IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received March 8, 2019, accepted March 30, 2019, date of publication April 9, 2019, date of current version April 22, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2910018

Control of a Time Delay System Arising From Linearized Conservation Laws

DANIELA DANCIU, (Senior Member, IEEE), DAN POPESCU, (Member, IEEE), AND VLADIMIR RĂSVAN, (Senior Member, IEEE)
Department of Automatic Control and Electronics, University of Craiova, 200585 Craiova, Romania

Corresponding author: Vladimir Răsvan (vrasvan@automation.ucv.ro)

This work was supported in part by the Romanian National Authority for Scientific Research and Innovation, CCCDI–UEFISCDI, under Project 78 BM/2017, and in part by the Fondazione Silvio Tronchetti Provera under Project 2C/27.01.2011.

ABSTRACT A structure of energy cogeneration with distributed parameters is considered, modeled by an initial boundary value problem for hyperbolic conservation laws with a nonlinear and nonstandard boundary condition. Under a standard simplifying assumption leading to linear partial differential equations, the model is studied by associating a system of functional differential and difference equations of neutral type: considered are basic theory (existence, uniqueness, and continuous data dependence), invariant sets (positiveness of some state variables), equilibria, and their inherent stability (without control). These properties are illustrated by simulation results. Furthermore, a control Lyapunov functional is constructed, and feedback stabilizing structure is designed. This paper ends with a conclusion section, where open problems, such as stability by the first approximation/robustness and stability preservation under singular perturbations, are pointed out.

INDEX TERMS Control system synthesis, delay systems, nonlinear control systems, power generation control.

I. INTRODUCTION

Since the very first papers on the subject [1], [2], the systems of conservation laws have turned out to be rather challenging for mathematicians, physicists and engineers. After more than half-century, the challenges in this field of research are still present - see for instance [3]: ‘‘the important class of hyperbolic conservation laws that includes the Euler equations for gas dynamics gathers all the difficulties’’ - poor regularity, necessity of an ‘‘entropy’’ criterion to select the relevant solution a.s.o. [4])

Control of systems described by or arising from conservation laws seems a more recent problem. Direct reference to conservation laws can be found in papers dealing with flow control in open canals [5]–[10] or with gas transport [11]–[13]. Other control applications are to be found in the relatively recent monographs [9], [14]–[16]. In fact all these references can be embedded in the larger class of controlling nonlinear systems with distributed parameters. *We have described in brief the first starting point of this paper. The second one arises from applications: among numerous and various applications of the conservation laws we may find those arising from power engineering*: thermal engineering – cogeneration (combined heat-electricity generation) control under the circumstances of long steam pipes; hydraulic power generation control under the water hammer phenomena; water level and flow control in water channels which are such that propagation phenomena have to be taken into account. To the aforementioned applications we may add the one concerning gas distribution pipes: steam and gas transport deals with compressible fluids while water transport with incompressible fluids.

All these engineering applications have in common 1-dimensional space distributed parameters: they are modeled by hyperbolic partial differential equations (hPDEs) in two variables (time and space) subject to non-standard boundary conditions (BCs). *The BCs are called non-standard because they are subject to an input signal arising from a dynamical subsystem with lumped parameters, i.e., described by ordinary differential equations (ODEs).* In turn this aforementioned subsystem is subject to a signal arising from the boundary conditions. Moreover, the control of the distributed parameter object is a boundary control: the control signals generated by the controllers are fed to the subsystem with

The associate editor coordinating the review of this manuscript and approving it for publication was Jin-Liang Wang.

lumped parameters. Consequently, *the stabilization problem becomes an urgent task since the aforementioned structure displays an internal feedback between the boundary conditions and the system of ODEs; as observed in [17], existence of internal feedbacks might generate instability*.

The third aspect that is worth mentioning in this introductory part concerns nonlinearities. Conservation laws are nonlinear (in fact, quasilinear) partial differential equations. The BCs which describe local dependencies (flow/pressure for incompressible (water) or compressible (steam) fluids, i.e., local hydraulic impedances) are, generally speaking, nonlinear. Moreover, the ODEs which control the boundaries are usually nonlinear. The resulting non-standard boundary value problems are, thus, nonlinear. Consequently, the rather complicated mathematical models of the aforementioned applications from power engineering require validation from the mathematical point of view. We give here just a single motivation for this necessary model validation. The hydraulic impedances are usually modeled and measured in steady state. Nevertheless, their description is considered valid and applied *tale quale* in modeling the transients also, just assuming the validity of the steady state description during transients (the so-called *als ob* principle in constructing models).

Such heuristic extensions can obviously lead to models which require a sound mathematical basis. The first step in this direction is to establish well-posedness in the sense of J. Hadamard: existence, uniqueness and smooth data (parameter and initial conditions) dependence. The significance of the well-posedness is best expressed in the reflections of Courant [18] (see also [19], [20]). The next step is given by displaying invariant sets with physical significance: for instance, certain state variables account for such physical quantities like absolute temperature, absolute pressure, substance concentrations, which have definite sign, here positive. These properties also need to be a consequence of the model properties. In the case of dynamics, the aforementioned properties should be valid along system's trajectories. Another aspect which is also resulting in invariant sets arises from the models of the hydraulic impedances: their description contains square roots of some expressions which need to be nonnegative along system's trajectories. The significance of such conditions might be, e.g., that the fluid flows from the higher pressure to the lower one, heat is transmitted from higher temperature to the lower one and so on. Moreover, failing to emphasize such invariant sets may affect numerical integration of the trajectories.

Another step in model validation – the third one – arises from the so-called *Stability Postulate* of N. G. Četaev – see, e.g., [21]: *only stable trajectories are observable and measurable* and this property holds for various physical objects including artifacts.

The "aggregate" of the aforementioned validation steps was called *augmented validation* [19].

We are in position to state now the subject, the structure and, therefore, the contributions of this paper: augmented validation in the sense of [19] will be applied to a model occurring in cogeneration and arising from conservation laws. The necessary simplifying assumptions will be pointed out together with the necessary steps to validate these simplifications. The paper will also present the next step in the study of cogeneration. Usually, the inherent stability as arising from the *Stability Postulate* is non-asymptotic while, in practice asymptotic (even exponential) stability is required. Therefore, feedback stabilization will be considered, based on the construction of a suitable control Lyapunov functional. Here, also some simplifying assumptions will be considered and their effect discussed. Finally, the conclusion section will enumerate all problems remaining unsolved throughout the paper, thus, pointing out research perspectives worth to be followed.

Summarizing, the aforementioned augmented validation is considered to be the main methodological contribution of the paper. A mathematically validated model is more trustable than one manipulated formally. Within this methodology, our approach is to associate a system of functional differential equations (FDEs) whose solutions are in one-to-one correspondence with those of the initial (basic) system - in fact a development of the method of d'Alembert in PDEs. This approach has certain advantages such as using the body of achievements established mainly for ODEs; it is also the illustration of a rather natural way of introducing the equations with deviated argument, in particular those of *neutral type*.

In brief, the contributions of the paper are as follows: i) starting from a model of cogeneration described by a system of hyperbolic conservation laws with nonstandard boundary conditions - linearized no more than necessary - the aforementioned augmented validation is done; the "instrument" is namely the associated system of FDEs and the one-to-one correspondence of the solutions of the two mathematical objects; ii) the Lyapunov synthesis of the stabilizing feedback control; the feedback control structure thus obtained ensures asymptotic stability from the very beginning, due to the Lyapunov functional approach; iii) extension of an earlier result concerning the same cogeneration structure but without propagation phenomena [22].

II. MATHEMATICAL MODEL(S)

We shall start with the following model

$$
\begin{aligned}
&\psi_c T_c\, \partial_t \xi_\rho + \partial_\lambda \xi_w = 0\\
&\psi_c T_c \partial_t \xi_w + \partial_\lambda \left(\xi_\rho + \frac{\xi_w^2}{\psi_c^2 \xi_\rho}\right) = 0, \quad t > 0,\ 0 < \lambda < 1\\
&\xi_w(0,t) = \pi_s(t)\Phi\left(\xi_\rho(0,t)/\pi_s(t)\right)\\
&\xi_w(1,t) = \sqrt{2}\psi_s \xi_\rho(1,t)\\
&T_a \frac{\mathrm{d}s}{\mathrm{d}t} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g, \quad 0 < \alpha < 1\\
&T_1 \frac{\mathrm{d}\pi_1}{\mathrm{d}t} = \mu_1(t) - \pi_1, \quad 0 \le \mu_1(t) \le 1\\
&T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1 - \mu_2(t)\pi_s - (\beta/\psi_c)\xi_w(0,t), \quad 0 < \beta < 1\\
&T_2 \frac{\mathrm{d}\pi_2}{\mathrm{d}t} = \mu_2(t)\pi_s - \pi_2, \quad 1-\beta \le \mu_2(t) \le 1, \qquad (1)
\end{aligned}
$$

where the function $\Phi(z)$, describing the subcritical/critical steam flow, is defined by

$$\Phi(z) = \begin{cases} \sqrt{2(1-z)}, & \nu_{cr} \leq z \leq 1 \\ \sqrt{2(1-\nu_{cr})}, & 0 < z \leq \nu_{cr} \\ 0, & z \geq 1 \end{cases} \tag{2}$$

and it is called sometimes the ASME formula for the subcritical/critical steam flow [23]; it replaces the Saint Venant formula for the same flow. The critical ratio ν_{cr} results from the Saint Venant formula and for the isothermal processes equals $1/\sqrt{e} \approx 0.606$. For overheated steam, the experimental data give $\nu_{cr} \approx 0.577$. For $\kappa = 1.3$ (the polytropic exponent of the steam), $\nu_{cr} \approx 0.546$. Therefore we shall take (for simulations) $\nu_{cr} = 0.6$.

As a remark, is to be mentioned that equation (2) was obtained in steady state; extending its validity during transients requires, at least from the mathematical point of view, to extend its validity for $z \geq 1$ by taking $\Phi(z) = 0$; this extension makes sense since it signifies there is no flow when the downstream pressure is larger than the upstream one.

This model, which also appears in other papers, has been elaborated starting from the one in [24]. It describes the co-generation dynamics. The significance of the state variables is as follows:

- state variables: s – rated (to the synchronous speed) deviation of the turbine rotation speed from the synchronous one; π_1, π_s, π_2 – rated (to some significant steady state values of the) steam pressures in the HP (high pressure), extraction chamber and LP (low pressure) turbine cylinder respectively; $\xi_\rho(\lambda, t)$, $\xi_w(\lambda, t)$ – steam pressure and steam mass flow along the steam pipe connecting the turbine and the steam consumer;
- inputs and references: μ_k, $k = 1, 2$ – rated (to their maximal values) control inputs (valve cross-sections); ν_g – rated mechanical power required at turbine's shaft;
- space independent variable: rated length of the pipe $0 < \lambda < 1$;
- parameters (coefficients): T_a, T_1, T_p, T_2, T_c – time constants of the turbine-turbogenerator turning masses, HP cylinder, extraction, chamber, LP cylinder and steam pipe respectively; $\alpha \in (0, 1)$, $\beta \in (0, 1)$ – mechanical power fraction contribution of the HP cylinder and extracted steam fraction respectively; ψ_c – velocity ratio for the steam flow along the pipe as rated to the sound velocity; $\psi_s \in (0, 1)$ – the load coefficient of the steam terminal consumer.

Some comments on this model seem necessary. The first group of comments concerns the PDEs. The starting point of adopting this model was the paper [25] where the modeling began from the *equations of the barotropic isentropic compressible fluid* [26], p. 429. Since barotropic means that the fluid pressure depends on density (specific volume) only, this shows that the processes are viewed as *isothermal*. In thermal power engineering it is accepted as an assumption that during short period transients the processes are indeed isothermal (unlike in gas distribution [27] or gas expansion in gas turbines [28]). For incompressible fluids, e.g. water, the processes are isothermal. The equations in [26] are in fact the Euler equations in *Fluid Mechanics* and, with the isothermal assumption, they are almost the same for water canals, penstocks and tunnels of hydropower engineering, steam or gas pipes: even the Darcy-Weisbach losses term is met in hydraulics and also in gas transport (in steam pipes it is usually neglected).

A second remark concerns the variables which are *scaled* (rated): as previously mentioned, they are rated to some reference, steady state significant (e.g. maximal) values. The advantages of this approach are threefold. Firstly, the model becomes somehow independent of a specific application (e.g. of the power level, pipe length, valve opening etc). Then these variables are dimensionless being nevertheless compatible and comparable. Also a sensible improvement *of the numerical conditioning is obtained*. The PDEs of (1), which are central in the model, display the form of conservation laws:

$$\partial_t U + \partial_\lambda f(U) = 0 \tag{3}$$

with U and f vectors of sufficiently smooth functions [29], [30]. The PDEs correspond to the case of cogeneration for a thermal consumer located sufficiently far away from the power block. The steam turbine is considered in the simplest case (from the point of view of the dynamics): a single regulated steam extraction and no intermediate re-heating hence, a steam turbine having a relatively small power output and relatively small steam pressures. The application is such that the PDEs do not contain distributed sources: the only steam source for propagation along the pipe is the regulated steam extraction of the turbine. In turn, this steam source arises from the boiler of the power block - the term $\mu_1(t)$: the steam is there super-critical, being proportional to the product of the controlled admission (cross section area) and the upstream pressure assumed constant - as in all studies on steam turbine control [31].

The steam turbine is located at the boundary $\lambda = 0$ and the flow is subcritical in steady state; this explains the use of Φ as defined by (2) – it has also been taken into account the assumption that the flow enthalpy does not change, hence the Saint Venant formula [32] does not apply [23]. The flow at the steam consumer ($\lambda = 1$) is critical, which explains the linearity of this boundary condition. Further comments on the model are also useful here, for the convenience of the reader.

The first one concerns the controlling structure: in a block representation as depicted in Fig. 1, the steam turbine dynamics is a system with lumped parameters, while the steam pipe connecting it to the thermal consumer is a nonlinear system with distributed parameters. The BCs are specific for this last block and have to be considered separately: they "receive" the signal $\pi_s(t)$ and provide the feedback signal $\xi_w(0, t)$. The feedback occurs at the level of turbine's extraction chamber and, consequently, the control signals $\mu_k(t)$, $k = 1, 2$ provide an indirect boundary control to the block with distributed parameters (the steam pipe).

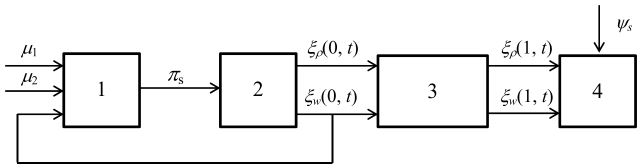


FIGURE 1. Block diagram for the cogeneration system: (1) - the turbine block; (2) and (4) - the boundary conditions blocks at λ = 0 and λ = 1, respectively; (3) - the distributed parameters (propagation) block.

