IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received February 1, 2019, accepted March 14, 2019, date of publication April 3, 2019, date of current version April 15, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2906470

Higher Order Numerical Approaches for Nonlinear Equations by Decomposition Technique

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The work of F. A. Shah and S. Sherbaz was supported in part by the Higher Education Commission (HEC), Pakistan, through under Grant 5418/Federal/NRPU/ R&D/HEC/2016, and in part by the COMSATS University, Islamabad (CUI) through under Grant 16-75/CRGP/CIIT/ATK/17/1141.

ABSTRACT In this paper, a unique decomposition technique is implemented along with an auxiliary function for the best implementation. Some new and efficient techniques are introduced and analyzed for nonlinear equations. These techniques are higher ordered in approaching to the root of nonlinear equations. Some existing classical methods such as the Newton method, Halley method, and Traub's approach and their various modified forms are the special cases of these newly purposed schemes. These new iterative schemes are a good addition in existing methods and are also a comprehensive and generalized form for finding the solution of nonlinear equations.

INDEX TERMS Decomposition technique, iterative scheme, convergence analysis, newton method, numerical examples, coupled system of equations.

I. INTRODUCTION

Nonlinear equations arises extensively in various fields of science and engineering. Such type of problems can be studied in a specific framework of $h(x) = 0$, and their solutions are of great importance. For investigating the approximate solutions for such nonlinear equations, a variety of methods are being derived by implementing various techniques. Techniques for generating numerical schemes include quadrature rules, Taylor series, homotopy and various other decomposition techniques, see [1]–[27] and the references therein.

In this article, we implement a decomposition technique which is mainly purposed by Chun [5] to develop the numerical schemes of high order. First of all, we rearrange the supposed nonlinear equation accompanied by the auxiliary function as an equivalent coupled system of equations by applying the Taylor series. This approach permits us to express the given nonlinear equation as sum of two linear and nonlinear parts. This approach of writing the given equation is known as the decomposition technique and plays the significant role in modifying different iterative schemes for obtaining the required solution of nonlinear equations $h(x) = 0$.

In this research work, we use coupled equations to exhibit the nonlinear problem as a sum of linear and nonlinear operators which also involve the auxiliary function. The value of auxiliary function helps to obtain several numerical schemes for solving nonlinear equations.

From the obtained methods and their numerical results presented in the article, we recommend that this new technique of decomposition is a powerful and novel tool for this purpose. It is also studied in literature that this decomposition technique is very useful for the solution of initial and boundary value problems. In section 2, we design the main idea of this decomposition technique and propose one-step and two-step numerical methods for solving nonlinear equations. One can observe that if the derivative of the function disappears, i.e $h'(x_n) = 0$, through the iterative procedure, then the sequence of approximation after each iteration created by the Newton method and the methods studied in the literature [1]–[25] are not defined. Due to serious sin of division results in a mathematical failure. This is another strong inspiration and motivation of this article that the derived higher order schemes also converge even if the derivative disappears

The associate editor coordinating the review of this manuscript and approving it for publication was Xiao-Jun Yang.

during the numerical computation process by using any software. We also demonstrate that the new methods comprise Newton, Traub and Halley methods and their alternative variant forms having accelerated order of convergence as special cases. Some numerical examples are given as a practical exhibition and explanation to demonstrate the efficiency and the performance of the new iterative schemes. Our results can be considered as significant development and enhancement of the earlier known results.

II. ITERATIVE METHODS

Consider the following nonlinear equation of the type

$$h(x) = 0. \tag{1}$$

Assume that r is a simple root of (1) and β is the initial guess for the approximation which is sufficiently close to the actual root r. Let us assume the auxiliary function $g(x)$, just as

$$h(x)g(x) = 0, \tag{2}$$

where $g(x)$ is a differentiable function. By using the technique of Taylor series one can rephrase the nonlinear equation (2) as a coupled system

$$h(\beta)g(\beta) + \left[h'(\beta)g(\beta) + h(\beta)g'(\beta)\right](x - \beta) + H(x) = 0. \tag{3}$$

It follows from (3) and (1), that

$$H(x) = h(x)\psi(x, \beta) - h(\beta)g(\beta) - \left[h'(\beta)g(\beta) + h(\beta)g'(\beta)\right](x - \beta), \tag{4}$$

where

$$\psi(x, \beta) = \frac{g(\beta)\left[h(\beta)g'(\beta) + h'(\beta)g(\beta)\right]}{[h(x)g'(\beta) + h'(x)g(\beta)]}, \tag{5}$$

$H(x)$ in equation (4) is derived from equation (3), and it is important to convey that $\psi(x, \beta)$ is an auxiliary arbitrary function which will be helpful for finding higher order iterative schemes for solving nonlinear equations. Here β is the initial estimate for the requirement of root of (1). We can rephrase equation (4) in the following form

$$x = \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]} - \frac{H(x)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]}. \tag{6}$$

Now we assume (6), in the subsequent form

$$x = c + N(x), \tag{7}$$

where

$$c = \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}, \tag{8}$$

and

$$N(x) = -\frac{H(x)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}. \tag{9}$$

Here $N(x)$ is a nonlinear function.

