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Optimal Measurement Matrix of Partial Polarimeter for Measuring Ellipsometric Parameters With Eight Intensity Measurements

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ABSTRACT We demonstrate that the ellipsometric parameters of the isotropic samples can be measured by a partial polarimeter with only eight intensity measurements. Under the Gaussian and Poisson noises, we propose the optimal measurement matrices for the polarization state generator and the polarization state analyzer that minimize the total estimation variance of eight nonzero Mueller elements related to the ellipsometric parameters. In addition, since considering only four of the eight elements, the estimation variance can be further reduced, and we also introduce two proper criteria to find the optimal measurement matrices that robustly minimize the total estimation variance for these four Mueller elements. Compared with the existing measurement matrices based on 16 intensity measurements, our proposed matrices can effectively decrease the total estimation variance and, thus, improve the measurement precision of the Mueller matrix and ellipsometric parameters with only eight measurements. Furthermore, the proposed optimal measurement matrices do not depend on the Mueller elements of samples and thus not on the ellipsometric measurements.

INDEX TERMS Optical polarization, polarimetry, ellipsometry, ellipsometric parameters, Mueller matrix, isotropic sample, statistical analysis.

I. INTRODUCTION

Polarimeter/ellipsometer consists in measuring the changes in the polarization state of reflected or transmitted light beam by specified samples. For example, it can accurately characterize the physical properties and characteristics of samples, such as the complex refractive index and thickness of the thin films [1]–[6]. Besides, polarimeter can be also used for measuring several plasma parameters in fusion problem, such as Faraday rotation angle and Cotton-Mouton angle [7]–[9] in a magnetized plasma. Therefore, polarimeter, as a powerful tool, has been applied widely in many fields [10]–[12].

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As an important application of the polarimeter, the characterization of films and surfaces mostly focuses on the measurement of smooth and isotropic samples, whose Mueller matrix is block diagonal, and two ellipsometric parameters are determined by the eight nonzero elements in this diagonal block of Mueller matrix [2], [13], [14]. In these cases, the complete Mueller polarimeter, which requires 16 intensity measurements, could no longer be the best choice for measuring ellipsometric parameters [15], [16]. Moreover, it seems that eight intensity measurements are enough for measuring these eight nonzero Mueller elements. Therefore, it is interesting to develop an optimized strategy with only eight intensity measurements to measure these eight Mueller elements. In the previous works, Savenkov *et al.* [17] and

Anna *et al.* [18] demonstrated that the times of intensity measurement can be reduced to less than 16 with partial polarimetry. However, to our knowledge, the optimization for the measurement matrices of the polarization state generator (PSG) and the polarization state analyzer (PSA) has not been considered yet in the presence of different types of noise. Furthermore, the optimal measurement matrices that minimize the total variance of these four of eight Mueller elements with eight intensity measurements have not been considered either, which can further reduce the estimation variance and thus improve the measurement precision of ellipsometric parameters.

In this paper, considering the measurement for ellipsometric parameters of isotropic samples in the presence of two common types of noise (Gaussian additive noise and Poisson shot noise) [13], [15], [19], we optimize the measurement matrices of partial polarimeter based on only eight intensity measurements. The proposed optimal measurement matrices minimize the total variance of the elements related to the ellipsometric measurements. Besides, we also introduce a proper criterion to find the optimal measurement matrices that robustly reduce the total variance for four of eight elements, which can further reduce the total variance. Compared with the existing measurement matrices, the proposed ones lead to the best performance, which considers both total estimation variance and intensity measurement times, for the ellipsometric measurements. In addition, the proposed optimal measurement matrices do not dependent on the Mueller matrix of samples and thus not on the ellipsometric parameters under both Gaussian and Poisson noise.