The second remark concerns the nonlinear character of (1): the conservation laws are nonlinear, the boundary conditions at $\lambda = 0$ are nonlinear (for subcritical flow) and the turbine model is nonlinear (in fact bilinear), as well, since the control signal modifies the cross-section for the critical steam flow which leaves the extraction chamber of the steam turbine to enter the LP cylinder of the steam turbine. A further remark concerning system (1) specifies that the time constants T_1 and T_2 are small with respect to T_a and T_p and, sometimes, they are neglected, e.g. [24], [33]. Consequently, a new system of reduced order is obtained

$$\begin{aligned}
&\psi_c T_c\, \partial_t \xi_\rho + \partial_\lambda \xi_w = 0\\
&\psi_c T_c \partial_t \xi_w + \partial_\lambda \left(\xi_\rho + \frac{\xi_w^2}{\psi_c^2 \xi_\rho}\right) = 0\\
&\xi_w(0,t) = \pi_s(t)\Phi\left(\xi_\rho(0,t)/\pi_s(t)\right)\\
&\xi_w(1,t) = \sqrt{2}\psi_s \xi_\rho(1,t)\\
&T_a \frac{\mathrm{d}s}{\mathrm{d}t} = \alpha\mu_1(t) + (1-\alpha)\mu_2(t)\pi_s - \nu_g\\
&T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \mu_1(t) - \mu_2(t)\pi_s - (\beta/\psi_c)\xi_w(0,t).
\end{aligned} \tag{4}$$

Taking into account the experience from the field of ODEs, system (1) appears as having two time scales: the fast dynamics (connected to T_1 and T_2) which defines the boundary layer and the slow dynamics, also called outer dynamics. The theory of singular perturbations is now standard for ODEs and also for time-delay systems in the retarded case. Less is known about this subject for systems described by neutral functional differential equations (FDEs) or by such boundary value problems like (1) – where the singular perturbation occurs in the subsystem of ODEs. There are however some interesting references [34], [35].

Looking ahead, we precise that the reduced system can be tackled in the same way as the complete one. It will be considered however only in the control design part, the general theory being done for the complete system (1).

A final remark concerns the nonlinear term in the conservation laws: if we consider the three applications arising from power engineering, two of them are almost always studied by neglecting the aforementioned term. In (1) this is the term $\xi_w^2/\ \psi_c^2\xi_\rho$ accounting for certain distributed losses whose space variation had been assumed from the very beginning of the analysis as negligible [25]. In the case of hydraulic plants, the same term of losses is assumed to have negligible space variations. Interesting to mention, registered data from several hundreds power plants of the former USSR, confirm that this is indeed the case [36]. For other applications this might not be the case.

In the following we shall adopt this simplifying but far-going assumption of neglecting the distributed nonlinear losses variation. Consequently, we shall focus on the following model

$$\begin{aligned}
&\psi_c T_c\, \partial_t \xi_\rho + \partial_\lambda \xi_w = 0\\
&\psi_c T_c \partial_t \xi_w + \partial_\lambda \xi_\rho = 0\\
&\xi_w(0,t) = \pi_s(t)\Phi\left(\xi_\rho(0,t)/\pi_s(t)\right)\\
&\xi_w(1,t) = \sqrt{2}\psi_s \xi_\rho(1,t)\\
&T_a \frac{\mathrm{d}s}{\mathrm{d}t} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g, \quad 0 < \alpha < 1\\
&T_1 \frac{\mathrm{d}\pi_1}{\mathrm{d}t} = \mu_1(t) - \pi_1, \quad 0 \le \mu_1(t) \le 1\\
&T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1 - \mu_2(t)\pi_s - (\beta/\psi_c)\xi_w(0,t), \quad 0 < \beta < 1\\
&T_2 \frac{\mathrm{d}\pi_2}{\mathrm{d}t} = \mu_2(t)\pi_s - \pi_2, \quad 1-\beta \le \mu_2(t) \le 1.
\end{aligned} \tag{5}$$

The PDEs are now linear but the boundary conditions are not. Moreover, (2) is sending to a rather recent problem of controlling systems described by hyperbolic PDEs with switched boundary conditions [37]–[39]. To this we add the internal feedback structure and the bilinear character of the ODEs modeling the steam turbine. The complexity of the model is obvious (*even with the aforementioned simplifications*) and this explains our option for augmented validation before synthesizing a control structure.

III. AUGMENTED VALIDATION – THE APPROACH

In the aforementioned references concerning control of systems with conservation laws, the problem of the well posedness in the sense of Hadamard is also tackled, mostly in the class of classical solutions, based on the second order compatibility conditions as in [40] (this approach is however not the only one in the cited references). The present paper will nevertheless follow another, complementary line which goes back to the 50's and 60's of the past century. The papers of A. D. Myshkis and his co-workers [41]–[44] deal with linear and quasi-linear hyperbolic PDEs containing rather general nonlinear Volterra type operator terms and distributed sources ("forcing" terms); the same type of terms occurs in the boundary conditions. The Volterra operator signifies a functional dependence "on the past" and, if the Riemann invariants are viewed along the characteristics and substituted in the boundary conditions, a system of functional equations (FEs) having a rather general form is obtained.

This system of FEs appears as an independent mathematical object incorporating even non-standard, dynamic BCs. Examination of the BCs of (1), (4) and/or (5) shows that the dynamics and the standard BCs cannot be separated (see also the feedback connection of Fig. 1) and it is felt that introducing the system of FEs is helpful. Moreover, a while later the papers of K. L. Cooke were published, [45], [46], containing nevertheless a simpler case of two linear hyperbolic PDEs with dynamic BCs containing linear ODEs. The approach is the same as in the case of the aforementioned papers of Myshkis and the result is a one-to-one correspondence, rigorously and completely proven in [19] between the solutions of the two mathematical objects. The system of FEs is now *a system of differential equations with deviated argument*; moreover, simple arithmetic conditions were given for the previously mentioned system with deviated argument to be of delayed, neutral or advanced type - a rather natural way of introducing the systems with deviated argument (in the spirit of the very first paper on the subject - the paper of J. Bernoulli of 1728 [47]). A result connected to the ones mentioned above is that of [48], cited after [49].

The advantage of this approach relies on the fact that some problems of applied dynamics such as existence of invariant sets, stability, oscillations, are better studied for systems of differential equations (possibly with deviated argument). Under these circumstances, the fact that the aforementioned systems of FEs are known can be only useful. For this reason (and several other) our development from the simple linear case (as the one of Cooke), then bilinear [24] and now with nonlinear BCs can give the possibility of tackling new classes of functional differential equations (FDEs) but also new types of boundary value problems (due to the aforementioned one-to-one correspondence of the solutions for the two mathematical objects).

Summarizing, it is the system of FDEs which allows developing validation and augmented validation for the model.

We recall here the steps of the augmented validation as defined in [19]: basic theory that is: (1) well-posedness in the sense of J. Hadamard (existence, uniqueness and continuous data – parameters and initial conditions – dependence), (2) existence of some significant invariant sets and (3) inherent stability of some equilibria as significant trajectories (*Stability Postulate* of Četaev).

The approach of the present paper will rely on the method established under rather general assumptions in [44]; we shall be using however the simpler version of [45] – version which was completely proven in [19]. Starting from (5), we shall associate to it a system of FDEs, obtained by considering the Riemann invariants along the characteristics. This consideration will also allow to associate some initial conditions to the aforementioned system of FDEs (starting from the initial conditions of (5)). Worth mentioning is that the method was successfully applied starting with the early papers [24], [33] then in several papers that followed [4], [50]–[52].

The novelty of the paper at this point consists in the nonlinear BCs at $\lambda = 0$; moreover, this condition contains uncontrolled commutations from the subcritical to critical branch and vice-versa – see (2). For this reason, a preliminary result is required.

Theorem 1: Consider system (4) *with some initial conditions* $s(0)$, $\pi_k(0)$ *(*$k = 1, s, 2$*),* ξ_ρ^0, ξ_w^0 *(*$0 \leq \lambda \leq 1$*) and assume existence of a piecewise classical solution of* (4)*. Then, if* $\pi_k(0) \geq 0$, $k = 1, s, 2$*, the following alternative holds* for each $\pi_k(t)$*: either* $\pi_k(t) \equiv 0$ *or* $\pi_k(t) > 0$ *for all* $t > 0$.

The proof is given in Appendix 1.

We are now in position to introduce the Riemann invariants and construct the associated system of FDEs as follows. We introduce the Riemann invariants by

$$\xi^{\pm}(\lambda, t) = \xi_\rho(\lambda, t) \pm \xi_w(\lambda, t) \tag{6}$$

together with the converse relations

$$\begin{aligned} \xi_\rho(\lambda, t) &= \frac{1}{2}(\xi^+(\lambda, t) + \xi^-(\lambda, t)) \\ \xi_w(\lambda, t) &= \frac{1}{2}(\xi^+(\lambda, t) - \xi^-(\lambda, t)). \end{aligned} \tag{7}$$

The boundary value problem for the Riemann invariants is obtained from (5):

$$\begin{aligned} &\psi_c T_c\, \partial_t \xi^{\pm}(\lambda, t) \pm \partial_\lambda \xi^{\pm}(\lambda, t) = 0 \\ &\frac{1}{2}(\xi^+(0, t) - \xi^-(0, t)) = \pi_s(t)\Phi\left(\frac{\xi^+(0, t) + \xi^-(0, t)}{2\pi_s(t)}\right) \\ &\xi^-(1, t) = \rho_1 \xi^+(1, t), \quad \rho_1 := \frac{1 - \sqrt{2}\psi_s}{1 + \sqrt{2}\psi_s} \\ &T_a \frac{\mathrm{d}s}{\mathrm{d}t} = \alpha\pi_1 + (1 - \alpha)\pi_2 - \nu_g \\ &T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1 - \mu_2(t)\pi_s - \frac{\beta(\xi^+(0, t) - \xi^-(0, t))}{2\psi_c} \\ &T_1 \frac{\mathrm{d}\pi_1}{\mathrm{d}t} = \mu_1(t) - \pi_1 \\ &T_2 \frac{\mathrm{d}\pi_2}{\mathrm{d}t} = \mu_2(t)\pi_s - \pi_2. \end{aligned} \tag{8}$$

We shall first re-write (8) based on the properties of $\Phi : [0, \infty) \mapsto \mathbb{R}$ defined by (2). Since $\pi_s(t) > 0$, the boundary condition at $\lambda = 0$ can be written as

$$z(t) - \Phi(z(t)) = x(t), \tag{9}$$

where we denote

$$z(t) := \frac{\xi^+(0, t) + \xi^-(0, t)}{2\pi_s(t)}, \quad x(t) := \frac{\xi^-(0, t)}{\pi_s(t)}. \tag{10}$$

Taking into account (2) we obtain that the mapping $\Psi := z - \Phi(z)$ is increasing and maps the intervals $[0, \nu_{cr}]$, $[\nu_{cr}, 1]$, $[1, \infty]$ onto $[-\sqrt{2(1 - \nu_{cr})}, \nu_{cr} - \sqrt{2(1 - \nu_{cr})}]$, $[\nu_{cr} - \sqrt{2(1 - \nu_{cr})}, 1]$ and $[1, \infty)$, respectively. The inverse mapping $h_p(x)$ exists and is defined by $x + g_p(x)$ where

$$g_p(x) = \begin{cases} \sqrt{2(1 - \nu_{cr})}, & \textit{for } -\sqrt{2(1 - \nu_{cr})} \leq x \\ & \quad \leq \nu_{cr} - \sqrt{2(1 - \nu_{cr})} \\ -1 + \sqrt{3 - 2x}, & \textit{for } \nu_{cr} - \sqrt{2(1 - \nu_{cr})} \leq x \leq 1 \\ 0, & \textit{for } x \geq 1. \end{cases} \tag{11}$$

As a consequence, we can re-write (8) as follows

$$\begin{aligned} &\psi_c T_c\, \partial_t \xi^{\pm}(\lambda, t) \pm \partial_\lambda \xi^{\pm}(\lambda, t) = 0 \\ &\xi^+(0, t) = \xi^-(0, t) + 2g_p(\xi^-(0, t)/\pi_s(t))\pi_s(t) \end{aligned}$$

$$\begin{aligned}
&\xi^-(1,t) = \rho_1 \xi^+(1,t), \quad \rho_1 := \frac{1-\sqrt{2}\psi_s}{1+\sqrt{2}\psi_s}\\
&T_a \frac{ds}{dt} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g\\
&T_p \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + (\beta/\psi_c) g_p(\xi^-(0,t)/\pi_s(t)))\pi_s\\
&T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1\\
&T_2 \frac{d\pi_2}{dt} = \mu_2(t)\pi_s - \pi_2. \qquad (12)
\end{aligned}$$

Let (λ, t) be some point such that $0 < \lambda < 1, t > 0$ and let $t^\pm(\sigma; \lambda, t) = t \pm \psi_c T_c(\sigma - \lambda)$ be the two characteristics crossing that point (see Fig. 2 – the straight lines $1a$ and $2a$, respectively). We integrate $\xi^+(\sigma, t^+(\sigma; \lambda, t)) \equiv \xi^+(\sigma; t + \psi_c T_c(\sigma - \lambda))$ with respect to σ from $\sigma = \lambda$ to $\sigma = 1$; since $\xi^+(\cdot, t^+(\cdot\,; \lambda, t))$ is constant, we obtain

$$\xi^+(\lambda, t) = \xi^+(1;\ t + \psi_c T_c(1-\lambda)) \qquad (13)$$

and, in particular, for $\lambda = 0$, $\xi^+(0,t) = \xi^+(1, t+\psi_c T_c)$ provided the characteristic $t^+(\sigma; \lambda, t)$ crosses the vertical $\sigma = 0$, i.e., if $t - \psi_c T_c > 0$.

In a similar way, we integrate $\xi^-(\sigma, t^-(\sigma; \lambda, t)) \equiv \xi^-(\sigma; t - \psi_c T_c(\sigma - \lambda))$ from $\sigma = \lambda$ to $\sigma = 0$ to obtain

$$\xi^-(\lambda, t) = \xi^-(0;\ t + \psi_c T_c \lambda) \qquad (14)$$

and, in particular, for $\lambda = 1$, $\xi^-(1,t) = \xi^-(0;\ t+\psi_c T_c)$ provided the characteristic $t^-(\sigma;\ \lambda, t)$ crosses the vertical $\sigma = 0$, i.e., if $t + \psi_c T_c > 0$.

At this point two comments are useful. The first one concerns formulae (13), (14): they are *a representation of the solution of* (8) – assumed to exist – *in function of the boundary conditions*. The second comment concerns the other set of equalities, i.e., $\xi^+(0,t) = \xi^+(1,\ t+\psi_c T_c)$ and $\xi^-(1,t) = \xi^-(0,\ t+\psi_c T_c)$. Since (8) as well as (5) describe a two-point boundary value problem, the aforementioned equalities describe the connections between the values at those two boundaries. From the physical point of view, it appears quite obviously that $\xi^+(\lambda, t)$ is the forward wave which propagates from $\lambda = 0$ to $\lambda = 1$ while, $\xi^-(\lambda, t)$ is the backward wave which propagates from $\lambda = 1$ to $\lambda = 0$.

We denote first

$$\begin{aligned}
&w^+(t) := \xi^+(1,t) \Rightarrow \xi^+(0,t) = w^+(t+\psi_c T_c)\\
&w^-(t) := \xi^-(0,t) \Rightarrow \xi^-(1,t) = w^-(t+\psi_c T_c) \qquad (15)
\end{aligned}$$

and next

$$y^+(t) := w^+(t+\psi_c T_c), \quad y^-(t) := w^-(t+\psi_c T_c) \qquad (16)$$

to obtain

$$\begin{aligned}
&\xi^+(0,t) = y^+(t), \quad \xi^+(1,t) = y^+(t-\psi_c T_c)\\
&\xi^-(1,t) = y^-(t), \quad \xi^-(0,t) = y^-(t-\psi_c T_c). \qquad (17)
\end{aligned}$$

Substituting (17) in (8) we obtain the following system of coupled delay differential and difference equations

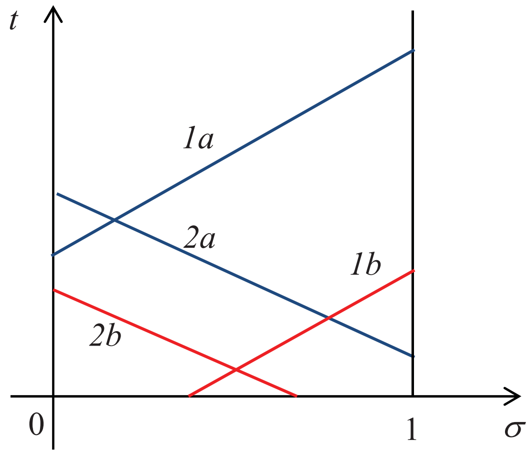


FIGURE 2. Forward and backward characteristics.

$$\begin{aligned}
&T_a \frac{ds}{dt} = \alpha\pi_1 + (1-\alpha)\pi_2 - \nu_g\\
&T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1\\
&T_p \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + (\beta/\psi_c) g_p(y^-(t-\psi_c T_c)/\pi_s(t)))\pi_s\\
&T_2 \frac{d\pi_2}{dt} = \mu_2(t)\pi_s - \pi_2\\
&y^+(t) = y^-(t-\psi_c T_c) + 2g_p(y^-(t-\psi_c T_c)/\pi_s(t))\pi_s(t)\\
&y^-(t) = \rho_1 y^+(t-\psi_c T_c). \qquad (18)
\end{aligned}$$

This system has thus been obtained starting from a classical solution of the boundary value problem (4), in fact of its version (12) written in the Riemann invariants. In order to treat it as a mathematical object which is interesting in itself, its solution should be defined and constructed.