We now build a sequence of higher order iterative schemes by using the following decomposition technique, which is actually due to Chun [5]. This decomposition of the nonlinear function $N(x)$ is quite different from that of Adomain decomposition or other decomposition techniques.

The central idea of this technique is to seek a solution having the series form

$$x = \sum_{i=0}^{\infty} x_i. \tag{10}$$

The nonlinear operator N can be decomposed as

$$N(x) = N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right\}. \tag{11}$$

Combining (7), (10) and (11), we have

$$\sum_{i=0}^{\infty} x_i = c + N(x_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} x_j\right) - N\left(\sum_{j=0}^{i-1} x_j\right)\right\}. \tag{12}$$

Thus we have the following iterative scheme:

$$\begin{cases} x_0 = c, \\ x_1 = N(x_0), \\ x_2 = N(x_0 + x_1) - N(x_0), \\ \vdots \\ x_{m+1} = N\left(\sum_{j=0}^{m} x_j\right) - N\left(\sum_{j=0}^{m-1} x_j\right). \quad m = 1, 2, \ldots \end{cases} \tag{13}$$

Then

$$x_1 + x_2 + \cdots + x_{m+1} = N(x_0 + x_1 + \cdots + x_m), \quad m = 1, 2, \ldots \tag{14}$$

and

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{15}$$

It can be shown that the series $\sum_{i=0}^{\infty} x_i$ converges absolutely and uniformly to a unique solution of equation (7).

From (8) and (13), we get

$$x_0 = c = \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}, \tag{16}$$

From (4) and (16), it can be obtained easily

$$H(x_0) = h(x_0)\,\psi(x_0, \beta). \tag{17}$$

$$x_1 = N(x_0) = -\frac{H(x_0)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]} = -\frac{h(x_0)\,\psi(x_0, \beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}. \tag{18}$$

Note that, x is approximated by

$$X_m = x_0 + x_1 + x_2 + \cdots + x_m, \tag{19}$$

where

$$\lim_{m\to\infty} X_m = x. \tag{20}$$

For $m = 0$,

$$x \approx X_0 = x_0 = c = \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}, \tag{21}$$

This formulation allows us to suggest the following one-step iterative method for solving the nonlinear equation (1).

Algorithm 2.1: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{h(x_n)g(x_n)}{[h(x_n)g'(x_n) + h'(x_n)g(x_n)]}, \quad n = 0, 1, 2, \cdots$$

This is one of the main recurrence relation which is also introduced by Hasanov *et al.* [8] and Noor and Noor [12] for generating different iterative methods for solving nonlinear equations by using a different basic technique.

For $m = 1$,

$$\begin{aligned} x \approx X_1 &= x_0 + x_1 = c + N(x_0) \\ &= \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]} \\ &\quad - \frac{h(x_0)\psi(x_0, \beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]}, \\ &= \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta) + h'(\beta)g(\beta)]} \\ &\quad - \frac{h(x_0)g(\beta)}{[h(x_0)g'(\beta) + h'(x_0)g(\beta)]}. \end{aligned} \tag{22}$$

This formulation permits us the following iterative scheme for solving nonlinear equations.

Algorithm 2.2: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)g(x_n)}{[h(x_n)g'(x_n) + h'(x_n)g(x_n)]}, \\ x_{n+1} &= y_n - \frac{h(y_n)g(x_n)}{[h(y_n)g'(x_n) + h'(y_n)g(x_n)]}, \quad n = 0, 1, 2, \cdots \end{aligned}$$

In a similar way, for $m = 2$, we have

$$\begin{aligned} x &\approx X_2 \\ &= x_0+x_1+x_2 = c+N(x_0+x_1) \\ &= \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]} - \frac{H(x_0+x_1)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]}, \\ &= \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]} - \frac{h(x_0+x_1)\,\psi(x_0+x_1, \beta)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]}, \\ &= \beta - \frac{h(\beta)g(\beta)}{[h(\beta)g'(\beta)+h'(\beta)g(\beta)]} - \frac{h(x_0)g(\beta)}{[h(x_0)g'(\beta)+h'(x_0)g(\beta)]} \\ &\quad - \frac{h(x_0+x_1)g(\beta)}{[h(x_0+x_1)g'(\beta)+h'(x_0+x_1)g(\beta)]}. \end{aligned} \tag{23}$$

This formulation enables us to suggest an iterative method for solving nonlinear equations given as follows.

