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Passivity Criterions of Networks With General Fractional Order Coupled Inductors

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ABSTRACT Fractional order coupled inductor (FCI) was proposed recently which can enrich existing circuit element system. In this paper, the passivity conditions of FCI are discussed. The study shows that different from the traditional passive coupled inductor, passive fractional order coupled inductor (PFCI) can be non-reciprocal. Traditional integer order coupled inductors are special cases of FCI, and the passivity conditions between them have great differences. Based on passivity conditions of FCI, modified multivariate positive real definition is proposed and multivariate passivity of general FCIs also be proved. Finally, this paper proposes passivity criterions of networks with general FCIs in the complex frequency domain and multivariate domain, respectively.

INDEX TERMS Fractional order, passive circuit, circuit theory, coupled inductor.

I. INTRODUCTION

In recent decades, the fast-developing fractional calculus provides users a powerful tool for interpretation of complex phenomena, processes and dynamical systems [1], [2]. Within electrical and electronic engineering field, there already exist many reports concerning on fractional order modeling of equipment (such as transformer [3], cable [4], DC-DC converter [5], supercapacitor [6], [7], etc.). These shows that fractional order models can describe skin effect and frequency dependent characteristics of dielectric materials more concisely and accurately. As the base of fractional order modeling, fractional order elements are proposed by means of fractional derivatives of current or voltage in their constitutive relations [8]. Gradually, a great number of novel fractional order circuits (such as oscillator [9], [10], filter [11], etc.) have also been proposed which not only can provide users extra design freedoms but also enhance the circuit performance [12]–[16]. At the same time, some new circuit elements including fractional order coupled inductor (FCI) [17] are proposed by researchers. After Soltan *et al.* [17] proposed the concept of FCI, many properties of FCIs are discussed, such as the impedance and phase responses [18], and sensitivity analysis of networks containing FCI [19], [20], etc.

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And multivariate domain passivity of reciprocal FCI has also been studied by paper [21].

There are many benefits to studying the passivity of FCI and its networks. On the one hand, passive networks have many excellent characteristics which have been deduced and proved, for example, the linear passive networks are definitely stable. On the other hand, passivity is also an important prerequisite for synthesizing fractional order passive networks [22], [23]. But until now, there are just two studies on the passivity of fractional order networks, one in the W-domain [24] and the other in the multivariate domain (just contains reciprocal FCIs with same element order) [21], the s-domain passivity of fractional order networks has not been discussed. Therefore, it is very important to study the passivity of FCI and its network.

The works of this paper are as follows:

1) The works in complex frequency domain. The passivity conditions of general FCI are proposed. And the passivity criterion of fractional order linear network is given for the first time.
2) The works in multivariate domain. We propose modified multivariate positive real definition. Then the multivariate passivity of general FCI is proved, this work extends the application-scope of multivariate network synthesis methods. And we also discuss the passivity

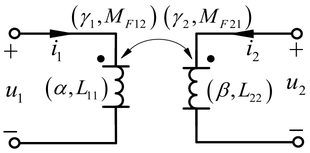

FIGURE 1. Notation of FCI.

TABLE 1. Passive conditions of FCIs.

	Conditions
(i)	$1 \geq \alpha,\, \beta \geq 0$
(ii)	$\gamma_1 = \gamma_2 = \frac{\alpha+\beta}{2}$
(iii)	$L_{11} \geq 0,\, L_{22} \geq 0$
(iv)	$4L_{11}L_{22} - (M_{12}+M_{21})^2 \geq 0$
(v)	$4L_{11}L_{22}cos\left(\frac{\alpha\pi}{2}\right)cos\left(\frac{\beta\pi}{2}\right) - M_{12}^2 - M_{21}^2 - 2M_{12}M_{21}cos\left(\frac{\alpha+\beta}{2}\pi\right) \geq 0$

criterion of fractional order linear network by expanding the multivariate domain criterion in [21].

This paper is organized as follows. Section II studies the passivity of general FCI. Section III presents passivity criteria of fractional order networks containing general passive FCI (PFCI) in complex frequency domain and the multivariate domain, respectively. Finally, we draw some conclusion in Section IV.

II. PASSIVITY OF GENERAL FCIs

A. PASSIVE CONDITONS OF GENERAL FCIs

The constitutive equations of FCI are as follows [17].

$$u_1(t) = L_{11}\frac{d^{\alpha} i_1(t)}{dt^{\alpha}} + M_{12}\frac{d^{\gamma_1} i_2(t)}{dt^{\gamma_1}} \tag{1a}$$

$$u_2(t) = M_{21}\frac{d^{\gamma_2} i_1(t)}{dt^{\gamma_2}} + L_{22}\frac{d^{\beta} i_2(t)}{dt^{\beta}} \tag{1b}$$

where α, β, γ_1, γ_2 are orders of fractional coupled inductors. L_{11}, L_{22} are pseudo self-inductances; M_{12}, M_{21} are pseudo mutual-inductances. Details of the definitions of fractional calculus may be found in [1]. The notation of general FCI is depicted in Fig. 1.

The impedance matrix of general FCI in complex frequency domain at zero initial condition is

$$\mathbf{Z}_M(s) = \begin{bmatrix} L_{11}s^{\alpha} & M_{12}s^{\gamma_1} \\ M_{21}s^{\gamma_2} & L_{22}s^{\beta} \end{bmatrix} \tag{2}$$

Paper [21] just studies reciprocal FCIs with same element order, i.e.,

$$\alpha = \beta = \gamma_1 = \gamma_2 \tag{3a}$$

$$M_{12} = M_{21} \tag{3b}$$

Its impedance matrix is

$$\mathbf{Z}_M(s) = \begin{bmatrix} L_{11} & M \\ M & L_{22}s^{\beta} \end{bmatrix} s^{\alpha} \tag{4}$$

Theorem 1: The passive conditions of FCIs defined by equation (1) are shown in Table 1.

Obviously, when the condition (iv) is satisfied, $L_{11}L_{22} - M_{12}M_{21} \geq 0$ holds.

The proof of Theorem 1 is shown in Appendix A.

B. MODIFIED MULTIVARIATE POSITIVE REAL DEFINITION

Traditional multivariate positive real definition is as follows.

Definition 1 [25]: A $n \times n$ matrix $\mathbf{Z}(p_1, p_2, \ldots, p_k)$ is said to be a k-variable positive real matrix, if

1) For real $p_1, p_2, \ldots, p_k$, the elements of $\mathbf{Z}(p_1, p_2, \ldots, p_k)$ are real;
2) In the domain $Re[p_1] > 0, Re[p_2] > 0, \ldots, Re[p_k] > 0$, the elements of $\mathbf{Z}(p_1, p_2, \ldots, p_k)$ are analytic;
3) $\mathbf{Z}(p_1, p_2, \ldots, p_k) + \mathbf{Z}^H(p_1, p_2, \ldots, p_k)$ is a positive semidefinite matrix in the domain $Re[p_1] > 0, Re[p_2] > 0, \ldots, Re[p_k] > 0$.

where the superscript H indicates conjugate transpose operation.

