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Finite Vertex Bi-Primitive 2-Arc Transitive Graphs Admitting a Two-Dimensional Linear Group

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ABSTRACT A graph is said to be vertex bi-primitive, if it is a bipartite graph, and the setwise stabilizer of its automorphism group acts primitively on two bi-parts. In this paper, we not only classify vertex bi-primitive 2-arc transitive graphs admitting a two-dimensional linear group but also determine the automorphism groups and the number of non-isomorphic ones of such graphs.

INDEX TERMS Arc-transitive graph, coset graph, vertex bi-primitive.

I. INTRODUCTION

In this paper, we assume that all graphs are finite, simple and undirected.

Denote by X a finite connected graph with vertex set $V(X)$ and edge set $E(X)$. Let $Arc(X)$ and $\mathrm{Aut}(X)$ be the arc set and full automorphism group of X, respectively. For a positive integer s, an *s-arc* in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of $s+1$ vertices such that $(v_{i-1}, v_i) \in A(X)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. For a subgroup G of $\mathrm{Aut}(X)$, the graph X is said to be *(G, s)-arc-transitive* if G acts transitively on the set of s-arcs of X. A graph X is said to be *s-arc-transitive*, if it is $(\mathrm{Aut}(X), s)$-arc-transitive. Throughout this paper, we will denote by $\mathbb{Z}_n$ the cyclic group of order n, by D_{2n} the dihedral group of order $2n$, and by A_n and S_n the alternating group and the symmetric group of degree n, respectively.

Since Tutte (1947) [28] proved that there exist no finite s-transitive cubic graphs for $s \geq 6$, s-transitive graphs has received lots of attention. For example, Weiss [29] proved that there exists no 8-arc transitive graphs of valence at least 3. Praeger [25] started a general analysis of automorphism groups of 2-arc transitive graphs, and Fang and Praeger [9] classified finite 2-arc transitive graphs admitting a Suzuki simple group. Let p and q be primes. Li *et al.* [21] and Praeger and Xu [26] classified vertex primitive symmetric graphs of order kp with $k < p$, and Ivanov and Praeger [17] classified affine primitive 2-arc transitive graphs. Li [18] classified vertex primitive and vertex bi-primitive s-transitive graphs for $s \geq 4$, Fang *et al.* [8] classified vertex primitive 2-arc regular graphs, and Li *et al.* [19] classified vertex primitive and vertex bi-primitive s-arc-regular graphs for $s \geq 3$. Recently, Pan *et al.* [23] classified connected 2-arc-transitive primitive and bi-primitive graphs of fourth-power-free order.

Now we first introduce the so called coset graph (see [24], [27]). Let G be a finite group, H a subgroup of G, and D a union of several double-cosets of the form HgH with $g \notin H$ such that $D = D^{-1}$. The *coset graph* $X = \mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined as the graph with vertex set $V(X) = [G : H]$, the set of right cosets of H in G, and edge set $E(X) = \{\{Hg, Hdg\} \bigm| g \in G,\ d \in D\}$. It is easy to see that X is well defined and has valence $|D|/|H| = |H : H \cap H^g|$. Further, X is connected if and only if D generates G, and X is G-arc-transitive if and only if $D = HgH$, a single double coset. Denote by H_G the largest normal subgroup of G in H. Then $H_G = \bigcap_{g\in G} H^g$ and if $H_G = 1$, we say that H is *core-free* in G. In what follows, we always assume that H is core-free in G whenever we mention a coset graph $\mathrm{Cos}(G, H, D)$.

Next, we introduce the concept of standard double cover. Let X be a graph. The *standard double cover* $X^{(2)}$ of X is defined as the graph with vertex set $\{u_1, u_2 \mid u \in V(X)\}$ and edge set $\{\{u_1, v_2\}, \{u_2, v_1\} \mid \{u, v\} \in E(X)\}$. The finite vertex-primitive 2-arc transitive graphs admitting a two-dimensional linear group is completed by author in [15]. In this paper, we classify vertex bi-primitive 2-arc transitive graphs admitting a two-dimensional projective linear group. Furthermore, we determine the automorphism groups and enumerate the number of non-isomorphic ones of such graphs.

Finally, we state the main results of this paper.

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TABLE 1. The vertex bi-primitive $(G, 2)$-arc-transitive graphs.

	G	G^*	H	k
X_1	$\mathrm{PGL}(2,5)$	$\mathrm{PSL}(2,5)$	D_6	3
X_2	$\mathrm{P\Gamma L}(2,27)$	$\mathrm{P\Sigma L}(2,27)$	$D_{28}.\mathbb{Z}_3$	7
X_3	$\mathrm{PGL}(2,p)$ $p=11,19$	$\mathrm{PSL}(2,p)$	D_{p+1}	$\frac{p+1}{4}$
X_4	$\mathrm{PGL}(2,p)$ $p=13,37$	$\mathrm{PSL}(2,p)$	D_{p-1}	$\frac{p-1}{4}$
X_5	$\mathrm{PGL}(2,p)$ $p\equiv\pm3,\pm13 \pmod{40}$	$\mathrm{PSL}(2,p)$	A_4	4
X_6	$\mathrm{PGL}(2,3^f)$	$\mathrm{PSL}(2,3^f)$	A_4	4
X_7	$\mathrm{PGL}(2,p)$ $p\equiv\pm11,\pm19 \pmod{40}$	$\mathrm{PSL}(2,p)$	A_5	5
X_8	$\mathrm{PGL}(2,p)$ $p\equiv\pm9 \pmod{20}$	$\mathrm{PSL}(2,p)$	A_5	6
X_9	$\mathrm{PGL}(2,q)$ $q=p^2\equiv 9 \pmod{20}$	$\mathrm{PSL}(2,q)$	A_5	6
X_{10}	$M(1,q)$ $q=p^2\equiv 9 \pmod{20}$	$\mathrm{PSL}(2,q)$	A_5	6
X_{11}	$\mathrm{PGL}(2,p)$ $p\equiv\pm7 \pmod{24}$	$\mathrm{PSL}(2,p)$	S_4	
X_{12}	$\mathrm{PGL}(2,p)$ $p\equiv\pm7 \pmod{16}$	$\mathrm{PSL}(2,p)$	S_4	3
X_{13}	$\mathrm{PGL}(2,9)$	$\mathrm{PSL}(2,9)$	S_4	3
X_{14}	M_{10}	$\mathrm{PSL}(2,9)$	S_4	3
X_{15}	$\mathrm{P\Gamma L}(2,9)$	$\mathrm{P\Sigma L}(2,9)$	$S_4\times\mathbb{Z}_2$	3

TABLE 2. The automorphism group of $(G, 2)$-arc-transitive bi-primitive graphs.