Algorithm 2.3: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)g(x_n)}{[h(x_n)g'(x_n) + h'(x_n)g(x_n)]}, \\ z_n &= y_n - \frac{h(y_n)g(x_n)}{[h(y_n)g'(x_n) + h'(y_n)g(x_n)]}, \end{aligned}$$

Algorithm 2.1, Algorithm 2.2 and Algorithm 2.3 are newly derived main recurrence schemes which produces the iterative methods of higher order for different values of the auxiliary function $g(x_n)$.

To express the basic idea, we choose here the auxiliary function $g(x_n) = e^{-\lambda x_n}$, then from above main described Algorithms, we obtain the following iterative methods for solving nonlinear equations.

Algorithm 2.4: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{h(x_n)}{[h'(x_n) - \lambda h(x_n)]}, \quad n = 0, 1, 2, \cdots$$

Algorithm 2.4 was obtained by Hasanov *et al.* [8] and Noor and Noor [12].

For $\lambda = 0$, Algorithm 2.4 gets the form of well known Newton's method for nonlinear equations (1).

Algorithm 2.5: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{[h'(x_n) - \lambda h(x_n)]}, \\ x_{n+1} &= y_n - \frac{h(y_n)}{[h'(y_n) - \lambda h(y_n)]}, \quad n = 0, 1, 2, \cdots \end{aligned}$$

Traub [26] and Noor [13] are the special cases for $\lambda = 0$.

Algorithm 2.6: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{[h'(x_n) - \lambda\, h(x_n)]}, \\ z_n &= y_n - \frac{h(y_n)}{[h'(y_n) - \lambda\, h(y_n)]}, \\ x_{n+1} &= z_n - \frac{h(z_n)}{[h'(z_n) - \lambda\, h(z_n)]}, \quad n = 0, 1, 2, \cdots \end{aligned}$$

Remark 2.1: We would like to point out that if we take $\lambda = 0$, $\lambda = \frac{1}{2}$, $\lambda = 1, \cdots$, in above derived iterative methods, we can obtain various known and knew methods for solving nonlinear equations. All the methods derived above are the good alternate of the Newton method and the existing variants of Newton methods. It is important to say that one should select a value of λ which never makes the denominator zero during an iterative procedure.

III. CONVERGENCE ANALYSIS

In this section, we analyze the convergence criteria of the main iterative schemes established in section 2 and the rest of the schemes can also be studied at the same pattern.

Theorem 3.1: Let $r \in A$ be a simple root of differentiable function $f : A \subseteq R \to R$ for an open interval A. If x_0

is sufficiently close to r, then iterative method defined by Algorithm 2.3 has at least eighth-order convergence.

Proof: Let r be a simple zero of h. Then by expanding $h(x_n)$ and $h'(x_n)$, in Taylor's serie about r we have

$$h(x_n) = h'(r)\left[e_n + k_2 e_n^2 + k_3 e_n^3 + k_4 e_n^4 + k_5 e_n^5 + k_6 e_n^6 + O(e_n^7)\right], \tag{24}$$

and

$$h'(x_n) = h'(r)\left[1 + 2k_2 e_n + 3k_3 e_n^2 + 4k_4 e_n^3 + 5k_5 e_n^4 + 6k_6 e_n^5 + O(e_n^7)\right]. \tag{25}$$

where

$$k_k = \frac{1}{c!}\frac{h^{(k)}(r)}{h'(r)}, \quad c = 2, 3, \cdots. \text{ and } e_n = x_n - r.$$

Now expanding $h(x_n)g(x_n), h(x_n)g'(x_n)$ and $h'(x_n)g(x_n)$ in Taylor series, we acquire

$$h(x_n)g(x_n) = h'(r)\left[g(r)e_n + \left(g'(r) + k_2 g(r)\right) e_n^2 + g_1 e_n^3 + O(e_n^4)\right], \tag{26}$$

$$h(x_n)g'(x_n) = h'(r)\left[g'(r)e_n + \left(g''(r) + k_2 g'(r)\right) e_n^2 + g_1 e_n^3 + O(e_n^4)\right], \tag{27}$$

and

$$h'(x_n)g(x_n) = h'(r) \times \left[\left(g(r) + g'(r) + 2k_2 g(r)\right) e_n + g_1 e_n^2 + g_2 e_n^3 + O(e_n^4)\right], \tag{28}$$

where

$$g_1 = \frac{1}{2}g''(r) + k_3 g'(r) + k_2 g(r), \tag{29}$$

and

$$g_2 = \frac{1}{6}g'''(r) + 4k_4 g(r) + k_2 g''(r) + 3k_3 g'(r). \tag{30}$$

From (26), (27) and (28), we get

$$\frac{h(x_n)g(x_n)}{h'(x_n)g(x_n) + h(x_n)g'(x_n)} = e_n - \left(\frac{g'(r)}{g(r)} + k_2\right) e_n^2 - \left(\frac{g''(p)}{g(p)}k_2^2 + 2k_3 - 2\frac{g'(r)}{g(r)}k_2\right) \times e_n^3 + (O)e_n^4. \tag{31}$$