The form of (18) and our previous experience, e.g. [19], [52], show that the solution can be constructed by steps on intervals $((k-1)\psi_c T_c,\ k\psi_c T_c)$, k being an integer. The construction requires however the knowledge of some initial conditions. Obviously, $s(0)$, $\pi_1(0)$, $\pi_s(0)$, $\pi_2(0)$ will migrate from (4) or (12). Since $y^\pm(t)$ have been obtained from $\xi^\pm(\lambda, t)$ by integration along the characteristics, it is but natural to obtain the initial conditions $y_0^\pm(t)$ on $(-\psi_c T_c,\ 0)$ the same way. Let $\xi_\rho^0(\lambda) = \xi_\rho(\lambda, 0)$, $\xi_w^0(\lambda) = \xi_w(\lambda, 0)$ be the initial conditions for $(\xi_\rho(\lambda,t), \xi_w(\lambda,t))$ in (4). From (6) we deduce $\xi_0^\pm(\lambda) = \xi^\pm(\lambda, 0)$ – the initial conditions for the Riemann invariants.

Let now again consider $\xi^+(\sigma,\ t^+(\sigma;\ \lambda, t))$ as previously, hence equality (13); if the characteristic $t^+(\sigma;\ \lambda, t)$ is of the type $1b$ (Fig. 2), then there will exist some $\hat\sigma \in (0,1)$ such that $t^+(\hat\sigma;\ \lambda, t) = t + \psi_c T_c(\hat\sigma - \lambda) = 0$; we shall then have $\hat\sigma = \lambda - t/\psi_c T_c > 0$ hence, $t - \psi_c T_c \lambda < 0$ now, unlike previously. From (13) we obtain quite straightforwardly

$$y^+(t) = \xi^+(-t/\psi_c T_c, 0) = \xi_0^+(-t/\psi_c T_c), \quad -\psi_c T_c \le t < 0. \qquad (19)$$

Starting from (14) we deduce for characteristics of $2b$ type (Fig. 2):

$$y^-(t) = \xi^-(1+t/\psi_c T_c, 0) = \xi_0^-(1+t/\psi_c T_c), \quad -\psi_c T_c \le t < 0. \qquad (20)$$

Now, all initial conditions for (18) are gathered. Its solution can be constructed by steps as follows: consider the first interval $(0, \psi_c T_c)$; given $\mu_1(t)$ and $\pi_1(0)$, $\pi_1(t)$ follows by integration of the second equation in (18). Next, $\pi_1(t)$ and $\mu_2(t)$ are input signals for the subsystem

$$\begin{aligned} T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} &= \pi_1 - (\mu_2(t) - (\beta/\psi_c) g_p(y^-(t-\psi_c T_c)/\pi_s))\pi_s \\ y^+(t) &= y^-(t-\psi_c T_c) + 2g_p(y^-(t-\psi_c T_c)/\pi_s)\pi_s(t) \\ y^-(t) &= \rho_1 y^+(t-\psi_c T_c). \end{aligned} \tag{21}$$

The terms containing $y^{\pm}(t-\psi_c T_c)$ are known on $(0, \psi_c T_c)$ since they are given by the initial conditions. Therefore, we can obtain $\pi_s(t)$, $y^{\pm}(t)$ on $(0, \psi_c T_c)$. Afterwards, knowing $\pi_s(t)$ and $\mu_2(t)$, we can compute $\pi_2(t)$ and further, $s(t)$. The process can be now be iterated on $((k-1)\psi_c T_c,\ k\psi_c T_c)$, $k > 1$.

We do now the converse operation. Turning back to (13) and (14) which may be viewed as representation formulae for the Riemann invariants, we take into account (17) to define

$$\begin{aligned} \xi^+(\lambda, t) &= y^+(t - \psi_c T_c \lambda); \\ \xi^-(\lambda, t) &= y^-(t + \psi_c T_c(\lambda - 1)). \end{aligned} \tag{22}$$

By direct check it can be shown that $\xi^{\pm}(\lambda, t)$ thus defined are a (possibly discontinuous) classical solution of (12), together with $(s(t), \pi_1(t), \pi_s(t), \pi_2(t))$ taken from the solution of (18).

We have obtained in fact the following result

Theorem 2: Consider the boundary value problem (12) *with* g_p *the mapping defined by* (11). *Let* $(\xi^{\pm}(\lambda, t);\ s(t),\ \pi_1(t),\ \pi_s(t),\ \pi_2(t))$ *be a (possibly discontinuous) classical solution of it, corresponding to certain initial conditions* $(\xi_0^{\pm}(\lambda), 0 \le \lambda \le 1; s(0), \pi_1(0), \pi_s(0), \pi_2(0))$ *and to certain given control functions* $\mu_k(t)$, $k = 1, 2$. *Then the set of functions* $(s(t), \pi_1(t), \pi_s(t), \pi_2(t), y^{\pm}(t))$, *where* $(s(t), \pi_1(t), \pi_s(t), \pi_2(t)$ *are those from the solution of* (12) *and* $y^{\pm}(t)$ *are defined from* (15)-(17), *defines a solution of* (18) *with the initial conditions* $(s(0), \pi_1(0), \pi_s(0), \pi_2(0); y_0^{\pm}(t), -\psi_c T_c \le t \le 0)$, *where* $(s(0), \pi_1(0), \pi_s(0), \pi_2(0))$ *migrate from the set of initial conditions attached to* (12) *and* $y_0^{\pm}(t)$ *are obtained starting from* $\xi_0^{\pm}(\lambda)$ via (19)-(20).

Conversely, let $(s(t), \pi_1(t), \pi_s(t), \pi_2(t), y^{\pm}(t))$ *be a solution of* (18) *with the initial conditions* $(s(0), \pi_1(0), \pi_s(0), \pi_2(0);$ $y_0^{\pm}(t), -\psi_c T_c \le t \le 0)$ *and with certain given control functions* $\mu_k(t)$, $k = 1, 2$. *Then the set of functions* $(\xi^{\pm}(\lambda, t),$ $s(t), \pi_1(t), \pi_s(t), \pi_2(t))$ *is a (possibly discontinuous) classical solution of* (11), *where* $(s(t), \pi_1(t), \pi_s(t), \pi_2(t))$ *migrate from the aforementioned ones and* $\xi^{\pm}(\lambda, t)$ *are given by* (22) *with* $y^{\pm}(t)$ *- the aforementioned ones; the initial conditions of this solution are* $(s(0), \pi_1(0), \pi_s(0), \pi_2(0))$ *which migrate from those given for* (18) *and* $\xi_0^{\pm}(\lambda)$ *are obtained from* (22) *by taking* $\lambda = 0$; *the control functions are the same for both mathematical objects.*

Theorem 2 establishes a one-to-one correspondence between the solutions of two mathematical objects (12) and (18) as mentioned at the beginning of this section. It thus follows that any result concerning one of these objects will automatically be projected back on the other. In the following we shall focus on the system of coupled nonlinear time delay differential and difference equations (18).

IV. AUGMENTED VALIDATION FOR THE SYSTEM OF FUNCTIONAL DIFFERENTIAL EQUATIONS

A. BASIC THEORY AND INVARIANT SETS (NONNEGATIVE STATE VARIABLES)

As already mentioned, the solution of (18) can be constructed by steps on intervals $((k-1)\psi_c T_c, k\psi_c T_c)$ for k integer and this solution is unique for given initial conditions. The variables s, π_1, π_s, π_2 are continuous, at least piecewise differentiable and display continuity with respect to data (parameters and initial conditions). The variables $y^{\pm}$ are piecewise continuous but can display discontinuities at $t = k\psi_c T_c$. From (18) we can obtain, after a simple manipulation

$$\begin{aligned} & y^+(k\psi_c T_c + 0) - y^+(k\psi_c T_c - 0) \\ &\quad = y^-((k-1)\psi_c T_c + 0) - y^-((k-1)\psi_c T_c - 0) \\ &\qquad + 2[g_p(y^-((k-1)\psi_c T_c + 0)/\pi_s(k\psi_c T_c)) \\ &\qquad - g_p(y^-((k-1)\psi_c T_c - 0)/\pi_s(k\psi_c T_c))] \cdot \pi_s(k\psi_c T_c) \\ & y^-(k\psi_c T_c + 0) - y^-(k\psi_c T_c - 0) \\ &\quad = \rho_1[y^+((k-1)\psi_c T_c + 0) - y^-((k-1)\psi_c T_c - 0)]. \end{aligned} \tag{23}$$

These are recurrence relations showing that the discontinuities are generated by the discontinuity at $t = 0$ which propagates. In turn, the previously mentioned discontinuity at $t = 0$ is a consequence of the mismatch between initial and boundary conditions. Indeed, from (12) it follows

$$\xi^+(0, 0+) = \xi^-(0, 0+) + 2g_p(\xi^-(0, 0+)/\pi_s(0))\pi_s(0)$$

whence from (17)

$$\begin{aligned} y^+(0+) &= y_0^-(-\psi_c T_c) + 2g_p(y_0^-(-\psi_c T_c)/\pi_s(0))\pi_s(0) \\ &\neq y^+(0) = y^+(0-). \end{aligned} \tag{24}$$

Worth mentioning is that matching of the initial and boundary conditions is also known as *"compatibility condition"* [16], [40]; if a similar condition is added, with respect to the derivatives of the BCs, then a second order compatibility condition is obtained.

Matched initial and boundary conditions is a quite seldom situation and by no means is to be credited for stability studies where arbitrary short period perturbations generate arbitrary initial conditions. On the other hand, propagation of the discontinuities can be observed within the simulation results (see, further, Subsection IV-D, Figures 11, 12, where some "peaks" can be observed occurring exactly at 0, $\psi_c T_c (= 0.75)$, $2\psi_c T_c$ etc.).

Another aspect of the augmented validation is given by the existence of the invariant sets. We have already proven a result of this kind for the system (4) - Theorem 1. However, now we treat system (18) as a self-contained mathematical object, hence we shall prove the following

Theorem 3: Consider system (18) *under the aforementioned assumption on its parameters. Then, if* $\pi_1(0) \geq 0$, $\pi_s(0) \geq 0$, $\pi_2(0) \geq 0$ *the alternative of Theorem 1 is valid for the state variables* $\pi_1(t)$, $\pi_s(t)$, $\pi_2(t)$. *Moreover, if the initial conditions* $y_0^{\pm}(t) \geq 0$ *for* $-\psi_c T_c \leq t \leq 0$, *then* $y^{\pm}(t) \geq 0$ *for all* $t > 0$.

The proof is given in Appendix 2. Here we shall briefly discuss the significance of the result. Since π_1, π_s, π_2 are steam pressures, they have to be positive and the existence of the aforementioned invariant sets is an additional validation of system (18) as a mathematical model. Concerning positiveness of $y^{\pm}(t)$, the representation formulae (22) will give positiveness of the Riemann invariants provided their initial conditions are such

$$\begin{aligned} &\xi^+(\lambda, 0) = y_0^+(-\psi_c T_c \lambda) \geq 0, \quad 0 \leq \lambda \leq 1 \\ &\Rightarrow \xi^+(\lambda, t) = y^+(t - \psi_c T_c \lambda) \geq 0, \quad 0 \leq \lambda \leq 1, \ t > 0 \end{aligned} \tag{25}$$

and similarly for $\xi^-(\lambda, t)$. Consequently, (7) will give

$$\begin{aligned} \xi_\rho(\lambda, 0) \geq 0, \quad 0 \leq \lambda \leq 1 \Rightarrow \xi_\rho(\lambda, t) \geq 0, \\ 0 \leq \lambda \leq 1, \quad t > 0, \end{aligned} \tag{26}$$

what is only natural since $\xi_\rho(\lambda, t)$ is also a steam pressure. Also (7) results in

$$\xi_\rho(\lambda, 0) - \xi_w(\lambda, 0) \geq 0 \Rightarrow \xi_\rho(\lambda, t) - \xi_w(\lambda, t) \geq 0. \tag{27}$$

which shows a certain kind of steam flowing.

B. STEADY STATES (EQUILIBRIA)

The steady states are nothing more but the mathematical representation of the operating points of an engineering system. We shall discuss the steady states of (5) via the steady states of (18). Let $\mu_k(t) \equiv \bar{\mu}_k$, $k = 1, 2$ be constant control inputs and ν_g - the mechanical load be given. The steady state of (18), the set $\bar{\pi}_1, \bar{\pi}_s, \bar{\pi}_2, \bar{y}^+, \bar{y}^-$ is subject to

$$\begin{aligned} &\alpha\bar{\pi}_1 + (1-\alpha)\bar{\pi}_2 = \nu_g; \quad \bar{\pi}_1 = \bar{\mu}_1, \quad \bar{\mu}_2\bar{\pi}_s = \bar{\pi}_2 \\ &\bar{\pi}_1 - \bar{\mu}_2\bar{\pi}_s - (\beta/\psi_c)g_p(\bar{y}^-/\bar{\pi}_s)\bar{\pi}_s = 0 \\ &\bar{y}^+ = \bar{y}^- + 2g_p(\bar{y}^-/\bar{\pi}_s)\bar{\pi}_s. \end{aligned} \tag{28}$$

Taking into account the invariant sets whose existence has been just proven, we shall be looking for steady states satisfying $\bar{\pi}_1 > 0$, $\bar{\pi}_s > 0$, $\bar{\pi}_2 > 0$, $\bar{y}^{\pm} > 0$. Also we require $0 < \bar{\mu}_1 < 1$, $1 - \beta \leq \bar{\mu}_2 < 1$. The computation can be done in several steps. In the first step $\bar{\pi}_1, \bar{\pi}_2, \bar{y}^-$ are computed directly provided the following nonlinear system is solved

$$\begin{aligned} &\alpha\bar{\mu}_1 + (1-\alpha)\bar{\mu}_2\bar{\pi}_s = \nu_g \\ &\bar{\mu}_1 - \bar{\mu}_2\bar{\pi}_s - (\beta/\psi_c)g_p(\bar{y}^-/\bar{\pi}_s)\bar{\pi}_s = 0 \\ &(1/\rho_1 - 1)\bar{y}^- = 2g_p(\bar{y}^-/\bar{\pi}_s)\bar{\pi}_s. \end{aligned}$$