1) MULTIVARIABLE DOMAIN FORM OF GENERAL PFCI

According to the passive conditions of FCI in Table 1 and equation (2), the impedance matrix of general PFCIs is

$$\mathbf{Z}_M(s) = \begin{bmatrix} L_{11}s^{\alpha} & M_{12}s^{\frac{\alpha+\beta}{2}} \\ M_{21}s^{\frac{\alpha+\beta}{2}} & L_{22}s^{\beta} \end{bmatrix} \tag{5}$$

By using variable substitution $p_1 = s^{\alpha}$ and $p_2 = s^{\frac{\alpha+\beta}{2}}$, equation (5) becomes

$$\mathbf{Z}_M(p_1, p_2) = \begin{bmatrix} L_{11}p_1 & M_{12}p_2 \\ M_{21}p_2 & L_{22}\frac{p_2^2}{p_1} \end{bmatrix} \tag{6}$$

If the variable substitution takes $p_1 = s^{\alpha}$, $p_2 = s^{\beta}$, $p_3 = s^{\frac{\alpha+\beta}{2}}$, then we can obtain

$$\mathbf{Z}_M(p_1, p_2, p_3) = \begin{bmatrix} L_{11}p_1 & M_{12}p_3 \\ M_{21}p_3 & L_{22}p_2 \end{bmatrix} \tag{7}$$

It is clear that there are many variable substitution methods.

In order to facilitate network synthesis, the multivariate impedance matrix of PFCI should meet the following properties:

1) All the elements are odd function, so that it has multi-variable reactance element properties;
2) The number of variables should be as small as possible, thus reducing the types of multivariable element.

In summary, we take equation (6) as the multivariate impedance matrix of PFCI.

However, regardless of which variable substitution, these multivariate impedance matrices never meet traditional multivariable positive real definition. The proof is in Appendix B.

2) MODIFIED MULTIVARIATE POSITIVE REAL DEFINITION

Traditional multivariate positive real definition (see Definition 1) believes that each variable is independent of each other and there is no constraint between them. However, if these variables are obtained by variable substitution $p_i = s^{\alpha_i}(i = 1, \ldots, k)$, there must be some connection between these p_i. This connection must be considered when studying fractional

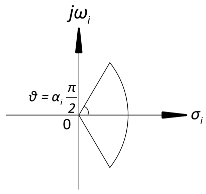

FIGURE 2. p_i**-Sector.**

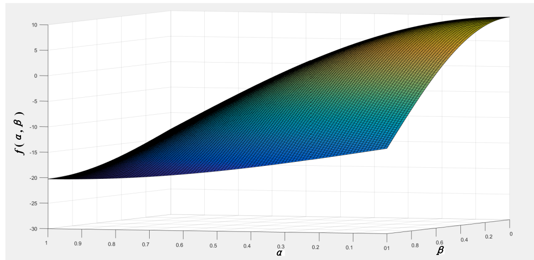

FIGURE 3. 3D plot of $f(\alpha, \beta)$**.**

order multi-port elements with multi-element orders. For this reason, this paper proposes modified multivariate positive real definition (see Definition 2).

Definition 2: A $n \times n$ matrix $\mathbf{Z}(p_1, p_2, \ldots, p_k)$ obtained by variable substitution $p_1 = s^{\alpha_1}, p_2 = s^{\alpha_2}, \ldots, p_k = s^{\alpha_k} (0 < \alpha_1, \ldots, \alpha_k \leq 1)$ is called positive-real if it satisfies all the following conditions.

1) For real $p_i (i = 1, \ldots, k)$, the elements of $\mathbf{Z}\left(p_1, p_2, \ldots, p_k\right)$ is real;
2) $\mathbf{Z}(p_1, p_2, \ldots, p_k)$ is analytic in all the p_i-sector region defined by $[-\alpha_i \frac{\pi}{2}, \alpha_i \frac{\pi}{2}]$, $(i = 1, \ldots, k)$;
3) $\mathbf{Z}\left(p_1, p_2, \ldots, p_k\right) + \mathbf{Z}^{\mathrm{H}}\left(p_1, p_2, \ldots, p_k\right)$ is a positive semidefinite matrix when $p_i (i = 1, \ldots, k)$ in the p_i-sector region.

Remark: Definition 2 can be applied to all the fractional order elements. And p_i-sector region means that when performing variable substitution, according to Euler's Formula,

$$p_i = s^{\alpha_i} = (\sigma^2 + \omega^2)^{\frac{\alpha_i}{2}} cos\alpha_i\theta_i + j(\sigma^2 + \omega^2)^{\frac{\alpha_i}{2}} sin\alpha_i\theta \quad (8)$$

If $\sigma > 0$, $\theta = tan^{-1}\frac{\sigma}{\omega} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $Re[p_i] > 0$ is in sector $\theta_i \in [-\alpha_i \frac{\pi}{2}, \alpha_i \frac{\pi}{2}]$. The p_i-sector is illustrated in Fig. 2.

C. MULTIVARIABLE DOMAIN PASSIVITY PROOF OF GENERAL PFCI

The multivariate impedance matrix of general PFCI in equation (6) is

$$\mathbf{Z}_M(p_1, p_2) = \begin{bmatrix} L_{11}p_1 & M_{12}p_2 \\ M_{21}p_2 & L_{22}\dfrac{p_2^2}{p_1} \end{bmatrix} \quad (9)$$

i) ***Real property:*** when p_1 and p_2 both take real, $\mathbf{Z}_M(p_1, p_2)$ is a real matrix.

ii) ***Positive property:*** set

$$p_1 = s^\alpha = A^\alpha cos\alpha\theta + jA^\alpha sin\alpha\theta \quad (10a)$$

$$p_2 = s^{\frac{\alpha+\beta}{2}} = A^{\frac{\alpha+\beta}{2}} cos\frac{\alpha+\beta}{2}\theta + jA^{\frac{\alpha+\beta}{2}} sin\frac{\alpha+\beta}{2}\theta \quad (10b)$$

$$\frac{p_2^2}{p_1} = s^\beta = A^\beta cos\beta\theta + jA^\beta sin\beta\theta \quad (10c)$$

where $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1; s = \sigma + j\omega; A = \sqrt{\sigma^2 + \omega^2}$; $\theta = arctan\frac{\omega}{\sigma}; \sigma > 0$ and ω are real. Obviously, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Hence, we can get $Re\left[\mathbf{Z}_M(p_1, p_2) + \mathbf{Z}_M^{\mathrm{H}}(p_1, p_2)\right]$ in equation (11) as shown at the bottom of this page. By using passive conditions of FCIs and equation (A-13), $Re\left[\mathbf{Z}_M(p_1, p_2) + \mathbf{Z}_M^{\mathrm{H}}(p_1, p_2)\right]$ is positive semidefinite matrix.

According to Definition 2, general FCI represented by $\mathbf{Z}_M(p_1, p_2)$ is passive.