II. POLARIMETER/ELLIPSOMETER

The Mueller matrix of the smooth and isotropic materials is block-diagonal containing only eight nonzero elements related to the ellipsometric parameters ψ and [1], and [2]:

$$M = r \begin{bmatrix} 1 & a & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & -c & b \end{bmatrix}, \tag{1}$$

where $a = -\cos 2\psi$, $b = \sin\, 2\psi \cos\Delta$, $c = \sin\, 2\psi \sin\Delta$, and r refers to the surface power reflectance. By measuring Mueller matrix, the ellipsometric parameters ψ and Δ are thus calculated by $1/2\cos^{-1}[-M_{12}/M_{11}]$ and $\tan^{-1}[M_{34}/M_{33}]$, respectively. Here, M_{ij} denotes the element in the ith row and jth column of matrix M. The typical measurement configuration for Muller polarimeter is shown in Fig. 1.

In practice, the PSG and the PSA in Mueller polarimeter are usually composed of a polarizer and a retarder as shown in Fig. 1 [18]–[20]. The measurement matrices (A, W) are composed of four eigenstate vectors of PSG and the PSA, and the intensity detection is given by:

$$\mathbf{V}_I = [W \otimes A]^T\, \mathbf{V}_M, \tag{2}$$

where $\otimes$ is Kronecker product, $\mathbf{V}_I$ and $\mathbf{V}_M$ are 16 dimensional vectors correspond to 16 intensities and 16 Mueller

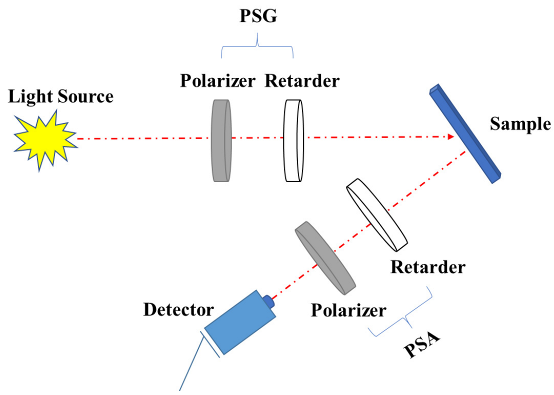

FIGURE 1. Typical measurement configuration for Mueller polarimeter.

elements, respectively. However, there are only eight nonzero elements in the Mueller matrix in Eq. (1), therefore, it needs to find a strategy to measure the eight Mueller elements with only eight intensity measurements.

A feasible strategy is that the matrices (A, W) containing two eigenstate vectors of the PSG while four of the PSA. Let us denote the two measurement matrices as:

$$A = \begin{bmatrix} 1 & \cos(2\alpha_1)\cos(2\varepsilon_1) & \sin(2\alpha_1)\cos(2\varepsilon_1) & \sin(2\varepsilon_1) \\ 1 & \cos(2\alpha_2)\cos(2\varepsilon_2) & \sin(2\alpha_2)\cos(2\varepsilon_2) & \sin(2\varepsilon_2) \end{bmatrix}^T$$
$$W = \begin{bmatrix} 1 & \cos(2\alpha_3)\cos(2\varepsilon_3) & \sin(2\alpha_3)\cos(2\varepsilon_3) & \sin(2\varepsilon_3) \\ 1 & \cos(2\alpha_4)\cos(2\varepsilon_4) & \sin(2\alpha_4)\cos(2\varepsilon_4) & \sin(2\varepsilon_4) \\ 1 & \cos(2\alpha_5)\cos(2\varepsilon_5) & \sin(2\alpha_5)\cos(2\varepsilon_5) & \sin(2\varepsilon_5) \\ 1 & \cos(2\alpha_6)\cos(2\varepsilon_6) & \sin(2\alpha_6)\cos(2\varepsilon_6) & \sin(2\varepsilon_6) \end{bmatrix}^T \tag{3}$$

with the azimuth $\alpha_i \in [-\pi/2, \pi/2]$, and the ellipticity $\varepsilon_i \in [-\pi/4, \pi/4]$, $i \in [1, 6]$ [1]. The intensity measurement is thus rewritten as:

