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Bipartite Consensus for Multi-Agent Systems With Time-Varying Delays Based on Method of Delay Partitioning

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ABSTRACT Bipartite consensus protocol is designed for multi-agent systems with time-varying delays. Then, the bipartite consensus problem is transformed into a corresponding stability problem by methods of gauge transformation and state transformation. The Lyapunov–Krasovskii functional is constructed, and the linear matrix inequality theory based on methods of delay partitioning and free matrix integral inequality is used to obtain sufficient conditions for a bipartite consensus of multi-agent systems. Both the first-order and second-order multi-agent systems are investigated. Finally, the effectiveness of the obtained results is illustrated by virtue of simulation results.

INDEX TERMS Bipartite consensus, multi-agent systems, time-varying delays, delay partitioning, linear matrix inequality, free matrix integral inequality.

I. INTRODUCTION

As one of the important research branches of control systems, consensus of multi-agent systems has aroused the research interest of a large number of scholars due to its wide applications in the formation of multi-robots, unmanned aerial vehicle joint reconnaissance, wireless sensor networks, controllability and stabilization of multi-agent systems [1], [2], and so on. Consensus means that, by designing the control protocol of systems, the positions(or velocities) of all agents can reach common value after the exchange of information.

In recent years, the research on consensus has developed rapidly and has obtained quite a lot of theoretical achievements [3]–[18]. Among these works, different issues of information transmission with delays were considered [10]–[18]. Necessary and sufficient conditions to guarantee the achievement of consensus were given for first-order and second-order multi-agent systems with constant delay in [3], [10] and [11], respectively. The local and global exponential synchronization of complex dynamical networks with switching topologies and time-varying coupling delays was studied in [12]. The delay partitioning method was used to give sufficient conditions for consensus by using the linear matrix inequality theory for the consensus of second-order multi-agent systems with constant communication delay in [13]. Reference [14] studied containment control for second-order multi-agent systems with time-varying delays. For general linear systems with time-varying delays, [15]–[18] gave corresponding theorems to guarantee that consensus of multi-agent systems could be achieved. In 2013, Altafini [19] gave the definition of consensus on networks with antagonistic interactions and investigated the bipartite consensus of multi-agent systems under the structurally balanced topology. For antagonistic networks with double integrator dynamics and constant communication delay, distributed consensus protocols were proposed and sufficient and necessary conditions for consensus achieving were given in [20]. It was proved that the largest tolerable communication delay was related to the largest eigenvalue of the graph Laplacian matrix. In [21], the bipartite consensus problem of second-order dynamics on coopetition networks with time-varying delays was investigated by using the LMI method. In [22] and [23], the bipartite consensus of high-order multi-agent dynamical systems with antagonistic interactions was considered.

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In [13], the method of partitioning was employed in dealing with the constant communication delay between the neighboring agents for second-order consensus of multi-agent systems under directed networks. The delay τ was partitioned into m small parts. Then, it was proved that the time delay tended to a constant with the increase of m. In [24], to investigate the stability of T-S fuzzy systems with interval time-varying delays and nonlinear perturbations, a novel delay partitioning method was proposed by partitioning the delay interval into a series of geometric progression based on subintervals under a common ratio α, i.e., the delay interval was unequally separated into multiple subintervals.

Based on the mentioned works, this paper investigates first-order and second-order consensus problems for multi-agent systems with antagonistic interactions and time-varying delays under directed topology, respectively. By applying the delay partitioning approach, the delay interval is separated into multiple subintervals with variable lengths which are related to the common ratio α, where α is a positive constant. If the the partition number is chosen as 1, using this delay partitioning approach, the obtained results include the works of [21]. Compared with the method of partitioning into equal m small parts, this partitioning method is more general. Further, by using *Lyapunov* stability theory and linear matrix inequality theory, sufficient conditions to guarantee bipartite consensus for multi-agent systems are given. Compared with the existing results, the obtained results have less conservatism in the sense that the permitted upper bound of delays can be larger.