The last equation can be solved separately with respect to the ratio $\bar{y}^-/\bar{\pi}_s$, being reduced to

$$\frac{1-\rho_1}{\rho_1}x = g_p(x)$$

where $x > 0$. Taking into account (11) we must take $g_p(x) = -1 + \sqrt{3 - 2x}$ and solve a second degree equation taking its positive root

$$\bar{x} = \frac{\bar{y}^-}{\bar{\pi}_s} = \frac{2\rho_1}{(1-\rho_1)^2}\left[-(1+\rho_1) + \sqrt{3 - 2\rho_1 + \rho_1^2}\right]. \tag{29}$$

which can be shown to be subject to $0 < \bar{x} < 1$. If we denote $\tilde{g}_p(\rho_1) := g_p(\bar{y}^-/\bar{\pi}_s)$ then we shall have the following two equations

$$\begin{aligned} \alpha\bar{\mu}_1 + (1-\alpha)\bar{\mu}_2\bar{\pi}_s &= \nu_g \\ \bar{\mu}_1 - \bar{\mu}_2\bar{\pi}_s - (\beta/\psi_c)\tilde{g}_p(\rho_1)\bar{\pi}_s &= 0. \end{aligned} \tag{30}$$

Observe that among the three unknowns one has to be imposed and this is $\bar{\pi}_s = \pi_s^0$ - the steam pressure at the steam extraction of the turbine. Consequently, $\bar{y}^- = \pi_s^0\bar{x}$ (from (29)) and $\bar{y}^+ = \bar{y}^-/\rho_1$. Moreover, system (30) becomes linear with respect to $\bar{\mu}_1$ and $\bar{\mu}_2$ which are obtained immediately as

$$\begin{aligned} \bar{\mu}_1 &= \nu_g + (1-\alpha)(\beta/\psi_c)\tilde{g}_p(\rho_1)\pi_s^0 \\ \bar{\mu}_2\pi_s^0 &= \nu_g - \alpha(\beta/\psi_c)\tilde{g}_p(\rho_1)\pi_s^0. \end{aligned} \tag{31}$$

We "read" now the physical significance of (31). As already mentioned, ν_g represents the mechanical load of the turbine, accounting for electricity generation. The term $\tilde{g}_p(\rho_1)\pi_s^0$ represents the thermal load, accounting for heat generation; it has been clearly stated that the thermal load coefficient was ψ_s hence ρ_1 is a measure of it. Therefore, $\tilde{g}_p(\rho_1)\pi_s^0$ is imposed *via* ψ_s and π_s^0. The two loads of the cogeneration are "covered" by the turbine steam consumption. In turn, the overall steam consumption is given by the steam flow at turbine's input - its rated value is $\bar{\mu}_1$ while the steam consumption for the thermal energy generation is given by the extracted steam flow - its rated value being $\bar{\mu}_2\pi_s^0$.

If the restrictions $0 < \bar{\mu}_1 < 1$, $1 - \beta < \bar{\mu}_2 < 1$ are taken into account, then it follows from (31) that

$$\begin{aligned} 0 < \nu_g + (1-\alpha)(\beta/\psi_c)\tilde{g}_p(\rho_1)\pi_s^0 < 1 \\ (1-\beta)\pi_s^0 < \nu_g - \alpha(\beta/\psi_c)\tilde{g}_p(\rho_1)\pi_s^0 < \pi_s^0. \end{aligned} \tag{32}$$

Inequalities (32) contain necessary restrictions on both mechanical and thermal loads. Since they reflect restrictions on steam flows through the steam turbine they are called by the design power engineers *consumption diagrams* (Fig. 3 - we denoted by $\nu_\theta := (\beta/\psi_c)\tilde{g}_p(\rho_1)\pi_s^0$) the rated thermal load, to have compatibility with ν_g - the rated mechanical load). In the sequel we shall consider only equilibria that are admissible with respect to the consumption diagrams.

C. THE SYSTEM IN DEVIATIONS AND INHERENT STABILITY

The purpose of this subsection is to verify for system (18) the Stability Postulate of Četaev, mentioned through Section I. Our first remark is that the steady state of (18) does not contain $\bar{s}$ since the state variable s does not appear in the RHS of (18). In fact, the first equation of (18) reduces to

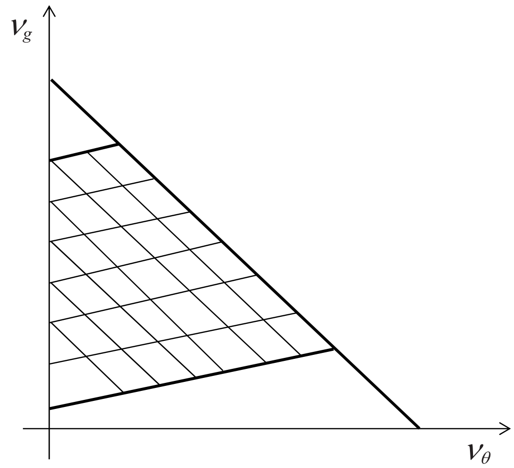


FIGURE 3. Consumption diagrams.

$\dot{s} = \varphi(t)$ and this suggests a zero eigenvalue of the linearized system. Therefore, the inherent stability of an equilibrium of (18) is at most non-asymptotic. Next, in order to illustrate the Postulate, it appears sufficient to obtain stability by the first approximation of the equilibrium. For this reason we shall introduce the system in deviations with respect to a certain steady state $(\bar{\pi}_1, \bar{\pi}_s^0, \bar{\pi}_2, \bar{y}^+, \bar{y}^-)$ and the associated constant control signals $(\bar{\mu}_1, \bar{\mu}_2)$. Introducing the deviations

$$\begin{aligned} &\zeta_1 = \pi_1 - \bar{\pi}_1; \quad \zeta_s = \pi_s - \bar{\pi}_s^0, \quad \zeta_2 = \pi_2 - \bar{\pi}_2 \\ &\zeta^\pm = y^\pm - \bar{y}^\pm; \quad u_i = \mu_i - \bar{\mu}_i, \quad i = 1, 2 \end{aligned} \tag{33}$$

we obtain the following system in deviations

$$\begin{aligned} T_a\frac{\mathrm{d}s}{\mathrm{d}t} &= \alpha\zeta_1 + (1-\alpha)\zeta_2, \quad T_1\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = u_1(t) - \zeta_1 \\ T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t} &= \zeta_1 - (\bar{\mu}_2 + (\beta/\psi_c)g_p(\bar{y}^-/\pi_s^0))\zeta_s - (\pi_s^0 + \zeta_s)u_2(t) \\ &\quad - (\beta/\psi_c)(\pi_s^0 + \zeta_s)\left[g_p\left(\frac{\bar{y}^- + \zeta^-(t-\psi_c T_c)}{\pi_s^0 + \zeta_s(t)}\right)\right. \\ &\qquad \left. - g_p(\bar{y}^-/\pi_s^0)\right] \\ T_2\frac{\mathrm{d}\zeta_2}{\mathrm{d}t} &= \bar{\mu}_2\zeta_s + (\pi_s^0 + \zeta_s(t))u_2(t) - \zeta_2 \\ \zeta^+(t) &= \zeta^-(t-\psi_c T_c) + 2g_p(\bar{y}^-/\pi_s^0))\zeta_s(t) + 2(\pi_s^0 + \zeta_s) \\ &\quad \times \left[g_p\left(\frac{\bar{y}^- + \zeta^-(t-\psi_c T_c)}{\pi_s^0 + \zeta_s(t)}\right) - g_p(\bar{y}^-/\pi_s^0)\right] \\ \zeta^-(t) &= \rho_1\zeta^+(t-\psi_c T_c). \end{aligned} \tag{34}$$

Inherent stability by the first approximation means letting $u_1(t) = u_2(t) \equiv 0$ and linearizing (34) around the resulting zero solution. The resulting system reads as follows

$$\begin{aligned} T_a\frac{\mathrm{d}s}{\mathrm{d}t} &= \alpha\zeta_1 + (1-\alpha)\zeta_2, \quad T_1\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} + \zeta_1 = 0 \\ T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t} &= \zeta_1 - (\bar{\mu}_2 + (\beta/\psi_c)g_p(\bar{y}^-/\pi_s^0))\zeta_s \\ T_2\frac{\mathrm{d}\zeta_2}{\mathrm{d}t} &= \bar{\mu}_2\zeta_s - \zeta_2 \\ \zeta^+(t) &= \zeta^-(t-\psi_c T_c) + 2g_p(\bar{y}^-/\pi_s^0)\zeta_s(t) \\ \zeta^-(t) &= \rho_1\zeta^+(t-\psi_c T_c). \end{aligned} \tag{35}$$

The "secular" (higher order, nonlinear) terms have been neglected since their contribution might perhaps strengthen the asymptotic stability of the corresponding subsystem. However they would not affect the critical subsystem hence the overall system remains stable but not asymptotically stable. The characteristic equation of (35) will be

$$\begin{aligned} &T_a\lambda(T_1\lambda + 1)(T_p\lambda + \bar{\mu}_2 + (\beta/\psi_c)g_p(\bar{y}^-/\pi_s^0)) \\ &\qquad \times (T_2\lambda + 1)(1 - \rho_1 \mathrm{e}^{-2\lambda\psi_c T_c}) = 0. \end{aligned} \tag{36}$$

One can see a simple zero root, three simple roots in $\mathbb{C}^-$ and an infinite chain of simple roots in $\mathbb{C}^-$ given by:

$$\lambda_k = -\frac{1}{2\psi_c T_c}\ln\rho_1^{-1} + \imath\frac{k\pi}{\psi_c T_c}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{37}$$

The system is thus in the critical case of a simple zero root, has non-asymptotic stability and, as all engineering applications, requires asymptotic if not exponential stability. We can now explain once more why, at this stage of model (augmented) validation, we restricted ourselves to inherent stability by the first approximation. Firstly, inherent stability as required by the Stability Postulate, can be quite weak hence its linear version can be accepted as sufficient at this stage of model discussion. Being in the first critical case (a simple zero root of the characteristic equation), the stability of the linearized system is non-asymptotic. For ODEs and FDEs of delayed type there exist theorems by the first approximation of the nonlinear systems in the first critical case - the Theorem of Malkin [53] for ODEs and of Halanay [54] for FDEs of delayed type. For neutral equations as (34) such a result is not known.

The aforementioned strengthened property of asymptotic/exponential stability is *achievable by designing a stabilizing feedback*. Worth mentioning that the aforementioned properties of system (18) are projected back onto the system (12) *via* Theorem 2.

D. SIMULATION RESULTS AND DISCUSSIONS

Several simulations have been performed aiming to illustrate the results on augmented validation obtained by rigorous proofs. The start models are represented by the equations (12), written with respect to the Riemann invariants, and the associated system (18) of functional differential equations. Since a one-to-one correspondence between the solutions of the two mathematical objects has been emphasized and rigorously proven (Theorem 2), the simulations will concern both systems, in parallel. Other properties, obtained by rigorous proofs, such as existence of invariant sets (positiveness of certain state variables) and inherent stability of equilibria will also be illustrated through simulations. We present next the simulation framework.

Examination of (12) shows that the subsystem which is essential for system's dynamics is given by

$$\begin{aligned}
&\psi_c T_c\, \partial_t \xi^{\pm}(\lambda, t) \pm \partial_\lambda \xi^{\pm}(\lambda, t) = 0\\
&\xi^{+}(0, t) = \xi^{-}(0, t) + 2g_p(\xi^{-}(0, t)/\pi_s(t))\pi_s(t)\\
&\xi^{-}(1, t) = \rho_1 \xi^{+}(1, t); \quad T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1\\
&T_p \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + (\beta/\psi_c) g_p(\xi^{-}(0, t)/\pi_s(t)))\pi_s
\end{aligned} \tag{38}$$

with the associated subsystem of differential difference equations

$$\begin{aligned}
&T_1 \frac{d\pi_1}{dt} = \mu_1(t) - \pi_1\\
&T_p \frac{d\pi_s}{dt} = \pi_1 - (\mu_2(t) + (\beta/\psi_c) g_p(y^{-}(t - \psi_c T_c)/\pi_s(t)))\pi_s\\
&y^{+}(t) = y^{-}(t - \psi_c T_c) + 2g_p(y^{-}(t - \psi_c T_c)/\pi_s(t))\pi_s(t)\\
&y^{-}(t) = \rho_1 y^{+}(t - \psi_c T_c).
\end{aligned} \tag{39}$$

These subsystems contain an internal feedback hence the stability problem is here important [17]. On the contrary, the equation for π_2 is simply a dynamical block controlled by the aforementioned one and the equation for s defines a dynamical output of the two previously mentioned systems.

We recall here the complexity of the mathematical model described by a system of hyperbolic conservation laws (even linearized) with nonlinear boundary conditions controlled by a system of ODEs containing bilinear terms. For such complex systems, the numerical methods as well as the computational procedures used for deriving the approximate numerical model are of high importance. More precisely, these tools have to provide the convergence of the approximate solution to the "real" solution and to preserve the basic properties of this solution and also its inherent Lyapunov stability, as already discussed in this paper. Taking into consideration these issues, the computational model of (38) was derived by using a recent computational procedure introduced in [55] and adapted in [56] for the class of systems of hyperbolic conservation laws with bilinear control terms on boundaries. This well-structured procedure is mainly based on a rigorously proven *convergent Method of Lines* combined with the advantages of certain devices belonging to the *Artificial Intelligence* field – the so-called *cell-based neural networks*. These AI devices consist of a network of elementary nonlinear dynamical subsystems having several types of inputs, local interconnections and one output [57]. Thus, the procedure not only provides convergence to the approximate solution, but also ensures the preservation of the basic properties of the "real" solution. In addition, it guarantees its inherent Lyapunov stability while at the same time ensures a reduced computational effort and time as well as a certain parallelization of the computation.

The system parameters considered in these simulations are as follows: $\Psi_c = 0.5$, $\Psi_s = 0.3487$, $T_c = 1.5$s, $T_p = 0.3$s, $\alpha = 0.5$, $\beta = 0.9$; also, $\nu_g = 0.6$, $\pi_s^0 = 0.9$. For verifying the inherent stability, the controls were considered constant and equal to their steady state values, i.e., $\mu_1(t) = \bar{\mu}_1 = 0.9601 \in [0, 1]$ and $\mu_2(t) = \bar{\mu}_2 = 0.2666 \in [0.1, 1]$, thus, confined within the proper intervals as required in (1).

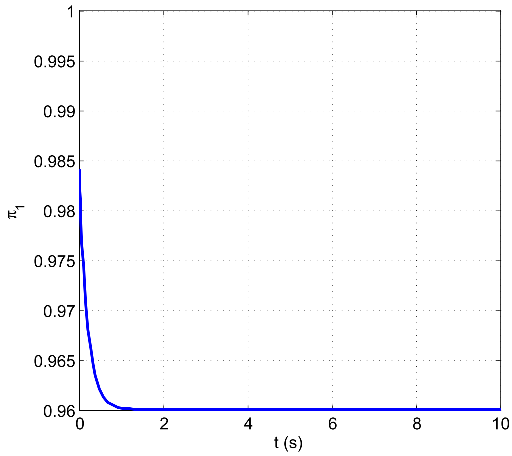


FIGURE 4. HP pressure $\pi_1(t)$.

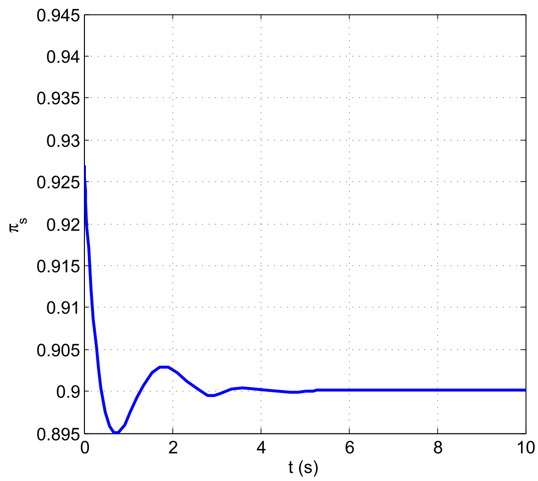


FIGURE 5. Extraction pressure $\pi_s(t)$.