$$\mathbf{V}_\mathbf{I}^8 = \left[Q_{A,W}^8\right]^T \mathbf{V}_M^8, \tag{4}$$

where $\mathbf{V}_\mathbf{I}^8$, $\mathbf{V}_M^8$ are eight dimensional vectors correspond to the eight intensity measurements and eight nonzero Mueller elements. $Q_{A,W}^8$ is an 8×8 matrix with its line is the vector of $[W \otimes A]^T$, from which these eight elements indexed corresponding to the set of nonzero elements in matrix M are kept. Therefore, the Mueller vector (matrix) $\mathbf{V}_M^8$ can be estimated by:

$$\hat{\mathbf{V}}_M^8 = \left\{\left[Q_{A,W}^8\right]^T\right\}^{-1} \mathbf{V}_\mathbf{I}^8. \tag{5}$$

However, this estimator could deviate from the true value because the intensities are always influenced by noise. Therefore, $\mathbf{V}_\mathbf{I}^8$ is a random variable (vector), and its covariance matrix $\Gamma_{\mathbf{V}_\mathbf{I}^8}$ is diagonal with each diagonal element denoting the estimation variance on each measured intensity. Subsequently, the covariance matrix $\Gamma_{\hat{\mathbf{V}}_M^8}$ of the estimated Mueller vector $\hat{\mathbf{V}}_M^8$ is calculated by [21]:

$$\Gamma_{\hat{\mathbf{V}}_M^8} = \left\{\left[Q_{A,W}^8\right]^{-1}\right\}^T \Gamma_{\mathbf{V}_\mathbf{I}^8} \left[Q_{A,W}^8\right]^{-1}. \tag{6}$$

The diagonal element $\left[\Gamma_{\hat{\mathbf{V}}_M^8}\right]_{ii}$ refers to the estimation variance σ_i^2, $i \in [1, 8]$ on each Mueller element:

$$\mathrm{Var}[M] = \begin{bmatrix} \sigma_1^2 & \sigma_3^2 & \bullet & \bullet \\ \sigma_2^2 & \sigma_4^2 & \bullet & \bullet \\ \bullet & \bullet & \sigma_5^2 & \sigma_7^2 \\ \bullet & \bullet & \sigma_6^2 & \sigma_8^2 \end{bmatrix}. \quad (7)$$

The optimal measurement matrices need to minimize the summation of variances for these eight elements:

$$\left(A_{8-opt}, W_{8-opt}\right) = \arg\min_{A,W} \sum_{i=1}^{8} \sigma_i^2. \quad (8)$$

In particular, when the dominant noise of measurement system is Gaussian additive noise, and the covariance matrix $\Gamma_{\mathbf{V}_\mathbf{I}^8}$ is diagonal with all its diagonal elements equal to σ^2 (denotes the variance of Gaussian additive noise) [13], [21]. Therefore, the variances σ_i^2 on each element of $\hat{\mathbf{V}}_M^8$ are calculated by:

$$\sigma_i^2 = \sigma^2 \left[\left\{ \left[Q_{A,W}^8 \right]^{-1} \right\}^T \left[Q_{A,W}^8 \right]^{-1} \right]_{ii}, \quad \forall i \in [1, 8]. \quad (9)$$

While in the case that the dominant noise is Poisson shot noise, the variance of intensity vector $\mathbf{V}_\mathbf{I}^8$ is equal to its true value [13], [15], [21], and thus the variance on $\hat{\mathbf{V}}_M^8$ depends on the true value of $\mathbf{V}_M^8$ as:

$$\sigma_i^2 = \sum_{k=1}^{8} \left[\mathbf{V}_M^8\right]_k \left[\sum_{n=1}^{8} \left(\left[Q_{A,W}^8 \right]_{nn}^{-1} \right)^2 \left[Q_{A,W}^8 \right]_{nk}^T \right]. \quad (10)$$