The paper is organized as follows: Some concepts about graph theory and some lemmas are described in Section 2. Bipartite consensus analyses for first-order and second-order multi-agent systems with antagonistic interactions and time-varying delays under directed topology are given in Section 3 and Section 4, respectively. Numerical examples are presented in Section 5 to illustrate the effectiveness of the obtained results. Finally, some conclusions are drawn in Section 6.

Notations: R^n and $R^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $I(\mathbf{0})$ is the identity(zero) matrix with appropriate dimension, A^T denotes the transpose of A, and $He(A) = A + A^T$. The $*$ denotes the elements below the main diagonal of a symmetric matrix. $sgn(\cdot)$ denotes sign function. $\mathbf{1}_n(\mathbf{0}_n)$ denotes a column vector with all 1(0) elements.

II. PRELIMINARIES

In this section, we introduce the graph theory. Define $G = (V, \mathcal{E}, \bar{A})$ as a weighted directed graph(digraph) with a set of nodes $V = \{1, 2, \ldots, n\}$, a set of edges $\mathcal{E}$, and a weighted adjacency matrix $\bar{A} = [a_{ij}] \in R^{n\times n}$, where $a_{ij} \neq 0$ iff $(j, i) \in \mathcal{E}$. The value of a_{ij} can be positive or negative since we consider multi-agent systems with antagonistic interactions in this paper. Here, we choose $a_{ii} = 0$ for all $i \in \{1, \ldots, n\}$. Define a node j as a neighbor of node i if $a_{ij} \neq 0$. The set of neighbors of node i is denoted by $N_i = \{j \in V : (j, i) \in \mathcal{E}\}$. A directed path between each pair of distinct nodes i and j is a finite-ordered sequence of distinct edges of G of the form $(i, k_1), (k_1, k_2), \ldots, (k_l, j)$. A digraph has a spanning tree if exists a node called root such that there exists a directed path from this node to every other node. The elements of Laplacian matrix $L = [l_{ij}]_{n\times n}$ are defined as

$$l_{ij} = \begin{cases} \sum_{j\in N_i} |a_{ij}|, & j = i; \\ -a_{ij}, & j \neq i. \end{cases}$$

Definition 1 (Structural Balance [19]): A digraph G is said to be structurally balanced if all the nodes of G can be partitioned into two nonempty subsets V_1 and V_2 such that $V_1 \bigcup V_2 = V$, $V_1 \bigcap V_2 = \Phi$, and the following two conditions hold:

1) $a_{ij} \geq 0$ if $i, j \in V_q (q \in \{1, 2\})$;

2) $a_{ij} \leq 0$ if $i \in V_q, j \in V_r (q \neq r, q, r \in \{1, 2\})$.

In order to facilitate the following analysis, we use an orthogonal matrix D to change the orthant order. The change of the orthant order is called a gauge transformation and the orthogonal matrix D is defined as follows

$$D \in \mathcal{D} \triangleq \{diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}, \sigma_i \in \{1, -1\}\}.$$

It is easy to see $D = D^{-1}$. When the digraph is structurally balanced, the elements of DAD can be guaranteed to be non-negative by selecting an appropriate D.

Lemma 1: For $n \times n$ matrix $Q > 0$, scalars $\tau \geq \tau_2 > \tau_1 > 0$, vector-valued function $x : [-\tau, 0] \to R^n$ such that the included integrations are well defined, the following inequalities hold

$$\begin{aligned} &-\tau \int_{t-\tau}^{t} \dot{x}^T(s) Q \dot{x}(s) ds \\ &\leq \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ * & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \\ &-\frac{\tau_2^2 - \tau_1^2}{2} \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \dot{x}^T(s) Q \dot{x}(s) ds d\theta \\ &\leq \begin{bmatrix} (\tau_2 - \tau_1) x(t) \\ \int_{t-\tau_2}^{t-\tau_1} x(s) ds \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ * & -Q \end{bmatrix} \begin{bmatrix} (\tau_2 - \tau_1) x(t) \\ \int_{t-\tau_2}^{t-\tau_1} x(s) ds \end{bmatrix}. \end{aligned}$$

Proof: It is easy to draw this conclusion from the idea in [25, Proof of Lemma 1]. □