The first set of tests concerning system (38) are illustrated by Figs. 4–7; the lumped variables π_1 and π_s are represented, as well as the boundary values of the Riemann invariants $\xi^{-}(0, t)$ and $\xi^{-}(1, t)$. *The basic qualitative results are confinement to the invariant sets (positiveness) and convergence of the transients to the constant steady states (equilibria) imposed by the constant inputs.*

We also note the oscillatory transients induced by the distributed parameters and the aforementioned internal feedback.

Next, Figs. 8–9 illustrate the transients in two dimensions of the Riemann invariants $\xi^{\pm}(\lambda, t)$. The process is oscillatory with respect to t and non-oscillatory with respect to λ: in fact $\xi^{\pm}(\lambda, t)$ approach quite quickly some quasi-constant - with respect to λ - values $\tilde{\xi}^{\pm}(t)$ and then they approach the steady states $\bar{\xi}^{\pm}$.

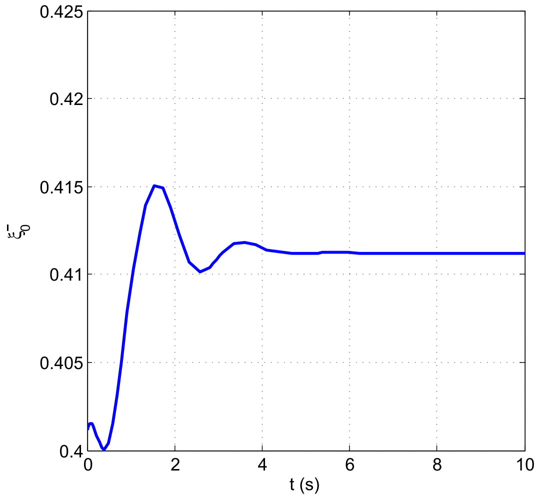


FIGURE 6. Time evolution of the backward wave at the end $\lambda = 0$.

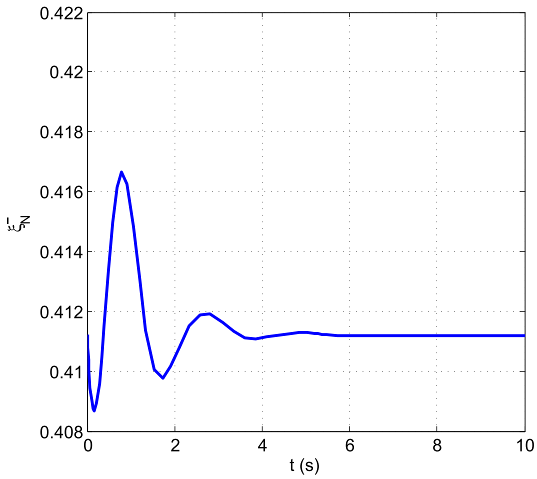


FIGURE 7. Time evolution of the backward wave at the end $\lambda = 1$.

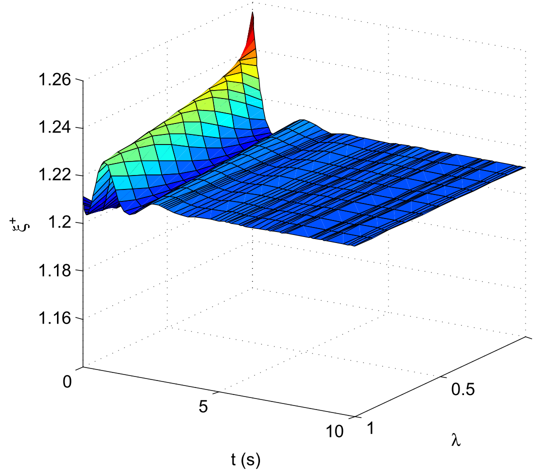


FIGURE 8. Space-time representation of the forward wave $\xi^+(\lambda, t)$.

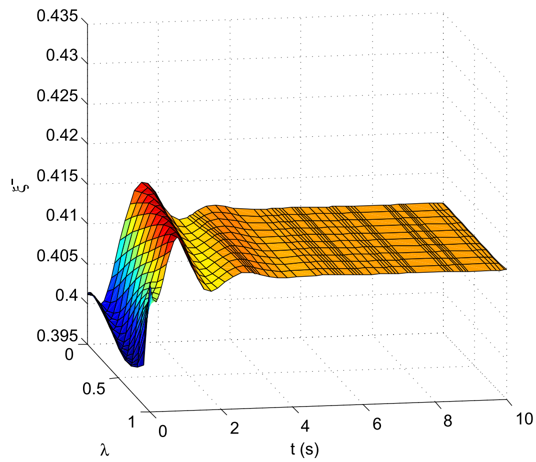


FIGURE 9. Space-time representation of the backward wave $\xi^-(\lambda, t)$.

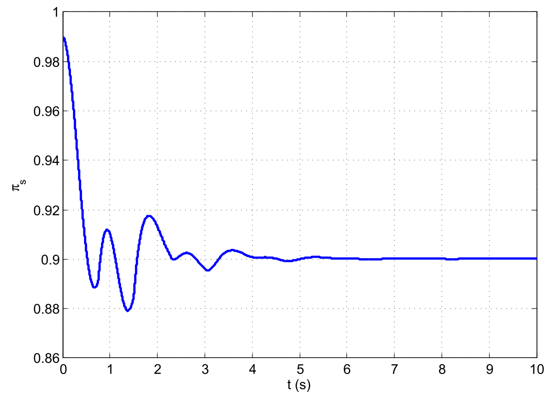


FIGURE 10. Extraction pressure $\pi_s(t)$ in the associated system of functional equations (39).

The following set of tests concerning system (39) are illustrated by Figs. 10–12, where we chose to figure $\pi_s(t)$ and $y^{\pm}(t)$, $-\psi_c T_c \leq t \leq t_1$ with t_1=15 sec. The state variable $\pi_1(t)$ was left aside since it does not interact with other state variables but just smoothes the constant input $\bar{\mu}_1$ which acts like a step signal for $t > 0$. The initial conditions for $y^{\pm}(t)$ on $(-\psi_c T_c, 0)$ were taken oscillatory, more precisely sinusoidal.

This way, an oscillatory behavior has been induced in the system aiming to test the inherent stability of the subsystem describing the state variables $(\pi_s(t), y^{\pm}(t))$. This inherent stability is confirmed, see oscillation quenching in figures 10–12.

From the mathematical point of view, this inherent stability is a consequence of (36) - the negative root of the factor accounting for the equation of π_s and the strongly stable difference operator resulting from the inequalities $0 < \rho_1 < 1$.

An interesting comment to be given concerns the peaks of figures 11 and 12. Due to the one-to-one correspondence between the solutions of the boundary value problem described by (38) and the system of FDEs described by (39) (Theorem 1), the discontinuities of the difference subsystem in (39) are projected back onto the solutions of (38). It has been stated from the beginning that the solutions of both

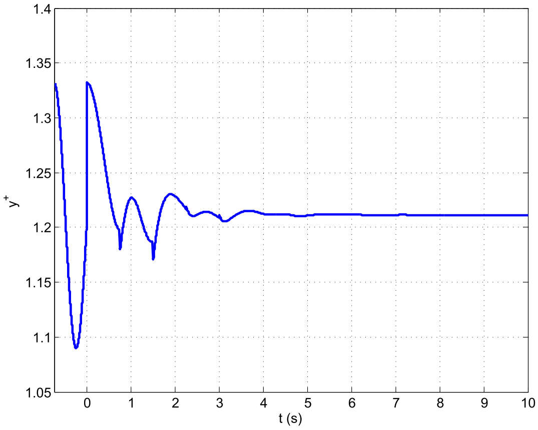


FIGURE 11. Representation function of the forward wave $y^+(t)$ in the associated system of functional equations (39).

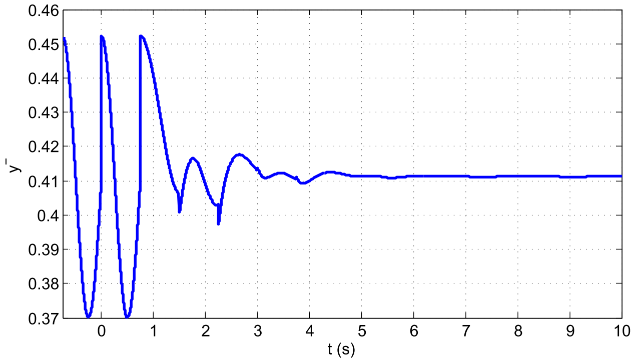


FIGURE 12. Representation function of the backward wave $y^-(t)$ in the associated system of functional equations (39).

mathematical objects can be discontinuous; the discontinuities of the difference subsystem occur at $t = k\psi_c T_c$ for k - an integer - see (23) - and they are generated by the discontinuity at 0. In turn, this discontinuity is a consequence of the mismatch between the boundary and the initial conditions - see (24). We have $\psi_c T_c = 0.75\ s$ and one can see from figures 11 and 12 the "peaks" occurring at $0.75, 1.5, \ldots$ Obviously their amplitude is damped since $0 < \rho_1 < 1$ ($\rho_1 \approx (1 - 0.387\sqrt{2})(1 + 0.387\sqrt{2})^{-1} = 0.2926$).

On the other hand, if one is seeking for a physical explanation, the boundary value problem described by (38) has to be considered. A careful inspection of its equations shows that, while the PDEs describe lossless propagation, the boundary conditions are "of the dissipative type" [58]. Therefore, oscillation quenching appears as a consequence of the aforementioned dissipation at the boundaries (which are also nonlinear - at least one of them); the nonlinear character of the boundaries is reflected in the shape of the damped oscillations - see figures 6-7 and 10-12.

As an overall, summarizing conclusion, the considered simulation results account for inherent stability but also for the connection between the solutions of the two mathematical objects (38) and (39); they even show the discontinuity propagation due to mismatching.

V. A STABILIZING FEEDBACK STRUCTURE

We have just shown that system (35) and, therefore, system (12) has an equilibrium which is but only stable, not asymptotically stable. Moreover, the equations (28) of the equilibrium give no information about the value of s at equilibrium. Normally, $\bar{s}$ i.e. the rotating speed of the steam turbine must equal the synchronous frequency of the Grid. Moreover, power engineers know that control of the rotating speed means also control of the active power exchanges within the Grid. For this reason the control structure must contain a rotating speed feedback component. Another aspect is given by the two time scales structure of (12) and, therefore, of (35). It has already been mentioned in Section II that the time constants T_1, T_2 are considered small with respect to T_a, T_p (and, here, even with respect to T_c) and neglected; consulting various references on steam turbine control e.g. [23], [28], [31] one can see that for small turbines the dynamics of the steam volumes enclosed in the turbine cylinders are not even mentioned.

It is but obvious that the reduction of the small T_1 and T_2 - done for (1) and leading to (4) - can be done for all its "avatars" (5), (8) and (12) as well as for the associated system of functional equations (18). The reduced form of (18) has the form

$$
\begin{aligned}
T_a\frac{\mathrm{d}s}{\mathrm{d}t} &= \alpha\mu_1(t) + (1-\alpha)\mu_2(t)\pi_s - \nu_g,\\
T_p\frac{\mathrm{d}\pi_s}{\mathrm{d}t} &= \mu_1(t) - \mu_2(t)\pi_s\\
&\quad + (\beta/\psi_c)g_p(y^-(t-\psi_cT_c)/\pi_s(t))\pi_s,\\
y^+(t) &= y^-(t-\psi_cT_c) + 2g_p(y^-(t-\psi_cT_c)/\pi_s(t))\pi_s(t),\\
y^-(t) &= \rho_1 y^+(t-\psi_cT_c).
\end{aligned}
\tag{40}
$$

We also mention that Theorems 1 and 2 can be adapted and proven for the reduced systems as well. It is thus possible to focus from now on system (40) whose equilibria coincide with those of (18); this last statement follows at once from the fact that the steady state equations for (40) can be obtained from those for (18) i.e. from (28) by taking into account that $\bar{\pi}_1 = \bar{\mu}_1$, $\bar{\mu}_2\bar{\pi}_s = \bar{\pi}_2$, now a consequence of taking $T_1 = T_2 = 0$.

The system in deviations deduced from (40) will be

$$
\begin{aligned}
T_a\frac{\mathrm{d}s}{\mathrm{d}t} &= \alpha u_1(t) + (1-\alpha)\bar{\mu}_2\zeta_s + (1-\alpha)(\pi_s^0+\zeta_s)u_2(t),\\
T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t} &= u_1(t) - (\bar{\mu}_2 + (\beta/\psi_c)g_p(\bar{y}^-/\pi_s^0))\zeta_s\\
&\quad - (\pi_s^0+\zeta_s)u_2(t) - (\beta/\psi_c)\times(\pi_s^0+\zeta_s)\\
&\quad \times\left[g_p\left(\frac{\bar{y}^- + \zeta^-(t-\psi_cT_c)}{\pi_s^0+\zeta_s(t)}\right) - g_p(\bar{y}^-/\pi_s^0)\right],\\
\zeta^+(t) &= \zeta^-(t-\psi_cT_c) + 2g_p(\bar{y}^-/\pi_s^0)\zeta_s(t) + 2(\pi_s^0+\zeta_s)\\
&\quad \times\left[g_p\left(\frac{\bar{y}^- + \zeta^-(t-\psi_cT_c)}{\pi_s^0+\zeta_s(t)}\right) - g_p(\bar{y}^-/\pi_s^0)\right],\\
\zeta^-(t) &= \rho_1\zeta^+(t-\psi_cT_c).
\end{aligned}
\tag{41}
$$

The characteristic equation of the linear subsystem will now be

$$T_a\lambda(T_p\lambda+\bar{\mu}_2+(\beta/\psi_c)g_p(\bar{y}^-/\pi_s^0))(1-\rho_1\mathrm{e}^{-2\lambda\psi_cT_c})=0 \tag{42}$$

and can be obtained either directly or by taking $T_1=T_2=0$ in (36). Consequently, (42) will have a simple zero root, a simple root in $\mathbb{C}^-$ and the same infinite chain of simple roots in $\mathbb{C}^-$ given by (37). In fact two roots in $\mathbb{C}^-$ were "reduced", located relatively far away from the imaginary axis $\imath\mathbb{R}$; these roots show that the boundary layer dynamics (i.e. the fast dynamics) is asymptotically (even exponentially) stable as required by the standard theory of the singular perturbations - see e.g. [59].