Considering the elements in the first row of both measurement matrices (A, W) are equal to 1/2, all the elements in the first column of matrix $\left[Q_{A,W}^8\right]^T$ are thus equal to 1/4. Then, the variances in Eq. (10) are rewritten as:

$$\sigma_i^2 = \frac{\left[\mathbf{V}_M^8\right]_1}{4} \sum_{n=1}^{8} \left(\left[Q_{A,W}^8 \right]_{nn}^{-1} \right)^2 + f\left(\mathbf{V}_M^8\right), \quad (11)$$

where

$$f\left(\mathbf{V}_M^8\right) = \sum_{k=2}^{8} \left[\mathbf{V}_M^8\right]_k \left[\sum_{n=1}^{8} \left(\left[Q_{A,W}^8 \right]_{nn}^{-1} \right)^2 \left[Q_{A,W}^8 \right]_{nk}^T \right]$$

for the convenience of discussion. From the second term $f\left(\mathbf{V}_M^8\right)$ in Eq. (11), we can see that the variances σ_i^2 do not only depend on the measurement matrices (A, W), but also on the Mueller matrix under the Poisson shot noise, which is different from the case of Gaussian additive noise. Therefore, an optimal set of measurement matrices being independent of the Mueller matrix should satisfy $f\left(\mathbf{V}_M^8\right) = 0$ [in Eq. (11)] [13], [15].

III. NUMERICAL OPTIMIZATION WITH GLOBAL ALGORITHM

In order to find the optimal set of measurement matrices (A, W) that minimizes the total variance based on eight intensity measurements, one can perform global optimization algorithm to search the optimal solutions numerically. It can be seen from Eq. (3) that, both eigenstate vectors for PSG and PSA include two parameters (azimuth α and ellipticity ε), the numerical search thus involves optimizing 12 parameters of PSG (four parameters) and PSA (eight parameters). A feasible algorithm is interior-point method [22], which is a common algorithm to solve both linear and nonlinear convex optimization problems. We have verified that the interior-point method can converge rapidly to the minimal variance of our optimization problem.

A. OPTIMAL MEASUREMENT MATRICES (A, W) FOR EIGHT-ELEMENTS

By employing interior-point method for the problem in Eq. (8), one can directly obtain the optimal set of 12 parameters of the PSG and the PSA under the Gaussian additive noise, and then the optimal measurement matrices $\left(A_{8-opt}^{Gau}, W_{8-opt}^{Gau}\right)$ are thus calculated by these 12 parameters as:

$$\begin{aligned} A_{8-opt}^{Gau} &= \frac{1}{2} \begin{bmatrix} 1 & 0.55 & 0.59 & 0.59 \\ 1 & -0.55 & 0.59 & -0.59 \end{bmatrix}^T, \\ W_{8-opt}^{Gau} &= \frac{1}{2} \begin{bmatrix} 1 & 0.55 & 0.59 & 0.59 \\ 1 & 0.55 & -0.59 & -0.59 \\ 1 & -0.55 & 0.59 & -0.59 \\ 1 & -0.55 & -0.59 & 0.59 \end{bmatrix}^T. \end{aligned} \quad (12)$$

The variances on these eight nonzero Mueller elements in Eq. (7) are also calculated by substituting Eq. (12) into Eq. (9), which are given by:

$$\mathrm{Var}[M] = \sigma^2 \begin{bmatrix} 2.0 & 6.6 & \bullet & \bullet \\ 6.6 & 21.9 & \bullet & \bullet \\ \bullet & \bullet & 16.4 & 16.4 \\ \bullet & \bullet & 16.4 & 16.4 \end{bmatrix}, \quad (13)$$

and the total variance is calculated to be $102.8\sigma^2$. Since σ^2 denotes the variance of Gaussian noise, which can be considered as a parameter value unrelated to the Mueller elements to be measured, the estimation variance does not depend on the Mueller matrix and thus not on the ellipsometric parameters.

However, in the presence of Poisson noise, the variance could depend on both the measurement matrices and the Mueller matrix to be measured. The goal is to find an optimal set of measurement matrices (A, W) that minimizes the total variance and be independent of the Mueller matrix. It means that (A, W) should make $f\left(\mathbf{V}_M^8\right) = 0$ [in Eq. (11)].