	G	$\mathrm{Aut}(X_i)$	n
X_1	$\mathrm{PGL}(2,5)$	$\mathrm{PGL}(2,5)$	1
X_2	$\mathrm{P\Gamma L}(2,27)$	$\mathrm{P\Gamma L}(2,27)$	1
X_3	$\mathrm{PGL}(2,p)$ $p=11,19$	$\mathrm{PGL}(2,p)$	1
X_4	$\mathrm{PGL}(2,p)$ $p=13,37$	$\mathrm{PGL}(2,p)$	1
X_5	$\mathrm{PGL}(2,p)$ $p\equiv\pm3,\pm13 \pmod{40}$	$\mathrm{PGL}(2,p)$	$\lceil\frac{p+\varepsilon}{12}\rceil-1$
X_6	$\mathrm{PGL}(2,3^f)(f>3)$ a prime	$\mathrm{PGL}(2,3^f)$	$\frac{3^f-3}{6f}$
	$\mathrm{PGL}(2,3^f)(f=3)$	$\mathrm{PGL}(2,3^3)$	1
	$\mathrm{PGL}(2,3^f)(f=3)$	$\mathrm{P\Gamma L}(2,3^3)$	1
X_7	$\mathrm{PGL}(2,p)$ $p\equiv\pm11,\pm19 \pmod{40}$	$\mathrm{PGL}(2,p)$	1
X_8	$\mathrm{PGL}(2,p)$ $p\equiv\pm9 \pmod{20}$	$\mathrm{PGL}(2,p)$	1
X_9	$\mathrm{PGL}(2,q)$ $q=p^2\equiv 9 \pmod{20}$	$\mathrm{P\Gamma L}(2,q)$	1
X_{10}	$M(1,q)$ $q=p^2\equiv 9 \pmod{20}$	$\mathrm{P\Gamma L}(2,q)$	1
X_{11}	$\mathrm{PGL}(2,p)$ $p\equiv\pm7 \pmod{24}$	$\mathrm{PGL}(2,p)$	1
X_{12}	$\mathrm{PGL}(2,p)$ $p\equiv\pm7 \pmod{16}$	$\mathrm{PGL}(2,p)$	1
X_{13}	$\mathrm{PGL}(2,9)$	$\mathrm{P\Gamma L}(2,9)$	1
X_{14}	M_{10}	$\mathrm{P\Gamma L}(2,9)$	1
X_{15}	$\mathrm{P\Gamma L}(2,9)$	$\mathrm{P\Gamma L}(2,9)$	1

Theorem 1: Let X be a $(G, 2)$-arc transitive bipartite graph of valence $k \geq 3$, and let G^* be a subgroup of G, which has the socle $\mathrm{PSL}(2, q)$, and acts primitively on each partite of X, where $q = p^f$ for some positive integer f and prime p. Then X is either a complete bipartite graph, a standard double cover of vertex primitive graphs, or a coset graph $\mathrm{Cos}(G, H, HgH)$ as given in Table 1. Furthermore, the automorphism groups and the number of non-isomorphic ones of such graphs as given in Table 2.

II. PRELIMINARY RESULTS

We first recall some group-theoretic results. By [16, Chapter II, Lemma 8.21] or [15, Proposition 2.4], one has the following proposition.

Proposition 2: Let G be a subgroup of $\mathrm{PSL}(2, p^f)$, and let $P \neq 1$ a Sylow p-subgroup of G, and $N = N_G(P)$. Then P is isomorphic to a subgroup of the additive group $GF(p^f)^+$, and there is a subgroup Z of G such that $N = PZ$ and $P \cap Z = 1$, where $Z = 1$ or Z is a maximal cyclic subgroup of G, $|Z| \mid |P| - 1$ and $|Z| \mid p^f - 1$.

Note: Let $f = sm$, $G = \mathrm{PSL}(2, p^s)$ and P be a Sylow-p subgroup of G. If both p and m are odd, then $|Z| \mid \frac{|P|-1}{2}$. Since the order of maximal cyclic subgroup of $\mathrm{PSL}(2, p^f)$ is $\frac{p^f-1}{2}$, if $|Z| = |P| - 1$ then $p^s - 1 \mid \frac{p^f-1}{2}$. It implies that the power of 2 of $p^f - 1$ is more than the power of 2 of $p^s - 1$, which is impossible because $p^f - 1 = (p^s)^m - 1 = (p^s - 1)[(p^s)^{m-1} + (p^s)^{m-2} + \cdots + p^s + 1]$, and $(p^s)^{m-1} + (p^s)^{m-2} + \cdots + p^s + 1$ is an odd.

Next, we give some characteristics of coset graph. By Godsil [12] and Dobson [7], one has the following proposition.

Proposition 3: Let G be a finite group, H be a core-free subgroup of G and D a union of several double-cosets HgH with $g \notin H$ such that $D = D^{-1}$. Let $X = \mathrm{Cos}(G, H, D)$, $A = \mathrm{Aut}(X)$ and $\sigma(g)$ the automorphism of G induced by the conjugate of $g \in G$ on G. Then $N_A(R(G)) = R(G)\mathrm{Aut}(G, H, D)$ and $R(G) \cap \mathrm{Aut}(G, H, D) = I(H)$, where $\mathrm{Aut}(G, H, D) = \{\alpha \in \mathrm{Aut}(G) \mid H^\alpha = H, D^\alpha = D\}$ and $I(H) = \{\sigma(h) \mid h \in H\}$.

Remark: If the coset graph $X = \mathrm{Cos}(G, H, D)$ is $R(G)$-arc transitive then $D = HgH$, a single double coset of H in G. In this case, $I(H)$ is transitive on the neighborhood of the vertex H in X and so is $\mathrm{Aut}(G, H, D)$.

As we known, if we use coset graph $\mathrm{Cos}(G, H, D)$ to characterize 2-arc-transitive graphs admitting a given group G, the most important thing is to find an 2-element g (or proving no existence of g) satisfying H acts 2-transitively on the set of cosets $[H : H \cap H^g]$. We give a condition to guarantee the 2-arc transitivity and bi-primitivity of the graph.

Lemma 4: Let X be a connected $(G, 2)$-arc-transitive graph with valency at least 3, where $G \leq Aut(X)$ is bi-primitive on the vertex set $V(X)$. Then there exists a index two subgroup G^* of G, a core-free maximal subgroup H of G^*, and a maximal subgroup L of H with index at least 3 satisfying the following condition.

Condition 1: The normalizer $N_G(L)$ has a subgroup $R \nsubseteq H$ such that $R \cong L.\mathbb{Z}_2$ and the action of H on $[H : L]$ by multiplication is 2-transitive, and there exists a 2-element $g \in R \setminus H$ such that $\langle H, g\rangle = G$

In particular, $X = \mathrm{Cos}(G, H, HgH)$, where $g \in R \setminus H$. Conversely, if G has a index two subgroup G^*, G^* has a core-free maximal subgroup H, and H has a maximal subgroup L with index at least 3 satisfying Condition $1'$, then the coset graph $\mathrm{Cos}(G, H, HgH)$ with $g \in R \setminus H$ is a connected $(G, 2)$-arc-transitive bi-primitive graph with valency $|H : L|$.

Proof: Let $u \in V(X)$ and $H = G_u$. By [9, Theorme 2.1], there exists a 2-element g such that $X = \mathrm{Cos}(G, H, HgH)$, where $g^2 \in H$. Let $L = H \cap H^g$. Then H acts 2-transitively on $[H : L]$ by multiplication, implying that L is maximal in H with $|H : L| = Val(X) \geq 3$. Further, as $L^g = H^g \cap H^{g^2} = H^g \cap H = L$, $g \in N_G(L)$. Since $\langle H, g\rangle = G$, $g \notin H$ and $g^2 \in L$. Then $R = \langle L, g\rangle = L.\mathbb{Z}_2 \notin H$ is a subgroup of $N_G(L)$.

For the converse, suppose that G has a index two subgroup G^*, and G^* has a core-free maximal subgroup H, and H has a maximal subgroup L with index at least 3 satisfying Condition 1. Then G^* is intransitive on $V(X)$ because if G^* is transitive then $G = G^*.H = G^*$. Take $g \in R \setminus H$, one has $g \in N_G(L)$, and $L \subseteq H \cap H^g$. If $H = H \cap H^g$, then g normalizes H, and $H \subseteq \langle H, g\rangle = H : \mathbb{Z}_2 = G$, which contradicts that $|G : H| = |V(X)| \geq 3$. Hence, by the maximality of L in H, one has $L = H \cap H^g$. It is easy to see that $\mathrm{Cos}(G, H, HgH)$ is a connected $(G, 2)$-arc-transitive vertex bi-primitive graph with valency $|H : L|$.