In the following we shall focus on feedback stabilization for (41). Observe however that for $t>\psi_cT_c$ the second difference equation can be taken into account in the other ones to obtain the following system which is valid for $t>\psi_cT_c$.

$$\begin{aligned}
T_a\frac{\mathrm{d}s}{\mathrm{d}t}&=\alpha u_1(t)+(1-\alpha)\bar{\mu}_2\zeta_s+(1-\alpha)(\pi_s^0+\zeta_s)u_2(t),\\
T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t}&=u_1(t)-(\bar{\mu}_2+(\beta/\psi_c)g_p(\rho_1\bar{y}^+/\pi_s^0))\zeta_s\\
&\quad-(\pi_s^0+\zeta_s)u_2(t)-(\beta/\psi_c)\times(\pi_s^0+\zeta_s)\\
&\quad\times\left[g_p\left(\rho_1\frac{\bar{y}^++\zeta^+(t-2\psi_cT_c)}{\pi_s^0+\zeta_s(t)}\right)-g_p(\rho_1\bar{y}^+/\pi_s^0)\right],\\
\zeta^+(t)&=\rho_1\zeta^+(t-2\psi_cT_c)+2g_p(\rho_1\bar{y}^+/\pi_s^0))\zeta_s(t)+2(\pi_s^0+\zeta_s)\\
&\quad\times\left[g_p\left(\rho_1\frac{\bar{y}^++\zeta^+(t-2\psi_cT_c)}{\pi_s^0+\zeta_s(t)}\right)-g_p(\rho_1\bar{y}^+/\pi_s^0)\right].
\end{aligned}\tag{43}$$

Following our experience in controlling cogeneration [4], [33], [50], [52], we associate to (43) the following Lyapunov functional $\mathcal{V}:\mathbb{R}\times\mathbb{R}\times L^2(-2\psi_cT_c,0;\mathbb{R})\mapsto\mathbb{R}_+$ defined by

$$\mathcal{V}(s,\zeta_s,\phi(\cdot))=\frac{1}{2}T_a\mathcal{L}(s,\zeta_s,\phi(\cdot))^2+\frac{1}{2}\delta_3T_p\zeta_s^2+\delta_4\int_{-2\psi_cT_c}^0\phi^2(\theta)\mathrm{d}\theta, \tag{44}$$

where the linear form $\mathcal{L}:\mathbb{R}\times\mathbb{R}\times L^2(-2\psi_cT_c,0;\mathbb{R}\mapsto\mathbb{R}$ is defined by

$$\mathcal{L}(s,\zeta_s,\phi(\cdot))=s+\frac{T_p}{T_a}\left(\delta_1\zeta_s+\frac{\delta_2}{T_p}\int_{-2\psi_cT_c}^0\phi(\theta)\mathrm{d}\theta\right) \tag{45}$$

with the real δ_1,δ_2 and the positive $\delta_3>0,\delta_4>0$ being free parameters.

The aforementioned experience showed that this quadratic functional is well suited to stabilize bilinear systems. Therefore we shall "bi-linearize" system (43) around the equilibrium. This means taking the linear terms in the expansion of

$$\begin{aligned}
f_p(\zeta_s,\zeta^+)&=(\pi_s^0+\zeta_s)\left[g_p\left(\rho_1\frac{\bar{y}^++\zeta^+}{\pi_s^0+\zeta_s}\right)-g_p(\rho_1\bar{y}^+/\pi_s^0)\right]\\
&=\rho_1g_p'(\rho_1\bar{y}^+/\pi_s^0)\left[-\frac{\bar{y}^+}{\pi_s^0}\zeta_s+\zeta^+\right]+o(|\zeta_s|+|\zeta^+|)
\end{aligned}$$

to obtain the following system

$$\begin{aligned}
T_a\frac{\mathrm{d}s}{\mathrm{d}t}&=\alpha u_1(t)+(1-\alpha)\bar{\mu}_2\zeta_s+(1-\alpha)(\pi_s^0+\zeta_s)u_2(t),\\
T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t}&=u_1(t)-(\bar{\mu}_2+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))\zeta_s\\
&\quad-(\pi_s^0+\zeta_s)u_2(t)-(\beta/\psi_c)\rho_1g_p'(\bar{x})\zeta^+(t-2\psi_cT_c),\\
\zeta^+(t)&=2(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))\zeta_s(t)\\
&\quad+\rho_1(1+2g_p'(\bar{x}))\zeta^+(t-2\psi_cT_c),
\end{aligned}\tag{46}$$

where we denoted $\bar{x}=\rho_1\bar{y}^+/\pi_s^0$. Since the linearization was done around the equilibrium $\bar{x}$ that is on the middle segment of (9), we shall have $g_p(\bar{x})>0$, $g_p'(\bar{x})<0$. We can first check that the inherent stability of (46) is non-asymptotic: by letting $u_1(t)=u_2(t)\equiv0$ the bilinear part is set to 0 and the characteristic equation of the remaining linear part is

$$\begin{aligned}
&T_a\lambda[(T_p\lambda+(\bar{\mu}_2+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x})))\\
&\quad\times(1-\rho_1(1+2g_p'(\bar{x}))\mathrm{e}^{-2\lambda\psi_cT_c})\\
&\quad+2(\beta/\psi_c)\rho_1g_p'(\bar{x})(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))\mathrm{e}^{-2\lambda\psi_cT_c}]=0.
\end{aligned}\tag{47}$$

Equation (47) has a simple zero root. The quasi-polynomial of (47) takes the form

$$\begin{aligned}
p(\lambda)&=(1-\rho_1(1+2g_p'(\bar{x})))\mathrm{e}^{-2\lambda\psi_cT_c})(T_p\lambda+\bar{\mu}_2)\\
&\quad+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))(1-\rho_1\mathrm{e}^{-2\lambda\psi_cT_c})
\end{aligned}$$

and, with the new variable $z:=\lambda\psi_cT_c$, it can be written as

$$\mathrm{e}^zp\left(\frac{z}{\psi_cT_c}\right)=(a_1z+a_0)\cosh\ z+(b_1z+b_0)\sinh\ z \tag{48}$$

In (48) we denoted

$$\begin{aligned}
a_1&=\frac{T_p}{\psi_cT_c}(1-\rho_1(1+2g_p'(\bar{x})));\\
b_1&=\frac{T_p}{\psi_cT_c}(1+\rho_1(1+2g_p'(\bar{x})))\\
a_0&=\bar{\mu}_2(1-\rho_1(1+2g_p'(\bar{x})))\\
&\quad+\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))(1-\rho_1)\\
b_0&=\bar{\mu}_2(1+\rho_1(1+2g_p'(\bar{x}))\\
&\quad+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))(1+\rho_1).
\end{aligned}\tag{49}$$

Taking into account (9), we find that $a_0>0$, $b_0>0$, also $a_1>0$, $b_1>0$. Therefore the necessary and sufficient conditions given in [60], page 99 are fulfilled hence the roots of $p(\lambda)$ are in $\mathbb{C}^-$. It is worth mentioning that the inequalities above are also based on the fact that $0<\rho_1<1$, accounting for the strong stability of the difference operator of all systems of functional differential equations throughout the paper, these systems being of neutral type.

The aforementioned stabilizing feedback structure will be synthesized as follows. The quadratic Lyapunov functional (44) is differentiated along the solutions of (46); after some simple manipulations the derivative function can be written as follows

$$\begin{aligned}
&\mathcal{W}(s,\zeta_s,\phi(\cdot))\\
&= \mathcal{L}(s,\zeta_s,\phi(\cdot))[(1-\alpha)\bar{\mu}_2\zeta_s\\
&\quad -\delta_1(\bar{\mu}_2+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x})))\zeta_s\\
&\quad -\delta_1(\beta/\psi_c)\rho_1 g_p'(\bar{x})\phi(-2\psi_c T_c)+2\delta_2(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))\zeta_s\\
&\quad +\delta_2(\rho_1-1+2\rho_1 g_p'(\bar{x}))]\\
&\quad +[(\alpha+\delta_1)\mathcal{L}(s,\zeta_s,\phi(\cdot))+\delta_3\zeta_s]u_1(t)\\
&\quad +[(1-\alpha-\delta_1)\mathcal{L}(s,\zeta_s,\phi(\cdot))-\delta_3\zeta_s](\pi_s^0+\zeta_s)u_2(t)\\
&\quad -\mathcal{Q}(\zeta_s,\phi(-2\psi_c T_c))
\end{aligned}\tag{50}$$

where the quadratic form $\mathcal{Q}(\zeta_s,\phi(-2\psi_c T_c))$ is given by

$$\begin{aligned}
&\mathcal{Q}(\zeta_s,\phi(-2\psi_c T_c))\\
&\quad =(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))[\delta_3(\bar{\mu}_2+\beta/\psi_c)\\
&\qquad -4\delta_4(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))]\zeta_s^2\\
&\qquad +[\delta_3(\beta/\psi_c)\rho_1 g_p'(\bar{x})-4\delta_4\rho_1(1+2g_p'(\bar{x}))\\
&\qquad \times(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))]\zeta_s\phi(-2\psi_c T_c)\\
&\qquad +\delta_4(1-\rho_1^2(1+2g_p'(\bar{x}))^2)\phi(-2\psi_c T_c)^2.
\end{aligned}\tag{51}$$

All necessary information that can be obtained from (50) and (51) is a consequence of the choice of the free parameters δ_k, $k=1,2,3,4$. Observe first that $\mathcal{L}(s,\zeta_s,\phi(\cdot))$ depends on s while the linear form multiplying it does not. To ensure $\mathcal{W}$ a definite sign, the linear form mentioned above must vanish. The choice

$$\begin{aligned}
\bar{\delta}_2 &= -\frac{(\beta/\psi_c)\rho_1 g_p'(\bar{x})}{1-\rho_1(1+2g_p'(\bar{x}))}\bar{\delta}_1\\
\bar{\delta}_1 &= \frac{(1-\alpha)\bar{\mu}_2}{\bar{\mu}_2-(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))}
\end{aligned}\tag{52}$$

ensures this vanishing. Next, the choice of the control signals as follows

$$\begin{aligned}
u_1 &= -\mathrm{Sat}\{k_1[(\alpha+\bar{\delta}_1)\mathcal{L}(s,\zeta_s,\phi(\cdot))+\delta_3\zeta_s]\},\\
-\bar{\mu}_1 &\le u_1(t)\le 1-\bar{\mu}_1\\
u_2 &= -\mathrm{Sat}\{k_2[(1-\alpha-\bar{\delta}_1)\mathcal{L}(s,\zeta_s,\phi(\cdot))-\delta_3\zeta_s]\},\\
\beta-\bar{\mu}_2 &\le u_2(t)\le 1-\bar{\mu}_2
\end{aligned}\tag{53}$$

will make the corresponding terms in (50) at least non-positive.

The third step is to consider the quadratic form $\mathcal{Q}(\zeta_s,\phi(-2\psi_c T_c))$. A rather straightforward (while tedious) manipulation shows that if the positive ratio δ_4/δ_3 is chosen smaller than the positive root of the equation

$$\begin{aligned}
&(2-\rho_1^2(1+2g_p'(\bar{x}))^2)X^2-(\beta/\psi_c)(1-\rho_1^2(1+2g_p'(\bar{x}))^2\\
&\qquad +2\rho_1 g_p'(\bar{x}))X+((\beta/\psi_c)\rho_1 g_p'(\bar{x}))^2=0
\end{aligned}\tag{54}$$

then

$$\mathcal{Q}(\zeta_s,\phi(-2\psi_c T_c))\ge\varepsilon_0(|\zeta_s|^2+|\phi(-2\psi_c T_c)|^2)\tag{55}$$

for some $\varepsilon_0>0$ sufficiently small. The overall result will be

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} &= \mathcal{W}(s(t),\zeta_s(t),\zeta^+(t-2\psi_c T_c))\\
&\le -\varepsilon_0(|\zeta_s(t)|^2+|\zeta^+(t-2\psi_c T_c)|^2).
\end{aligned}\tag{56}$$

From here the Lyapunov stability follows. Since system (46) has an infinite dimensional state space, the norm is important: stability is obtained in the sense of the norm induced by the Lyapunov functional (44) - (45). This stability is even asymptotic taking into account that the closed loop system (46) with the control functions chosen from (53) satisfies the structural conditions allowing application of the Barbashin-Krasovskii-LaSalle invariance principle for neutral functional differential-difference equations (Theorem 9.8.2 of [49]). It follows that the bilinear closed loop system

$$\begin{aligned}
T_a\frac{\mathrm{d}s}{\mathrm{d}t} &= (1-\alpha)\bar{\mu}_2\zeta_s+\alpha u_1(s,\zeta_s,\zeta_t^+(\cdot))\\
&\quad +(1-\alpha)(\pi_s^0+\zeta_s)u_2(s,\zeta_s,\zeta_t^+(\cdot))\\
T_p\frac{\mathrm{d}\zeta_s}{\mathrm{d}t} &= -(\bar{\mu}_2+(\beta/\psi_c)(g_p(\bar{x})-\bar{x}g_p'(\bar{x})))\zeta_s\\
&\quad -(\beta/\psi_c)\rho_1 g_p'(\bar{x})\zeta^+(t-2\psi_c T_c)+u_1(s,\zeta_s,\zeta_t^+(\cdot))\\
&\quad -(\pi_s^0+\zeta_s)u_2(s,\zeta_s,\zeta_t^+(\cdot))\\
\zeta^+(t) &= 2(g_p(\bar{x})-\bar{x}g_p'(\bar{x}))\zeta_s(t)\\
&\quad +\rho_1(1+2g_p'(\bar{x}))\zeta^+(t-2\psi_c T_c)
\end{aligned}\tag{57}$$

is globally asymptotically stable. This asymptotic stability might be even exponential provided a Persidskii type result [54] were true for neutral functional equations (linear or non-linear) as it is the case for delay differential equations [61].

In the following we shall discuss the specific features, the advantages and the drawbacks of the proposed control structure (53) where $\bar{\delta}_1$, $\bar{\delta}_2$ are taken from (52) and the linear form $\mathcal{L}(s,\zeta_s,\phi(\cdot))$ is defined in (45), with $\delta_2=\bar{\delta}_2$. If we refer to the standard control for steam turbines with one regulated extraction but without a long steam conduit (pipe) leading to the thermal consumer, the controller structure contains two linear controllers driven by the control errors of the rotating speed and the steam extraction pressure, in our notations - the state variables s and ζ_s. Due to position limiters of the controlled valves, the controllers are saturated. These controllers can be independent or coupled to achieve e.g. noninteracting control. The book [28] is a good classical reference in this field. The aforementioned controller structure is, generally speaking, stabilizing - at the level of the linearized models. Bilinear models were considered afterwards in order to obtain a better fitting to the consumption diagrams data. It was thus encouraging to re-discover the same structure (with linear saturated connected controllers) in the bilinear case, rigorously designed starting from a suitably found ("guessed") Lyapunov function [22]. Consequently, this structure offered the property of global asymptotic stability - even exponential stability, due to a nonlinear extension of K. P. Persidskii theorem [54].

The same approach was taken in this paper for the conduit (steam pipe) with distributed parameters. We started from a suitably "guessed" Lyapunov functional and obtained a once again linear feedback structure with saturation but with an additional term of distributed feedback. It is worth clarifying this feedback term, starting from (45). The generic $\phi(\cdot)$ accounts for the state variable $\zeta^+(\cdot)$ in the last (difference) equation of (43), defined on $(-2\psi_c T_c, 0)$. We have, from the second difference equation of (41)

$$\int_{-2\psi_c T_c}^{0} \zeta^+(\theta)\mathrm{d}\theta = \int_{-\psi_c T_c}^{0} [\zeta^+(\theta) + (1/\rho_1)\zeta^-(\theta)]\mathrm{d}\theta.$$

Using the representation formulae (17) and the definition (6) of the Riemann invariants we obtain that the distributed feedback is a feedback with respect to the average of a linear combination steam pressure/steam flow along the pipe

$$\int_{-\psi_c T_c}^{0} [\zeta^+(t+\theta) + (1/\rho_1)\zeta^-(t+\theta)]\mathrm{d}\theta = \frac{\psi_c T_c}{1-\psi_s\sqrt{2}} \int_0^1 (\zeta_\rho(\lambda, t) - \psi_s\sqrt{2}\zeta_w(\lambda, t))\mathrm{d}\lambda.$$

We did not see any such controller proposal in cogeneration. *Its basic advantage is global asymptotic stability for the closed loop*, deduced from the properties of the Lyapunov functional (44) and of its derivative - see (50) and (51). Implementation of the distributed term is a practical problem - to implement a finite memory integral - but preservation of stability under discretization is a theoretical one. One might conjecture necessity of a sufficiently small discretizing step but a sound scientific basis of this conjecture is obviously required.