Lemma 2 (Free-Matrix-Based Integral Inequality [26]): Let $x : [a, b] \to R^n$ be a differentiable vector-valued function. For $\hat{Z} = \hat{Z}^T \in R^{n\times n}$, $\hat{W}_1 = \hat{W}_1^T$, $\hat{W}_2 = \hat{W}_2^T \in R^{3n\times 3n}$ and $\hat{W}_3 \in R^{3n\times 3n}$, $\hat{N}_1, \hat{N}_2 \in R^{3n\times n}$ satisfying

$$\begin{bmatrix} \hat{W}_1 & \hat{W}_3 & \hat{N}_1 \\ * & \hat{W}_2 & \hat{N}_2 \\ * & * & \hat{Z} \end{bmatrix} \geq 0,$$

it holds that

$$-\int_a^b \dot{x}^T(s) \hat{Z} \dot{x}(s) ds \leq \varpi^T \Omega \varpi,$$

where $\varpi = \left[x^T(b)\ \ x^T(a)\ \ \frac{1}{b-a}\int_a^b x^T(s)ds\right]^T$, $\Omega = (b-a)(\hat{W}_1 + \frac{1}{3}\hat{W}_2) + He(\hat{N}_1\Lambda_1 + \hat{N}_2\Lambda_2)$, $\Lambda_1 = \bar{e}_1 - \bar{e}_2$, $\Lambda_2 = 2\bar{e}_3 - \bar{e}_1 - \bar{e}_2$, $\bar{e}_1 = [I\ \mathbf{0}\ \mathbf{0}]$, $\bar{e}_2 = [\mathbf{0}\ I\ \mathbf{0}]$, $\bar{e}_3 = [\mathbf{0}\ \mathbf{0}\ I]$.

III. BIPARTITE CONSENSUS OF FIRST-ORDER MULTI-AGENT SYSTEMS

Consider a first-order multi-agent system composed of n agents. Suppose that its topology is represented by a digraph with each agent as a node and the dynamics of each agent can be expressed as

$$\dot{x}_i(t) = u_i(t), \quad i \in \{1, \ldots, n\}, \tag{1}$$

where $x_i(t), u_i(t) \in R$ are the position and the control input of agent i, respectively. According to the definition in [19], we say that bipartite consensus of (1) with antagonistic relationship can be achieved if for any initial states, it holds that

$$\lim_{t\to\infty}(x_i(t) - sgn(a_{ij})x_j(t)) = 0, \quad i, j \in \{1, \ldots, n\}. \tag{2}$$

In order to solve the bipartite consensus problem of systems (1) with time-varying communication delays, we propose the following bipartite consensus protocol for each $i \in \{1, 2, \ldots, n\}$

$$u_i(t) = -\sum_{j\in N_i}|a_{ij}|(x_i(t-\tau(t)) - sgn(a_{ij})x_j(t-\tau(t))), \tag{3}$$

where $\tau(t)$ are the communication delays and satisfy

$$0 \le \tau(t) \le h, \quad \dot{\tau}(t) \le \mu < \infty. \tag{4}$$

The application of protocol (3) to system (1) results in

$$\dot{x}(t) = -Lx(t-\tau(t)), \tag{5}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^T$.

Next, we transform the bipartite consensus problem into a corresponding stability problem of error system. Let $\bar{x}(t) = [\bar{x}_1(t), \ldots, \bar{x}_n(t)]^T = Dx(t)$, where $D \in \mathcal{D}$ can guarantee DAD to be non-negative. Since $D = D^{-1}$, we have $x(t) = D\bar{x}(t)$. Putting it into (5), we can obtain

$$\dot{\bar{x}}(t) = -\bar{L}\bar{x}(t-\tau(t)), \tag{6}$$

where $\bar{L} = DLD$. Define $y(t) = [y_1(t), \ldots, y_{n-1}(t)]^T$, where $y_i(t) = \bar{x}_1(t) - \bar{x}_{i+1}(t)(i \in \{1, \ldots, n-1\})$. Then,

$$\dot{y}(t) = -E\bar{L}Fy(t-\tau(t)) = Ay(t-\tau(t)), \tag{7}$$

where $E = [\mathbf{1}_{n-1}, -I_{n-1}]$, $F = [\mathbf{0}_{n-1}, -I_{n-1}]^T$, $A = -E\bar{L}F$.