For convenience, we always suppose $X = \mathrm{Cos}(G, H, HgH)$, where $H = G_u, u \in V(X)$. Let $v = u^g$ and $L = G_{uv} = H \cap H^g$. Then H is 2-transitive on $[H : L]$ by right multiplication. In particular, L is a maximal subgroup of H with index ≥ 3. Furthermore, $N_G(L)$ has a subgroup $R \nsubseteq H$ such that $R = L.\mathbb{Z}_2$.

Next, by [15], we have the lemma about the number of non-isomorphic ones of such graph.

Lemma 5 [15, Lemma 2.7]: Let G be a finite group. Assume that G has a core-free subgroup H, and assume further that H contains the subgroup $L = H \cap H^g$ such that $N_G(L) = L.\mathbb{Z}_2$ and $N_H(L) = L$. If L has unique conjugate class in H, then there is unique non-isomorphic graph $X = \mathrm{Cos}(G, H, HgH)$ such that X is a connected G-arc transitive graph.

Finally, the lemma is crucial for determining the full automorphism group $\mathrm{Aut}(X)$.

Lemma 6: Let (G, H) be a pair given in Tables 1, and let $X = \mathrm{Cos}(G, H, HgH)$ for some $g \in R \setminus H$, where R is as defined in Lemma 4. Then G is normal in $\mathrm{Aut}(X)$, and $\mathrm{Aut}(X) \leq \mathrm{Aut}(T)$, where $T = \mathrm{soc}(G)$, the socle of G.

Proof: Let $A = \mathrm{Aut}(X)$. Set $n = |V(X)|$. Then $T \trianglelefteq G \leq A \leq Sym(n)$. Suppose that $\mathrm{soc}(A) \neq T$. Then there exist groups F_1 and F such that $G \leq F_1 < F \leq A \leq Sym(n)$, $T = \mathrm{soc}(F_1) \neq \mathrm{soc}(F)$, and F_1 is maximal in F. Note that G is almost simple and transititve on $V(X)$, all the possibilities for (F_1, F) are list in [22]. For Table 1, inspecting these pairs, there are not pairs $(T, \mathrm{soc}(F))$ satisfying these conditions. Therefore, for any $X = \mathrm{Cos}(G, H, HgH)$ in Table 1, one has $\mathrm{soc}(A) = T$, and $A \leq \mathrm{Aut}(T)$. Checking the groups listed in Table 1, it is easily shown that $G \trianglelefteq \mathrm{Aut}(T)$, so $G \trianglelefteq A$. □

III. VERTEX BI-PRIMITIVE GRAPHS

Suppose G is biprimitive on V with two parts Δ_1 and Δ_2. Let $G^* = G_{\Delta_1} = G_{\Delta_2}$.

Lemma 7: Let G^* be an almost simple with $T = \mathrm{soc}(G^*)$, and let X be a $(G, 2)$-arc transitive vertex bi-primitive graph. Then either G is an almost simple, or X is the standard double cover of a vertex-primitive $(G^*, 2)$-arc transitive graph.

Proof: Assume that G is not an almost simple group. It follows that $C := C_G(T) \neq 1$. Since $|G : G^*| = 2$ and $C_{G^*}(T) = 1$, we have that $C \cong \mathbb{Z}_2$. Now $C \lhd G$, and it then follows that $G = G^* \times \mathbb{Z}_2$. Let Γ_C be the quotient of Γ induced by C. By [13, Proposition 2.6], Γ is a standard double cover of Γ_C. Further $G^* \cong G/C \leq \mathrm{Aut}(\Gamma_C)$. Then G/C is primitive on $V(\Gamma_C)$, and by [25, Theorem 4.1], Γ_C is $(G/C, 2)$-transitive. □

Proof of Theorem 1: Let $q = p^f$. Since G^* is primitive on Δ_i, where $i = 1, 2$. Suppose that G^* acts unfaithfully on Δ_1. Then the pointwise stabilizer $K_1 = (G^*)_{(\Delta_1)} \neq 1$ and $K_1 \trianglelefteq G^*$. Since $(G^*)^{\Delta_2}$ is primitive, K_1 acts transitively on Δ_2. This implies that X is a complete bipartite graph $K_{t,t}$ for some $t \geq 3$, a contradiction. Thus, G^* acts faithfully and primitively on Δ_1 and Δ_2, which implies that T is transitive on $\Delta_i (i = 1, 2)$. Assume that X is not a standard double cover of vertex primitive graph. Then by Lemma 7, G is an almost simple group with $\mathrm{soc}(G) = \mathrm{soc}(G^*) = T$. Suppose T is primitive on Δ_i, T_u is a maximal subgroup of T. By [6, Section 239], $T_u \in \{D_{\frac{2(q\pm1)}{d}}, \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}, A_5, S_4, A_4, \mathrm{PGL}(2, p^r), \mathrm{PSL}(2, p^s)\}$, where $d = (2, q-1)$, $2r = f$, $sm = f$ with m an odd prime.

For $q = 5$, it is obvious that $G = \mathrm{PGL}(2, 5)$ and $H = A_4, \mathbb{Z}_5 \rtimes \mathbb{Z}_2$ or D_6. Then $V(X) = 10, 12$ or 20, respectively. Checking these graph of order 10, 12, there are not these biprimitive graphs. For $H = D_6$, one has $L = \mathbb{Z}_2$ and $N_G(L) = D_8$. Obviously, there is an involution $g \in N_G(L) \setminus L$ such that $N_G(H) < \langle H, g\rangle = G$. By lemma 4, the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected 2-arc transitive biprimitive graph which is a graph defined in row 1 of Table 1, which is isomorphic to the generalized Petersen graph $X = GP(10, 2)$, which is isomorphic to the coset graph $X = \mathrm{Cos}(G, H, HgH)$.

For $q = p^f > 5$, suppose that $T_u = \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{d}}$. Since G^* is at least 2-transitive on Δ_i, one has $X \cong K_{q+1,q+1}$, a contradiction.

Suppose that $T_u = D_{\frac{2(q\pm1)}{d}}$. Then $T_{uv} = D_{\frac{2(q\pm1)}{dt}}$. Let O be a subgroup of the outer automorphism group of G^*. By [23, Corollary 3.2], one has $H = D_{\frac{2(q\pm1)}{d}}.O$, and $L = D_{\frac{2(q\pm1)}{dt}}.O$, where t is an odd prime and divides $q \pm 1$. When $T_{uv} \ncong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$, we claim that there is no subgroup of G which is isomorphic to $L.\mathbb{Z}_2$. Suppose that there exists a subgroup R which is isomorphic to $L.\mathbb{Z}_2$. Let $R = L.\langle g\rangle = L \cup Lg$, where $g^2 \in L$. On the one hand, $\frac{|R\cap T|}{|T_{uv}|} = \frac{|O||R||T|}{|L|\cdot|RT|} = \frac{|R|}{|L|}\cdot\frac{|O||T|}{|RT|} = 2\frac{|G|}{|RT|} \geq 2$. On the other hand, $R\cap T = (L\cup Lg)\cap T = (L\cap T)\cup(Lg\cap T) = T_{uv} \cup (Lg \cap T)$. For any $h, t \in Lg \cap T$, since $g^2 \in L$, one has $ht^{-1} \in L \cap T$, that is, $ht^{-1} \in T_{uv}$, which implies that $T_{uv}h = T_{uv}t$. It follows that $|R \cap T| \leq |T_{uv} \cup T_{uv}t| \leq 2|T_{uv}|$. Thus, $|R\cap T| = 2|T_{uv}|$, and $T_{uv} \lhd R\cap T < T$, $R\cap T = T_{uv}.\mathbb{Z}_2$. If $T_{uv} \ncong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$, then there is an element $a \in T_{uv}$ such that $R \cap T \leq N_T(\langle a\rangle) = T_u$, which is contradict to $2 \nmid |T_u : T_{uv}|$.