Interestingly, the global optimization result shows that the optimal measurement matrices (A, W) in the presence of Poisson noise are equal to these of Gaussian noise:

$$\left(A_{8-opt}^{Poi}, W_{8-opt}^{Poi}\right) = \left(A_{8-opt}^{Gau}, W_{8-opt}^{Gau}\right). \quad (14)$$

The total estimation variance can be calculated according to Eq. (11) and is equal to $102.8[\mathbf{V}_M^8]_1/4$. Here, $[\mathbf{V}_M^8]_1$ is the first element in Mueller vector, and thus can be considered as the intensity of reflected light by the samples. The optimal azimuths α_i and ellipticities ε_i corresponding to the optimal measurement matrices in Eq. (12) for both Gaussian and Poisson noise are shown in Table 1.

TABLE 1. Optimal azimuths and ellipticities for the measurement of 8-elements in presence of both Gaussian and Poisson noise.

Parameters	PSG	PSA
$(\alpha_i, \varepsilon_i)$	(66.48°, -18.10°)	(-23.52°, -18.10°)
	(23.52°, 18.10°)	(66.48°, -18.10°)
		(-66.48°, 18.10°)
		(23.52°, 18.10°)

B. OPTIMAL MEASUREMENT MATRICES (A, W) FOR FOUR-ELEMENTS

Indeed, the two ellipsometric parameters can be even calculated by only four Mueller elements (M_{11}, M_{12}, M_{33}, M_{34}) [13], [23]. Is there any optimal set of measurement matrices leads to minimal total variance of these four elements? Therefore, the other goal is to find the optimal measurement matrices that satisfy:

$$\left(A_{4-opt}, W_{4-opt}\right) = \arg\min_{A,W} \sum_{i\in\Omega} \sigma_i^2, \quad \Omega = \{1,3,5,7\}. \quad (15)$$

Under the Gaussian noise, we also obtain the optimal measurement matrices by employing interior-point method:

$$A_{4-opt}^{Gau} = \frac{1}{2}\begin{bmatrix} 1 & 0.5773 & 0.6703 & -0.4663 \\ 1 & -0.5773 & -0.4663 & 0.6703 \end{bmatrix}^T,$$
$$W_{4-opt}^{Gau} = \frac{1}{2}\begin{bmatrix} 1 & -0.0003 & -1.0000 & 0.0001 \\ 1 & 0.0003 & -1.0000 & -0.0001 \\ 1 & -0.0001 & 1.0000 & 0.0003 \\ 1 & -0.0001 & 1.0000 & -0.0003 \end{bmatrix}^T, \quad (16)$$

and the minimal total variance is calculated to be $20\sigma^2$. However, the measurement matrix W_{4-opt}^{Gau} is singular and has a poor robustness, furthermore, the related matrix $Q_{A,W}^8$ ($= [W \otimes A]^T$) is thus singular. Indeed, it is unavailable in practice. As a compromise, it needs to find such a set of measurement matrices that may lead to a little higher estimation variance than the minimal one but be robust to the presence of variation. In general, the robustness of matrix Q can be described by its condition number [18]:

$$C(Q) = \|Q\| \left\|Q^{-1}\right\|, \quad (17)$$

where $\|\cdot\|$ refers to the 2-norm of matrix. The condition number $C(Q)$ equals to the ratio of the largest singular value of matrix Q to the smallest one. A "well-conditioned" matrix is associated with a value close to 1, while a "badly-conditioned" matrix with a high value [18]. The condition number of the matrix W_4^{Gau} in Eq. (16) equals to 6.42×10^3, and that of the matrix $Q_{A,W}^8$ equals to 1.11×10^4. It means that a minor disturbance would lead to a huge error in the practical measurement [24]. In other words, the optimal solution in Eq. (16) cannot be used in practice.

