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Noise-Tolerant Zeroing Neural Network for Solving Non-Stationary Lyapunov Equation

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ABSTRACT As a crucial means for stability analysis in control systems, the Lyapunov equation is applied in many fields of science and engineering. There are some methods proposed and studied for solving the non-stationary Lyapunov equation, such as the zeroing neural network (ZNN) model. However, a common drawback these methods have is that they rarely tolerate noises. Therefore, given that the existence of various types of noises during computation, a noise-tolerant ZNN (NTZNN) model with anti-noise ability is proposed for solving the non-stationary Lyapunov equation in this paper. For comparison, the conventional ZNN (CZNN) model is also applied to solve the same problem. Furthermore, theoretical analyses are provided to prove the global and exponential convergence performance of the proposed NTZNN model in the absence of noises. On this basis, the anti-noise performance of the proposed NTZNN model is proven. Finally, by adopting the proposed NTZNN model and the CZNN model to solve the non-stationary Lyapunov equation, computer simulations are conducted under the noise-free case and the noisy case, respectively. The simulation results indicate that the proposed NTZNN model is practicable for solving the non-stationary Lyapunov equation and superior to the CZNN model at the existence of noises.

INDEX TERMS Non-stationary Lyapunov equation, noise-tolerant zeroing neural network (NTZNN), conventional zeroing neural network (CZNN), global and exponential convergence.

I. INTRODUCTION

As a crucial means for stability analysis in control systems [1], the Lyapunov equation is applied in many fields of science and engineering, such as boundary value problems [2], model order reduction [3], multi-agent systems [4], and Markov jump linear systems [5]. Thus, solving the Lyapunov equation has been a focus in recent decades.

The typical methods to solve the Lyapunov equation are numerical algorithms, which include two types methods, i.e., direct methods and iterative methods [3], [6]. Some direct methods, such as Bartels-Stewart methods and Hammarling methods, which are based on the Schur decomposition, are feasible for solving non-large-scale Lyapunov equations. Then, iterative methods become popular after being employed in large scale sparse problems. Particularly, the alternating direction implicit (ADI) iteration presented by Eugene L. Wachspress is strongly competitive [7]. After that, some extended ADI iterations are presented and applied [8]–[10]. Another method that rivals the ADI iteration is the Krylov subspace method [11]–[13], which uses the projection strategy in the iterative process. Recently, rational Krylov subspace method appears in many literatures [14]–[17]. The generalized minimal residual method is employed to solve the Lyapunov equation in multi-agent systems [4]. In [1], Sun and Zhang propose quantum algorithms to exponentially accelerate the process of solving the Lyapunov equation.

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It is worth pointing out that the above methods are used to solve the stationary Lyapunov equation but may be inapplicable in non-stationary problems with parameters varying fast with time.

In recent decades, neural network has become a focus [18]–[23]. Due to the parallel-distributed processing property, recurrent neural network (RNN) is applied in many fields [24]–[27], especially for time-varying problems [28]–[32]. Furthermore, a lot of attention is paid to solving the Lyapunov equation by using various RNN models. The gradient-based neural network (GNN) model [33], [34] and the zeroing neural network (ZNN) model [34]–[36] are two typical RNN models to solve the Lyapunov equation. Thereinto, GNN model is designed for solving stationary problems, while ZNN model is used to solve stationary and non-stationary problems [34]. For non-stationary Lyapunov equations, ZNN model is able to converge to theoretical solution due to the use of time derivative information. In addition, there are some extensions of ZNN model [35], [36]. It is important to note that the above methods are employed in the absence of noises.

Given that the existence of noises generated by model implementation, measurement error, external interference and so on [37], this paper aims to solve the non-stationary Lyapunov equation at the existence of various types of noises, by proposing a noise-tolerant zeroing neural network (NTZNN) model. The remainder of the paper is divided into four parts. In Section II, the problem formulation is presented and the NTZNN model for this problem is proposed, with the conventional ZNN (CZNN) model introduced. In Section III, we give theoretical analysis for the convergence performance of the proposed NTZNN model under the noise-free case and the noisy case. In Section IV, computer simulations are conducted by adopting the proposed NTZNN model and the CZNN model respectively, and the results show that the proposed NTZNN model has a prominent superiority to the CZNN model in tolerating noises for the non-stationary Lyapunov equation solving.