Thus $N_G(L) = L$. By the claim and by Proposition 4, no graph aries. Assume that $T_{uv} \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. By the 2-transitivity of H on $[H : L] = [T_u : T_{uv}] = \Omega$, for even q, it is easy to see that $q = 8$ with $T_u = D_{14}, T_{uv} = \mathbb{Z}_2$. Since $|\mathrm{Aut}(T) : T| = 3$, there is no subgroup G of $\mathrm{Aut}(T)$ such that $|G : T| = 2$, a contradiction. For odd q, if $T_u = D_{q+1}$ then $q = p = 11, 19, 27$ with $T_{uv} = \mathbb{Z}_2^2$, and if $T_u = D_{q-1}$ then $q = p = 27$ with $T_{uv} = \mathbb{Z}_2$ or $q = p = 13, 37$ with $T_{uv} = \mathbb{Z}_2^2$. It follows that $G = \mathrm{PGL}(2, p), H = D_{p+1}, L = \mathbb{Z}_2^2$ with $p = 11, 19$ or $G = \mathrm{PGL}(2, p), H = D_{p-1}, L = \mathbb{Z}_2^2$ with $p = 13, 37$ or $G = P\Gamma L_2(27), H = D_{28}.\mathbb{Z}_3, L = \mathbb{Z}_2^2.\mathbb{Z}_3$. For $G = P\Gamma L_2(27), H = D_{28}.\mathbb{Z}_3, L = \mathbb{Z}_2^2.\mathbb{Z}_3$, one has $N_{G^*}(L) = A_4.\mathbb{Z}_3, N_G(L) = S_4.\mathbb{Z}_3$, and $N_G(H) = D_{26}.D_6$ by Magma [2]. Let $g \in N_G(L) \setminus L$ be an involution, one has $N_G(H) \lessdot \langle H, g\rangle = G$. It follows that the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected 2-arc transitive biprimitive graph by lemma 4, and X is a graph as in row 2 of Table 1. For $G = \mathrm{PGL}(2, p), H = D_{p+1}(p = 11, 19)$ or $D_{p-1}(p = 13, 37)$ and $L = \mathbb{Z}_2^2$, one has $N_{G^*}(L) = A_4, N_G(L) = S_4$ and $N_G(H) = D_{2(p+1)}$ or $D_{2(p-1)}$ depending on $H = D_{p+1}$ or D_{p-1}. Let $g \in N_G(L) \setminus L$ be an involution. Obviously, $N_G(H) \lessdot \langle H, g\rangle = G$. By lemma 4, the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected 2-arc transitive biprimitive graph, and defined in rows 3-4 of Table 1.

Suppose that $T_u = A_4$. Then $q = p \equiv 3, 13, 27, 37 \pmod{40}$, implying that $\mathrm{Out}(T) = \mathbb{Z}_2$, $G = \mathrm{PGL}(2, p)$ and $H = T_u$. If $L = T_{uv} = \mathbb{Z}_2^2$, then $N_{G^*}(L) = A_4 = H$, and $N_G(L) = S_4 = N_G(H)$. For any $g \in N_G(L) \setminus L$, one has $\langle g, H\rangle = N_G(H) \neq G$, no graph appears. Let $L = T_{uv} = \mathbb{Z}_3$. Then L is conjugate to a subgroup of D_{q+1} or D_{q-1}, depending on $3 \mid q + 1$ or $q - 1$. It follows that $N_{G^*}(L) = D_{q+1}$ or D_{q-1}, and $N_G(L) = D_{2(q+1)}$ or $D_{2(q-1)}$. There are 2-elements $g \in N_G(L) \setminus N_{G^*}(L)$ such that $\langle H, g\rangle = G$. By Lemma 4, $X = \mathrm{Cos}(G, H, HgH)$ is a $(G, 2)$-arc-transitive graph of valency 4, and X is a graph as in row 5 of Table 1.

Suppose that $T_u = \mathrm{PSL}(2, p^s)$, where $sm = f$ for some odd prime m. When $p^s = 2, f = m$ is an odd prime and $T_u = \mathrm{PSL}(2, 2) \cong S_3$. In this case, $T_u < D_{2(2^f+1)}$ is not a maximal subgroup of T, a contradiction. When $p^s = 3$, $f = m$ is an odd prime and $T_u = \mathrm{PSL}(2, 3) \cong A_4$. Since both G_u^* and T_u are 2-transitive groups of degree 4, one may assume $G^* = T$. It implies that $G = \mathrm{PGL}(2, 3^f), H = A_4$ and $L = \mathbb{Z}_3$. By the subgroup of $\mathrm{PSL}(2, q)$ (see [6]), one has $N_{G^*}(L) \leq \mathbb{Z}_3^f{:}\mathbb{Z}_{\frac{3^f-1}{2}}$. Obviously, $N_{G^*}(L) = \mathbb{Z}_3^f$, and $N_G(L) = N_{G^*}(L).\mathbb{Z}_2$. It follows that there are 2-elements $g \in N_G(L) \setminus N_{G^*}(L)$ such that $\langle H, g\rangle = G$, and X is a $(G, 2)$-arc-transitive graph of valency 4 which give rise to the examples as in row 6 of Table 1. When $p^s > 3$, one has $T_u = \mathrm{PSL}(2, p^s)$ is insolvable. By [4, Theorem 5.2, 5.3], G_u^* is an almost simple with $\mathrm{soc}(G_u^*) = T_u$, and $T_u^{\Gamma(u)}$ is 2-transitive except for $p^s = 8$ with degree 28. For $p^s = 8$, one has $H = P\Sigma L(2, 8)$, and $G^* = P\Sigma L(2, 2^{3m})$ with m odd prime. Since $\mathrm{Aut}(G^*) = G^*$, there is no an almost group G which has a index 2 subgroup G^*. Then we may assume that $G^* = T, H = T_u = \mathrm{PSL}(2, p^s)$. Since H has a 2-transitive permutation representation, by [4, Theorem 5.3] one has $L = \mathbb{Z}_p^s : \mathbb{Z}_{\frac{p^s-1}{d}}, L = A_5(p^s = 11, 9)$, or $L = A_4(p^s = 5)$. Let $L = \mathbb{Z}_p^s : \mathbb{Z}_{\frac{p^s-1}{d}}$. We claim that $N_{G^*}(L) = L$. Set $N = \mathbb{Z}_p^s$. Suppose that $L < N_{G^*}(L)$. By [11, Proposition 3.2], one has $N_{G^*}(L) \leq N_{G^*}(N) \leq \mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{d}}$. Let $N_{G^*}(L) = \mathbb{Z}_p^t : \mathbb{Z}_l$, where $t \geq s$. By Proposition 2 , one has $l \mid (p^s - 1)$. Since $L < N_{G^*}(L)$, and by the note of Proposition 2, $l = \frac{p^s-1}{d}$. Let $M = \mathbb{Z}_{\frac{p^s-1}{d}}$ be a subgroup of L. Since $N_L(M) = M$, the length of conjugacy class of M in L has p^s. Let Ω be the set of all the subgroups which conjugacy to M in L. Then we may consider the action of the subgroup $\mathbb{Z}_p^t$ of $N_{G^*}(L)$ on Ω. Suppose $t > s$. Then there is an element g of order p such that $M^g = M$, which is impossible. Thus, $t = s$, implying that $N_{G^*}(L) = L$. Next $N_G(L) \leq N_G(\mathrm{PSL}(2, p^s)) = N_G(H)$, no graph appears. Let $L = A_5(p^s = 11, 9)$ or $L = A_4(p^s = p = 5)$. Similarly, $N_{G^*}(L) = L$, and $N_G(L) \leq N_G(\mathrm{PSL}(2, p^s)) = N_G(H)$, no graph appears.