To end this discussion let us mention that the feedback control functions contain some free parameters as $\delta_3 > 0$ (only the ratio $\delta_4/\delta_3 > 0$ is restricted by some inequality hence there is enough freedom for choice) or the newly introduced gains $k_i > 0$, $i = 1, 2$. These free parameters can still be chosen to fulfill other control requirements besides stability (e.g. overshoot, settling time, steady state error, non-interacting control at least at the static level). In the case of the lumped parameters, the practical experience can offer enough information (procedures, data) for free parameter choice. With the additional integral term in our case, it is difficult to offer suitable know-how for the aforementioned parameter choice. Moreover, the care to avoid de-stabilization is also necessary. Other theoretical and applied aspects will be discussed in the following section.

VI. PERSPECTIVE: CONCLUSIONS AND OPEN PROBLEMS

We start by summarizing the previous sections of the paper. A dynamical system describing the transients and the steady states of a cogeneration system model was considered. This dynamical system relies on a system of conservation laws with nonlinear boundary conditions. The boundary conditions are non-standard due to their connection in an internal feedback with a bilinear system of ODEs describing the dynamics of the steam turbine. The model is assembled from various sources and under various assumptions. Such circumstances implied a necessary model validation. We undertook the so-called *augmented validation* [19] which includes the basic theory as summarized in [18] - existence, uniqueness and continuous data dependence (together known as well-posedness in the sense of Hadamard) - but also existence of certain invariant sets (here - positiveness of some state variables accounting for pressures) and fulfillment of the Stability Postulate of Četaev - inherent stability of steady states, in our case - equilibria.

We used the approach summarized in [19]: to associate a system of functional differential equations to the initial one, where the conservation laws were linearized to become linear hyperbolic PDEs and establish a one-to-one correspondence between the solutions of the two mathematical objects.

Further, taking into account that the inherent stability of the equilibria is at most non-asymptotic, feedback stabilization has been considered. The engineering idea of stabilization was the standard one in power engineering and recommended since the very first references for steam turbine control [62] and for the control of steam turbines with regulated steam extractions [63]: to control the steam turbine - rotating speed and the steam pressure at the steam extraction. The approach was to neglect the fast dynamics due to small time constants and synthesize the feedback control using a suitably constructed control Lyapunov functional. The closed loop system of first approximation - a bilinear (not linear!) system - was shown to possess an asymptotically stable equilibrium.

From now on, several theoretical as well as practical problems appear as yet unsolved. Consider first the theoretical problems. The closed loop system (57) is a bilinear system of first approximation describing the slow transients. But the control signals (53) act upon e.g. the nonlinear system (43). Therefore a theorem of stability by the first approximation is necessary to prove the asymptotic stability of the equilibrium of the aforementioned nonlinear system (43). Theorems on stability by the first approximation (with both the system of first approximation and the basic one - nonlinear) for systems of ODEs are to be found in [54]. For systems of equations with deviated arguments both of delayed or neutral type very little is known: a quite recent reference might be [64], [65]. We suggest use of comparison theorems.

The aforementioned problems arising from the first approximation by a bilinear system have been pointed out by a rather mathematical approach and/or point of view. There exists however another point of view - that *of the control engineer*. Its basics is *robustness as preservation of system's properties with respect to uncertainties*. It is worth mentioning that the feedback structure itself is an approach to robustness as pointed out in [66], 4th chapter "Feedback and oscillation". Robustness became an important issue in control later, in the 70's of the past century: an interesting survey, full of philosophical complements, is [67]. Dealing with robustness starts with the definition of the uncertainties. Linear control theory deals with structured and unstructured uncertainties. The bounded unstructured uncertainties in time domain can be considered in the case of nonlinear systems as well; another such class of uncertainties is the class of uncertainties subject to quadratic inequalities.

The essential difference (according to our point of view) between stability by the first approximation and robustness approaches is that, unlike the first approach, the second one can give results valid globally (not in finite state space domains only). A far going case is the one of the *absolute stability* (a.k.a. systems of Lurie type or systems with sector restricted nonlinearity). Since the very first paper on the subject [68] it became clear that in many control engineering problems the information on the nonlinear actuator is rather poor. The idea was to "embed" the nonlinear function in a class of such functions and to obtain stability criteria valid for the entire class of nonlinear functions; this was called *absolute stability*. Two competing approaches of the problem followed: the use of a special Lyapunov function of the type "quadratic form of the state variables plus the integral of the nonlinear function", which started with [69] and the frequency domain inequalities approach which started with the early paper of Popov [70]. The perfect equivalence of the two approaches was established later *via* the Positiveness (Yakubovich-Kalman-Popov) Lemma. On the other hand, it was discovered by Yakubovich, Brockett, J. L. Willems, G. Zames and P. L. Falb that more information concerning the nonlinear function (e.g. monotonicity, slope restrictions, parity etc) is known, less restrictive the frequency domain inequality becomes. This relaxation of the frequency domain inequality is achieved *via* the *multipliers* of the transfer function of the linear subsystem: no multiplier for the circle criterion, the non-causal multiplier $(\alpha + \beta s)$ for the Popov criterion, Yakubovich multiplier for the slope restriction criterion, Brockett-Willems and Zames-Falb multipliers for the case of the monotone nonlinearities etc.

At the same time, it was pointed out that the Lurie type sector restrictions can be expressed as quadratic constraints on the input and output variables of the nonlinearity and also that a lot of known information about the nonlinear functions can be "encoded" in some (static or dynamic) quadratic constraints [71]–[73]. Later, the time/frequency duality in the Fourier theory led to an augmentation of the quadratic restrictions list by including quadratic restrictions in the frequency domain [74]. All following development strongly relies on the complementarity of quadratic restrictions and multipliers. A recent reference is [75].

Another aspect concerns the neglected fast dynamics. As already mentioned, small time constants generate a singularly perturbed system e.g. (34). It is generally admitted (and, in certain cases rigorously proven) that if both the fast and the slow systems have stable equilibria then the equilibrium of the overall system is also stable. The proofs exist for systems of ODEs and systems with deviated argument of delayed type. For systems of neutral type such as (34) we do not know results of this type.

To end the open problems with mathematical character we shall mention exponential stability. Exponential stability is normally the type of stability mostly required in applications since it provides some estimation of the speed of perturbation quenching. For linear time invariant systems this property follows from the location of the roots of a characteristic equation. In the case of linear time varying systems it was proven by K. P. Persidskii - see e.g. [54] - that uniform asymptotic stability is always exponential. The result has been extended to a certain class of nonlinear systems of ODE (*op. cit.*) and to systems of equations with deviated argument of delayed type in [61]. For neutral equations the result is not known.

The practical (applied) problems are mainly concerned with implementation of the control signals (53): if s and ζ_s - the rotating speed deviation and the steam pressure deviation respectively - are known as measurable for feedback implementation, the linear form $\mathcal{L}(s, \zeta_s, \zeta_t^+(\cdot))$ contains, besides the aforementioned variables, an integral which is the average of a distributed state variable $\zeta_t^+(\sigma) := \zeta^+(t+\sigma)$ for $-2\psi_c T_c \leq t \leq 0$. Reintroducing $\zeta^-(t+\sigma)$ under the integral of $\mathcal{L}(s, \zeta_s, \zeta_t^+(\cdot))$ *via* the second difference equation of (8), then using the representation formulae (22) and (6) the aforementioned integral takes the form

$$\int_{-2\psi_c T_c}^{0} \zeta^+(t+\sigma)\mathrm{d}\sigma = \frac{2\psi_c T_c}{1-\psi_s\sqrt{2}} \int_0^1 (\zeta_\rho(\lambda, t) - \psi_s\sqrt{2}\zeta_w(\lambda, t))\mathrm{d}\lambda, \quad (58)$$

being now expressed in terms of the deviations of the steam pressure and flow along the distributed parameter pipe. The suitably discretized integral will lead to a controller with finite memory. The closed loop system will get a rather non-standard form. For this system the entire validation procedure has to be re-started. To it one has to add stability preservation. The inferred property would be the requirement of a sufficiently small discretizing step.

Summarizing, we have sketched an entire research program which is felt to be somehow complementary to the research on conservation laws control.

APPENDIX 1

Proof of Theorem 1: We have firstly

$$\pi_1(t) = \mathrm{e}^{-t/T_1}\pi_1(0) + \frac{1}{T_1}\int_0^t \mathrm{e}^{-(t-\tau)/T_1}\mu_1(\tau)\mathrm{d}\tau \quad (59)$$

with $0 \leq \mu_1(t) \leq 1$. Let $\pi_1(0) = 0$ and $\mu_1(t) \equiv 0$ on $(0, t)$; therefore $\pi_1(t) \equiv 0$. If $\pi_1(0) > 0$ and $\mu_1(t) \equiv 0$ on $(0, t)$, then $\pi_1(t) > 0$. If $\mu_1(t) \not\equiv 0$ on $(0, t)$, then $\pi_1(t) > 0$ regardless the value of $\pi_1(0)$.

Consider now the equation for π_s and assume first that $\pi_s(0) > 0$; since $\pi_1(t)$ and $\xi_w(0, t)$ are continuous, there will exist a sufficiently small interval $(0, \hat{t})$ such that $\pi_s(t) > 0$ on this interval. It follows that on it we can define

$$\Phi^\star(t) := \Phi(\xi_\rho(0, t)/\pi_s(t)) \geq 0 \quad (60)$$

hence

$$T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1(t) - (\mu_2(t) + \Phi^\star(t))\pi_s, \quad \mu_2(t) + \Phi^\star(t) > 0. \tag{61}$$

Since in any case $\pi_1(t) \geq 0$ the representation formula for the solution of the first order differential equation will give

$$\pi_s(t) = \exp\left\{-\int_0^t \alpha(\tau)\mathrm{d}\tau\right\}\left(\pi_s(0) + (1/T_p) \times \int_0^t \exp\left\{\int_0^\tau \alpha(\lambda)\mathrm{d}\lambda\right\}\pi_1(\tau)\mathrm{d}\tau\right), \tag{62}$$

where $\alpha(t) := (1/T_p)(\mu_2(t) + \Phi^\star(t))$. Define the set

$$\mathcal{M} = \{t|\pi_s(t) > 0, 0 \leq \sigma < t\}. \tag{63}$$

From continuity it follows that this set is non-void and, if it does not coincide with $\mathbb{R}_+$, there will exist a finite $\theta = \sup\mathcal{M}$. Writing down (62) for $t = \theta$ we find $\pi_s(\theta) > 0$ and, from continuity, we shall find some $\Delta > 0$ sufficiently small such that $\pi_s(\theta + \Delta) > 0$ - a contradiction to the definition of θ which, therefore, cannot be finite. Therefore $\pi_s(t) > 0$ in this case.

Let now $\pi_s(0) \geq 0$ and let $\pi_s^\varepsilon(t)$ be the solution of (61) for $\pi_s^\varepsilon(0) = \pi_s(0) + \varepsilon$. From the aforementioned development it follows $\pi_s^\varepsilon(t) > 0$ for all $t > 0$. Letting $\varepsilon \to 0$, $\pi_s^\varepsilon(t) \to \pi_s(t)$ (from the theorem of continuous data dependence of the solution of ODEs); here $\pi_s(t) \geq 0$ for all $t > 0$ is the solution of (61) corresponding to $\pi_s(0)$. Let now $\pi_s(0) = 0$: for $\theta > 0$ as above we obtain the following alternative: either $\pi_s(t) > 0$ if $\pi_1(\tau) \geq 0$ for $0 \leq \tau < \theta$ or $\pi_s(t) \equiv 0$; indeed if $\pi_s(\theta) = 0$ then $\pi_1(\tau) \equiv 0$ for $0 \leq \tau < \theta$ hence $\pi_s(t) \equiv 0$ on that interval; finally, the solutions being analytical, the aforementioned property holds for all $t > 0$.

Consider now the equation

$$T_2 \frac{\mathrm{d}\pi_2}{\mathrm{d}t} = \mu_2(t)\pi_s(t) - \pi_2 = \varphi_2(t) - \pi_2 \tag{64}$$

which is similar to (59) hence the proof follows the same line. This concludes the proof to Theorem 1.

APPENDIX 2

Proof of Theorem 3: For $\pi_1(t)$ representation (59) holds and the proof of Appendix 1 is valid here also. Since (64) also holds for $\pi_2(t)$, the property of $\pi_2(t)$ holds provided it holds for $\pi_s(t)$. For what is left of the proof (its main part) we consider subsystem (21). Let $0 \leq \psi_c T_c$. On this interval the differential equation for π_s takes the form

$$T_p \frac{\mathrm{d}\pi_s}{\mathrm{d}t} = \pi_1(t) - (\mu_2(t) + (\beta/\psi_c)g_p^\star(t))\pi_s, \tag{65}$$

where $g_p^\star(t) = g_p(y_0^-(t - \psi_c T_c)/\pi_s(t))$. Clearly (65) is like (61) hence we shall have $\pi_s(t) > 0$ on $(0, \psi_c T_c)$ - see Appendix 1. We can then write the difference equations of (21) as

$$\frac{y^+(t)}{\pi_s(t)} = \frac{y_0^-(t - \psi_c T_c)}{\pi_s(t)} + 2g_p\left(\frac{y_0^-(t - \psi_c T_c)}{\pi_s(t)}\right)$$

$$\frac{y^-(t)}{\pi_s(t)} = \rho_1 \frac{y_0^+(t - \psi_c T_c)}{\pi_s(t)}, \quad 0 \leq t \leq \psi_c T_c. \tag{66}$$

From (11) we obtain quite straightforwardly that the RHSs (Right Hand Sides) of (66) are nonnegative hence $y^\pm(t)/\pi_s(t) \geq 0$ and $y^\pm(t) \geq 0$ on $(0, \psi_c T_c)$. The proof can be iterated on $(k\psi_c T_c, (k+1)\psi_c T_c$ and this concludes it.