II. PROBLEM FORMULATION AND NTZNN MODEL

First of all, the problem formulation of the non-stationary Lyapunov equation is presented in this section. Then, in order to solve this problem, a NTZNN model is proposed and the existing CZNN model is introduced.

A. PROBLEM FORMULATION

In this paper, we consider the following non-stationary Lyapunov matrix equation:

$$M^{\mathrm{T}}(t)Y(t) + Y(t)M(t) + N(t) = 0, \tag{1}$$

where $M(t) \in \mathbb{R}^{n\times n}$ and $N(t) \in \mathbb{R}^{n\times n}$ denote non-stationary coefficient matrices and $M^{\mathrm{T}}(t) \in \mathbb{R}^{n\times n}$ is the transposed matrix of $M(t)$. $Y(t)$ is an unknown non-stationary matrix for the solution of (1). Note that, (1) has a unique solution [34]. For solving the non-stationary Lyapunov equation (1), we propose the NTZNN model with anti-noise ability in the ensuing Section II-B.

B. NTZNN MODEL

We define the following error function:

$$E(t) = M^{\mathrm{T}}(t)Y(t) + Y(t)M(t) + N(t). \tag{2}$$

Next, the evolution formula is designed as [38]:

$$\dot{E}(t) = -\zeta\, E(t) - \eta \int_0^t E(\delta)\mathrm{d}\delta, \tag{3}$$

where $\zeta > 0 \in \mathbb{R}$ and $\eta > 0 \in \mathbb{R}$. For the above linear system, the superposition principle can be applied to decompose the output when input is decomposable. Combining (2) and (3), the following NTZNN model can be given as follows:

$$\begin{aligned} M^{\mathrm{T}}(t)\dot{Y}(t) + \dot{Y}(t)M(t) = &-\dot{M}^{\mathrm{T}}(t)Y(t) - Y(t)\dot{M}(t) - \dot{N}(t) \\ &- \zeta[M^{\mathrm{T}}(t)Y(t) + Y(t)M(t) + N(t)] \\ &- \eta \int_0^t [M^{\mathrm{T}}(\delta)Y(\delta) + Y(\delta)M(\delta) \\ &+ N(\delta)]\mathrm{d}\delta, \end{aligned} \tag{4}$$

where $Y(t) \in \mathbb{R}^{n\times n}$ starts from the initial state $Y(0)$. Besides, the conventional ZNN model can be obtained from [39] as

$$\begin{aligned} M^{\mathrm{T}}(t)\dot{Y}(t) &+ \dot{Y}(t)M(t) \\ &= -\dot{M}^{\mathrm{T}}(t)Y(t) - Y(t)\dot{M}(t) - \dot{N}(t) \\ &\quad - \zeta\,[M^{\mathrm{T}}(t)Y(t) + Y(t)M(t) + N(t)]. \end{aligned} \tag{5}$$

Considering the existence of various types of noises, the proposed NTZNN model (4) turns to

$$\begin{aligned} M^{\mathrm{T}}(t)\dot{Y}(t) + \dot{Y}(t)M(t) = &-\dot{M}^{\mathrm{T}}(t)Y(t) - Y(t)\dot{M}(t) - \dot{N}(t) \\ &- \zeta\,[M^{\mathrm{T}}(t)Y(t) + Y(t)M(t) + N(t)] \\ &- \eta \int_0^t [M^{\mathrm{T}}(\delta)Y(\delta) \\ &+ Y(\delta)M(\delta) + N(\delta)]\mathrm{d}\delta + T(t), \end{aligned} \tag{6}$$

where $T(t) \in \mathbb{R}^{n\times n}$ stands for noises in matrix form.

III. THEORETICAL ANALYSES OF NTZNN

In this section, we discuss the convergence performance of the CZNN model (5) and the proposed NTZNN model (4). It has been presented in [39] that the output $Y(t)$ of the CZNN model for solving the non-stationary Lyapunov equation converges globally and exponentially to the theoretical solution of (1). In addition, in the absence of noises, the global and exponential convergence performance of the proposed NTZNN model is proven. Furthermore, the robustness of the proposed NTZNN model is studied at the existence of noises.

A. CONVERGENCE OF NTZNN

In this subsection, we discuss the proposed NTZNN model (4) and prove its global and exponential convergence performance in the absence of noises.

Theorem 1: The output $Y(t)$ of the proposed NTZNN model (4), which is randomly initialized, converges globally and exponentially to the theoretical solution of the non-stationary Lyapunov equation (1) as time evolves.