Assume that $T_u = A_5$. Then $q = p \equiv \pm 1 \pmod 5$, or $q = p^2 \equiv -1 \pmod 5$ with p an odd prime. By [4, Theorem 5.3], $T_u^{\Gamma(u)}$ is 2-transitive. So we may suppose $G^* = T$ and $H = T_u$. It follows that $L \in \{A_4, D_{10}\}$. Assume that $q = p \equiv \pm 1 \pmod 5$. Then $G = \mathrm{PGL}(2, p)$. If $L = A_4$, then $N_G(L) = S_4$. By Lemma 4, if $N_{G^*}(L) = S_4$, no graph appear. Since G^* contains a subgroup isomorphic to S_4 if and only if $16 \mid q^2 - 1$. Then $L = N_{G^*}(L) < N_G(L) = S_4$ if and only if $16 \nmid q^2 - 1$. Note that $q = p \equiv \pm 1 \pmod 5$. Then $q = p \equiv \pm 11, \pm 19 \pmod{40}$. It follows that there is a 2-element $g \in N_G(L) \setminus N_{G^*}(L)$ such that $\langle H, g\rangle = G$. By Proposition 4, the coset graph $\mathrm{Cos}(G, H, HgH)$ is a connected pentavalent vertex bi-primitive $(G, 2)$-arc transitive graph as in row 7 of Table 1. If $L = D_{10}$, then $N_G(L) = D_{20}$ by the subgroup of $\mathrm{PGL}(2, q)$ (see [3, Theorem 2]). Since G^* contains a subgroup isomorphic to D_{20} if and only if $20 \mid q \pm 1$, and G contains a subgroup isomorphic to D_{20} if and only if $20 \mid 2(q \pm 1)$, one has $L = N_{G^*}(L) < N_G(L) = D_{20}$ if and only if $q \neq \pm 1 \pmod{20}$ and $20 \mid 2(q \pm 1)$. Note that $q = p \equiv \pm 1 \pmod 5$. Then $q = p = \pm 9 \pmod{20}$. Thus, when $q = p \equiv \pm 9 \pmod{20}$, there are 2-elements $g \in N_G(L) \setminus L$ such that $\langle H, g\rangle = G$. By Lemma 4, the coset graph $\mathrm{Cos}(G, H, HgH)$ is a connected pentavalent 2-arc transitive biprimitive graph as in row 8 of Table 1. Assume that $q = p^2 \equiv -1 \pmod 5$. Let $L = A_4$. Since $16 \mid q^2 - 1 = p^4 - 1$, one has $N_{G^*}(L) = S_4$. By [11, Theorem 3.5], if $G = \mathrm{PGL}(2, q)$ then $N_G(L) = N_{G^*}(L) = S_4$, no graph appears. Again by [11, Theorem 1.3, 1.5], for $G = M(1, q)$ or $P\Sigma L_2(q)$, either $N_G(L) = S_4$ or $N_G(L) \leq N_G(\mathrm{PSL}(2, p))$. For $G = M(1, q)$, one has $N_G(\mathrm{PSL}(2, p)) = \mathrm{PGL}(2, p)$. Then $N_G(L) = N_{G^*}(L) = S_4$, no graph appears. For $G = P\Sigma L_2(q)$, one has $N_G(\mathrm{PSL}(2, p)) = \mathrm{PGL}(2, p) \times \langle \phi \rangle$, where ϕ is a field automorphism of order 2. Then $N_G(L) = S_4 \times \langle \phi \rangle$. There is an involution $g \in N_G(L) \setminus N_{G^+}(L)$ such that $\langle H, g\rangle = G$. It follows that the coset graph $\mathrm{Cos}(G, H, HgH)$ is a connected pentavalent 2-arc transitive biprimitive graph. We claim that it is a standard double cover. Let $\Sigma = \{g|g \in N_G(L)\setminus N_{G^*}(L)$, and g is a 2-elements$\}$. Let $K = N_G(H)\cap N_G(L) = S_5\cap(S_4 \times \langle\phi\rangle) = S_4$. Obviously, K transitively acts on Σ. Furthermore, for any $g = a_i\phi \in \Sigma$ $(a_i \in S_4)$, there is $k \in K = S_4$ such that $(HgH)^k = (Ha_i\phi H)^k = (Ha_i^k\phi H) = (Ha_i^{-1}\phi H) = H(a_i\phi)^{-1}H$. Since the order of a_i is 2 or 4, one may assume that k is an involution. Then k may induce a inner automorphism $\sigma(k)$ of $R(G)$ by conjugate, and then $R(G) \rtimes \sigma(k) = R(G) \times \langle k'\rangle \leq \mathrm{Aut}(X)$, where $k' = R(k)\sigma(k) \in L(G)$, the left regular representation $L(G)$. By [13, Proposition 2.6], X is a standard double cover as the claim. Let $L = D_{10}$. Then G^* contains a subgroup of isomorphic to $L.\mathbb{Z}_2$ if and only if $20 \mid q \pm 1$. Recall that $q = p^2 \equiv -1 \pmod 5$, one has $N_{G^*}(L) = L$ and $q = p^2 \equiv 9 \pmod{20}$ because there is no prime p such that $q = p^2 \equiv -1 \pmod{20}$. By [11, Theorem 1.3,1.5, 3.5], $N_G(L) = L.\mathbb{Z}_2 \leq G$ if and only if $N_G(L) \leq S_5$ with $G = P\Sigma L(2, q)$ or $N_G(L) \leq N_G(D_{q\pm1})$ with $10|q \pm 1$. Obviously, if $G = P\Sigma L_2(q)$ and $N_G(L) \leq S_5$, then no graph appears because $H \leq S_5$, implying that there is no element $g \in N_G(L) \setminus H$ such that $\langle H, g\rangle = G$. Thus, for $q = p^2 \equiv 9 \pmod{20}$, $G = \mathrm{PGL}(2, q)$ or $M(1, q)$, one has $N_G(L) = D_{20}$ and there is an involution $g \in N_G(L) \setminus H$ such that $\langle H, g\rangle = G$. It follows that the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected $(G, 2)$-arc transitive biprimitive graph of valency 6, and X is a graph as in row 9-10 of Table 1.

