) + \sum_{l=0}^{k-1} \ln \mu_{r(t_l)} \\ + \int_0^t \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))} \right. \\ \left. - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]\mathrm{d}t \\ + \int_0^t \frac{\mathcal{H}V(x(t), t, r(t))}{V(x(t), t, r(t))}\mathrm{d}B(t). \quad (25)\end{aligned}$$

For any $k \geq 1$, it is easy to know that

$$\begin{aligned}\sum_{l=0}^{k-1} \ln \mu_{r(t_l)} &= \sum_{l=0}^{k-1}\sum_{i\in\mathcal{I}} \ln \mu_i I(r(t_l) = i) \\ &= \sum_{i\in\mathcal{I}} \ln \mu_i \sum_{l=0}^{k-1} I(r(t_l) = i) \\ &= \sum_{i\in\mathcal{I}} N_i(t_{k-1}) \ln \mu_i, \quad (26)\end{aligned}$$

which together with (25) implies that for any $t \geq 0$,

$$\begin{aligned}\ln V(x(t), t, r(t)) \leq \sum_{i\in\mathcal{I}} \Big[N_i(t) - I(r(t) = i)\Big] \ln \mu_i \\ + \ln V(x_0, 0, r_0) + \int_0^t \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))}\right. \\ \left. - \frac{1}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]\mathrm{d}t + M(t), \quad (27)\end{aligned}$$

in which

$$M(t) = \int_0^t \frac{\mathcal{H}V(x(t), t, r(t))}{V(x(t), t, r(t))}\mathrm{d}B(t), \quad (28)$$

is a continuous local martingale vanishing at $t = 0$, and its quadratic variation is given by

$$\langle M(t), M(t)\rangle = \int_0^t \frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\mathrm{d}t. \quad (29)$$

Setting $\varepsilon \in (0, 1)$, by comprehensively utilizing the exponential martingale inequality and the Borel-Cantelli lemma (see [15]), there exists an integer K_0, such that for all $m > K_0$,

$$\begin{aligned}M(t) &\leq \frac{2}{\varepsilon}\ln m + \frac{\varepsilon}{2}\langle M(t), M(t)\rangle \\ &= \frac{2}{\varepsilon}\ln m + \int_0^t \frac{\varepsilon}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\mathrm{d}t, \ \ a.s.\end{aligned}$$

holds for all $0 \leq t \leq m$. Substituting the above inequality into (27) implies that for any $0 \leq t \leq m$,

$$\begin{aligned}\ln V(x(t), t, r(t)) \leq \sum_{i\in\mathcal{I}} \Big[N_i(t) - I(r(t) = i)\Big] \ln \mu_i \\ + \ln V(x_0, 0, r_0) + \int_0^t \left[\frac{\mathcal{L}V(x(t), t, r(t))}{V(x(t), t, r(t))}\right. \\ \left. - \frac{1-\varepsilon}{2}\frac{|\mathcal{H}V(x(t), t, r(t))|^2}{V^2(x(t), t, r(t))}\right]\mathrm{d}t \\ + \frac{2}{\varepsilon}\ln m, a.s.. \quad (30)\end{aligned}$$

For each $i \in \mathcal{I}$, there exists a constant $\gamma_i \geq 0$, such that

$$\begin{aligned}&\frac{\mathcal{L}V(x(t), t, i)}{V(x(t), t, i)} - \frac{1-\varepsilon}{2}\frac{|\mathcal{H}V(x(t), t, i)|^2}{V^2(x(t), t, i)} \\ &\leq \frac{\mathcal{L}V(x(t), t, i)}{V(x(t), t, i)} - (1-\varepsilon)\gamma_i\frac{|\mathcal{H}V(x(t), t, i)|}{V(x(t), t, i)} \\ &\quad + \frac{1-\varepsilon}{2}\gamma_i^2 \\ &= \frac{\mathcal{L}V(x(t), t, i)}{V(x(t), t, i)} - (1-\varepsilon)\gamma_i\frac{|\mathcal{H}V(x(t), t, i)|}{V(x(t), t, i)} \\ &\quad + \frac{(1-\varepsilon)^2}{2}\gamma_i^2 + \frac{\varepsilon(1-\varepsilon)}{2}\gamma_i^2, \quad (31)\end{aligned}$$