D. APPLICATION EXAMPLES

Consider the following impedance matrix of an FCI.

$$\mathbf{Z}_M(s) = \begin{bmatrix} s^\alpha & -s^{\frac{\alpha+\beta}{2}} \\ 3.5s^{\frac{\alpha+\beta}{2}} & 4s^\beta \end{bmatrix} \quad (12)$$

From the expression of $\mathbf{Z}_M(s)$, we know that $L_{11} = 1$, $L_{22} = 4, M_{12} = -1, M_{21} = 3.5$. Hence, passive conditions (i) to (iv)of FCIs is satisfied, if the condition (v) is also satisfied, then this FCI is passive, otherwise, it is active. According condition (v), set

$$f(\alpha, \beta) = 16cos\left(\frac{\alpha\pi}{2}\right)cos\left(\frac{\beta\pi}{2}\right) - 13.25 + 7cos\left(\frac{\alpha+\beta}{2}\pi\right) \quad (13)$$

The 3D plot of $f(\alpha, \beta)$ as shown in Fig. 3.

Specifically, when $f(\alpha, \beta) \geq 0$, its 3D plot and its top view are shown in Fig. 4.

From Fig. 4(b), we know that when $0 \leq \alpha, \beta \leq 0.6112$, then $f(\alpha, \beta) \geq 0$, approximately. In order to judge the passivity of this FCI and verify the correctness of the passive conditions, we consider two cases that condition (v) is satisfied ($\alpha = 0.5$, $\beta = 0.4$) and not satisfied ($\alpha = 0.7$, $\beta = 0.9$), respectively.

Case 1: $\alpha = 0.5$, $\beta = 0.4$. We terminate current sources $i_1(t) = \sin\left(t + \frac{\pi}{6}\right)$ A and $i_2(t) = \sin(t + \frac{\pi}{2})$ A on the ports of FCI, respectively. The voltages and instantaneous power absorbed of port 1 and port 2 are depicted in Fig. 5.

$$Re\left[\mathbf{Z}_M(p_1, p_2) + \mathbf{Z}_M^{\mathrm{H}}(p_1, p_2)\right] = \begin{bmatrix} 2L_{11}A^\alpha cos\alpha\theta & (M_{12} + M_{21})A^{\frac{\alpha+\beta}{2}} cos\dfrac{\alpha+\beta}{2}\theta \\ (M_{12} + M_{21})A^{\frac{\alpha+\beta}{2}} cos\dfrac{\alpha+\beta}{2}\theta & 2L_{22}A^\beta cos\beta\theta \end{bmatrix} \quad (11)$$

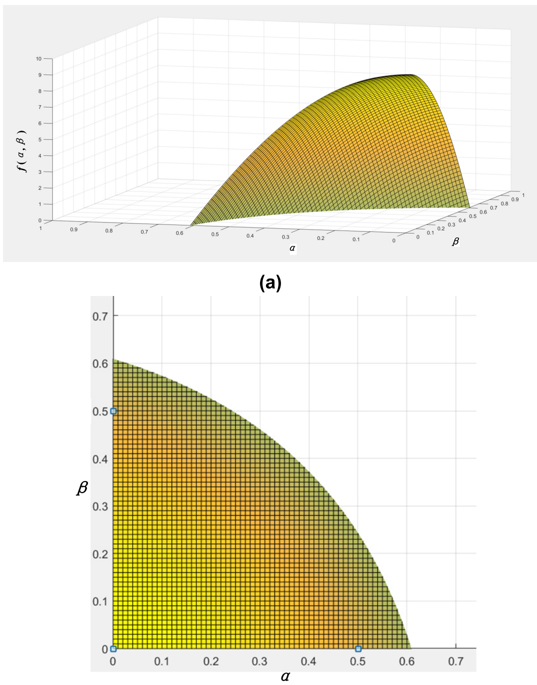

FIGURE 4. (a) 3D plot of $f(\alpha, \beta) \geq 0$. (b) Top view.

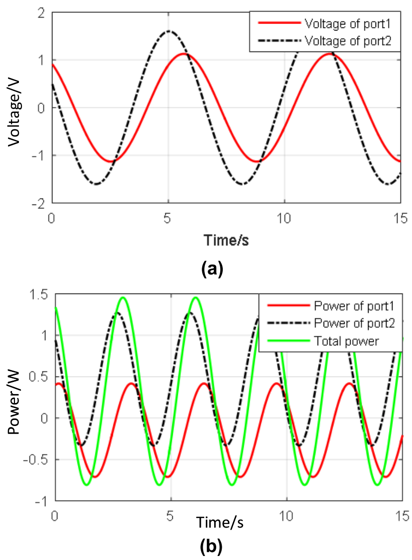

FIGURE 5. (a) The voltages of port 1 and port 2. (b) The power of FCI with $\alpha = 0.5$, $\beta = 0.4$.

As can be seen from Fig. 5, the FCI with orders $\alpha = 0.5$ and $\beta = 0.4$ absorbs power from current sources. Furthermore, the average power in a period is

$$\begin{aligned} P_{av} &= \frac{1}{\sqrt{2}} \times \frac{1.29}{\sqrt{2}} \cos\left(133.59^\circ - 30^\circ\right) \\ &\quad + \frac{1}{\sqrt{2}} \times \frac{1.63}{\sqrt{2}} \cos\left(146.46^\circ - 90^\circ\right) \\ &= 0.478 > 0 \end{aligned} \tag{14}$$

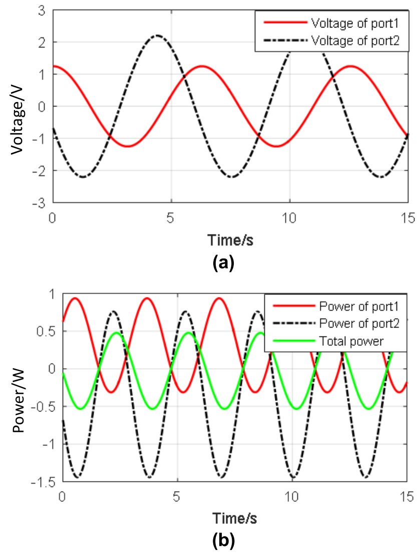

FIGURE 6. (a) The voltages of port 1 and port 2. (b) The power of FCI with $\alpha = 0.7$, $\beta = 0.9$.

Hence, this FCI is passive. The conclusion is consistent with the passive condition.

Case 2: $\alpha = 0.7, \beta = 0.9$. Like Case1, the same current sources are used. The voltages and instantaneous power absorbed of port 1 and port 2 are depicted in Fig. 6.

As can be seen from Fig. 6, the FCI with orders $\alpha = 0.7$ and $\beta = 0.9$ provides power. The average power in a period is

$$\begin{aligned} P_{av} &= \frac{1}{\sqrt{2}} \times \frac{1.25}{\sqrt{2}} \cos\left(92.69^\circ - 30^\circ\right) \\ &\quad + \frac{1}{\sqrt{2}} \times \frac{2.21}{\sqrt{2}} \cos\left(-162.65^\circ - 90^\circ\right) \\ &= -0.027 < 0 \end{aligned} \tag{15}$$

Hence, this FCI is active.

III. PASSIVITY CRITERIA OF LINEAR FRACTIONAL ORDER NETWORKS

In this section, this paper takes one-port as example, the proof of multi-port is similar.