Using (31), we have

$$y_n = r + \left(\frac{g'(r)}{g(r)} + k_2\right) e_n^2 - \left(\frac{g''(r)}{g(r)}k_2^2 + 2k_3 - 2\frac{g'(r)}{g(r)}k_2 + 2k_2^2 \quad + 2\frac{g'(r)}{g(r)}\right) e_n^3 + O(e_n^4). \tag{32}$$

Expanding $h(y_n)$, in Taylor's series about r and using (30), we have

$$h(y_n) = h'(r)\left[\left(\frac{g'(r)}{g(r)} + k_2\right) e_n^2 - \left\{2k_2^2 - 2k_3 - \frac{g''(r)}{g(r)}k_2^2 - 2\frac{g'(r)}{g(r)}k_2 + 2\left(\frac{g'(r)}{g(r)}\right)^2\right\} e_n^3 + O(e_n^4)\right]. \tag{33}$$

Now using (33), we obtain

$$h(y_n)g(x_n) = h'(r)\left[\left(g'(r) + k_2 g(r)\right) e_n^2 - \left(\frac{\left(g'(r)\right)^2}{g(r)} + k_2 g'(r) - g''(r) + 2k_2^2 g(r) - 2k_3 g(r)\right) \times e_n^3 + O(e_n^4)\right]. \tag{34}$$

Expanding $h'(y_n)$, in Taylor's series about r and using (32), we have

$$h'(y_n) = h'(r)\left[1 + 2k_2\left(\frac{g'(r)}{g(r)} + k_2\right) e_n^2 - 2k_2 \times \left\{2k_2^2 - 2k_3 - \frac{g''(r)}{g(r)} - 2k_2\frac{g'(r)}{g(r)} + 2\left(\frac{g'(r)}{g(r)}\right)^2\right\} e_n^3 + O(e_n^4)\right]. \tag{35}$$

$$h'(y_n)g(x_n) = h(r)\left[g(r) + g'(r)\, e_n + \frac{1}{2}\left(4k_2 g'(r) + g''(r) + 4k_2^2 g(p)\right) e_n^2 + O(e_n^3)\right]. \tag{36}$$

Now using (34), (35) and (36), we get

$$\frac{h(y_n)g(x_n)}{h'(y_n)g(x_n) + h(y_n)g'(x_n)} = \left[\left(\frac{g'(r)}{g(r)} + k_2\right) e_n^2 + \left\{2k_3 - 2k_2^2 + \frac{g''(r)}{g(r)} - 2k_2\frac{g'(r)}{g(r)} - 2\left(\frac{g'(r)}{g(r)}\right)\right\} e_n^3 + O(e_n^4)\right]. \tag{37}$$

Now (32) and (37), we obtain

$$z_n = \left[3k_2\left(\frac{g'(r)}{g(r)}\right)^2 + 3k_2^2\left(\frac{g'(r)}{g(r)}\right) + \left(\frac{g'(r)}{g(r)}\right)^3 + k_2^3\right] \times, e_n^4 + O(e_n^5). \tag{38}$$

Expanding $h(z_n)$, in Taylor's series about r and using (38), we have

$$h(z_n) = h'(r)\left[\left\{k_2^3 e_n^4 + 3k_2^2\frac{g'(r)}{g(r)} + 3k_2\left(\frac{g'(r)}{g(r)}\right)^2 + \left(\frac{g'(r)}{g(r)}\right)^3\right\} e_n^4 + O(e_n^5)\right]. \tag{39}$$

Expanding $h'(z_n)$, in Taylor's series about r and using (38), we have

$$h'(z_n) = h'(r)\left[1 + 2\left\{3k_2^2\left(\frac{g'(r)}{g(r)}\right)^2 + 3k_2^3\frac{g'(r)}{g(r)} + \left(\frac{g'(r)}{g(r)}\right)^3 k_2^3\right\} e_n^4 + O(e_n^5)\right]. \quad (40)$$

Now using (39) and (40), we obtain

$$\frac{h(z_n)g(x_n)}{h'(z_n)g(x_n) + h(z_n y_n)g'(x_n)} = \left[3k_2\left(\frac{g'(r)}{g(r)}\right)^2 + 3k_2^2\frac{g'(r)}{g(r)} + \left(\frac{g'(r)}{g(r)}\right)^3 + k_2^3\right] \times e_n^4 + O(e_n^5). \quad (41)$$

From (38) and (41), we obtain

$$x_{n+1} = r + \left(\frac{g'(r)}{g(r)} + k_2\right)^7 e_n^8 + O(e_n^9). \quad (42)$$

Finally, the error equation is

$$e_{n+1} = \left(\frac{g'(r)}{g(r)} + k_2\right)^7 e_n^8 + O(e_n^9). \quad (43)$$

From (43), we analyze that the discussed main scheme has at least eighth-order convergence.