Proof: Let $\sigma(t) = \int_0^t E(\delta)\mathrm{d}\delta$, with $e_{ij}(t)$, $\sigma_{ij}(t)$, $\dot{\sigma}_{ij}(t)$, $\ddot{\sigma}_{ij}(t)$ respectively being the ijth element of $E(t)$, $\sigma(t)$, $\dot{\sigma}(t)$, $\ddot{\sigma}(t)$. In this way, the ijth subsystem of the evolution formula (3) turns to

$$\ddot{\sigma}_{ij}(t) = -\zeta\,\dot{\sigma}_{ij}(t) - \eta\,\sigma_{ij}(t), \quad \forall i,j \in 1,\ldots,n. \tag{7}$$

$\lambda_1 = (-\zeta + \sqrt{\zeta^2 - 4\eta})/2$ and $\lambda_2 = (-\zeta - \sqrt{\zeta^2 - 4\eta})/2$ are characteristic roots of (7), located on the left half-plane with $\zeta > 0$ and $\eta > 0$. Thus, the second-order system (7) is stable. Moreover, in terms of different ζ and η, the following analyses are given with the initial values $\sigma_{ij}(0) = 0$ and $\dot{\sigma}_{ij}(0) = e_{ij}(0)$.

1) For $\zeta^2 > 4\eta$, $\lambda_1 \neq \lambda_2$, we get the solution to (7):

$$\sigma_{ij}(t) = C_1\exp(\lambda_1 t) + C_2\exp(\lambda_2 t),$$

where C_1 and C_2 are unknown coefficients that can be obtained by using initial values $\sigma_{ij}(0) = 0$ and $\dot{\sigma}_{ij}(0) = e_{ij}(0)$. Next, we have

$$\sigma_{ij}(t) = \frac{e_{ij}(0)[\exp(\lambda_1 t) - \exp(\lambda_2 t)]}{\sqrt{\zeta^2 - 4\eta}},$$

of which the derivative is

$$e_{ij}(t) = \frac{e_{ij}(0)[\lambda_1\exp(\lambda_1 t) - \lambda_2\exp(\lambda_2 t)]}{\sqrt{\zeta^2 - 4\eta}}.$$

Finally, we get the error in matrix form:

$$E(t) = \frac{E(0)[\lambda_1\exp(\lambda_1 t) - \lambda_2\exp(\lambda_2 t)]}{\sqrt{\zeta^2 - 4\eta}}.$$

2) For $\zeta^2 = 4\eta$, $\lambda_1 = \lambda_2$, we get the error in matrix form in the similar way:

$$E(t) = E(0)[\exp(\lambda_1 t) + \lambda_1 t\exp(\lambda_1 t)].$$

3) For $\zeta^2 < 4\eta$, $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, we get the error in matrix form in the similar way:

$$E(t) = E(0)\exp(\alpha t)[\frac{\alpha}{\beta}\sin(\beta t) + \cos(\beta t)].$$

It can be generalized from the proof in [40, Th. 1] that the error $E(t)$, which is randomly initialized, converges globally and exponentially to zero while $\zeta > 0$ and $\eta > 0$. Furthermore, combining the analyses of the above three cases, we draw the conclusion that the output $Y(t)$ of the proposed NTZNN model (4) for solving the non-stationary Lyapunov equation converges globally and exponentially to the theoretical solution of (1). The proof is thus completed. ■

B. NTZNN AT THE EXISTENCE OF NOISES

In this section, we discuss the robustness of the proposed NTZNN model (4) at the existence of various types of noises, i.e., linear noises, and random noises.

a) Linear Noises: To discuss the robustness of the proposed NTZNN model (4) at the existence of linear noises, we present the following theorem.

Theorem 2: At the existence of the linear noise $T(t) = Tt + \tau \in \mathbb{R}^{n\times n}$, where T and τ are constants, the upper bound of the steady-state error $\lim_{t\to\infty}\|E(t)\|_\mathrm{F}$ of the noise-polluted NTZNN model (6) is $\|T\|_\mathrm{F}/\eta$, and $\lim_{t\to\infty}\|E(t)\|_\mathrm{F}$ approaches to zero while η approaches to positive infinity. In addition, when $T = 0$, the linear noise $T(t) = Tt + \tau$ degrades to constant noise $T(t) = \tau$, and the residual error converges to zero globally.