Therefore, a proper criterion is needed to find an optimal set of measurement matrices $\left(A_{4-opt}^{Gau}, W_{4-opt}^{Gau}\right)$ that robustly minimizes the total variance of these four elements. We thus introduce a new cost function, which is a linear combination of the total estimation variance and the condition number of $Q_{A,W}^8$:

$$\varepsilon_1 = \sum_{i\in\Omega} \sigma_i^2, \quad \Omega = \{1,3,5,7\}, \quad \varepsilon_2 = C\left(Q_{A,W}^8\right), \quad (18)$$

and

$$\left(A_{4-opt}^{Gau}, W_{4-opt}^{Gau}\right) = \arg\min_{A,W} \{\omega_1\varepsilon_1 + \omega_2\varepsilon_2\}, \quad (19)$$

where $\omega_1 = \varepsilon_1/(\varepsilon_1+\varepsilon_2)$ and $\omega_2 = \varepsilon_2/(\varepsilon_1+\varepsilon_2)$ are two adaptive weight factors. It should be noted that the first term $\omega_1\varepsilon_1$ in Eq. (19) minimizes the total variance of these four elements, while the second term $\omega_2\varepsilon_2$ minimizes the condition number of $Q_{A,W}^8$ and leads to a robust solution. By employing interior-point method, one obtains the optimal solution of Eq. (19) as:

$$A_{4-opt}^{Gau} = \frac{1}{2}\begin{bmatrix} 1 & 0.5748 & 0.6150 & -0.5398 \\ 1 & -0.5748 & 0.5398 & 0.6150 \end{bmatrix}^T,$$
$$W_{4-opt}^{Gau} = \frac{1}{2}\begin{bmatrix} 1 & -0.2059 & 0.9716 & -0.1166 \\ 1 & 0.2059 & 0.9716 & 0.1166 \\ 1 & -0.1174 & -0.9718 & 0.2046 \\ 1 & 0.1174 & -0.9718 & -0.2045 \end{bmatrix}^T. \quad (20)$$

The total estimation variance of those four elements is equal to $20.71\sigma^2$. Indeed, it is slightly higher (~3.6%) than the minimal one of 20 σ^2. The condition number of $Q_{A,W}^8$ corresponding to the optimal matrices $\left(A_{4-opt}^{Gau}, W_{4-opt}^{Gau}\right)$ [in Eq. (20)] is calculated to be 3.31. It means that by using the measurement matrices shown in Eq. (20), one can get an optimal set of measurement matrices robustly minimizes the total variance. The optimal azimuths and ellipticities corresponding to the optimal measurement matrices are shown in Table 2.

TABLE 2. Optimal azimuths and ellipticities for the measurement of 4-elements in presence of Gaussian noise.

Parameters	PSG	PSA
$(\alpha_i, \varepsilon_i)$	(46.93°, -32.67°)	(101.97°, -6.70°)
	(136.80°, 37.95°)	(78.03°, 6.70°)
		(-96.90°, 11.81°)
		(-83.11°, -11.80°)

In order to compare with the existing measurement matrices, the total estimation variance of the four elements obtained by using different existing measurement matrices are shown in Table 3. In addition, the measurement times (refers to the number of intensity measurements) are also presented for comparison.

In addition, our method is based on eight intensity measurements, while others are based on 16 measurements. Indeed, with 16 measurements, we can perform our method (with eight measurements) two times, and the total estimation variance can be further decreased by half. Considering this fact, we introduce a proper optimization ratio [24]:

$$R = \frac{V_{ref} - 0.5 \cdot V_{pro}}{V_{ref}}. \tag{21}$$

where V_{ref} refers to the total estimation variance with the referenced measurement matrices, while V_{pro} refers to the total estimation variance with our proposed optimal measurement matrices in Eq. (20). The optimization ratios related to different referenced measurement matrices are presented in Table 3 for comparison.

TABLE 3. The total variance of the four elements obtained by different measurement matrices. (Gaussian noise).

Instrument matrix	Ref.[20]	Ref.[15]	Ref.[13]	Proposed
Total variance	80	22	16.54	20.73
Measurement times	16	16	16	8
Optimization ratio	87.04%	52.89%	37.33%	/

It can be seen from Table 3 that, considering both total estimation variance and measurement times, the proposed optimal set of measurement matrices (*A, W*) in Eq. (20) leads to the best performance. Specifically, comparing with the regular tetrahedron measurement matrix in [15], which minimizes the total variance of full 16 Mueller elements, the optimization ratio of our method is 52.89%, while with the measurement matrix in [13], which minimizes the total variance of the four Mueller elements based on 16 intensity measurements, the optimization ratio of our method is 37.33%.