Assume that $T_u = S_4$. Then $q = p \equiv \pm1 \pmod 8$ and $G = \mathrm{PGL}(2, p)$, and then $H = S_4$ and $L = D_6$ or D_8. Suppose that $L = D_6$. Then $N_{G^*}(L)$ contains a subgroup isomorphic to $L.\mathbb{Z}_2$ if and only if $12 \mid (p \pm 1)$, that is, $p \equiv \pm1 \pmod{12}$. Thus, if $q = p \equiv \pm1 \pmod 8$ and $p \equiv \pm1 \pmod{12}$ then $N_{G^*}(L) = N_G(L)$, no graph appears. If $q = p \equiv \pm1 \pmod 8$ and $p \neq \pm1 \pmod{12}$, then $N_{G^*}(L) = L$. By [11, Theorem 3.5], $L.\mathbb{Z}_2 \leq G$ if and only if $12 \mid 2(p \pm 1)$, that is $p \equiv \pm1 \pmod 6$. So, $N_G(L) = L.\mathbb{Z}_2 > N_{G^*}(L) = L$ if and only if $p \equiv \pm7, \pm17 \pmod{48}$. Let $g \in N_G(L) \setminus L$ be an involution. By [11, Theorem 3.5], $\langle H, g\rangle = G$. It follows that the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected $(G, 2)$-arc transitive biprimitive graph, and X is a graph as in row 11 of Table 1. Suppose that $L = D_8$. Then $N_{G^*}(L)$ contains a subgroup isomorphic to $L.\mathbb{Z}_2$ if and only if $16 \mid (p \pm 1)$, that is, $p \equiv \pm1 \pmod{16}$. Thus, if $q = p \equiv \pm1 \pmod 8$ and $p \equiv \pm1 \pmod{16}$ then $N_{G^*}(L) = N_G(L)$, no graph appear. If $q = p \equiv \pm1 \pmod 8$ and $p \neq \pm1 \pmod{16}$, then $N_{G^*}(L) = L$. By [11, Theorem 3.5], $L.\mathbb{Z}_2 \leq G$ if and only if $16 \mid 2(p \pm 1)$, that is $p \equiv \pm1 \pmod 8$. So, $N_G(L) = L.\mathbb{Z}_2 > N_{G^*}(L) = L$ if and only if $p \equiv \pm7 \pmod{16}$. Let $g \in N_G(L) \setminus L$ be an involution. By [11, Theorem 3.5], $\langle H, g\rangle = G$. It follows that the coset graph $\mathrm{Cos}(G, H, HgH)$ is a connected 2-arc transitive biprimitive graph, and X is a graph as in row 12 of Table 1.

Assume that $T_u = \mathrm{PGL}(2, p^r)$. Then $f = 2r$ and p is an odd prime because $\mathrm{PGL}(2, p^r) \cong \mathrm{PSL}(2, p^r)$ for $p = 2$. When $p^r = 3$, one has $r = 1, f = 2, T = \mathrm{PSL}(2, 9) \cong A_6$, and $T_u = \mathrm{PGL}(2, 3) \cong S_4$. Since $\mathrm{Out}(T) = 2^2$, and M_{10} or $\mathrm{PGL}(2, 9)$ has no subgroup of order $2|T_u|$, one has either $G^* = T$ and $G = \mathrm{PGL}(2, 9), P\Sigma L(2, 9), M_{10}$ or $G^* = P\Sigma L(2, 9)$ and $G = P\Gamma L(2, 9)$. Suppose $G^* = T$. Then $H = S_4, L = S_3$ or D_8. By Magma [2], whether $L = S_3$ or D_8, one had $N_{G^*}(L) = L$. For $G = \mathrm{PGL}(2, 9)$ or M_{10}, again by Magma [2], if $L = S_3$ then $N_G(L) = L$, no graph appears, and if $L = D_8$ then $N_G(L) = L.\mathbb{Z}_2$. It follows that the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected $(G, 2)$-arc transitive biprimitive graph of valency 3, and X is a graph as in rows 13-14 of Table 1. For $G = P\Sigma L(2, 9)$, one has $N_G(H) = H \times \mathbb{Z}_2$, and by Magma [2], one has $N_G(L) = L \times \mathbb{Z}_2$ and $N_{G^*}(L) = L$ no matter what $L = S_3$ or $L = D_8$. Obviously, $N_G(L) \leq N_G(H)$. It follows that there is no 2-element g such that $\langle H, g\rangle = G$, no graph appears. Suppose $G^* = P\Sigma L(2, 9)$. Then $G = P\Gamma L(2, 9)$, $H = S_4 \times \mathbb{Z}_2$, and $L = S_3 \times \mathbb{Z}_2$ or $D_8 \times \mathbb{Z}_2$. If $L = S_3 \times \mathbb{Z}_2$, then by Magma [2], $N_G(L) = N_{G^*}(L)$, no graph arises. If $L = D_8 \times \mathbb{Z}_2$ then by Magma [2], $N_G(L) = L.\mathbb{Z}_2$ and $N_{G^*}(L) = L$. Again by Magma [2], there exists $g \in N_G(L) \setminus L$ be an involution such that $\langle H, g\rangle = G$. It follows that the coset graph $X = \mathrm{Cos}(G, H, HgH)$ is a connected $(G, 2)$-arc transitive vertex bi-primitive graph of valency 3, and X is a graph as in row 15 of Table 1. When $p^r \geq 5$, T_u is insolvable, by [4, Theorem 5.3 and Note 2], $T_u^{\Gamma(u)}$ is 2-transitive, so we may suppose $G^+ = T$, and $H = \mathrm{PGL}(2, p^r)$. And by [4, Theorem 5.3], $L \cong \mathbb{Z}_p^r \rtimes \mathbb{Z}_{p^r-1}$. Let $N = \mathbb{Z}_p^s$. Then we claim $N_G^*(L) = L$. Suppose that $L < N_G(L)$. By [11, Proposition 3.2], one has $N_G^*(L) \leq N_G^*(N) \leq \mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{d}}$. Let $N_G^*(L) = \mathbb{Z}_p^t : \mathbb{Z}_l$, where $t \geq s$. By Proposition 2, one has $l \mid (p^s - 1)$. Since $L < N_G^*(L)$, and by the note of Proposition 2, $l = \frac{p^s-1}{d}$. Let $M = \mathbb{Z}_{\frac{p^s-1}{d}}$ be a subgroup of L. Since $N_L(M) = M$, the length of conjugacy class of M in L has p^s. Let Ω be the set of all the subgroups which conjugacy to M in L. Then we may consider the action of the subgroup $\mathbb{Z}_p^t$ of $N_G^*(L)$ on Ω. Suppose $t > s$. Then there is an element g of order p such that $M^g = M$, which is impossible. Thus, $t = s$, implying that $N_{G^*}(L) = L$. Note that $|G : G^*| = 2$, one has $G = \mathrm{PGL}(2, p^f), M(1, p^f)$ or $G = P\Sigma L_2(p^f)$. If $G = \mathrm{PGL}(2, p^f)$ or $M(1, q)$, then $N_G(L) = L$, no graph appears. If $G = P\Sigma L_2(p^f)$ then $N_G(L) = L.\mathbb{Z}_2$ and $N_G(L) \leq N_G(\mathrm{PSL}(2, p^r))$, there is no elements $g \in N_G(L)\setminus H$ such that $\langle H, g\rangle = G$ because $H \leq N_G(\mathrm{PSL}(2, p^r))$.