Proof: The Laplace transform [41] of the linear-noise-polluted NTZNN model $\dot{E}(t) = -\zeta\,E(t) - \eta\int_0^t E(\delta)\mathrm{d}\delta + Tt + \tau$ is

$$sE(s) - E(0) = -\zeta\,E(s) - \frac{\eta}{s}E(s) + \frac{T}{s^2} + \frac{\tau}{s}, \tag{8}$$

where $T/s^2 + \tau/s$ is the Laplace transform of the linear noise $T(t) = Tt + \tau$. Then, (8) can be rewritten as

$$E(s) = \frac{s[E(0) + T/s^2 + \tau/s]}{s^2 + s\zeta + \eta},$$

of which the poles are $s_1 = (-\zeta + \sqrt{\zeta^2 - 4\eta})/2$ and $s_2 = (-\zeta - \sqrt{\zeta^2 - 4\eta})/2$, located on the left half-plane with $\zeta > 0$ and $\eta > 0$. Thus, this system is stable. According to the final value theorem [41], we obtain

$$\lim_{t\to\infty} E(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s^2[E(0) + T/s^2 + \tau/s]}{s^2 + s\zeta + \eta} = \frac{T}{\eta}.$$

Thus, we draw the conclusion that $\lim_{t\to\infty}\|E(t)\|_\mathrm{F} = \|T\|_\mathrm{F}/\eta$ and $\lim_{t\to\infty}\|E(t)\|_\mathrm{F} \to 0$ as $\eta \to \infty$. When $T = 0$, $\lim_{t\to\infty}\|E(t)\|_\mathrm{F} = 0$. That is, the output $Y(t)$ of the linear-noise-polluted NTZNN model arbitrarily approaches to the theoretical solution of the non-stationary Lyapunov equation (1) as long as η is sufficiently large. In addition, at the existence of constant noises, the steady-state error of the proposed NTZNN model converges to zero. The proof is completed. ■

As for the linear-noise-polluted CZNN model, we study its performance in the similar way. The Laplace transform [41] of the linear-noise-polluted CZNN model $\dot{E}(t) = -\zeta\,E(t) + Tt + \tau$ is

$$sE(s) - E(0) = -\zeta\,E(s) + \frac{T}{s^2} + \frac{\tau}{s}.$$

Similar to the proof to Theorem 2, we obtain

$$\lim_{t\to\infty} E(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{T/s + \tau}{\zeta} = \begin{cases}\infty, & T \neq 0,\\ \dfrac{\tau}{\zeta}, & T = 0.\end{cases}$$

Thus, we draw the conclusion that when $T \neq 0$, $\lim_{t\to\infty}\|E(t)\|_\mathrm{F} = \infty$ and when $T = 0$, $\lim_{t\to\infty}\|E(t)\|_\mathrm{F} = \|\tau\|_\mathrm{F}/\zeta$.

That is, the steady-state error of the linear-noise-polluted CZNN model goes to infinite when $T \neq 0$. In addition, when $T = 0$, the linear noise becomes constant noise $T(t) = \tau$, with the steady-state error converging to the constant $\|\tau\|_{\mathrm{F}}/\zeta$.

Based on the above analyses, it is obvious that the proposed NTZNN model (4) is superior to the CZNN model (5) at the existence of linear noises.

b) Bounded Random Noises: In practical applications, random noises are more common than linear noises. To discuss the performance of the proposed NTZNN model (4) at the existence of bounded random noises, we present the following theorem.

Theorem 3: At the existence of the bounded random noise $T(t) = \omega(t) \in \mathbb{R}^{n\times n}$, the steady-state error $\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}}$ of the noise-polluted NTZNN model (6) is bounded. Furthermore, $\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}}$ can be arbitrarily small, while ζ is sufficiently large and η is appropriate.

Proof: The ijth subsystem of the bounded random noise-polluted NTZNN model $\dot{E}(t) = -\zeta E(t) - \eta \int_0^t E(\delta)\mathrm{d}\delta + \omega(t)$ can be rewritten as

$$\dot{e}_{ij}(t) = -\zeta e_{ij}(t) - \eta \int_0^t e_{ij}(\delta)\mathrm{d}\delta + \omega_{ij}(t), \quad \forall i,j \in 1,\ldots,n. \tag{9}$$

Next, upper bounds of steady-state errors can be obtained by the following analyses in terms of different ζ and η. Note that λ_1, λ_2 and α, β have been defined in Theorem 1.