which together with the condition (17) implies that for any $i \in \mathcal{I}$,

$$\begin{aligned}&\frac{\mathcal{L}V(x(t), t, i)}{V(x(t), t, i)} - \frac{1-\varepsilon}{2}\frac{|\mathcal{H}V(x(t), t, i)|^2}{V^2(x(t), t, i)} \\ &\quad < \lambda_i - \frac{\ln \mu_i}{\mathbf{E}(\tau_i)} + \frac{\varepsilon(1-\varepsilon)}{2}\gamma_i^2, \quad (32)\end{aligned}$$

substituting above inequality into (30) implies that for any $0 \leq t \leq m$,

$$\begin{aligned}&\ln V(x(t), t, r(t)) \\ &\leq \sum_{i\in\mathcal{I}} \Big[N_i(t) - I(r(t) = i)\Big] \ln \mu_i + \ln V(x_0, 0, r_0) \\ &\quad + \int_0^t \left[\lambda_{r(t)} - \frac{\ln \mu_{r(t)}}{\mathbf{E}(\tau_{r(t)})} + \frac{\varepsilon(1-\varepsilon)}{2}\gamma_{r(t)}^2\right]\mathrm{d}t \\ &\quad + \frac{2}{\varepsilon}\ln m \\ &= \ln V(x_0, 0, r_0) + \sum_{i\in\mathcal{I}} \Big[N_i(t) - I(r(t) = i)\Big] \ln \mu_i \\ &\quad + \sum_{i\in\mathcal{I}} \left[\lambda_i - \frac{\ln \mu_i}{\mathbf{E}(\tau_i)} + \frac{\varepsilon(1-\varepsilon)}{2}\gamma_i^2\right]\Pi_i(t) \\ &\quad + \frac{2}{\varepsilon}\ln m, a.s.. \quad (33)\end{aligned}$$

Consequently, if $m-1 \le t \le m$ and $m > K_0$, there is probability 1 that

$$\begin{aligned}\frac{1}{t}\ln V(x(t),t,r(t)) &\le \frac{1}{m-1}\left(\ln V(x_0,0,r_0)+\frac{2}{\varepsilon}\ln m\right)\\ &\quad+\frac{1}{t}\sum_{i\in\mathcal{I}}\Big[N_i(t)-I(r(t)=i)\Big]\ln\mu_i\\ &\quad+\frac{1}{t}\sum_{i\in\mathcal{I}}\left[\lambda_i-\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}+\frac{\varepsilon(1-\varepsilon)}{2}\gamma_i^2\right]\Pi_i(t). \quad (34)\end{aligned}$$

By (6), (7) and (8), it is easy to show that for any $i \in \mathcal{I}$,

$$\begin{aligned}\lim_{t\to\infty}\frac{\Pi_i(t)}{t} &= \lim_{t\to\infty}\frac{\frac{\Pi_i(t)}{N_i(t)}\frac{N_i(t)}{N(t)}}{\sum_{j\in\mathcal{I}}\frac{\Pi_j(t)}{N_j(t)}\frac{N_j(t)}{N(t)}}\\ &= \frac{\widetilde{\pi}_i\mathbf{E}(\tau_i)}{\sum_{j\in\mathcal{I}}\widetilde{\pi}_j\mathbf{E}(\tau_j)}=\pi_i, \quad a.s., \quad (35)\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\frac{N_i(t)}{t} &= \lim_{t\to\infty}\frac{N_i(t)}{\Pi_i(t)}\frac{\Pi_i(t)}{t}\\ &= \frac{\widetilde{\pi}_i}{\sum_{j\in\mathcal{I}}\widetilde{\pi}_j\mathbf{E}(\tau_j)}=\frac{\pi_i}{\mathbf{E}(\tau_i)}, \quad a.s.. \quad (36)\end{aligned}$$

Substituting the above two inequalities into (34), we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\ln V(x(t),t,r(t)) \le \sum_{i\in\mathcal{I}}\pi_i\left[\lambda_i+\frac{\varepsilon(1-\varepsilon)}{2}\gamma_i^2\right], \quad a.s.. \quad (37)\end{aligned}$$

Letting $\varepsilon \to 0$, by condition (15), we finally get the required assertion (18), Thus, we complete the proof of this theorem.

Remark 3.2: 1. If there exists a common Lyapunov function $V(x,t)$, such that $V(x,t)=V(x,t,i)=V(x,t,j)$ for all $i,j \in \mathcal{I}$, then we can take value $\mu_i = 1$ in condition (16) for all $i \in \mathcal{I}$.