Lemma 1 [26]: Assuming that network N is a linear time-invariant network and its impedance parameter $\boldsymbol{Z}(s)$ exists, then network N is passive if $\boldsymbol{Z}(s) + \boldsymbol{Z}^{\mathrm{H}}(s)$ is positive semidefinite in right half plane (RHP) of complex frequency domain.

Lemma 1 proposed by Otto Brune is traditional complex frequency domain positive real definition. However, the real property of Lemma 1, a condition for judging whether the coefficients of $\boldsymbol{Z}(s)$ are real, cannot be applied to fractional order networks. Thus, this paper extends the real property of Lemma 1 for fractional order network, then Definition 3 is obtained.

Definition 3: The impedance parameter $\boldsymbol{Z}(s)$ of a linear fractional order network with k element orders is said to be positive real, if

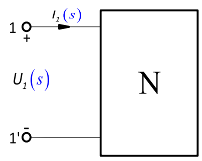

FIGURE 7. One-port N.

1) For real $s^{\alpha_i}(i = 1, \cdots, k)$, the elements of $\boldsymbol{Z}(s)$ is real;
2) $\boldsymbol{Z}(s)$ is analytic in right half plane;
3) $\boldsymbol{Z}(s) + \boldsymbol{Z}^{\mathrm{H}}$ (s) is a positive semidefinite matrix.

A. PASSIVITY CRITERIA IN COMPLEX FREQUENCY DOMAIN

Theorem 2: Given a fractional order one-port composed of passive elements, its immittance function is positive real.

Proof: the one-port N is shown in Fig. 7. In which $U_1(s)$ and $I_1(s)$ represent the Laplace transforms of port-voltage and port-current, respectively.

1) Sufficiency:

According to Tellegen's theorem, the input impedance $Z(s)$ of one-port N can be written as [27]

$$Z(s) = \frac{U_1(s)}{I_1(s)} = \frac{U_1(s)\, I_1^*(s)}{I_1(s)\, I_1^*(s)} = \frac{1}{|I_1(p)|^2} \sum_{l=2}^{b} U_l(p)\, I_l^*(p) \tag{16}$$

where $U_l(s)$ and $I_l(s)$ are the Laplace transforms of branch voltage and branch current inside the network N, respectively; b represents the branch. Superscript $*$ indicates conjugate operation. Equation (16) can be rewritten as

And $\sum_{l=2}^{b} U_l(p)\, I_l^*(p)$ can be divided into four parts.

$$\sum\nolimits_{l=2}^{b} U_l(s)\, I_l^*(s) = \xi_R(s) + \xi_L(s) + \xi_C(s) + \xi_M(s) \tag{17}$$

where

$$\xi_R(s) = \sum\nolimits_R R\, |I_R(s)|^2 \tag{18a}$$

$$\xi_C(s) = \sum\nolimits_C \frac{1}{s^{\beta_i}} C_i\, |I_C(s)|^2 \tag{18b}$$

$$\xi_M(s) = \sum\nolimits_M \boldsymbol{I}_M^{\mathrm{H}}(s)\, Z_M(s)\, \boldsymbol{I}_M(s) \tag{18c}$$

and $\boldsymbol{I}_r(s)$ and $\boldsymbol{I}_M(s)$ are current vector. The subscript R refers to resistor part; L to inductor part; C to capacitor part; M to coupled inductor part.

The positive real property proof of $Z(s)$ is as follows.

i) *Real property:* since all the elements are passive, when s takes real, $\xi_R(s)$, $\xi_L(s)$, $\xi_M(s)$ are real. Obviously, $Z(s)$ is real.

ii) *Positive property:* since all the elements are passive, R, $s^{\alpha_i} L_i$, $\frac{1}{s^{\beta_i}} C_i$, $Z_M(s)$ are positive. By using complex quadratic form property, we can find

$$\begin{aligned}\zeta_R &= Re\left[\xi_R(s) + \xi_R^{\mathrm{H}}(s)\right] \\ &= 2\sum\nolimits_R R\, |I_R(s)|^2 \geq 0\end{aligned} \tag{19a}$$

$$\begin{aligned}\zeta_L &= Re\left[\xi_L(s) + \xi_L^{\mathrm{H}}(s)\right] \\ &= 2\sum\nolimits_L Re[s^{\alpha_i}] L_i\, |I_L(s)|^2 \geq 0\end{aligned} \tag{19b}$$

$$\begin{aligned}\zeta_C &= Re\left[\xi_C(s) + \xi_C^{\mathrm{H}}(s)\right] \\ &= 2\sum\nolimits_C Re[\frac{1}{s^{\beta_i}}] C_i\, |I_C(s)|^2 \geq 0\end{aligned} \tag{19c}$$

$$\begin{aligned}\zeta_M &= Re\left[\xi_M(\boldsymbol{p}) + \xi_M^{\mathrm{H}}(\boldsymbol{p})\right] \\ &= Re\left[\sum\nolimits_M \boldsymbol{I}_M^{\mathrm{H}}(s)\, [\boldsymbol{Z}_M(s) + \boldsymbol{Z}_M^{\mathrm{H}}(s)] \boldsymbol{I}_M(s)\right] \geq 0\end{aligned} \tag{19d}$$

Obviously,

$$\begin{aligned}Re\left[Z(s) + Z^{\mathrm{H}}(s)\right] = \frac{1}{|I_1(s)|^2}[\zeta_R(s) + \zeta_L(s) \\ + \zeta_C(s) + \zeta_M(s)] \geq 0\end{aligned} \tag{20}$$

In summary, $Z(s)$ is positive real function by referring to Definition 3.

2) Necessity:

The energy of network N is

$$\bar{W}(t) = \int_{-\infty}^{+\infty} u_1(t)\, i_1(t) dt \tag{21}$$

Based on Parseval's theorem [28] and Fourier transform, $\bar{W}(t)$ is transformed into frequency domain, as shown in equation (22).

$$\begin{aligned}W(j\omega) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} U_1(j\omega)\, I_1^*(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega)\, |I_1(j\omega)|^2\, d\omega\end{aligned} \tag{22}$$

Dividing $W(j\omega)$ into real and imaginary parts respectively, then we can get

$$\begin{aligned}W(j\omega) &= \frac{1}{2\pi} \mathrm{Re}\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega)\, |I_1(j\omega)|^2\, d\omega\right] \\ &\quad + j\frac{1}{2\pi} \mathrm{Im}\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega)\, |I_1(j\omega)|^2\, d\omega\right] \\ &\triangleq \mathrm{Re}[W(j\omega)] + j\mathrm{Im}[W(j\omega)]\end{aligned} \tag{23}$$

where $\mathrm{Re}[W(j\omega)]$ is even; $\mathrm{Im}[W(j\omega)]$ is odd [29]. Hence,

$$W(j\omega) = \frac{1}{2\pi} \mathrm{Re}\left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(j\omega)\, |I_1(j\omega)|^2\, d\omega\right] \tag{24}$$

Since $Z(s)|_{s=j\omega} \geq 0$, then $W(j\omega) \geq 0$.

In summary, the network N is passive.