It is easy to see that (2) is equivalent to $\lim_{t\to\infty} y(t) = \mathbf{0}_{n-1}$, i.e., the bipartite consensus problem of system (5) is equivalent to the stability problem of error system (7).

Next, an approach of delay partitioning is introduced based on the geometric sequence division method. The delay interval $[0, h]$ is partitioned into N geometric subintervals with a positive common ratio α, where N can be any positive integer and $\alpha = \frac{\delta_i}{\delta_{i-1}}(i \in \{2, \ldots, N\})$, and $\delta_i(i \in \{1, 2, \ldots, N\})$ is the length of the ith subinterval. FIGURE 1 gives an illustration with $\alpha > 1$.

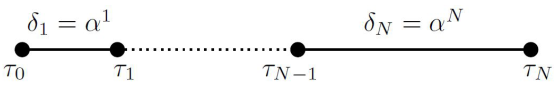

FIGURE 1. Geometric sequence partitioning of the delay interval ($\alpha > 1$).

It is obvious that $\delta_i = \tau_i - \tau_{i-1} = \alpha^i$, where $\tau_i = \tau_{i-1} + \alpha^i (i \in \{1, \ldots, N\})$ with $\tau_0 = 0$. Hence, $[0, h] = \bigcup_{i=1}^N H_i$, where $H_i = [\tau_{i-1}, \tau_i)(i \in \{1, \ldots, N-1\})$ and $H_N = [\tau_{N-1}, \tau_N]$. For notational simplicity, define the following matrix

$$e_j = \Bigg[\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{j-1}\ I\ \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{3N-j+2}\Bigg]^T \in R^{(3N+2)(n-1)\times(n-1)},$$

where $j \in \{1, \ldots, 3N+2\}$.

Based on the above mentioned method of delay partitioning, the following theorem for the bipartite consensus problem of (1) with time-varying delays can be given.

Theorem 1: Assume that the topology of (1) has a directed spanning tree and is structurally balanced. Bipartite consensus of system (1) using (3) can be asymptotically achieved if there exist positive definite matrices $Q, Z_i, Q_i, R_{2i}, R_{3i} \in R^{(n-1)\times(n-1)}(i = 1, \ldots, N)$, $P \in R^{(n-1)\times(n-1)}$, $S \in R^{N(n-1)\times N(n-1)}$, symmetric matrices $W_1, W_2 \in R^{3(n-1)\times 3(n-1)}$, and matrices $W_3 \in R^{3(n-1)\times 3(n-1)}$, $N_1, N_2 \in R^{3(n-1)\times(n-1)}$, such that the following linear matrix inequalities hold

$$\begin{bmatrix} W_1 & W_3 & N_1 \\ * & W_2 & N_2 \\ * & * & Z_i \end{bmatrix} \ge 0, i \in \{1, \ldots, N\}, \tag{8}$$

$$\Psi_1 + \Psi_2 + \Psi_{3k} + \Psi_4 < 0, k \in \{1, \ldots, N\}, \tag{9}$$

where

$$\Psi_1 = He(e_2A^TPe_1^T) + He\left(\begin{bmatrix} e_{N+3}^T \\ \vdots \\ e_{2N+2}^T \end{bmatrix}^T S \begin{bmatrix} e_2^T - e_3^T \\ \vdots \\ e_2^T - e_{N+2}^T \end{bmatrix}\right),$$

$$\Psi_2 = e_2Qe_2^T - (1-\mu)e_1Qe_1^T + \sum_{i=1}^N e_{i+1}Q_ie_{i+1}^T - \sum_{i=1}^N e_{i+2}Q_ie_{i+2}^T,$$