In the presence of Poisson noise, we calculate $f\left(\mathbf{V}_M^8\right)$ in Eq. (11) by substituting the optimal measurement matrices $\left(A_{4-opt}^{Gau}, W_{4-opt}^{Gau}\right)$ in Eq. (20). Unfortunately, it is equal to 1.91×10^5, which means that the proposed measurement matrices depend on the Mueller matrix significantly. Furthermore, these measurement matrices make the estimation variance depend on the ellipsometric parameters.

Therefore, it needs to introduce another proper cost function to find optimal measurement matrices that minimize the total variance as well as to make the total variance be independent of the Mueller matrix of sample [$f\left(\mathbf{V}_M^8\right) = 0$ in Eq. (11)] as:

$$\varepsilon_1 = \sum_{i \in \Omega} \sigma_i^2, \quad \Omega = \{1,3,5,7\}, \quad \varepsilon_2 = 10^{\mathrm{m}} \cdot f\left(\mathbf{V}_M^8\right), \tag{22}$$

and

$$\left(A_{4-opt}^{poi}, W_{4-opt}^{poi}\right) = \arg \min_{A,W} \{\omega_1 \varepsilon_1 + \omega_2 \varepsilon_2\}, \tag{23}$$

where $\omega_1 = \varepsilon_1/(\varepsilon_1 + \varepsilon_2)$ and $\omega_2 = \varepsilon_2/(\varepsilon_1 + \varepsilon_2)$ are two adaptive weight factors, and m is a penalty coefficient. We can obtain the optimal numerical solution that satisfies $f\left(\mathbf{V}_M^8\right)$ =0 by increasing the penalty coefficient m [24]. Therefore, in these cases, the total estimation variance can be considered to be unrelated to the measured Mueller matrix. In other words, in Eq. (23), the first term minimizes the total variance of the four elements in Mueller matrix, while the second term ensure the optimal solution is unrelated to the measured Mueller elements for Poisson shot noise. By employing interior-point method with different values of penalty coefficient m, which ranges from 0.1 to 30, the trends of total estimation variance, the value of second term $f\left(\mathbf{V}_M^8\right)$ in Eq. (11) and the condition number for $Q_{A,W}^8$ are presented in Fig. 2.

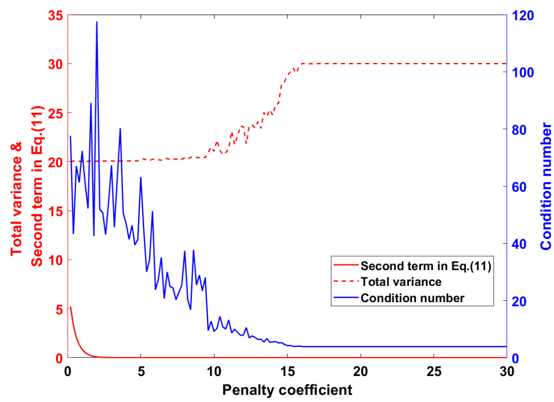

FIGURE 2. The total estimation variance, the value of second term $f\left(V_M^8\right)$ in Eq. (11) and the condition number for $Q_{A,W}^8$ in functions of penalty coefficient.

According to Fig. 2, when $m > 16$, the condition number and the total variance tended towards stable when $f\left(\mathbf{V}_M^8\right) \approx 0$, and the corresponding optimal measurement matrices are what we want, which are given by:

$$A_{4-opt}^{poi} = \frac{1}{2}\begin{bmatrix} 1 & -0.5419 & 0.6036 & -0.5848 \\ 1 & 0.5419 & -0.6036 & -0.5848 \end{bmatrix}^T,$$
$$W_{4-opt}^{poi} = \frac{1}{2}\begin{bmatrix} 1 & 0.4748 & -0.7315 & -0.4894 \\ 1 & -0.4748 & 0.7315 & -0.4894 \\ 1 & 0.4748 & 0.7315 & 0.4894 \\ 1 & -0.4748 & -0.7315 & 0.4894 \end{bmatrix}^T. \tag{24}$$