B. PASSIVITY CRITERIA IN MULTIVARIABLE DOMAIN

Theorem 3: The necessary and sufficient condition for a fractional order one-port composed of passive elements to be passive network is that its multivariable immittance function is positive real.

Proof: This paper uses Fig. 7 to illustrate the proof process.

1) Sufficiency:

In Fig. 7, fractional order passive elements are converted from complex frequency domain to multivariate domain. The variable substitution method is

$$s^{\alpha_i} L_i \xrightarrow{s^{\alpha_i}=p_i} p_i L_i \tag{25a}$$

$$\frac{1}{s^{\alpha_i}} C_i \xrightarrow{s^{\alpha_i}=p_i} \frac{1}{p_i} C_i \tag{25b}$$

$$Z_M(s) = \begin{bmatrix} L_{11} s^{\alpha_i} & M_{12} s^{\frac{\alpha_i+\beta_i}{2}} \\ M_{21} s^{\frac{\alpha_i+\beta_i}{2}} & L_{22} s^{\beta_i} \end{bmatrix} \xrightarrow{s^{\alpha_i}=p_1,\, s^{\frac{\alpha_i+\beta_i}{2}}=p_2} \begin{bmatrix} L_{11} p_1 & M_{12} p_2 \\ M_{21} p_2 & L_{22} \frac{p_2^2}{p_1} \end{bmatrix} \tag{25c}$$

where $0 < \alpha_i \leq 1$, $0 < \beta_i \leq 1$, and i is positive integer. When these fractional order element orders are equal to 1, they are traditional integer order elements. According to Section 2, it is obvious that the impedance parameters of above multivariable elements are positive real.

After variable substitution, using generalized Tellegen's theorem [29], the impedance function of network N is

$$Z(\boldsymbol{p}) = \frac{U_1(\boldsymbol{p})}{I_1(\boldsymbol{p})} = \frac{1}{|I_1(\boldsymbol{p})|^2} \sum_{l=2}^{b} U_l(\boldsymbol{p}) I_l^*(\boldsymbol{p}) \tag{26}$$

where $U_1(\boldsymbol{p})$ and $I_1(\boldsymbol{p})$ are the port-voltage and port-current in the multivariate domain, respectively; $\boldsymbol{p} = [p_1, p_2, \ldots, p_k]$; b represents the branch.

i) *Real property:* when $p_1, p_2, \ldots, p_k$ all take real, $Z(\boldsymbol{p})$ is real by using Definition 2.

ii) *Positive property:* similar to Theorem 2,

$$Re\left[Z(\boldsymbol{p}) + Z^{\mathrm{H}}(\boldsymbol{p})\right] \geq 0 \tag{27}$$

In summary, $Z(\boldsymbol{p})$ is positive real matrix by Definition 2.

2) Necessity:

Firstly, $Z(s)$ is converted to $Z(\boldsymbol{p})$ by appropriate variable substitution. At this time, $Z(\boldsymbol{p})$ is a multivariable positive real function.

After that, the energy of network N is

$$\bar{W}(t) = \int_{-\infty}^{+\infty} u_1(t)\, i_1(t) dt \tag{28}$$

Based on Parseval's theorem [28] and multidimensional Fourier transform [30], $\bar{W}(t)$ is transformed into frequency domain, as shown in equation (29).

$$\begin{aligned} W(j\boldsymbol{\omega}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} U_1(j\boldsymbol{\omega}) I_1^*(j\boldsymbol{\omega}) d\omega_1 \cdots d\omega_k \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Z(j\boldsymbol{\omega}) I_1^2(j\boldsymbol{\omega}) d\omega_1 \cdots d\omega_k \end{aligned} \tag{29}$$

where $\boldsymbol{\omega} = [\omega_1, \omega_2, \ldots, \omega_k]$.

Dividing $W(j\boldsymbol{\omega})$ into real and imaginary parts respectively, i.e.,

$$\begin{aligned} W(j\boldsymbol{\omega}) &= \frac{1}{2\pi} \mathrm{Re}\left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Z_1(j\boldsymbol{\omega}) I_1^2(j\boldsymbol{\omega}) d\omega_1 \cdots d\omega_k\right] \\ &\quad + j\frac{1}{2\pi} \mathrm{Im}\left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Z_1(j\boldsymbol{\omega}) I_1^2(j\boldsymbol{\omega}) d\omega_1 \cdots d\omega_k\right] \\ &\triangleq \mathrm{Re}[W(j\boldsymbol{\omega})] + j\mathrm{Im}[W(j\boldsymbol{\omega})] \end{aligned} \tag{30}$$

where $Re[W(j\boldsymbol{\omega})]$ is even; $\mathrm{Im}[W(j\boldsymbol{\omega})]$ is odd [31]. Hence, $W(j\boldsymbol{\omega})$

$$= \frac{1}{2\pi} \mathrm{Re}\left[\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} Z_1(j\boldsymbol{\omega}) I_1^2(j\boldsymbol{\omega}) d\omega_1 \cdots d\omega_k\right] \tag{31}$$

Since $Z(\boldsymbol{p})|_{\boldsymbol{p}=j\boldsymbol{\omega}} \geq 0$, then $W(j\boldsymbol{\omega}) \geq 0$.

In summary, the network N is passive.

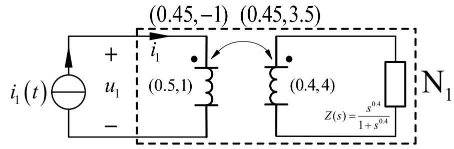

FIGURE 8. The network N_1 with its port current source.

C. APPLICATION EXAMPLES

Example 1: Given the network N_1 and a port current source $i_1(t) = sin\left(t + \frac{\pi}{6}\right)$ A as shown in Fig. 8.

The impedance function of network N_1 is

$$Z_1(s) = \frac{1.5s^{0.9} + 1.7s^{0.5}}{0.8s^{0.4} + 1} \tag{32}$$

Next, we prove the positive real properties of $Z_1(s)$.

i) For complex frequency domain. Obviously, real property of $Z_1(s)$ is holds. If we set $s = \sigma + j\omega$ and use Euler's formula, $s^{0.4}$, $s^{0.5}$, $s^{0.9}$ can be divided into real and imaginary parts, i.e.

$$\begin{aligned} s^{0.9} &= A_{1,1} + jB_{1,1} = (\sigma^2 + \omega^2)^{\frac{9}{20}} \cos(0.9\theta) \\ &\quad + j(\sigma^2 + \omega^2)^{\frac{9}{20}} \sin(0.9\theta) \end{aligned} \tag{33a}$$

$$\begin{aligned} s^{0.5} &= A_{1,2} + jB_{1,2} = (\sigma^2 + \omega^2)^{\frac{1}{4}} \cos(0.5\theta) \\ &\quad + j(\sigma^2 + \omega^2)^{\frac{1}{4}} \sin(0.5\theta) \end{aligned} \tag{33b}$$

$$\begin{aligned} s^{0.4} &= A_{1,3} + jB_{1,3} = (\sigma^2 + \omega^2)^{\frac{1}{5}} \cos(0.4\theta) \\ &\quad + j(\sigma^2 + \omega^2)^{\frac{1}{5}} \sin(0.4\theta) \end{aligned} \tag{33c}$$

where $\theta = tan^{-1}\frac{\sigma}{\omega}$. Then the function $f_1(s) = \frac{1}{2}\mathrm{Re}\left[Z_1^{\mathrm{H}}(s) + Z_1(s)\right] = \mathrm{Re}[Z_1(s)]$ shown in equation (35a) can be obtained. Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $A_{1,1}, A_{1,2}$ both are greater than 0, $f_1(s) > 0$. From Definition 3, $Z_1(s)$ is positive real.