$$\Psi_{3k} = e_1\left(\sum_{i=1}^N \delta_i^2 A^TZ_iA\right)e_1^T + \sum_{i=1, i\ne k}^N \begin{bmatrix} e_{i+1}^T \\ e_{i+2}^T \\ e_{2N+2+i}^T \end{bmatrix}^T \Omega_{3i} \begin{bmatrix} e_{i+1}^T \\ e_{i+2}^T \\ e_{2N+2+i}^T \end{bmatrix} + \begin{bmatrix} e_1^T \\ e_{k+1}^T \\ e_{k+2}^T \end{bmatrix}^T \begin{bmatrix} -2Z_k & Z_k & Z_k \\ * & -Z_k & 0 \\ * & * & Z_k \end{bmatrix} \begin{bmatrix} e_1^T \\ e_{k+1}^T \\ e_{k+2}^T \end{bmatrix},$$

$$\Omega_{3i} = \delta_i^2(W_1 + \frac{1}{3}W_2) + \delta_i He(N_1\Lambda_1 + N_2\Lambda_2),$$

$$\Psi_4 = e_1\left(\sum_{i=1}^{N}\frac{1}{4}\tau_i^4 A^T R_{2i} A + \sum_{i=1}^{N}\frac{1}{4}(\tau_i^2-\tau_{i-1}^2)^2 A^T R_{3i} A\right) \times e_1^T + \sum_{i=1}^{N}\begin{bmatrix}\tau_i e_2^T\\ e_{N+2+i}^T\end{bmatrix}^T\begin{bmatrix}-R_{2i} & R_{2i}\\ * & -R_{2i}\end{bmatrix}\begin{bmatrix}\tau_i e_2^T\\ e_{N+2+i}^T\end{bmatrix} + \sum_{i=1}^{N}\delta_i^2\begin{bmatrix}e_2^T\\ e_{2N+2+i}^T\end{bmatrix}^T\begin{bmatrix}-R_{3i} & R_{3i}\\ * & -R_{3i}\end{bmatrix}\begin{bmatrix}e_2^T\\ e_{2N+2+i}^T\end{bmatrix}.$$

Proof: When $\tau(t)\equiv 0$, system (5) can be written as $\dot{x}(t) = -Lx(t)$. According to [19], when the topology of (1) has a directed spanning tree and is structurally balanced, the system (1) using (3) can achieve bipartite consensus, which implies that the LMIs (8) and (9) are feasible.

When $\tau(t)\not\equiv 0$ for $t\geq 0$, there exists an integer $k\in\{1,\ldots,N\}$ such that $\tau(t)\in H_k$. Define an augmented vector as

$$\xi(t) = \left[y^T(t-\tau(t)), y(t), \eta^T(t), \eta_1^T(t), \eta_2^T(t)\right]^T,$$

where $\xi(t)\in R^{(3N+2)(n-1)}$, $\eta(t) = \left[y^T(t-\tau_1),\ldots,y^T(t-\tau_N)\right]^T$, $\eta_1(t) = \left[\int_{t-\tau_1}^{t} y^T(s)ds,\ldots,\int_{t-\tau_N}^{t} y^T(s)ds\right]^T$, $\eta_2(t) = \left[\frac{1}{\delta_1}\int_{t-\tau_1}^{t-\tau_0} y^T(s)ds,\ldots,\frac{1}{\delta_N}\int_{t-\tau_N}^{t-\tau_{N-1}} y^T(s)ds\right]^T$.

A *Lyapunov* − *Krasovskii* functional is chosen as

$$V(t) = V_1(t)+V_2(t)+V_3(t)+V_4(t),$$

where

$$V_1(t) = y^T(t)Py(t)+\eta_1^T(t)S\eta_1(t),$$
$$V_2(t) = \int_{t-\tau(t)}^{t} y^T(s)Qy(s)ds + \sum_{i=1}^{N}\int_{t-\tau_i}^{t-\tau_{i-1}} y^T(s)Q_iy(s)ds,$$
$$V_3(t) = \sum_{i=1}^{N}\delta_i\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\beta}^{t}\dot{y}^T(s)Z_i\dot{y}(s)dsd\beta,$$
$$V_4(t) = \sum_{i=1}^{N}\frac{\tau_i^2}{2}\int_{-\tau_i}^{0}\int_{\theta}^{0}\int_{t+\beta}^{t}\dot{y}^T(s)R_{2i}\dot{y}(s)dsd\beta d\theta + \sum_{i=1}^{N}\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau_i}^{-\tau_{i-1}}\int_{\theta}^{0}\int_{t+\beta}^{t}\dot{y}^T(s)R_{3i} \times\dot{y}(s)dsd\beta d\theta.$$