2. The condition (17) can be used to describe the stability degrees subsystems. For each $i \in \mathcal{I}$, if $\lambda_i - \frac{\ln\mu_i}{\mathbf{E}(\tau_i)} < 0$, the i-th subsystem is almost surely exponentially stable.

3. The condition (19) implies that the stationary distribution of semi-Markov process plays a very important role to guarantee the stability of the whole system.

Theorem 3.3: Suppose that there exist a function $V \in C^{2,1}(\mathbf{R}^n\times\mathbf{R}^+\times\mathcal{I};\mathbf{R}^+)$, and constants $c>0$, $p>0$, $\mu_i>0$, $\alpha_i\in\mathbf{R}$, $\beta_i \ge 0$, $i \in \mathcal{I}$, such that the conditions (15), (16) and the following inequalities

$$\mathcal{L}V(x,t,i) \le \alpha_i V(x,t,i), \quad (38)$$
$$|\mathcal{H}V(x,t,i)|^2 \ge \beta_i V^2(x,t,i), \quad (39)$$

are satisfied. Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln(|x(t;x_0)|)\le\frac{1}{p}\sum_{i\in\mathcal{I}}\pi_i\left[(\alpha_i-0.5\beta_i)+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right] \; a.s. \quad (40)$$

for all $x_0 \in \mathbf{R}^n$. In particular, if

$$\sum_{i\in\mathcal{I}}\pi_i\left[(\alpha_i-0.5\beta_i)+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]<0, \quad (41)$$

then system (9) switched by semi-Markov process with stationary distribution π is almost surely exponentially stable.

Proof. For any $i \in \mathcal{I}$, by (38) and (39), we have

$$\begin{aligned}\mathcal{L}V(x,t,i)-\gamma_i|\mathcal{H}V(x,t,i)|+\left[\frac{1}{2}\gamma_i^2+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]V(x,t,i)\\ \le\left[\alpha_i-\gamma_i\sqrt{\beta_i}+\frac{1}{2}\gamma_i^2+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]V(x,t,i). \quad (42)\end{aligned}$$

Let $\gamma_i = \sqrt{\beta_i}$ and $\lambda_i = \alpha_i - \frac{1}{2}\beta_i + \frac{\ln\mu_i}{\mathbf{E}(\tau_i)}$. Thus, (42) and (41) can guarantee that the conditions (17) and (19) of Theorem 3.1 are satisfied respectively. Thus the assertion of this theorem follows from the conclusion of Theorem 3.1.

Next, we consider the deterministic case of system (9) with $g(x(t),t,r(t)) \equiv 0$ of the form:

$$\begin{aligned}\mathrm{d}x(t) &= f(x(t),r(t))\mathrm{d}t, \quad (43)\\ x(0) &= x_0.\end{aligned}$$

For any $V \in C^1(\mathbf{R}^n\times\{i\};\mathbf{R}^+)$, $i\in\mathcal{I}$, (11) and (12) reduce to

$$\mathcal{L}V(x,i)=V_x(x,i)f(x,i) \quad (44)$$

and

$$\mathcal{H}V(x,i)=0, \quad (45)$$

respectively.

Corollary 3.4: (see Corollary 1 of [31]) Assume that there exist constants $c>0$, $\mu_i>1$, and $\alpha_i\in\mathbf{R}$, $i\in\mathcal{I}$ such that the conditions (15), (16), and the following inequality

$$V_x(x,i)f(x,i)\le\alpha_i V(x,i), \quad (46)$$

are satisfied. Then

$$\limsup_{t\to\infty}\frac{1}{t}\ln(|x(t;x_0)|)\le\sum_{i\in\mathcal{I}}\pi_i\left[\alpha_i+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right], \; a.s.. \quad (47)$$

In particular, if

$$\sum_{i\in\mathcal{I}}\pi_i\left[\alpha_i+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]<0, \quad (48)$$

then system (43) switched by semi-Markov process with stationary distribution π is almost surely exponentially stable.