ii) For multivariate domain. We can take

$$s^{0.9} = p_1 = |p_1|\, cos\theta_1 + j\,|p_1|\, sin\theta_1 \tag{35a}$$

$$s^{0.5} = p_2 = |p_2|\, cos\theta_2 + j\,|p_2|\, sin\theta_2 \tag{35b}$$

$$s^{0.4} = p_3 = |p_3|\, cos\theta_3 + j\,|p_3|\, sin\theta_3 \tag{35c}$$

where $\theta_1 = 0.9\theta$; $\theta_2 = 0.5\theta$; $\theta_3 = 0.4\theta$. The positive property of $f_1(p_1, p_2, p_3) = \frac{1}{2}\mathrm{Re}[Z_1^{\mathrm{H}}(p_1,$

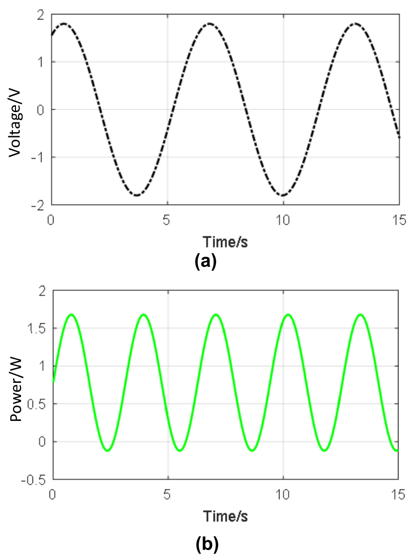

FIGURE 9. (a) The voltages of port 1. (b) The power of network N_1.

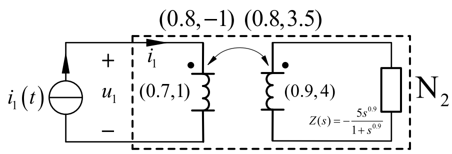

FIGURE 10. The network N_2 with its port current source.

$p_2, p_3) + Z_1(p_1, p_2, p_3)] = \mathrm{Re}[Z_1(p_1, p_2, p_3)]$ can be judged and its simplification result is given in equation (36) as shown at the bottom of this page. It is clear that $f_1(p_1, p_2, p_3) > 0$ and its real property are hold. From Definition 2, $Z_1(s)$ is positive real.

On the other hand, the network N_1 is composed of a PFCI and a positive real impedance function. Hence, the network N_1 is passive. The voltage and instantaneous power absorbed of N_1 is depicted in Fig. 9.

And the average power in a period is

$$P_{av} = \frac{1}{\sqrt{2}} \times \frac{1.8}{\sqrt{2}} \cos\left(60^\circ - 30^\circ\right) = 0.779 > 0 \tag{37}$$

Example 2: Given the network N_2 and a port current source $i_1(t) = sin\left(t + \frac{\pi}{6}\right)$ A as shown Fig. 10.

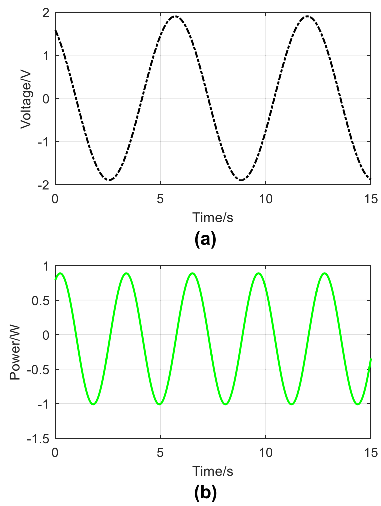

FIGURE 11. (a) The voltages of port 1. (b) The power of network N_2.

The impedance function of network N_2 is

$$Z_2(s) = \frac{7.5s^{1.6} + 2.5s^{0.7}}{4s^{0.9} - 1} \tag{38}$$

Next, the positive real properties of $Z_1(s)$ is judged.

i) For complex frequency domain. Obviously, real property of $Z_2(s)$ is holds. Similarly, we also use Euler's formula and set

$$s^{1.6} = A_{2,1} + jB_{2,1} = (\sigma^2 + \omega^2)^{\frac{4}{5}} \cos(1.6\theta) + j(\sigma^2 + \omega^2)^{\frac{9}{20}} \sin(1.6\theta) \tag{39a}$$

$$s^{0.7} = A_{2,2} + jB_{2,2} = (\sigma^2 + \omega^2)^{\frac{7}{20}} \cos(0.7\theta) + j(\sigma^2 + \omega^2)^{\frac{7}{20}} \sin(0.7\theta) \tag{39b}$$

$$s^{0.9} = A_{2,3} + jB_{2,3} = (\sigma^2 + \omega^2)^{\frac{9}{20}} \cos(0.9\theta) + j(\sigma^2 + \omega^2)^{\frac{9}{20}} \sin(0.9\theta) \tag{39c}$$

The function $f_2(s) = \frac{1}{2}\mathrm{Re}\left[Z_2^{\mathrm{H}}(s) + Z_2(s)\right] = \mathrm{Re}\left[Z_2(s)\right]$ shown in equation (41) can be obtained. However, when $A_{2,2}$ is big enough, $f_2(s)$, as shown at the top the next page, is not always positive. Thus, $Z_2(s)$ doesn't meet positive real property.

ii) For multivariate domain. The variable substitution $p_1 = s^{0.7}$; $p_2 = s^{0.8}$ is considered. $Z_2(s)$ is converted

$$f_1(s) = \frac{1.5A_{1,1} + 1.7A_{1,2} + 1.2A_{1,1}A_{1,3} + 1.36A_{1,2}A_{1,3} + 1.2B_{1,1}B_{1,3} + 1.36B_{1,2}B_{1,3}}{\left(0.8A_{1,3} + 1\right)^2 + 0.64B_{1,3}^2}$$
$$= \frac{1.5A_{1,1} + 1.7A_{1,2} + 1.2\left(\sigma^2 + \omega^2\right)^{\frac{13}{20}} \cos(0.5\theta) + 1.36\left(\sigma^2 + \omega^2\right)^{\frac{9}{20}} \cos(0.1\theta)}{\left(0.8A_{1,3} + 1\right)^2 + 0.64B_{1,3}^2} \tag{34}$$

$$f_1(p_1, p_2, p_3) = \frac{1.5|p_1| cos\theta_1 + 1.7|p_2| cos\theta_2 + 1.2|p_1||p_3|\cos(0.5\theta) + 1.36|p_2||p_3|\cos(0.1\theta)}{(0.8|p_3| cos\theta_2 + 1)^2 + 0.64|p_3|^2 sin^2\theta_2} \tag{36}$$