1) For $\zeta^2 > 4\eta$, $\lambda_1 \neq \lambda_2$, we get the solution to (9):

$$\begin{aligned} e_{ij}(t) = {} & \frac{e_{ij}(0)[\lambda_1 \exp(\lambda_1 t) - \lambda_2 \exp(\lambda_2 t)]}{\sqrt{\zeta^2 - 4\eta}} + \frac{1}{\sqrt{\zeta^2-4\eta}} \\ & \times \int_0^t \{\lambda_1 \exp[\lambda_1(t-\delta)] \\ & - \lambda_2 \exp[\lambda_2(t-\delta)]\}\omega_{ij}(\delta)\mathrm{d}\delta. \end{aligned}$$

According to the triangle inequality, we get

$$\begin{aligned} |e_{ij}(t)| \le {} & \frac{|e_{ij}(0)[\lambda_1 \exp(\lambda_1 t) - \lambda_2 \exp(\lambda_2 t)]|}{\sqrt{\zeta^2-4\eta}} \\ & + \frac{\int_0^t |\lambda_1 \exp[\lambda_1(t-\delta)]\omega_{ij}(\delta)|\mathrm{d}\delta}{\sqrt{\zeta^2-4\eta}} \\ & + \frac{\int_0^t |\lambda_2 \exp[\lambda_2(t-\delta)]\omega_{ij}(\delta)|\mathrm{d}\delta}{\sqrt{\zeta^2-4\eta}}. \end{aligned}$$

Furthermore, we get

$$\begin{aligned} |e_{ij}(t)| \le {} & \frac{|e_{ij}(0)[\lambda_1 \exp(\lambda_1 t) - \lambda_2 \exp(\lambda_2 t)]|}{\sqrt{\zeta^2-4\eta}} \\ & + \frac{2\int_0^t |\lambda_2 \exp[\lambda_2(t-\delta)]|\mathrm{d}\delta}{\sqrt{\zeta^2-4\eta}} \max_{0\le\delta\le t} |\omega_{ij}(\delta)|. \end{aligned}$$

Due to

$$\begin{aligned} \int_0^t |\lambda_2 \exp[\lambda_2(t-\delta)]|\mathrm{d}\delta &= \int_0^t \{-\lambda_2 \exp[\lambda_2(t-\delta)]\}\mathrm{d}\delta \\ &= 1 - \exp(\lambda_2 t) \le 1, \end{aligned}$$

we derive

$$\begin{aligned} |e_{ij}(t)| \le {} & \frac{|e_{ij}(0)[\lambda_1 \exp(\lambda_1 t) - \lambda_2 \exp(\lambda_2 t)]|}{\sqrt{\zeta^2-4\eta}} \\ & + \frac{2}{\sqrt{\zeta^2-4\eta}} \max_{0\le\delta\le t} |\omega_{ij}(\delta)|. \end{aligned}$$

Finally, in accordance with the proof in [40, Th. 1], we get

$$\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}} \le \frac{2n}{\sqrt{\zeta^2-4\eta}} \sup_{0\le\delta\le t} |\omega_{ij}(\delta)|.$$

2) For $\zeta^2 = 4\eta$, $\lambda_1 = \lambda_2$, we get the solution to (9):

$$\begin{aligned} e_{ij}(t) = {} & e_{ij}(0)[\exp(\lambda_1 t) + \lambda_1 t \exp(\lambda_1 t)] \\ & + \int_0^t \{\exp[\lambda_1(t-\delta)] \\ & + \lambda_1(t-\delta)\exp[\lambda_1(t-\delta)]\}\omega_{ij}(\delta)\mathrm{d}\delta. \end{aligned}$$

From the proof in [40, Th. 1], there exist $\iota > 0$ and $\kappa > 0$, so that $|\lambda_1| t \exp(\lambda_1 t) \le \iota \exp(-\kappa t)$. Combining the above inequality and the proof of the case 1), we get

$$\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}} \le \left(\frac{\iota}{\kappa} - \frac{1}{\lambda_1}\right) n \sup_{0\le\delta\le t} |\omega_{ij}(\delta)|.$$