Proof. For any $i \in \mathcal{I}$, by combining (44), (45) and the condition (46), we have

$$\begin{aligned}\mathcal{L}V(x,i)-\gamma_i|\mathcal{H}V(x,i)| +\left[\frac{1}{2}\gamma_i^2+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]V(x,i)\\ \le\left[\alpha_i+\frac{1}{2}\gamma_i^2+\frac{\ln\mu_i}{\mathbf{E}(\tau_i)}\right]V(x,i). \quad (49)\end{aligned}$$

Let $\gamma_i = 0$ and $\lambda_i = \alpha_i + \frac{\ln\mu_i}{\mathbf{E}(\tau_i)}$, (49) and (48) can guarantee that the conditions (17) and (19) of Theorem 3.1 are satisfied, thus the assertion of this corollary follows from the conclusion of Theorem 3.1.

For each $i \in \mathcal{I}$, if the sojourn time τ_i of semi-Markov process follows exponential distribution with a parameter $\theta_i > 0$, that is for $x \geq 0$, $\mathcal{P}(\tau_i \leq x) = 1 - e^{-\frac{x}{\theta_i}}$, then $\mathbf{E}(\tau_i) = \theta_i$ and the corresponding semi-Markov process reduces to Markov process. We can obtain the following two corollaries to provide the sufficient conditions of almost sure exponential stability of stochastic system (9) switched by Markov process, which are the classical results in some published papers.

Corollary 3.5: (Theorem 2.2 of [15]) Suppose that there exist a common Lyapunov function $V \in C^{2,1}(\mathbf{R}^n \times \mathbf{R}^+; \mathbf{R}^+)$, and constants $c > 0, p > 0$, $\beta_i \geq 0$, $\alpha_i \in \mathbf{R}$, $i \in \mathcal{I}$, such that the conditions (15), (38) and (39) of Theorem 3.3 are satisfied. Then,

$$\limsup_{t\to\infty} \frac{1}{t} \ln(|x(t; x_0)|) \leq \frac{1}{p} \sum_{i\in\mathcal{I}} \pi_i(\alpha_i - 0.5\beta_i), \quad a.s. \tag{50}$$

for all $x_0 \in \mathbf{R}^n$. In particular, if

$$\sum_{i\in\mathcal{I}} \pi_i(\alpha_i - 0.5\beta_i) < 0, \tag{51}$$

then system (9) switched by Markov process with stationary distribution π is almost surely exponentially stable.

Proof: By item 1 of Remark 3.2, the conditions (40) and (41) of Theorem 3.3 reduce to (50) and (51), thus the assertion of this corollary follows from the conclusion of Theorem 3.3.

Corollary 3.6: (Theorem 3.3 of [8]) For each $i \in \mathcal{I}$, there are constants $\widetilde{\alpha}_i$, ρ_i and σ_i such that the following inequalities

$$x^T f(x, t, i) \leq \widetilde{\alpha}_i |x|^2, \tag{52}$$

$$|g(x, t, i)| \leq \rho_i |x|, \tag{53}$$

$$|x^T g(x, t, i)| \geq \sigma_i |x|^2, \tag{54}$$

are satisfied for all $(x, t) \in \mathbf{R}^n \times \mathbf{R}^+$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln(|x(t; x_0)|) \leq \frac{1}{p} \sum_{i\in\mathcal{I}} \pi_i(\widetilde{\alpha}_i + 0.5\rho_i^2 - \sigma_i^2), \ a.s. \tag{55}$$

for all $x_0 \in \mathbf{R}^n$. In particular, if

$$\sum_{i\in\mathcal{I}} \pi_i(\widetilde{\alpha}_i + 0.5\rho_i^2 - \sigma_i^2) < 0, \tag{56}$$

then system (9) switched by Markov process with stationary distribution π is almost surely exponentially stable.

Proof: We choose the common Lyapunov function $V(x) = V(x, t, i) = |x|^2$ for all $i \in \mathcal{I}$, which means that $\mu_i = 1$ for all $i \in \mathcal{I}$ in the condition (16) of Theorem 3.1. Substituting (52), (53) and (54) into (11) and (12), it is obtained that

$$\mathcal{L}V(x, t, i) = 2x^T f(x, t, i) + |g(x, t, i)|^2 \leq (2\widetilde{\alpha}_i + \rho_i^2)V(x), \tag{57}$$

and

$$|\mathcal{H}V(x, t, i)|^2 = 4|x^T g(x, t, i)|^2 \geq 4\sigma_i^2 V^2(x). \tag{58}$$

Let $\alpha_i = 2\widetilde{\alpha}_i + \rho_i^2$ and $\beta_i = 4\sigma_i^2$, the above two inequalities can guarantee that the conditions (38) and (39) of Theorem 3.3 are satisfied. Thus, (55) and (56) can guarantee that (40) and (41) of Theorem 3.3 are satisfied. The assertion of this corollary follows from Theorem 3.3 immediately.