$$f_2(s) = \frac{-7.5A_{2,1} - 2.5A_{2,2} + 30A_{2,1}A_{2,3} + 10A_{2,2}A_{1,3} + 30B_{2,1}B_{2,3} + 10B_{2,2}B_{2,3}}{(0.8A_{1,3}+1)^2 + 0.64B_{1,3}^2} = \frac{7.5|A_{2,1}| - 2.5A_{2,2} + 30(\sigma^2+\omega^2)^{\frac{5}{4}}\cos(-0.7\theta) + 10(\sigma^2+\omega^2)^{\frac{4}{5}}\cos(0.2\theta)}{(4A_{3,3}-1)^2 + 16B_{3,3}^2} \tag{40}$$

to

$$Z_2(p_1,p_2) = \frac{2.5p_1^2 + 7.5p_1p_2^2}{4p_2^2 - p_1} \tag{41}$$

When $p_1 = 5$; $p_2 = 1$ (if p_1 and p_2 are positive-real numbers, they must be in their respective sectors),

$$f_2(p_1,p_2) = \frac{1}{2}\mathrm{Re}\left[Z_2^{\mathrm{H}}(p_1,p_2) + Z_2(p_1,p_2)\right] = \mathrm{Re}\left[Z_2(p_1,p_2)\right] = -100 < 0 \tag{42}$$

Hence, $Z_2(p_1,p_2)$ does not meet Definition 2. Similarly, if the variable substitution $p_1 = s^{0.7}$; $p_2 = s^{0.9}$ is considered, $Z_2(s)$ will be converted to

$$Z_2(p_1,p_2) = \frac{2.5p_1 + 7.5p_1p_2}{4p_2 - 1} \tag{43}$$

When $p_1 = 1$; $p_2 = 0.2$, $f_2(p_1,p_2) = -20 < 0$. It also does not meet Definition 2.

On the other hand, the network N_2 is composed of an active FCI and a non-positive real impedance function. Hence, the network N_2 is active. The voltage and instantaneous power absorbed of port 1 is depicted in Fig. 11.

The average power in a period is

$$P_{av} = \frac{1}{\sqrt{2}} \times \frac{1.875}{\sqrt{2}}\cos(123.75^\circ - 30^\circ) = -0.061 < 0 \tag{44}$$

IV. CONCLUSION

Traditional passive coupled inductors are reciprocal, but PFCIs can be non-reciprocal. In this paper, passivity conditions of general FCI are proposed, then modified multivariate positive real definition and multivariate passivity proof of general FCIs is given, this work can expand the application-range of fractional order passive network synthesis methods. In addition, passivity criteria of fractional order linear networks in complex frequency domain and multivariate domain are proposed, respectively.

APPENDIX A
THE PROOF OF PASSIVE CONDITIONS FOR FCIs

According to Lemma 1, this paper can obtain the passivity conditions of FCIs by positive semidefinite property of $\boldsymbol{Z}(s) + \boldsymbol{Z}^{\mathrm{H}}(s)$.

Set $s = \sigma + j\omega$, $A = \sqrt{\sigma^2+\omega^2}$ and $\theta = arctan\frac{\omega}{\sigma}$. Then this paper uses Euler's formula and gets the equivalent expression of $\boldsymbol{Z}_M(s)$ in equation (A-1) as shown at the bottom of this page. And then $\boldsymbol{Z}_M(s) + \boldsymbol{Z}_M^{\mathrm{H}}(s)$ in equation (A-2) as shown at the bottom of this page.

Condition (iii) and (iv): the conditions which ensures $\boldsymbol{Z}_M(s) + \boldsymbol{Z}_M^{\mathrm{H}}(s)$ semidefinite in RHP can be expressed by equation (A-3) to (A-5) from equation (A-2).

$$L_{11}A^{\alpha}\cos(\alpha\theta) \geq 0 \tag{A-3}$$

$$L_{22}A^{\beta}\cos(\beta\theta) \geq 0 \tag{A-4}$$

$$L_{11}L_{22}A^{\alpha+\beta}cos(\alpha\theta)\cos(\beta\theta) - \frac{1}{4}A^{2\gamma_1}M_{12}^2 - \frac{1}{4}A^{2\gamma_2}M_{21}^2 - \frac{1}{2}M_{12}M_{21}A^{\gamma_1+\gamma_2}\cos(\gamma_1\theta + \gamma_2\theta) \geq 0 \tag{A-5}$$

Set $\sigma = 1$ and $\omega = 0$, then $A = \sqrt{\sigma^2+\omega^2} = 1$, $\theta = arctan\frac{\omega}{\sigma} = 0$. Using equations (A-3) to (A-5), we can get

$$L_{11} \geq 0, \quad L_{22} \geq 0 \tag{A-6}$$

$$4L_{11}L_{22} - (M_{12}+M_{21})^2 \geq 0 \tag{A-7}$$

which are same to the Condition (iii) and (iv).

Condition (i): taking equation (A-6) and the fact that $\frac{\pi}{2} \geq |\theta|$ into consideration, then equation (A-8) is obtained.

$$1 \geq \alpha, \quad \beta \geq 0 \tag{A-8}$$

which coincides exactly with Condition (i).

$$\boldsymbol{Z}_M(s) = \begin{bmatrix} A^{\alpha}L_{11}[\cos(\alpha\theta) + jsin(\alpha\theta)] & A^{\gamma_1}M_{12}[cos(\gamma_1\theta) + jsin(\gamma_1\theta)] \\ A^{\gamma_2}M_{21}[cos(\gamma_2\theta) + jsin(\gamma_2\theta)] & A^{\beta}L_{22}[cos(\beta\theta) + jsin(\beta\theta)] \end{bmatrix} \tag{A-1}$$

$$\boldsymbol{Z}_M(s) + \boldsymbol{Z}_M^{\mathrm{H}}(s) = 2\begin{bmatrix} L_{11}A^{\alpha}\cos(\alpha\theta) & \frac{1}{2}[A^{\gamma_1}M_{12}cos(\gamma_1\theta) + A^{\gamma_2}M_{21}cos(\gamma_2\theta)] \\ \frac{1}{2}[A^{\gamma_1}M_{12}cos(\gamma_1\theta) + A^{\gamma_2}M_{21}cos(\gamma_2\theta)] & L_{22}A^{\beta}\cos(\beta\theta) \end{bmatrix} + 2j\begin{bmatrix} 0 & \frac{1}{2}[A^{\gamma_1}M_{12}sin(\gamma_1\theta) - A^{\gamma_2}M_{21}sin(\gamma_2\theta)] \\ -\frac{1}{2}[A^{\gamma_1}M_{12}cos(\gamma_1\theta) - A^{\gamma_2}M_{21}cos(\gamma_2\theta)] & 0 \end{bmatrix} \tag{A-2}$$