The total variance of these four elements ε_1 is equal to $30[\mathbf{V}_M^8]_1/4$, and $f\left(\mathbf{V}_M^8\right)$ is equal to $\sim 10^{22}$, which can be considered as zero in practice. Besides, the condition number of $Q_{A,W}^8$ is equal to 3.89. All these results show that, by using the proposed measurement matrices $\left(A_{4-opt}^{poi}, W_{4-opt}^{poi}\right)$ in Eq. (24), one can get a robust lower-variance estimation be independent of Mueller matrix to be measured. The optimal azimuths and ellipticities corresponding to the optimal measurement matrices in Eq. (24) are shown in Table 4.

It should be noted that, the total variance related to the measurement matrices in Eq. (24) are unrelated to the measured Mueller elements and thus the ellipsometric measurements, while those related to the measurement matrices in Eq. (20) are not. Besides, the total variance and the measurement times

TABLE 4. Optimal azimuths and ellipticities for the measurement of 4-elements in Mueller matrix in presence of Poisson shot noise.

Parameter	PSG	PSA
$(\alpha_i, \varepsilon_i)$	(131.92°, -35.79°)	(-57.01°, -29.30°)
	(-48.08°, -35.79°)	(129.99°, -29.30°)
		(57.01°, 29.30°)
		(-129.99°, 29.30°)

by using different existing referenced measurement matrices are presented in Table 5 for comparison.

TABLE 5. The total variance of the four Mueller elements obtained by different measurement matrices. (Poisson noise.)

Instrument matrix	Ref.[20]	Ref.[15]	Ref.[13]	Proposed
Total variance $\bullet[\mathbf{V}_M^8]_1/4$	$>$ 57.4	22	16.54	30
Measurement times	16	16	16	8
Optimization ratio	$>$ 73.87%	31.82%	9.31%	/

Table 5 shows that, considering both total estimation variance and measurement times [and thus the optimization ratio], the proposed measurement matrices in Eq. (24) lead to the best performance. In particular, compared with the existing measurement matrices in [13] and [15], the optimization ratios of the proposed measurement matrices are 31.82% and 9.31%, respectively. In short, by using the proposed measurement matrices in Eq. (24), one can obtain the best performance of the estimation for these four Mueller elements with eight intensity measurements, and consequently, the ellipsometric measurement precision is thus improved.

IV. CONCLUSION

In conclusion, we demonstrate that the ellipsometric parameters can be measured by a partial polarimetry with only eight intensity measurements. In the presence of Gaussian additive and Poisson shot noise, we propose the optimal set of measurement matrices for the PSG and the PSA that minimizes the noise propagation, and thus reduces the estimation variance. Furthermore, since the estimation variance can be further reduced if we only consider four Mueller elements related to the ellipsometric measurements, in this case, we also propose two robust optimal sets of measurement matrices that minimize the total variance of these four Mueller elements by introducing proper criterion under Gaussian and Poisson noise, respectively. Besides, by using our proposed measurement matrices, the total estimation variance is independent of the Mueller matrix to be measured, and thus of the ellipsometric measurements. In addition, compared with the existing referenced measurement matrices, the proposed measurement matrices effectively reduce the total estimation variance and require less measurement times.

Of course, there are other sources of noise in the ellipsometric system, such as the instrumental noise due to the imperfection of optical device [25], [26], the proposed optimization approach can be also applied for the error reduction in these cases. In addition, for the further work, the idea of the minimizes the noise propagation in polarimeter/ellipsometer with less intensity measurements can be also extended to other common types of Mueller matrix, such as the one exhibiting symmetries [27].

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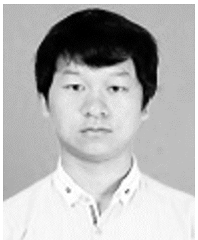

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