Subsequently, the methods produced from this main scheme have at least eighth-order of convergence.

In a similar way, one can prove the convergence of Algorithm 2.2 and also the methods developed from this scheme.

Remark 3.1: Several iterative methods can be developed form various specific values of auxiliary functions $g(x)$, from developed main Algorithms.

If $\frac{g'(x_n)}{g(x_n)} = -\frac{h''(x_n)}{2h'(x_n)}$, then the Algorithm 2.1 reduces to the following iterative method.

Algorithm 3.1: For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{2h(x_n)h'(x_n)}{2\,[h'(x_n)]^2 - h''(x_n)h(x_n)},$$

which is well known Halley method of third-order convergence [26]. Similarly, the Algorithm 2.2 and Algorithm 2.3 reduce to the following iterative methods for the above mentioned particular value of the auxiliary function.

Algorithm 3.2: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{2h(x_n)h'(x_n)}{2\,[h'(x_n)]^2 - h''(x_n)h(x_n)},$$
$$x_{n+1} = y_n - \frac{2h(y_n)h'(x_n)}{2h'(y_n)h'(x_n) - h''(x_n)h'(y_n)},$$

which is seventh-order convergent method for solving nonlinear equations and appears to be new one with the following error equation.

$$e_{n+1} = (8k_2^2k_3^2 - 7k_3k_2^4 + 2k_2^6 - 3k_3^3)e_n^7 + O(e_n^8).$$

Algorithm 3.3: For a given x_0, compute the approximate solution x_{n+1} by the following iterative schemes:

$$y_n = x_n - \frac{2h(x_n)h'(x_n)}{2\,[h'(x_n)]^2 - h''(x_n)h(x_n)},$$
$$z_n = y_n - \frac{2h(y_n)h'(x_n)}{2h'(y_n)h'(x_n) - h''(x_n)h(y_n)},$$
$$x_{n+1} = z_n - \frac{2h(z_n)h'(x_n)}{2h'(x_n)h'(z_n) - h''(x_n)h(z_n)},$$

which is fifteenth-order convergent method for solving nonlinear equations and also appears to be new one with the following error equation.

$$e_{n+1} = \left(162k_2^2k_3^6 - 414k_3^5k_2^4 - 27k_3^7 + 246k_3^2k_2^{10} + 584k_3^4k_2^6 - 49k_3^3k_2^8 + 8k_2^{14} - 68k_3k_2^{12}\right)e_n^{15} + O(e_n^{16}).$$

IV. NUMERICAL RESULTS

Some examples are presented to illustrate the efficiency of the newly developed one-step, two-step and three-step iterative methods in this article. We try to compare the Newton's method (NM), the method of Chun [5] (CN), the method of Bi *et al.* [2] (BM), Algorithm 2.4 (FS1), the Algorithm 2.5 (FS2) and the Algorithm 2.6 (FS3) presented in this article. All the numerical experiments are performed with Intel (R) Core[TM] 2 × 2.1 GHz, 1 GB of RAM and computer uses 32 bit real numbers and all the codes are written in Maple. In Tables 4.1 and 4.2, results are shown for all considered examples. IT stands for number of iterations, δ is the difference of the last two consecutive iterations $\delta = |x_{n+1} - x_n|$. We use $\varepsilon = 10^{-300}$. Stopping criteria used is mentioned below for computer programs:

(i). $|x_{n+1} - x_n| < \varepsilon$,
(ii). $|h(x_n)| < \varepsilon$.

The computational order of convergence p approximated by means of

$$\rho \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}.$$

We consider the following nonlinear equations as trial for different methods for comparison.

$$h_1(x) = \sin^2 x - x^2 + 1,$$
$$h_2(x) = x^2 - e^x + x + 2,$$
$$h_3(x) = (x-1)^3 - 1,$$
$$h_4(x) = x^3 - 10,$$
$$h_5(x) = xe^{x^2} - \sin^2 x + 3\cos x - 5,$$
$$h_6(x) = e^{x^2+7x-30} - 1.$$

Remark 4.1: We would like to mention that CPU time displayed by Maple software of all the methods in all examples is as follows: Time: 0.1s.