The time derivative of $V(t)$ is given as

$$\dot{V}(t) = \dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)+\dot{V}_4(t).$$

Invoking (7), we have

$$\begin{aligned}\dot{V}_1(t) &= \dot{y}^T(t)Py(t)+y^T(t)P\dot{y}(t)+\dot{\eta}_1^T(t)S\eta_1(t) + \eta_1^T(t)S\dot{\eta}_1(t)\\ &= y^T(t-\tau(t))A^TPy(t)+y^T(t)PAy(t-\tau(t)) + \begin{bmatrix}y^T(t)-y^T(t-\tau_1)\\ \vdots\\ y^T(t)-y^T(t-\tau_N)\end{bmatrix}^T S\eta_1(t) + \eta_1^T(t)S\begin{bmatrix}y^T(t)-y^T(t-\tau_1)\\ \vdots\\ y^T(t)-y^T(t-\tau_N)\end{bmatrix}\\ &= \xi^T(t)\Psi_1\xi(t),\end{aligned}$$

$$\begin{aligned}\dot{V}_2(t) &\leq \sum_{i=1}^{N}[y^T(t-\tau_{i-1})Q_iy(t-\tau_{i-1})-y^T(t-\tau_i)Q_i \times y(t-\tau_i)] - (1-\mu)y^T(t-\tau(t))Qy(t-\tau(t)) + y^T(t)Qy(t)\\ &= \xi^T(t)\Psi_2\xi(t).\end{aligned}$$

The time derivative of $V_3(t)$ is expressed as

$$\begin{aligned}\dot{V}_3(t) &= \sum_{i=1}^{N}\delta_i^2\dot{y}^T(t)Z_i\dot{y}(t) - \sum_{i=1}^{N}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds\\ &= y^T(t-\tau(t))\left(\sum_{i=1}^{N}\delta_i^2A^TZ_iA\right)y(t-\tau(t)) - \sum_{i=1}^{N}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds. \qquad (10)\end{aligned}$$

For the case of $\tau(t)\in H_k(1\leq k\leq N)$, the second term in (10) can be written as

$$\begin{aligned}&-\sum_{i=1}^{N}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds\\ &\quad = -\sum_{i=1,i\neq k}^{N}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds - \delta_k\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds. \qquad (11)\end{aligned}$$

Applying Lemma 2 to deal with the first term of (11), it follows that

$$-\sum_{i=1,i\neq k}^{N}\delta_i\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{y}^T(s)Z_i\dot{y}(s)ds \leq \sum_{i=1,i\neq k}^{N}\varpi_{3i}^T(t)\Omega_{3i}\varpi_{3i}(t), \qquad (12)$$

where $\varpi_{3i}(t) = [y^T(t-\tau_{i-1}), y^T(t-\tau_i), \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}} y^T(s)ds]^T$. Applying Lemma 1 to deal with the second term of (11), it follows that

$$\begin{aligned}&-\delta_k\int_{t-\tau_k}^{t-\tau_{k-1}}\dot{y}^T(s)Z_k\dot{y}(s)ds\\ &\leq -(\tau_k-\tau(t))\int_{t-\tau k}^{t-\tau(t)}\dot{y}^T(s)Z_k\dot{y}(s)ds - (\tau(t)-\tau_{k-1})\int_{t-\tau(t)}^{t-\tau_{k-1}}\dot{y}^T(s)Z_k\dot{y}(s)ds\\ &\leq \begin{bmatrix}y(t-\tau_k)\\ y(t-\tau(t))\end{bmatrix}^T\begin{bmatrix}-Z_k & Z_k\\ * & -Z_k\end{bmatrix}\begin{bmatrix}y(t-\tau_k)\\ y(t-\tau(t))\end{bmatrix}\end{aligned}$$