REFERENCES

[1] E. Hopf, "The partial differential equation $u_t + uu_x = \mu u_x x$," *Commun. Pure Appl. Math.*, vol. 3, pp. 201–230, Sep. 1950.
[2] O. A. Oleinik, "On Cauchy's problem for nonlinear equations in a class of discontinuous functions (in Russian)," *Dokl. Akad Nauk SSSR (NS)*, vol. 95, no. 3, pp. 451–454, 1954.
[3] D. Serre, "Systems of conservation laws: A challenge for the XXIst century," in *Mathematics unlimited: Beyond*, B. Engquist and W. Schmid, Eds. Berlin, Germany: Springer, 2000, pp. 1061–1080.
[4] V. Răsvan, D. Danciu, and D. Popescu, "Time delay and wave propagation in controlling systems of conservation laws," in *Proc. 21st Int. Conf. Syst. Theory, Control Comput.* Piscataway, NJ, USA: IEEE Press, 2017.
[5] G. Leugering and J. P. G. Schmidt, "On the modelling and stabilization of flows in networks of open canals," *SIAM J. Contr. Optim.*, vol. 41, no. 1, pp. 164–180, 2002.
[6] P. O. Malaterre and B. P. Baume, "Modeling and regulation of irrigation canals: Existing applications and ongoing researches," in *Proc. Conf. IEEE Int. Conf. Syst., Man, Cybern. (SMC)*, vol. 4, Oct. 1998, pp. 3850–3855.
[7] J. de Halleux, C. Prieur, J.-M. Coron, B. D'Andréa-Novel, and G. Bastin, "Boundary feedback control in networks of open channels," *Automatica*, vol. 39, no. 8, pp. 1365–1376, Aug. 2003.
[8] C. Prieur, J. Winkin, and G. Bastin, "Boundary control of non-homogeneous systems of two conservation laws," in *Proc. 44th IEEE Conf. Decis. Control (CDC)*, Dec. 2005, pp. 1899–1904.
[9] J.-M. Coron and B. D'Andréa-Novel, and G. Bastin, "A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws," *IEEE Trans. Aut. Control*, vol. 52, no. 1, pp. 2–11, Jan. 2007.
[10] S. Blandin, X. Litrico, M. L. D. Monache, B. Piccoli, and A. Bayen, "Regularity and Lyapunov stabilization of weak entropy solutions to scalar conservation laws," *IEEE Trans. Aut. Control*, vol. 62, no. 4, pp. 1620–1635, Apr. 2017.
[11] M. Gugat, M. Dick, and G. Leugering, "Classical solutions and feedback stabilization for the gas flow in a sequence of pipes," *Netw. Heterogeneous Media*, vol. 5, no. 4, pp. 691–709, 2010.
[12] M. Gugat and M. Herty, "Existence of classical solutions and feedback stabilization for the flow in gas networks," *ESAIM, Control, Optim. Calculus Variat.*, vol. 17, no. 1, pp. 28–51, 2011.
[13] M. Gugat, M. Dick, and G. Leugering, "Gas flow in fan-shaped networks: Classical solutions and feedback stabilization," *SIAM J. Control Optim.*, vol. 49, no. 5, pp. 2101–2117, 2011.
[14] M. Krstić and A. S. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs* (Advances in Design and Control), vol. 16. Philadelphia, PA, USA: SIAM Publications, 2008.
[15] M. Gugat, *Optimal Boundary Control and Boundary Stabilization of Hyperbolic Systems* (SpringerBriefs in Electrical and Computer Engineering). Basel, Switzerland: Birkhäuser, 2015.
[16] G. Bastin and J. M. Coron, *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*. Basel, Switzerland: Birkhäuser, 2016.
[17] Y. I. Neymark, *Dynamical Systems and Controlled Processes* (in Russian). Moscow, Russia: Nauka, 1978.
[18] R. Courant, "Hyperbolic partial differential equations," in *Modern Mathematics for the Engineer: First Series*, E. Beckenbach, Ed. New York, NY, USA: McGraw Hill, 1956, pp. 92–109.
[19] V. Răsvan, "Augmented validation and a stabilization approach for systems with propagation," in *Systems Theory: Perspectives, Applications and Developments*, F. Miranda, Ed. New York, NY, USA: Nova Science Publishers, 2014, pp. 125–169.
[20] D. Danciu and V. Răsvan, "Delays and propagation: Control liapunov functionals and computational issues," in *Low-Complexity Controllers for Time-Delay Systems*, A. Seuret, C. Bonnet, H. Mounier, and H. Ozbay, Eds. Berlin, Germany: Springer, 2014, pp. 141–154.
[21] V. Răsvan, "The stability postulate of N.G. Cetaev and the augmented model validation," *IFAC PapersOnLine*, vol. 50, pp. 7450–7455, 2017.

[22] A. Halanay and V. Răsvan, "Stabilization of a class of bilinear control systems with applications to steam turbine regulation," *Tohoku Math. J.*, vol. 32, no. 2, pp. 299–308, 1980.

[23] T. J. Kearton, *Steam Turbine Theory and Practice*. London, U.K.: Pitman, 2004.

[24] V. Răsvan, "Stability of bilinear control systems occurring in combined heat electricity generation I: The mathematical models and their properties," *Revue Roumaine des Sci. Techn.-Serie Électrotechnique et Énergétique*, vol. 26, no. 3, pp. 455–465, 1981.

[25] I. P. Kabakov, "Concerning the control process for the steam pressure," (in Russian), *Inzh. Sbornik*, vol. 2, no. 2, pp. 27–60, 1946.

[26] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II. New York, NY, USA: Wiley, 1962.

[27] H. Pascal, *Écoulement Non Permanent Dans Les Gazoducs*. Paris, France: Éditions Technip, 1972.

[28] I. I. Kirillov, *Automatic Control of Steam and Gas Turbines* (in Russian). Moscow, Russia: Mashinostroenie, 1988.

[29] P. D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. Philadelphia, PA, USA: SIAM Publications, 1973.

[30] P. D. Lax, *Hyperbolic Partial Differential Equations*. New York, NY, USA: AMS, 2006.

[31] T. Force, "Dynamic models for turbine-governors in power system studies," IEEE Power Energy Soc., Piscataway, NJ, USA, Tech. Rep. PES-TR1-2013, 2013.

[32] H. Czichos and M. Hennecke, *Hütte-Das Ingenieurwissen*. Berlin, Germany: Springer, 2013.

[33] V. Răsvan, "Stability of bilinear control systems occurring in combined heat electricity generation II: Stabilization of the reduced models," *Rev. Roumaine Sci. Techn. Série Electrotechn. Energ.*, vol. 29, no. 4, pp. 423–432, 1984.

[34] Y. Tang, C. Prieur, and A. Girard, "Stability analysis of a singularly perturbed coupled ode-pde system," in *Proc. 54th IEEE Conf. Decision Control (CDC)*. Piscataway, NJ, USA: IEEE Press, Dec. 2015, pp. 4591–4596.

[35] Y. Tang, C. Prieur, and A. Girard, "Singular perturbation approximation of linear hyperbolic systems of balance laws," *IEEE Trans. Aut. Control*, vol. 61, no. 10, pp. 3031–3037, Oct. 2016.

[36] G. V. Aronovich, N. A. Kartvelishvili, and Y. K. Lyubimtsev, *Hydraulic Shock and Surge Tanks* (in Russian). Moscow, Russia: Nauka, 1968.

[37] S. Amin, F. M. Hante, and A. M. Bayen, "Exponential stability of switched linear hyperbolic initial-boundary value problems," *IEEE Trans. Autom. Control*, vol. 57, no. 2, pp. 291–301, Feb. 2012.

[38] C. Prieur, A. Girard, and E. Witrant, "Stability of switched linear hyperbolic systems by Lyapunov techniques," *IEEE Trans. Aut. Control*, vol. 59, no. 8, pp. 2196–2202, Aug. 2014.

[39] C. Prieur, A. R. Teel, and L. Zaccarian, "Relaxed persistent flow/jump conditions for uniform global asymptotic stability," *IEEE Trans. Autom. Control*, vol. 59, no. 10, pp. 2766–2771, Oct. 2014.

[40] T. Li, *Global Classical Solutions for Quasilinear Hyperbolic Systems*. Paris, France: Wiley, 1994.

[41] A. D. Myshkis and A. S. Shlopak, "Mixed problem for systems of differential-functional equations with partial derivatives and Volterra type operators," *Mat. Sbornik*, vol. 41, no. 83, pp. 239–256, 1957.

[42] V. E. Abolinia and A. D. Myshkis, "Mixed problem for an almost linear hyperbolic system in the plane (Russian)," *Mat. Sbornik*, vol. 50, no. 92, pp. 423–442, Apr. 1960.

[43] A. D. Myshkis and A. M. Filimonov, "Continuous solutions of quasilinear hyperbolic systems with two independent variables," *Differ. Equ.*, vol. 17, no. 3, pp. 488–500, 1981.

[44] A. D. Myshkis and A. M. Filimonov, "On the global continuous solvability of the mixed problem for one-dimensional hyperbolic systems of quasilinear equations," *Differ. Equ.*, vol. 44, pp. 413–427, Mar. 2008.

[45] K. L. Cooke, "A linear mixed problem with derivative boundary conditions," in *Seminar on Differential Equations and Dynamical Systems (III)* (Lecture Notes), vol. 51, D. Sweet and J. Yorke, Eds. College Park, MD, USA: Univ. Maryland, 1970, pp. 11–17.

[46] K. L. Cooke and D. W. Krumme, "Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations," *J. Math. Anal. Appl.*, vol. 24, pp. 372–387, Nov. 1968.

[47] J. Bernoulli, "Meditationes. Dechordis vibrantibis." in *Proc. 18th Comment. Acad. Sci. Imper. Petropol.*, vol. 3, 1728, pp. 13–28.

[48] A. F. Neves, H. Ribeiro, and O. Lopez, "On the spectrum of evolution operators generated by hyperbolic systems," *J. Funct. Anal.*, vol. 67, pp. 320–344, Jul. 1986.

[49] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Applied Mathematical Sciences), vol. 99. New York, NY, USA: Springer, 1993.

[50] V. Răsvan, E. Bobaşu, D. Danciu, and D. Popescu, "Control and stabilization of a linearized system of conservation laws," in *Proc. 17th Int. Carpathian Control Conf. (ICCC)*. Piscataway, NJ, USA: IEEE Press, 2016.

[51] V. Răsvan, D. Popescu, and D. Danciu, "Controller synthesis for a system of conservation laws," in *Proc. 20th Int. Conf. Syst. Theory, Control Comput. (ICSTCC)*. Piscataway, NJ, USA: IEEE Press, 2016.

[52] D. Danciu and V. Răsvan, "Neutral functional differential equations and systems of conservation laws," *IFAC-PapersOnLine*, vol. 50, pp. 13336–13341, Jul. 2017.

[53] I. G. Malkin, *Theory Stability Motion*. Washington, DC, USA: U.S. Atomic Energy Commission, 1952.

[54] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags* (Mathematics in Science and Engineering), vol. 23. New York, NY, USA: Academic, 1993.

[55] D. Danciu, "A CNN-based approach for a class of non-standard hyperbolic partial differential equations modeling distributed parameters (nonlinear) control systems," *Neurocomputing*, vol. 164, pp. 56–70, Sep. 2015.

[56] D. Danciu, D. Popescu, and E. Bobaşu, "Neural networks-based computational modeling of bilinear control systems for conservation laws: Application to the control of cogeneration," *IEEE Trans. Ind. Appl.*, vol. 54, no. 6, pp. 6498–6507, Nov./Dec. 2018.

[57] L. O. Chua and T. Roska, "The CNN paradigm," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 40, no. 3, pp. 147–156, Mar. 1993.

[58] S. K. Godunov, *Équations De La Physique Mathématique*. Moscow, Russia: Mir, 1978.

[59] J. C. Neu, *Singular Perturbation in the Physical Sciences*. Providence, RI, USA: American Mathematical Society, 2015.

[60] N. G. Čebotarev and N. N. Meiman, *The Routh-Hurwitz Problem for Polynomials and Entire Functions* Works of 'Steklov' Mathematical Institute), no. XXVII. Moscow USSR: USSR Academy Publishing House, 1949.

[61] V. Răsvan, "Slope restrictions and monotonicity in nonlinear systems: Almost linear behavior," in *Proc. Electron. J. Qual. Theory Differ. Equ.*, Szeged, Hungary, 2012, pp. 1–19.

[62] M. Tolle, *Die Regelung der Kraftmaschinen*. Berlin, Germany: Springer-Verlag, 1905.

[63] T. Stein, *Regelung und Ausgleich Dampfanlagen*. Berlin, Germany: Springer-Verlag, 1926.

[64] M. I. Gil, *Stability of Vector Differential Delay Equations* (Frontiers in Mathematics). Basel, Switzerland: Birkhäuser, 2013.

[65] M. I. Gil, *Stability of Neutral Functional Differential Equations* (Atlantis Studies in Differential Equations). Berlin, Germany: Atlantis Press, 2014.

[66] N. Wiener, *Cybernetics*. Paris, France: Hermann, 1948.

[67] M. G. Safonov, "Origins of robust control: Early history and future speculations," *Ann. Rev. Control*, vol. 36, pp. 173–181, Dec. 2012.

[68] B. V. Bulgakov, "Self-sustained oscillations of control systems," *Dokl. AN SSSR*, vol. 37, no. 9, pp. 283–287, 1942.

[69] A. I. Lurie and V. N. Postnikov, "Sliding Mode Control in Electro-Mechanical Systems," *Prikl. Mat. Mekh. (PMM)*, vol. 8, pp. 246–248, Jun. 1944.

[70] V. M. Popov, "Stability criteria for non-linear systems of automatic control based on the utilization of the Laplace transform," (in Romanian), *Power Stud. Res.*, vol. IX, no. 1, pp. 119–136, 1959.

[71] V. A. Yakubovich, "Frequency conditions for the absolute stability of control systems with several nonlinear or linear nonstationary blocks," *Avtom. I Telemekh.*, vol. 28, no. 6, pp. 5–30, 1967.

[72] V. Răsvan, *Absolute Stability of Time Lag Control Systems* (Russian). Moscow, Russia: Nauka, 1983.

[73] A. S. Shiriaev, "Some remarks on 'systems analysis via IQC,'" *IEEE Trans. Autom. Control*, vol. 45, no. 8, pp. 1527–1532, Aug. 2000.

[74] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Control*, vol. 42, no. 6, pp. 819–830, Jun. 1997.

[75] V. V. Kulkarni, L. Y. Pao, and M. G. Safonov, "Positivity preservation properties of the Rantzer multipliers," *IEEE Trans. Aut. Control*, vol. 56, no. 1, pp. 190–194, Jan. 2011.

DANIELA DANCIU (M'06–SM'13) received the Dipl.Eng. degree in automation and computers and the Ph.D. degree *(cum laude)* degree in automatic control from the University of Craiova, Romania, in 1990 and 2003, respectively, where she is currently an Associate Professor with the Department of Automatic Control and Electronics.

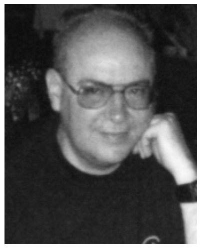
She is the author of two books and author/coauthor of more than 70 journal papers, book chapters, and conference proceedings. Her research interests include modeling, qualitative analysis, control and simulation of dynamical systems, distributed parameters control systems, as well as computational methods based on artificial intelligence.

Dr. Danciu is a member of SIAM and IFAC-TC2.6 Distributed Parameter Systems and an Invited Reviewer for zbMath, *AMS Mathematical Reviews* (MathSciNet), and more than ten scientific journals. Her awards and honors include the Romanian National Research Council Award for research results, in 2013, the Honor Diploma from the University of Craiova for outstanding research results, in 2015, and the IEEE Region 8's Clementina Saduwa Award 2017.

DAN POPESCU (M'99) received the Dipl.Eng. degree in automation and computers and the Ph.D. degree in automatic control from the University of Craiova, Craiova, Romania, in 1977 and 1997, respectively.

From 2004 to 2008, he was the Dean of the Faculty of Automation, Computers and Electronics, University of Craiova, Romania, where he was the Vice-Rector, from 2008 to 2016, and is currently a Professor with the Department of Automatic Control and Electronics. His research interests include control engineering, modeling and simulation, robust control, and time-delay systems.

Dr. Popescu has been a member of the IFAC Robust Control Technical Committee, since 2008.



VLADIMIR RĂSVAN (M'92–SM'94) received the Diploma Engineer and Ph.D. degrees in automatic control engineering from the Bucharest Polytechnic Institute, Romania, in 1967 and 1972, respectively.

After ten years of applied research at the Power Equipment Research and Development Institute, Bucharest, Romania, he joined the Faculty of Electrical Engineering, University of Craiova, as an Associate Professor. Since 1990, he has been a Professor of systems and control with the School of Automation, Computers and Electronics, University of Craiova, where he was the Head of School, from 1990 to 2004. Among his publications, there are some 100 journal papers and book chapters, more than 120 conference papers, and seven books, including *Absolute Stability of Time Lag Control Systems* (Russia: Nauka, Leningrad USSR, 1983), *Applications of Liapunov Methods in Stability* (Kluwer Academic, 1993), and *Stability and Stable Oscillations in Discrete Time Systems* (Gordon & Breach, 2000). His research interests include nonlinear systems stability, nonlinear oscillations (including synchronization), time lag, and distributed parameter systems. He is a Full Member of the Romanian Academy of Engineering Sciences. He is also a member AMS, SIAM, and AAAS. He is a recipient of the Romanian Academy Award for distinguished research.