TABLE 1. (Numerical examples for $\lambda = 1$).

	IT	x_n	$h(x_n)$	δ	ρ
$h_1,\ x_0 = 1.0$					
NM	11	1.4044916480	0.00e–001	1.78e–404	2.00000
CN	8	1.4044916480	0.00e–001	0.00e–001	4.00000
BM	4	1.4044916480	0.00e–001	7.28e–423	8.06265
FS1	9	1.4044916480	0.00e–001	2.74e–337	2.00000
FS2	5	1.4044916480	0.00e–001	2.74e–337	4.00000
FS3	4	1.4044916480	0.00e–001	0.00e–001	7.78101
$h_2,\ x_0 = 2.0$					
NM	10	0.2575302850	0.00e–001	8.05e–449	2.00000
CN	6	0.2575302850	0.00e–001	4.68e–046	4.00000
BM	4	0.2575302850	0.00e–001	3.00e–500	8.09301
FS1	13	0.2575302850	0.00e–001	1.52e–324	2.00000
FS2	7	0.2575302850	0.00e–001	1.52e–324	7.91241
FS3	4	0.2575302850	0.00e–001	3.00e–500	7.90379
$h_3,\ x_0 = 3.5$					
NM	12	2.0000000000	0.00e–001	4.88e–338	2.00000
CN	7	2.0000000000	0.00e–001	3.20e–374	4.00000
BM	4	2.0000000000	0.00e–001	0.00e–001	7.99559
FS1	11	2.0000000000	0.00e–001	0.00e–001	3.00000
FS2	6	2.0000000000	0.00e–001	0.00e–001	3.99929
FS3	4	2.0000000000	0.00e–001	0.00e–001	8.17560
$h_4,\ x_0 = 1.5$					
NM	11	2.1544346900	1.00e–498	1.06e–441	2.00000
CN	7	2.1544346900	1.00e–498	4.39e–351	4.00000
BM	4	2.1544346900	7.00e–499	1.00e–499	7.92915
FS1	11	2.1544346900	7.00e–499	1.00e–499	2.00000
FS2	6	2.1544346900	1.00e–498	0.00e–001	4.00000
FS3	4	2.1544346900	7.00e–499	1.00e–499	8.00000
$h_5,\ x_0 = -2$					
NM	13	–1.20764782710	0.00e–001	4.20e–327	2.00000
CN	8	–1.20764782710	0.00e–001	0.00e–001	4.00000
BM	5	–1.20764782710	0.00e–001	0.00e–001	7.99583
FS1	12	–1.20764782710	0.00e–001	2.34e–413	2.00000
FS2	7	–1.20764782710	0.00e–001	0.00e–001	4.00000
FS3	5	–1.20764782710	0.00e–001	0.00e–001	8.01955
$h_6,\ x_0 = 3.5$					
NM	17	3.0000000000	0.00e–001	1.83e–378	2.00000
CN	10	3.0000000000	0.00e–001	9.85e–348	4.00000
BM	6	3.0000000000	0.00e–001	0.02e–001	7.99999
FS1	17	3.0000000000	0.00e–001	0.00e–001	2.00000
FS2	9	3.0000000000	0.00e–001	0.00e–001	4.00000
FS3	6	3.0000000000	0.00e–001	0.00e–001	8.00000

TABLE 2. (Numerical examples for $\lambda = 0.5$).