$$D = \frac{1}{\sigma_1^2+\omega_1^2}[[4L_{11}L_{22}-(M_{12}+M_{21})^2]\sigma_1^2\sigma_2^2 + 4L_{11}L_{22}\left(-\sigma_1^2\omega_2^2+2\sigma_1\sigma_2\omega_1\omega_2\right) - (M_{12}+M_{21})^2\sigma_2^2\omega_1^2] \quad \text{(B-2)}$$

$$D = \begin{bmatrix}\sigma_1\sigma_2\\ \sigma_1\omega_2\\ \sigma_2\omega_1\\ \omega_1\omega_2\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} c & 0 & 0 & \frac{4L_{11}L_{22}}{\sigma_1^2+\omega_1^2}\\ 0 & -\frac{4L_{11}L_{22}}{\sigma_1^2+\omega_1^2} & 0 & 0\\ 0 & 0 & -\frac{(M_{12}+M_{21})^2}{\sigma_1^2+\omega_1^2} & 0\\ \frac{4L_{11}L_{22}}{\sigma_1^2+\omega_1^2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix}\sigma_1\sigma_2\\ \sigma_1\omega_2\\ \sigma_2\omega_1\\ \omega_1\omega_2\end{bmatrix} \quad \text{(B-3)}$$

Condition (ii): now this paper will give the derivation of the condition (ii) to ensure equation (A-5) holds in RHP.

Let $\omega = 0$ and $\sigma \geq 0$, then $A = \sqrt{\sigma^2+\omega^2} = \sigma$, $\theta = arctan\frac{\omega}{\sigma} = 0$. Equation (A-5) becomes

$$4L_{11}L_{22}\sigma^{\alpha+\beta} - (M_{12}\sigma^{\gamma_1}+M_{21}\sigma^{\gamma_2})^2 \geq 0 \quad \text{(A-9)}$$

which can be equivalently expressed as follows:

$$4L_{11}L_{22}\sigma^{\alpha+\beta} \geq (M_{12}\sigma^{\gamma_1-\frac{\alpha+\beta}{2}} + M_{21}\sigma^{\gamma_2-\frac{\alpha+\beta}{2}})^2 \triangleq \Delta^2 \quad \text{(A-10)}$$

where $\Delta = M_{12}\sigma^{\gamma_1-\frac{\alpha+\beta}{2}} + M_{21}\sigma^{\gamma_2-\frac{\alpha+\beta}{2}}$. This paper can get the following conclusions from equation (A-10).

a) When $\gamma_1 > \frac{\alpha+\beta}{2}$ or $\gamma_2 > \frac{\alpha+\beta}{2}$, and if $\sigma\to\infty$ (big enough), then $\Delta\to\infty$ (big enough), hence equation (A-10) cannot hold;

b) When $\gamma_1 < \frac{\alpha+\beta}{2}$ or $\gamma_2 < \frac{\alpha+\beta}{2}$, and if $\sigma\to 0$ (small enough), then $\Delta\to\infty$ (big enough), hence equation (A-10) cannot hold.

Hence, only when $\gamma_1 = \gamma_2 = \frac{\alpha+\beta}{2}$, equation (A-10) can hold. The condition (ii) must be satisfied.

Condition (v): based on condition (ii), equation (A-5) can be equivalently rewritten as

$$f(\theta) = L_{11}L_{22}cos(\alpha\theta)\cos(\beta\theta) - \frac{1}{4}M_{12}^2 - \frac{1}{4}M_{21}^2 - \frac{1}{2}M_{12}M_{21}cos[(\alpha+\beta)\theta] \quad \text{(A-11)}$$

Obviously, $f(\theta)$ is an even function within the range $\frac{\pi}{2}\geq|\theta|$, so this paper just considers $\theta\in[0,\frac{\pi}{2}]$. When $\theta = 0$, we can get

$$f(0) = L_{11}L_{22} - \frac{1}{4}(M_{12}+M_{21})^2 \geq 0 \quad \text{(A-12)}$$

By using trigonometric formulas, $f(\theta)$can be equivalently expressed as equation (A-13).

$$f(\theta) = \frac{1}{2}(L_{11}L_{22}-M_{12}M_{21})\cos[(\alpha+\beta)\theta] + \frac{1}{2}L_{11}L_{22}\cos[(\alpha-\beta)\theta] - \frac{1}{4}M_{12}^2 - \frac{1}{4}M_{21}^2 \quad \text{(A-13)}$$

And the derivative of equation (A-13) is

$$f'(\theta) = -\frac{\alpha+\beta}{2}(L_{11}L_{22}-M_{12}M_{21})\sin[(\alpha+\beta)\theta] - \frac{\alpha-\beta}{2}L_{11}L_{22}\sin[(\alpha-\beta)\theta] \quad \text{(A-14)}$$

Since $L_{11}L_{22}-M_{12}M_{21}\geq 0$ and $f'(\theta)\leq 0, f(\theta)$ decreases monotonically in $\theta\in[0,\frac{\pi}{2}]$. Hence, considering $f(0)\geq 0$, if this paper ensures $f\left(\frac{\pi}{2}\right)\geq 0$, then $f(\theta)\geq 0$ will hold in $\theta\in[0,\frac{\pi}{2}]$. The equation (A-11) yields the condition $f\left(\frac{\pi}{2}\right)\geq 0$, i.e. Condition (v).

APPENDIX B
THE PROOF OF NOT MEETING TRADITIONAL MULTIVARIATE POSITIVE REAL DEFINITION

Proof: This paper takes equation (9) as an example, the proof of other multivariate forms is similar.

Let $p_1 = \sigma_1 + j\omega_1$; $p_2 = \sigma_2 + j\omega_2$, where $\sigma_1, \sigma_2 > 0$ and ω_1, ω_2 are both real. Hence,

$$Re[\mathbf{Z}_M(p_1,p_2)+\mathbf{Z}_M^{\mathrm{H}}(p_1,p_2)]\Big|_{\substack{p_1=\sigma_1+j\omega_1\\ p_2=\sigma_2+j\omega_2}} = \begin{bmatrix} 2L_{11}\sigma_1 & (M_{12}+M_{21})\sigma_2\\ (M_{12}+M_{21})\sigma_2 & \frac{2L_{22}(\sigma_1\sigma_2^2-\sigma_1\omega_2^2+2\sigma_2\omega_1\omega_2)}{\sigma_1^2+\omega_1^2}\end{bmatrix} \quad \text{(B-1)}$$

Obviously, $2L_{11}\sigma_1\geq 0$. The determinant D of equation (B-2) is equation (B-3), as shown at the top of this page. The quadratic form of D is equation (B-4). Where

$$c = \frac{4L_{11}L_{22}-(M_{12}+M_{21})^2}{\sigma_1^2+\omega_1^2} \geq 0 \quad \text{(B-4)}$$

by referring passive condition (iv). It is obvious that the quadratic matrix in equation (B-3), as shown at the top of this page, is not a positive semidefinite matrix, so $D\geq 0$ does not hold.

In summary, $\mathbf{Z}_M(p_1,p_2)$ does not meet traditional multivariate positive real definition.

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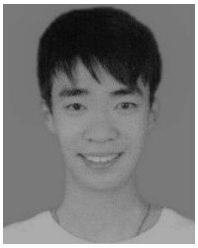

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