3) For $\zeta^2 < 4\eta$, $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, we get the solution to (9):

$$\begin{aligned} e_{ij}(t) = {} & e_{ij}(0)\exp(\alpha t)[\frac{\alpha}{\beta}\sin(\beta t) + \cos(\beta t)] \\ & + \int_0^t \{\exp[\alpha(t-\delta)]\frac{\alpha}{\beta}\sin[\beta(t-\delta)] \\ & + \exp[\alpha(t-\delta)]\cos[\beta(t-\delta)]\}\omega_{ij}(\delta)\mathrm{d}\delta. \end{aligned}$$

In the similar way, we get

$$\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}} \le \frac{4n\sqrt{\eta}}{\zeta\sqrt{4\eta-\zeta^2}} \sup_{0\le\delta\le t} |\omega_{ij}(\delta)|.$$

Based on the analyses of the above three cases, we draw the conclusion that at the existence of the bounded random noise $T(t) = \omega(t)$, the steady-state error $\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}}$ of the noise-polluted NTZNN model (6) is bounded. Furthermore, $\lim_{t\to\infty}\|E(t)\|_{\mathrm{F}}$ can be arbitrarily small, while ζ is sufficiently large and η is appropriate. The proof is completed. ■

IV. ILLUSTRATIVE EXAMPLE

In previous sections, we have proposed the NTZNN model (4) for solving the non-stationary Lyapunov equation (1), and analyzed its convergence and robustness at the existence of noises. In this section, in order to substantiate the superiority of the proposed NTZNN model (4) to the CZNN model (5), comparative computer simulations are conducted at the existence of various types of noises.

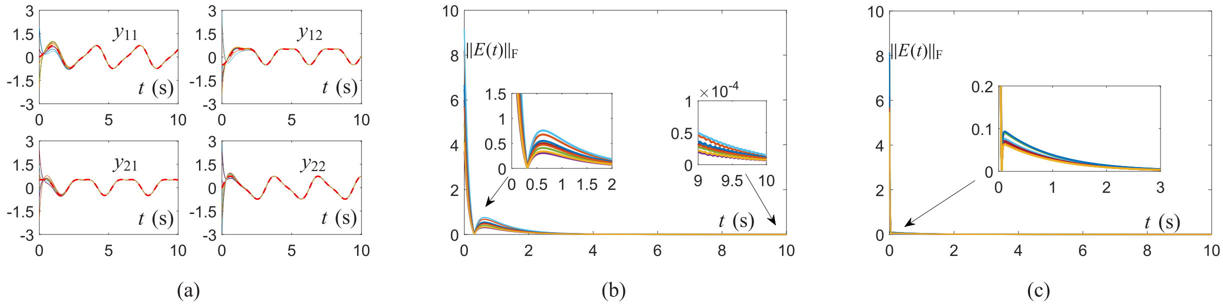


FIGURE 1. Outputs $Y(t)$ starting from ten randomly initial states $Y(0) \in [-3, 3]^{2\times 2}$ and residual errors $\|E(t)\|_F$ of the proposed NTZNN model (4) for solving the non-stationary Lyapunov equation (1) in the absence of noises. (a) $Y(t)$ with $\zeta = 8$ and $\eta = 8$, where the theoretical solution $Y^*(t)$ is denoted by red dotted line and colored solid lines stand for outputs $Y(t)$. (b) $\|E(t)\|_F$ with $\zeta = 8$ and $\eta = 8$. (c) $\|E(t)\|_F$ with $\zeta = 80$ and $\eta = 80$.