	IT	x_n	$h(x_n)$	δ	ρ
$h_1,\ x_0 = 1.0$					
NM	11	1.4044916480	0.00e–001	1.78e–404	2.00000
CN	8	1.4044916480	0.00e–001	0.00e–001	4.00000
BM	4	1.4044916480	0.00e–001	7.28e–423	8.06265
FS1	10	1.4044916480	0.00e–001	7.28e–423	2.00000
FS2	6	1.4044916480	0.00e–001	0.00e–001	4.00000
FS3	4	1.4044916480	0.00e–001	7.28e–423	8.16225
$h_2,\ x_0 = 2.0$					
NM	10	0.257530280	0.00e–001	8.05e–449	2.00000
CN	6	0.257530280	0.00e–001	4.68e–461	4.00000
BM	4	0.257530280	0.00e–001	3.00e–500	8.09301
FS1	12	0.257530280	1.00e–499	1.65e–343	2.00000
FS2	7	0.257530280	0.00e–001	3.00e–500	4.00000
FS3	4	0.257530280	0.00e–001	3.00e–500	7.99379
$h_3,\ x_0 = 3.5$					
NM	12	2.000000000	0.00e–001	4.88e–338	2.00000
CN	7	2.000000000	0.00e–001	3.20e–374	4.00000
BM	4	2.000000000	0.00e–001	0.00e–001	7.99559
FS1	11	2.000000000	0.00e–001	0.00e–001	2.00000
FS2	6	2.000000000	0.00e–001	0.00e–001	4.00000
FS3	4	2.000000000	0.00e–001	0.00e–001	7.99999
$h_4,\ x_0 = 1.5$					
NM	11	2.1544346900	1.00e–498	1.06e–441	2.00000
CN	7	2.1544346900	1.00e–498	4.39e–351	4.00000
BM	4	2.1544346900	7.00e–499	1.00e–499	7.92915
FS1	9	2.1544346900	1.00e–498	9.99e–492	2.00000
FS2	5	2.1544346900	7.00e–499	9.99e–492	4.00000
FS3	4	2.1544346900	7.00e–499	1.00e–499	7.99994
$h_5,\ x_0 = -2$					
NM	13	–1.20764782710	0.00e–001	4.20e–327	2.00000
CN	8	–1.20764782710	0.00e–001	0.00e–001	4.00000
BM	5	–1.20764782710	0.00e–001	0.00e–001	7.99583
FS1	13	–1.20764782710	0.00e–001	0.00e–001	2.00000
FS2	7	–1.20764782710	0.00e–001	0.00e–001	4.00000
FS3	5	–1.20764782710	0.00e–001	0.00e–001	8.01583
$h_6,\ x_0 = 3.5$					
NM	17	3.000000000000	0.00e–001	1.83e–378	2.00000
CN	10	3.000000000000	0.00e–001	9.85e–348	4.00000
BM	6	3.000000000000	0.00e–001	0.02e–001	7.99999
FS1	17	3.000000000000	0.00e–001	5.98e–467	2.00000
FS2	9	3.000000000000	0.00e–001	5.98e–467	4.00000
FS3	6	3.000000000000	0.00e–001	0.00e–001	7.99999

A. COMPARISON WITH SOME WELL KNOWN METHODS

Let us examine some known iterative schemes from literature for computational purpose and compare these methods to the resent derived methods. Shah *et al.* [22] suggested three-step numerical method of sixth-order convergence consisting of three parameters (RWB)

$$x_{n+1} = x_n - \frac{2h(x_n)}{3h'(x_n)},$$

$$z_n = x_n - \frac{3h'(y_n) + h'(x_n)}{6h'(y_n) - 2h'(x_n)} \frac{h(x_n)}{h'(x_n)},$$

$$x_{n+1} = z_n - \frac{(2a-b)h'(x_n) + bh'(y_n) + ch(x_n)}{(-a-b)h'(x_n) + (3a+b)h'(y_n) + ch(x_n)} \frac{h(z_n)}{h'(x_n)},$$

where $a,\ b,\ c \in R$ and $a \neq 0$.

Noor [16] also purposed a sixth-order convergent scheme by modifying the well-known Householder method. Their method consists of two-steps (NNS)

$$y_n = x_n - \frac{h(x_n)}{h'(x_n)},$$

$$x_{n+1} = y_n - \frac{h(y_n)}{h'(y_n)} - \frac{[h(y_n)]^2\, h''(y_n)}{2\,[h'(y_n)]^3},$$

TABLE 3. (Number of iterations and COC for various iterative methods for $\lambda = 1$).

$h(x)$	x_0	FS1	FS2	FS3	NM	RWB	NNS
$h_1(x)$	0.0	(11,2.0)	(6,4.0)	(5,8.0)	div	div	div
$h_2(x)$	0.2	(15,2.0)	(8,3.9)	(6,8.1)	div	div	div
$h_3(x)$	0.0	(10,2.1)	(6,4.0)	(4,8.0)	div	div	div
$h_4(x)$	100	(111,2.0)	(56,4)	(38,8.0)	div	div	div
$h_5(x)$	0.6	(11,2.0)	(6,4.0)	(5,8.0)	div	div	div
$h_6(x)$	0.5	(18,2.0)	(10,4.0)	(7,8.0)	div	div	div
$h_7(x)$	0.0	(10,2.0)	(6,4.1)	(3,8.2)	div	div	div
$h_8(x)$	1.0	(2,2.0)	(2,4.0)	(2,8.0)	div	div	div

div stands for divergence of the method.

TABLE 4. (Number of iterations and COC for various iterative methods for $\lambda = 1$).