If we further assume that there exists a positive number λ, such that for any $k \geq 0$,

$$\mathcal{P}(t_{k+1} - t_k \geq x) = e^{-\lambda x}, \tag{59}$$

then $\mathbf{E}(\tau_i) = \frac{1}{\lambda}$ for every $i \in \mathcal{I}$, and the occurrence switching number $N(t)$ of $\{r(t), t \geq 0\}$ can be regard as Poisson process with exponent λ. We denote δ_λ for the set of all such kind of switching processes obeying Poisson distribution with the exponent λ. In the following, we consider the corresponding switched stochastic linear system with the following form:

$$\begin{aligned} dx(t) &= A(r(t))x(t)dt + G(r(t))x(t)dB(t), \\ x(0) &= x_0, \end{aligned} \tag{60}$$

in which, $\{r(t), t \geq 0\}$ is the switching process belongs to δ_λ.

Corollary 3.7: (see Theorem 1 of [18]) Suppose that there exist positive-definite matrices P_i and constants $\mu_i > 0$, $\gamma_i \geq 0$ such that for all $i, j \in \mathcal{I}$, the following inequalities

$$A_i^T P_i + P_i A_i + G_i^T P_i G_i - \gamma_i (G_i^T P_i + P_i G_i) + \frac{1}{2}\gamma_i^2 P_i + \lambda \ln \mu_i P_i < 0, \tag{61}$$

$$P_j \leq \mu_i P_i, \quad j \neq i, \tag{62}$$

are satisfied. Then the stochastic linear system (60) is almost surely exponentially stable for all switching process belonging to δ_λ.

Proof: We construct the multiple Lyapunov functions as $V(x(t), i) = x^T(t)P_i x(t), i \in \mathcal{I}$. it is obtained that for any $i \in \mathcal{I}$,

$$\mathcal{L}V(x, i) = x^T(A_i^T P_i + P_i A_i + G_i^T P_i G_i)x, \tag{63}$$

and

$$\mathcal{H}V(x, i) = x^T(G_i^T P_i + P_i G_i)x, \tag{64}$$

which together with the condition (61) implies that for any $i \in \mathcal{I}$,

$$\mathcal{L}V(x, i) - \gamma_i |\mathcal{H}V(x, i)| + \left[\frac{1}{2}\gamma_i^2 + \frac{\ln \mu_i}{\mathbf{E}(\tau_i)}\right] V(x, i) < 0, \tag{65}$$

which implies that the condition (19) of Theorem 3.1 is satisfied for arbitrary stationary distribution π. Thus the assertion of this corollary follows from Theorem 3.1 immediately.

IV. AN EXAMPLE

In this section, we shall present an example to demonstrate the results derived in this paper.

Example: Consider the following semi-Markovian jump system:

$$dx(t) = f(x(t), r(t))dt + b(r(t))x(t)dB(t), \tag{66}$$

where $B(t)$ is a one dimensional Brownian motion, $\{r(t), t \geq 0\}$ is a semi-Markov process taking values in the state space $\mathcal{I} = \{1, 2\}$, and

$$f(x, 1) = \frac{-x}{|\sin x|}, \quad b(1) = 1,$$
$$f(x, 2) = x, \quad b(2) = 1.$$

Then system (66) can be regard as the result of the following two subsystems

$$dx(t) = \frac{-x(t)}{|\sin x(t)|}dt + x(t)dB(t) \tag{67}$$

and

$$dx(t) = x(t)dt + x(t)dB(t) \tag{68}$$

switching form one to the other according to the transition of the semi-Markov process $\{r(t), t \geq 0\}$.