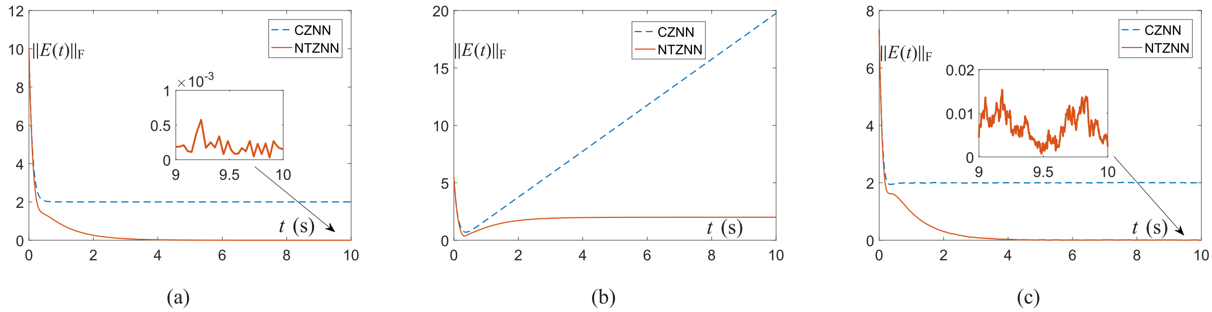


FIGURE 2. Residual errors $\|E(t)\|_F$ of the proposed NTZNN model (4) and the CZNN model (5) for solving the non-stationary Lyapunov equation (1) with $\zeta = 8$ and $\eta = 8$. (a) $\|E(t)\|_F$ at the existence of the constant noise $T(t) = [8]^{2\times 2}$. (b) $\|E(t)\|_F$ at the existence of the linear noise $T(t) = [8t]^{2\times 2}$. (c) $\|E(t)\|_F$ at the existence of the random noise $T(t) = [7.5, 8.5]^{2\times 2}$.

In this example, time-varying coefficient matrices $M(t)$ and $N(t)$ are defined as

$$M(t) = \begin{bmatrix} -1 - \frac{1}{2}\mathcal{C} & \frac{1}{2}\mathcal{S} \\ \frac{1}{2}\mathcal{S} & -1 + \frac{1}{2}\mathcal{C} \end{bmatrix}, \; N(t) = \begin{bmatrix} \mathcal{S} & \mathcal{C} \\ -\mathcal{C} & \mathcal{S} \end{bmatrix},$$

where $\mathcal{C} = \cos(2t)$, $\mathcal{S} = \sin(2t)$. For such a simple example, it is easy to get the following theoretical solution $Y^*(t)$ to (1):

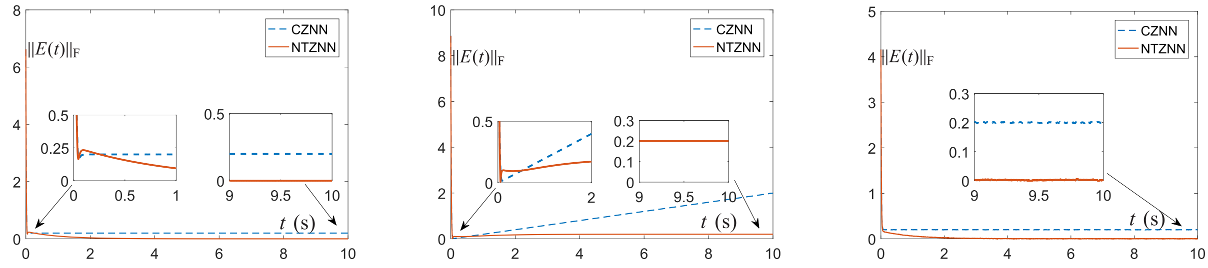
$$Y^*(t) = \begin{bmatrix} \frac{\mathcal{S}(2-\mathcal{C})}{3} & \frac{(1-2\mathcal{C})(2+\mathcal{C})}{6} \\ \frac{(1+2\mathcal{C})(2-\mathcal{C})}{6} & \frac{\mathcal{S}(2+\mathcal{C})}{3} \end{bmatrix}.$$

In order to reveal the superiority of the proposed NTZNN model (4) to the CZNN model (5) in tolerating noises, we adopt them to perform computer simulations under the noise-free case and the noisy case, respectively. Corresponding simulation results are displayed in Fig. 1 through Fig. 3.

After adopting the proposed NTZNN model (4) to solve the non-stationary Lyapunov equation (1) in the absence of noises, the corresponding simulation results are obtained as shown in Fig. 1. To be specific, with $\zeta = 8$ and $\eta = 8$, Fig. 1(a) displays the outputs $Y(t)$ of the proposed NTZNN model (4), while Fig. 1(b) displays the corresponding residual errors $\|E(t)\|_F$. As can be seen from Fig. 1(a), the outputs $Y(t)$ from ten randomly initial states $Y(0) \in [-3, 3]^{2\times 2}$ tend to the theoretical solution $Y^*(t)$ denoted by red dotted line. Besides, it is shown in Fig. 1(b) that residual errors $\|E(t)\|_F$ converge to zero in 6 s. Furthermore, Fig. 1(c) displays residual errors $\|E(t)\|_F$ with $\zeta = 80$ and $\eta = 80$. According to Fig. 1(b) and Fig. 1(c), it is obvious that residual errors $\|E(t)\|_F$ with $\zeta = 80$ and $\eta = 80$ converge faster than $\|E(t)\|_F$ with $\zeta = 8$ and $\eta = 8$. There is no doubt that the proposed NTZNN model (4) is viable and valid.