$$+\begin{bmatrix} y(t-\tau(t)) \\ y(t-\tau_{k-1}) \end{bmatrix}^T \begin{bmatrix} -Z_k & Z_k \\ * & -Z_k \end{bmatrix} \begin{bmatrix} y(t-\tau(t)) \\ y(t-\tau_{k-1}) \end{bmatrix}$$
$$= \eta_0^T(t) \begin{bmatrix} -2Z_k & Z_k & Z_k \\ * & -Z_k & 0 \\ * & * & -Z_k \end{bmatrix} \eta_0(t), \quad (13)$$

where $\eta_0(t) = \left[y^T(t-\tau(t)), y^T(t-\tau_{k-1}), y^T(t-\tau_k)\right]^T$. Then, it follows from (10)-(13) that

$$\dot{V}_3(t) \leq y^T(t-\tau(t)) \left(\sum_{i=1}^{N} \delta_i^2 A^T Z_i A \right) y(t-\tau(t))$$
$$+ \sum_{i=1, i\neq k}^{N} \varpi_{3i}^T(t) \Omega_{3i} \varpi_{3i}(t)$$
$$+ \eta_0^T(t) \begin{bmatrix} -2Z_k & Z_k & Z_k \\ * & -Z_k & 0 \\ * & * & -Z_k \end{bmatrix} \eta_0(t)$$
$$= \xi^T(t) \Psi_{3k} \xi(t). \quad (14)$$

The time derivative of $V_4(t)$ is presented as

$$\dot{V}_4(t) = \sum_{i=1}^{N} \frac{1}{4} \tau_i^4 \dot{y}^T(t) R_{2i} \dot{y}(t)$$
$$+ \sum_{i=1}^{N} \frac{1}{4} (\tau_i^2 - \tau_{i-1}^2)^2 \dot{y}^T(t) R_{3i} \dot{y}(t)$$
$$- \sum_{i=1}^{N} \frac{\tau_i^2}{2} \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \dot{y}^T(\beta) R_{2i} \dot{y}(\beta) d\beta d\theta$$
$$- \sum_{i=1}^{N} \frac{\tau_i^2 - \tau_{i-1}^2}{2} \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta}^{t} \dot{y}^T(\beta) R_{3i} \dot{y}(\beta) d\beta d\theta$$
$$= y^T(t-\tau(t)) \left[\sum_{i=1}^{N} \frac{1}{4} \tau_i^4 A^T R_{2i} A \right] y(t-\tau(t))$$
$$+ y^T(t-\tau(t)) \left[\sum_{i=1}^{N} \frac{1}{4} \left(\tau_i^2 - \tau_{i-1}^2\right)^2 A^T R_{3i} A \right]$$
$$\times y(t-\tau(t)) - \sum_{i=1}^{N} \frac{\tau_i^2}{2} \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \dot{y}^T(\beta) R_{2i}$$
$$\times \dot{y}(\beta) d\beta d\theta - \sum_{i=1}^{N} \frac{\tau_i^2 - \tau_{i-1}^2}{2} \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta}^{t} \dot{y}^T(\beta)$$
$$\times R_{3i} \dot{y}(\beta) d\beta d\theta. \quad (15)$$

By using Lemma 1, the last two terms of (15) are deduced as

$$- \sum_{i=1}^{N} \frac{\tau_i^2}{2} \int_{-\tau_i}^{0} \int_{t+\theta}^{t} \dot{y}^T(\beta) R_{2i} \dot{y}(\beta) d\beta d\theta$$
$$\leq \sum_{i=1}^{N} \begin{bmatrix} \tau_i y(t) \\ \int_{t-\tau_i}^{t} y(\beta) d\beta \end{bmatrix}^T \begin{bmatrix} -R_{2i} & R_{2i} \\ * & -R_{2i} \end{bmatrix} \begin{bmatrix} \tau_i y(t) \\ \int_{t-\tau_i}^{t} y(\beta) d\beta \end{bmatrix},$$
$$- \sum_{i=1}^{N} \frac{\tau_i^2 - \tau_{i-1}^2}{2} \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta}^{t} \dot{y}^T R_{3i} \dot{y}(\beta) d\beta d\theta$$
$$\leq \sum_{i=1}^{N} \begin{bmatrix} (\tau_i - \tau_{i-1}) y(t) \\ \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}^T \begin{bmatrix} -R_{3i} & R_{3i} \\ * & -R_{3i} \end{bmatrix} \begin{bmatrix} (\tau_i - \tau_{i-1}) y(t) \\ \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}$$
$$\leq \sum_{i=1}^{N} (\tau_i - \tau_{i-1})^2 \begin{bmatrix} y(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}^T \begin{bmatrix} -R_{3i} & R_{3i} \\ * & -R_{3i} \end{bmatrix}$$
$$\times \begin{bmatrix} y(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}.$$