Suppose T is not primitive on Δ_i. Then $G_u^* \cap T$ is not a maximal subgroup of T. By [11, Theorem 1.1], we only consider these pairs $(G^*, H) = (\mathrm{PGL}(2, 7), D_{12})$, $(\mathrm{PGL}(2, 7), D_{16})$, $(\mathrm{PGL}(2, 9), D_{20})$, $(\mathrm{PGL}(2, 9), D_{16})$, $(M_{10}, \mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, $(M_{10}, \mathbb{Z}_8 \rtimes \mathbb{Z}_2)$, $(P\Gamma L(2, 9), \mathbb{Z}_{10} \rtimes \mathbb{Z}_4)$, $(P\Gamma L(2, 9), N_G(D_8))$, $(\mathrm{PGL}(2, 11), D_{20})$ or $(\mathrm{PGL}(2, p), S_4)$, with $p = \pm11, 19, \pmod{40}$. If $G^* = \mathrm{PGL}(2, 7)$, $P\Gamma L_2(9)$, $\mathrm{PGL}(2, 11)$ or $\mathrm{PGL}(2, p)$, then $G = G^* \times \mathbb{Z}_2$. By [13, Proposition 2.6], X is the standard double cover of a vertex-primitive $(G^*, 2)$-arc transitive graph.

If $(G^*, H) = (\mathrm{PGL}(2, 9), D_{20})$ or $(G^*, H) = (M_{10}, \mathbb{Z}_5 \rtimes \mathbb{Z}_4)$, then by Magma [2], X is a standard double cover of a vertex-primitive graph.

If $(G^*, H) = (\mathrm{PGL}(2,9), D_{16})$ or $(G^*, H) = (M_{10}, \mathbb{Z}_8 \rtimes \mathbb{Z}_2)$, then $H = D_{16}$ or $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$. Because the order of all the maximal subgroup of H is 8, there is no maximal subgroup L of index k, where $k \geq 3$, a contradiction.

Finally, we determine the automorphism groups and enumerate the number of non-isomorphic ones of such graphs.

Let X_i be the coset graph corresponding to the i-th row of Table 1, and let n be the number of non-isomorphic graph X_i. Next, we give the non-isomorphic number and automorphism group of graph X_i. By Lemma 6, $G \leq \mathrm{Aut}(X_i) \leq \mathrm{Aut}(T)$. For the graphs $X_i (i = 1, 2, 3, 4)$, one has $\mathrm{Aut}(X_i) = G$, and by Magma [2], there exists unique these graph for every $i = 1, 2, 3, 4$.

For the graph X_5, one has $G = \mathrm{PGL}(2, p) = \mathrm{Aut}(T)$. By Lemma 6, $\mathrm{Aut}(X_5) = G$. Let $\varepsilon = \pm 1$.

Claim 1: $n = \lceil \frac{p+\varepsilon}{12} \rceil - 1$, that is, if $12 \mid p+\varepsilon$ then $n = \frac{p+\varepsilon}{12} - 1$, and if $6 \mid p+\varepsilon$ and $12 \nmid p+\varepsilon$ then $n = \frac{\frac{p+\varepsilon}{6} - 1}{2}$.

Recall that $H = A_4, L = \mathbb{Z}_3$. Obviously, $N_G(H) = S_4$ and $N_G(L) = D_{2(p+\varepsilon)}$. Set $N_G(L) = \langle a, b \mid a^{p+\varepsilon} = b^2 = 1, a^b = a^{-1} \rangle$. Then $L = \mathbb{Z}_3 = \langle a^{\frac{p+\varepsilon}{3}} \rangle$ and $N_G(H) \cap N_G(L) = N_{N_G(H)}(L) = S_3 = \langle a^{\frac{p+\varepsilon}{3}}, b \rangle$. Since $|N_G(L) : N_{G^*}(L)| = 2$, and $N_{G^*}(L) \leq G^*$, one has $N_{G^*}(L) = D_{p+\varepsilon} = \langle a^2, ba \rangle$. By Lemma 4, g is a 2-element in $N_G(L) \setminus N_{G^*}(L)$.

First, we claim that $g \neq a^i$. Suppose to the contrary that $g = a^i$. Then $(Ha^iH)^b = (Ha^{-i}H)$. Obviously, b may induce a inner automorphism $\sigma(b)$ of $R(G)$ by conjugate. Then $A := \mathrm{Aut}(X) = R(G) \rtimes \sigma(b)$. Take $b' = R(b)\sigma(b)$. Note that $o(b) = 2$, it is easy to see that $b' \in L(G)$, the left regular representation $L(G)$ of G on $V(X)$, and then $b' \in C_A(R(G))$. It follows that $A = R(G) \times \langle b' \rangle$, and by [13, Proposition 2.6], X is a standard double cover. Furthermore, $g \notin N_{N_G(H)}(L)$. Otherwise, $\langle H, g \rangle \neq G$, a contradiction. Thus, $g \in \Sigma = \{a^t b \mid t \text{ is an even, and } t \neq 0, \frac{p+\varepsilon}{3}, \frac{2(p+\varepsilon)}{3}\}$, and then $|\Sigma| = \frac{p+\varepsilon}{2} - 3$. Since $S_3 \leq N_G(L)$, one has S_3 fixes Σ by conjugate. Assume that $\frac{p+\varepsilon}{2}$ is odd, that is, $4 \nmid p+\varepsilon$. Since the centralizer of every element of order 2 in $N_G(L)$ is either itself or $a^{\frac{p+\varepsilon}{2}}$, and since $a^{\frac{p+\varepsilon}{2}} \notin S_3$, it follows that the orbit length of S_3 on Σ is 6, and then the number of S_3-orbits on Σ is $\frac{\frac{p+\varepsilon}{2} - 3}{6} = \frac{\frac{p+\varepsilon}{6} - 1}{2}$. Assume that $\frac{p+\varepsilon}{2}$ is even, that is $4 \mid p+\varepsilon$. Then $a^{\frac{p+\varepsilon}{2}} b \in \Sigma$. If $g = a^{\frac{p+\varepsilon}{2}} b$, then $(Ha^{\frac{p+\varepsilon}{2}} bH)^b = (Ha^{-\frac{p+\varepsilon}{2}} bH)$. Similarly, X is a standard double cover. Since $\langle b \rangle$ is the stabilizer of S_3 on $a^{\frac{p+\varepsilon}{2}} b$, one has the length of S_3-orbit containing $a^{\frac{p+\varepsilon}{2}} b$ is 3, and others is 6. Thus, if X is not a standard double cover, then the number of length 6 of S_3-orbits on Σ is $\frac{\frac{p+\varepsilon}{2} - 3 - 3}{6} = \frac{p+\varepsilon}{12} - 1$. If g, g' belong to the same S_3-orbit, then it is easy to see that $\mathrm{Cos}(G, H, HgH) \cong \mathrm{Cos}(G, H, Hg'H)$. Conversely, assume that g, g' belong to the difference S_3-orbit, and $\mathrm{Cos}(G, H, HgH) \cong \mathrm{Cos}(G, H, Hg'H)$. Note that $\mathrm{Aut}(X) = G$. Then there exists $m \in \mathrm{Aut}(G, H) = S_4$ such that $(HgH)^m = Hg'H$. Since for any a G-arc transitive coset graph $\mathrm{Cos}(G, H, HgH)$, $I(H) \leq \mathrm{Aut}(G, H, D)$ is transitive on the neighborhood of H by Proposition 3. Then one may assume that $(Hg)^m = Hg'$. Recall that $g, g' \in N_G(L)$, one has $H \cap H^g = H \cap H^{g'} = L$. Note that $\mathrm{Cos}(G, H, HgH) \cong \mathrm{Cos}(G, H, Hg'H)$, one has $m \in N_G(L) \cap S_4 = S_3$. By the character of $N_G(L)$, one has the multiplication of any two elements of order 2 in $N_G(L)$ contains in $\langle a \rangle$. Then $g^m g' \in H \cap \langle a \rangle = L$. One may assume that $g^m = g'c^2$, where $c \in L$. It follows that $g^m = c^{-1} g' c = g'^c$. This implies that g, g' belong the same L-orbits. Then the number of length 6 of S_3-orbits on Σ is the number of non-isomorphic graphs, that is, $n = \lceil \frac{p+\varepsilon}{12} \rceil - 1$.