As for noisy cases, Fig. 2 shows the corresponding residual errors $\|E(t)\|_F$ synthesized separately by the CZNN model (5) and the proposed NTZNN model (4) with $\zeta = 8$ and $\eta = 8$. Next, we discuss the following three cases separately.

1) For constant noises, as shown in Fig. 2(a), the residual error $\|E(t)\|_F$ of the proposed NTZNN model (4) still tends to zero, while the CZNN model (5) converges to a non-zero constant. It is the convergence performance of them that highlights the prominent superiority of the proposed NTZNN model (4) to the CZNN model (5).
2) For linear noises, as shown in Fig. 2(b), there is a great difference between the CZNN model (5) and the proposed NTZNN model (4). That is, the residual error $\|E(t)\|_F$ of the proposed NTZNN model (4) tends to a constant related to the parameter η. To be specific, according to Theorem 2, so long as the η is sufficiently large, the residual error $\|E(t)\|_F$ may tend to zero. On the contrary, the residual error $\|E(t)\|_F$ of the CZNN model



(a) (b) (c)

FIGURE 3. Residual errors $\|E(t)\|_F$ of the proposed NTZNN model (4) and the CZNN model (5) for solving the non-stationary Lyapunov equation (1) with $\zeta = 80$ and $\eta = 80$. (a) $\|E(t)\|_F$ at the existence of the constant noise $T(t) = [8]^{2\times2}$. (b) $\|E(t)\|_F$ at the existence of the linear noise $T(t) = [8t]^{2\times2}$. (c) $\|E(t)\|_F$ at the existence of the random noise $T(t) = [7.5, 8.5]^{2\times2}$.

(5) increases rapidly, and there is no tendency to stop. There is no doubt that the proposed NTZNN model (4) is superior to the CZNN model (5) under linear noises.

3) For random noises, as shown in Fig. 2(c), the proposed NTZNN model (4) performs better than the CZNN model (5) with the random noise $T(t) = [7.5, 8.5]^{2\times2}$. Thereinto, the random noise $T(t) = [7.5, 8.5]^{2\times2}$ can be decomposed into a random noise $T(t) = [-0.5, 0.5]^{2\times2}$ and a constant noise $T(t) = [8]^{2\times2}$. The residual error $\|E(t)\|_F$ of the proposed NTZNN model (4) tends to zero and stays near 0.01, whlie the residual error $\|E(t)\|_F$ of the CZNN model (5) stays at a high level. The results also indicate the superiority in handling random noises of the proposed NTZNN model (4) to the CZNN model (5).

Furthermore, Fig. 3 displays residual errors $\|E(t)\|_F$ synthesized separately by the CZNN model (5) and the proposed NTZNN model (4) at the existence of noises with $\zeta = 80$ and $\eta = 80$. Compared with Fig. 2, simulation results in Fig. 3 are similar, except the convergence speed of residual errors $\|E(t)\|_F$. To be specific, residual errors $\|E(t)\|_F$ with $\zeta = 80$ and $\eta = 80$ converge faster than $\|E(t)\|_F$ with $\zeta = 8$ and $\eta = 8$.

In a word, we draw the conclusion that the proposed NTZNN model (4) is effective for solving the non-stationary Lyapunov equation (1) and is superior to the CZNN model (5) at the existence of various types of noises.

V. CONCLUSION

For solving the non-stationary Lyapunov equation (1) at the existence of various types of noises, this paper has proposed the NTZNN evolution formula (3), from which the NTZNN model (4) has been derived. Furthermore, the global and exponential convergence performance of the proposed NTZNN model (4) has been proven in the absence of noises. Then, the anti-noise ability of the proposed NTZNN model (4) has been proven. To be specific, at the existence of linear noises and random noises, the steady-state error of the noise-polluted NTZNN model (6) is bounded. At last, computer simulations have been conducted by adopting the proposed NTZNN model (4) and the CZNN model (5) to solve the non-stationary Lyapunov equation (1) under the noise-free case and the noisy case, respectively. The simulation results reveal the practicality and prominent superiority of the proposed NTZNN model (4) to the CZNN model (5). In a word, the proposed NTZNN model (4) has a great contribution to solve the non-stationary Lyapunov equation at the existence of noises. Regarding the future research direction, we intend to construct models which have better performance than the NTZNN model proposed in this paper, and apply them specifically.

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