$h(x)$	x_0	FS1	FS2	FS3	SG	CH	BHQ	LMMW
$h_1(x)$	0.0	(11,2.0)	(6,4.0)	(5,8.0)		div	div	div
$h_2(x)$	0.2	(15,2.0)	(8,3.9)	(6,8.1)		div	div	div
$h_3(x)$	0.0	(10,2.1)	(6,4.0)	(4,8.0)		div	div	div
$h_4(x)$	100	(111,2.0)	(56,4)	(38,8.0)		div	div	div
$h_5(x)$	0.6	(11,2.0)	(6,4.0)	(5,8.0)		div	div	div
$h_6(x)$	0.5	(18,2.0)	(10,4.0)	(7,8.0)		div	div	div
$h_7(x)$	0.0	(10,2.0)	(6,4.1)	(3,8.2)		div	div	div
$h_8(x)$	1.0	(2,2.0)	(2,4.0)	(2,8.0)		div	div	div

div stands for divergence of the method.

Zhang *et al.* [27] developed a sixth-order convergent varrient of the Jarrat's method. Their method is given below (WKL)

$$\begin{aligned} x_{n+1} &= x_n - \frac{2h(x_n)}{3h'(x_n)}, \\ z_n &= x_n - \frac{3h'(y_n)+h'(x_n)}{6h'(y_n)-2h'(x_n)}\frac{h(x_n)}{h'(x_n)}, \\ x_{n+1} &= z_n - \frac{(5a+3b)h'(x_n)-(3a+b)h'(y_n)}{2a\,h'(x_n)+2\,bh'(y_n)}\frac{h(z_n)}{h'(x_n)}. \end{aligned}$$

where $a,\ b \in R$ and $a+b \neq 0$.

Sharma and Guha (SG) [23] developed a method by modifimg the Ostrowski method having sixth-order convergence containing three-steps and one parameter

We may notice that,

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ z_n &= y_n - \frac{h(x_n)}{h'(x_n)}\frac{h(y_n)}{h(x_n)-2h(y_n)}, \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(x_n)}\frac{h(x_n)+a\,h(y_n)}{h(x_n)+(a-2)h(y_n)}, \end{aligned}$$

where $a \in R$.

Adomian *et al.* [3] also developed a sixth-order modification of the Ostroski's method. Their family of method consists of the following three steps (CH)

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ z_n &= y_n - \frac{h(x_n)}{h'(x_n)}\frac{h(y_n)}{h(x_n)-2h(y_n)}, \\ x_{n+1} &= z_n - H(u_n)\frac{h(z_n)}{h'(x_n)}, \end{aligned}$$

where $u_n = \frac{h(y_n)}{h'(x_n)}$, and $H(t)$ represents a real-valued function satisfying $H(0) = 1, H'(0) = 2$.

Similarly, Bi *et al.* [2] suggested a three-step iterative method of eighth-order convergent method as (BHQ):

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ z_n &= y_n - \frac{h(y_n)}{h'(x_n)}\frac{h(x_n)+\alpha h(y_n)}{h(x_n)-(\alpha-2)h(y_n)}, \\ x_{n+1} &= z_n - H(\mu_n)\frac{h(z_n)}{h[z_n,y_n]+h[z_n,y_n,x_n](z_n-y_n)}, \end{aligned}$$

where $\mu_n = \frac{h(z_n)}{h'(x_n)}$, and $\alpha \in R$.

Based upon the well-known King's method and the Newton method recently He [10] established a three-step and sixteenth-order iterative method (LMMW)

$$\begin{aligned} y_n &= x_n - \frac{h(x_n)}{h'(x_n)}, \\ z_n &= y_n - \frac{h(y_n)}{h'(x_n)}\frac{2h(x_n)-h(y_n)}{2h(x_n)-5h(y_n)}, \\ x_{n+1} &= z_n - \frac{h(z_n)}{h'(z_n)} - \frac{2h(z_n)-h\left(z_n-\frac{h(z_n)}{h'(z_n)}\right)}{2h(z_n)-5h\left(z_n-\frac{h(z_n)}{h'(z_n)}\right)}\frac{h\left(z_n-\frac{h(z_n)}{h'(z_n)}\right)}{h'(z_n)}. \end{aligned}$$

We test these iterative methods and the derived new methods for the following functions.

$$h_1(x) = x^3 + 4x^2 + 10, \beta \approx 1.365.$$
$$h_2(x) = x^2 e^{x^2} - \sin^2(x) + 3\cos(x) + 5, \beta \approx -1.207.$$
$$h_3(x) = \sin^2 x - x^2 + 1, \beta \approx 1.404.$$
$$h_4(x) = \tan^{-1}(x), \beta \approx 0.$$
$$h_5(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5, \beta \approx 1.424.$$
$$h_6(x) = e^{x^2+7x-30} - 1, \beta \approx -1.0.$$
$$h_7(x) = x^5 + x^4 + 4x^2 - 15, \beta \approx 1.347.$$

ACKNOWLEDGMENT

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions, which improved the quality of this paper.

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