Choosing the common Lyapunov function $V(x, 1) = V(x, 2) = \frac{1}{2}x^2$, it implies $\mu_1 = \mu_2 = 1$. A simple calculation shows that

$$\mathcal{L}V(x, 1) = \frac{-x^2}{|\sin x|} + \frac{x^2}{2}, \quad \mathcal{H}V(x, 1) = x^2,$$

and

$$\mathcal{L}V(x, 2) = \frac{3x^2}{2}, \quad \mathcal{H}V(x, 2) = x^2.$$

By choosing $\gamma_1 = \gamma_2 = 0$, we get that

$$\mathcal{L}V(x, 1) - \gamma_1|\mathcal{H}V(x, 1)| + \frac{1}{2}\gamma_1^2 V(x, 1) \leq -\frac{x^2}{2},$$
$$\mathcal{L}V(x, 2) - \gamma_2|\mathcal{H}V(x, 2)| + \frac{1}{2}\gamma_2^2 V(x, 2) \leq \frac{3}{2}x^2,$$

it means that we can take values $\lambda_1 = -1$ and $\lambda_2 = 3$. Since $\lambda_1 < 0$, the subsystem (67) is almost surely exponentially stable. The simulation result of the state trajectory of subsystem (67) is shown in Fig.1.

On the other hand, the subsystem (68) has the explicit solution of the form

$$x(t) = \pm e^{t+B(t)},$$

it is obviously that

$$\lim_{t\to\infty} \ln \frac{|x(t)|}{t} = 1, \quad a.s.,$$

thus, subsystem (68) is not almost surely exponential stable, the simulation result of the state trajectory of subsystem (68) is shown in Fig.2.

Next, we will setup the proper stationary distribution of semi-Markov process such that the whole system (66) is almost surely exponentially stable. Let the transition probability matrix

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{pmatrix},$$

and let $\tau_1 \sim$ Weibull$(1, 2)$ and $\tau_2 \sim$ Weibull$(1, 2)$, which implies that the stationary distribution of $\{r(t), t \geq 0\}$ is $\pi = (\frac{4}{5}, \frac{1}{5})$. This stationary distribution can guarantee that the condition (19) of Theorem 3.1 is satisfied, thus whole system (66) is almost surely exponentially stable. The simulation result of the state trajectory of the whole system is shown in Fig.3.

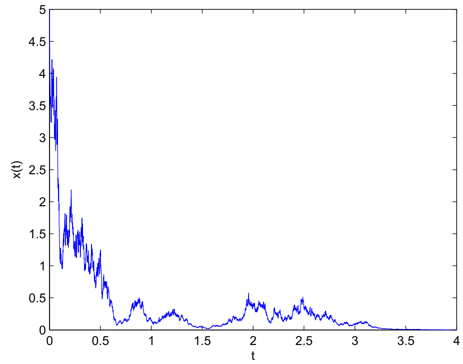

FIGURE 1. Computer simulation of the path of $x(t)$ for the subsystem (67).

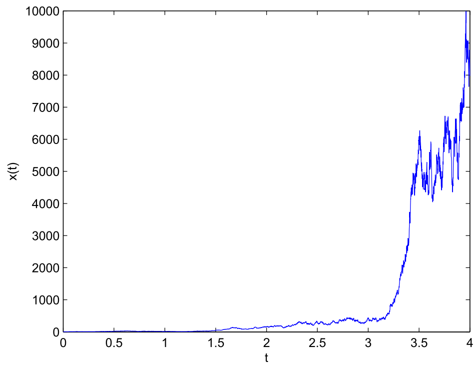

FIGURE 2. Computer simulation of the path of $x(t)$ for the subsystem (68).

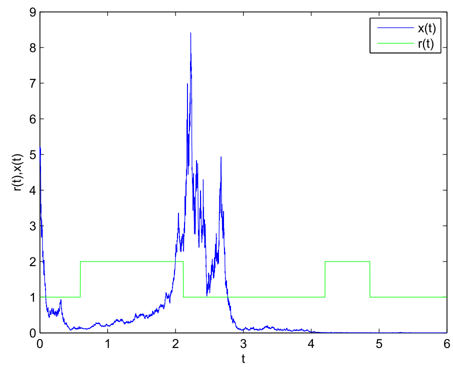

FIGURE 3. Computer simulation of the paths of $r(t)$ and $x(t)$ for the whole system (66).

V. CONCLUSION

In this paper, we studied the almost sure exponential stability of semi-Markovian jump stochastic systems. By a novel approach based on the stochastic analysis method and the ergodic property of semi-Markov process, the sufficient conditions described by inequalities based on the stationary distribution of semi-Markov process are obtained. In particular, our results generalize some classical results in [8], [15], [18]. Finally, an example is given to illustrate the effectiveness of the obtained results.

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