For the graph X_6,

Claim 2: When $f = 3$, there are two non-isomorphic graphs, and $\mathrm{Aut}(X_6) = G$ or $P\Gamma L_2(3^f)$, respectively. When $f > 3$, one has $n = \frac{3^f - 3}{6f}$, and $\mathrm{Aut}(X_6) = G$.

Note that $H = A_4, L = \mathbb{Z}_3, N_G(H) = S_4, N_G(L) = \mathbb{Z}_3^f \rtimes \mathbb{Z}_2$ and $N_{G^*}(L) = \mathbb{Z}_3^f$, $N_{N_G(H)}(L) = S_3 \leq N_G(L)$. Since $g \in N_G(L) \setminus N_{G^*}(L)$ is a 2-element, one has g is a element of order 2 in $N_G(L)$. If $g \in N_{N_G(H)}(L) = S_3$, then $\langle H, g \rangle \neq G$, which contradicts the connectivity of X_6. Set $\Omega := \{g \mid g \in N_G(L), g \notin N_{G^*}(L) \cup S_3 \text{ and } g \text{ is a 2-element}\}$. Then $|\Omega| = 3^f - 3$. Since $S_3 \leq N_G(L)$, S_3 induces an action by conjugate on Ω.

Take $M = \langle x, y \rangle \leq GL_2(3^f)$, where

$$x := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \quad y := \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}, \; a \in GF(3^f).$$

Obviously, M is a subgroup of order $2 \cdot 3^f$ of $GL_2(3^f)$. Let Z be the center of $GL_2(3^f)$. By computation, for any two elements a, b of M, if $aZ = bZ$ then $a = b$. It implies that $M \leq GL_2(3^f)/Z = G$. Since $N_G(L) \leq \mathbb{Z}_3^f \rtimes \mathbb{Z}_{\frac{3^f-1}{2}}$, the subgroup of order 2 of $\mathbb{Z}_{\frac{3^f-1}{2}}$ is unique and Sylow-3 subgroup of G has unique conjugacy class, one has $M = N_G(L)$, and $L = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle$.

First, we consider the orbits of S_3 on Ω. Since $N_G(L)$ has no elements of order 6 and no subgroup $\mathbb{Z}_2^2$, one has the length of S_3-orbits on Ω is 6, implying that the number of S_3 on Ω is $\frac{3^f-3}{6}$. Let ϕ be a field automorphism of order f. Obviously, ϕ fixes the set of all the elements of order 2 in M. We may consider the orbits of $\langle \phi \rangle$ on Ω. Since ϕ is a field automorphism of $GF(3^f)$, it fixes the subfield $GF(3)$. It implies that $a^3 = a$, that is, there are three fixed points $0, 1, -1$ of ϕ. Since $o(\phi) = f$ is a prime, the length of $\langle \phi \rangle$-orbits is 1 or f. Thus, $f \mid 3^f - 3$. When $f = 3$, the number of S_3-orbits on Ω is 4. By computation, ϕ may connect three of S_3-orbits, and fix the last S_3-orbit. Thus, $n = 2$. When ϕ fixes S_3-orbits, one has $\phi \in \mathrm{Aut}(G, H, D)$, and $\mathrm{Aut}(X) = P\Gamma L_2(3^f)$, and when ϕ connects three of S_3-orbits, $\mathrm{Aut}(X) = G$. (For $f = 3$, we may check it by Magma [2].) When $f > 3$, obviously, $\langle \phi \rangle$ can not fix any the S_3-orbits because the length of S_3-orbits and the length of $\langle \phi \rangle$-orbits are coprime. Then the number of $S_3 \times \langle \phi \rangle$-orbits on Ω is $n = \frac{3^f-3}{6f}$. By Lemma 6, $\mathrm{Aut}(X) \leq \mathrm{Aut}(T)$, and $\phi \notin \mathrm{Aut}(G, H, D)$, one has $\mathrm{Aut}(X) = G$. If g, g' belong to the same S_3-orbits or $\langle \phi \rangle$-orbits, one has

$\mathrm{Cos}(G,H,HgH) \cong \mathrm{Cos}(G,H,Hg'H)$. Conversely, if g,g' belong to the different $S_3 \times \langle\phi\rangle$-orbits, and $\mathrm{Cos}(G,H,HgH) \cong \mathrm{Cos}(G,H,Hg'H)$, then there exists $m \in \mathrm{Aut}(G,H) = S_4 \times \langle\phi\rangle$ such that $(HgH)^m = Hg'H$ because $\mathrm{Aut}(X) = G$. Since for any a G-arc transitive coset graph $\mathrm{Cos}(G,H,HgH)$, $I(H) \leq \mathrm{Aut}(G,H,D)$ is transitive on the neighborhood of H, one may assume that $(Hg)^m = Hg'$, implying that $m \in N_{\mathrm{Aut}(G)}(L) \cap \mathrm{Aut}(G,H) = S_3 \times \langle\phi\rangle$. Since multiplication of any two elements of order 2 in $N_G(L)$ contains in $\mathbb{Z}_3^f$, one has $g^m g' \in (H \cap \langle x\rangle) = L$. One may assume that $g^m = g'c^2$, where $c \in L$. It follows that $g^m = g'^c$. This implies that g,g' belong the same L-orbits. Then the number n of non-isomorphic graph equals to the number of $S_3 \times \langle\phi\rangle$-orbits on Ω, that is, $n = \frac{3^f-3}{6f}$.

For the graphs $X_i (i = 7, 8, 11, 12, 15)$, one has $\mathrm{Aut}(X_i) = G$ by Lemma 6. Obviously, $N_H(L) = L$, $N_G(L) = L.\mathbb{Z}_2$ and L has a unique conjugate class in H. By Lemma 5, there exists unique non-isomorphic graph.

For the graphs X_9 and X_{10}, one has $H = \mathrm{A}_5$, $L = D_{10}$, $N_G(L) = D_{20}$. Obviously, $N_H(L) = L$ and L has a unique conjugate class in H. Then by Lemma 5, there exists unique non-isomorphic graph, and by Lemma 6, $\mathrm{Aut}(X_i) \leq \mathrm{Aut}(T)$, and then $\mathrm{Aut}(X_i) = G$ or $P\Gamma L_2(q)$. Let ϕ be a field automorphism of order 2. Then ϕ normalizer H and L, that is, $\phi \in N_{\mathrm{Aut}(G)}(L) = N_G(L) \rtimes \langle\phi\rangle$, implying that ϕ fix the set of all the 2-elements in $N_G(L)$. Then $\phi \in \mathrm{Aut}(G,H,D) < \mathrm{Aut}(X_i)$. Thus, $\mathrm{Aut}(X_i) = P\Gamma L_2(q)$ for $i = 8$ or 9.

For the graphs X_{13} and X_{14}. Obviously, $N_H(L) = L$, $N_G(L) = L.\mathbb{Z}_2$ and L has a unique conjugate class in H. By Lemma 5, there exists unique non-isomorphic graph. It is similar to the above, one has $\mathrm{Aut}(X_i) = P\Gamma L_2(9)$ for $i = 13$ or 14. (We may also check it by Magma [2].)

Thus, we get the automorphism groups and the number of non-isomorphic ones of such graphs as in Table 2.

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