Hence,

$$\dot{V}_4(t) \leq y^T(t-\tau(t)) \left[\sum_{i=1}^{N} \frac{1}{4} \tau_i^4 A^T R_{2i} A \right] y(t-\tau(t))$$
$$+ y^T(t-\tau(t)) \left[\sum_{i=1}^{N} \frac{1}{4} \left(\tau_i^2 - \tau_{i-1}^2\right)^2 A^T R_{3i} A \right]$$
$$\times y(t-\tau(t)) + \sum_{i=1}^{N} \begin{bmatrix} \tau_i y(t) \\ \int_{t-\tau_i}^{t} y(\beta) d\beta \end{bmatrix}^T$$
$$\times \begin{bmatrix} -R_{2i} & R_{2i} \\ * & -R_{2i} \end{bmatrix} \begin{bmatrix} \tau_i y(t) \\ \int_{t-\tau_i}^{t} y(\beta) d\beta \end{bmatrix}$$
$$+ \sum_{i=1}^{N} (\tau_i - \tau_{i-1})^2 \begin{bmatrix} y(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}^T$$
$$\times \begin{bmatrix} -R_{3i} & R_{3i} \\ * & -R_{3i} \end{bmatrix} \begin{bmatrix} y(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} y(\beta) d\beta \end{bmatrix}$$
$$= \xi^T(t) \Psi_4 \xi(t).$$

Therefore, we can get that

$$\dot{V}(t) \leq \xi^T(t) \left(\Psi_1 + \Psi_2 + \Psi_{3k} + \Psi_4\right) \xi(t).$$

Obviously, $\dot{V}(t) < 0$ if (8) and (9) hold, which means that the error system (7) is asymptotically stable, i.e., the system (1) can achieve bipartite consensus. □

IV. BIPARTITE CONSENSUS OF SECOND-ORDER MULTI-AGENT SYSTEMS

In this section, suppose the dynamics of each agent can be expressed as

$$\dot{x}_i(t) = v_i(t), \dot{v}_i(t) = u_i(t), \quad i \in \{1, \ldots, n\}, \quad (16)$$

where $x_i(t), v_i(t), u_i(t) \in R$ are the position, velocity and control input of agent i, respectively. We say that the bipartite consensus of (16) with antagonistic relationship can be achieved if for any initial states, the following conclusions hold

$$\lim_{t\to\infty} (x_i(t) - sgn(a_{ij}) x_j(t)) = 0,$$
$$\lim_{t\to\infty} (v_i(t) - sgn(a_{ij}) v_j(t)) = 0, \quad i, j \in \{1, \ldots, n\}. \quad (17)$$

In order to solve the bipartite consensus problem of system (16) with time-varying communication delays, we propose the following bipartite consensus protocol for each $i \in \{1, 2, \ldots, n\}$

$$u_i(t) = -\beta \sum_{j \in N_i} |a_{ij}|(v_i(t-\tau(t)) - sgn(a_{ij})v_j(t-\tau(t))) - \sum_{j \in N_i} |a_{ij}|(x_i(t-\tau(t)) - sgn(a_{ij})x_j(t-\tau(t))), \quad (18)$$

where $\beta > 0$ is the relative damping gain and $\tau(t)$ are the communication delays satisfying (4). Denote $\zeta_i = [x_i, v_i]^T$, $\zeta(t) = \left[\zeta_1^T, \ldots, \zeta_n^T\right]^T$. Let $x_D(t) = Dx(t) = [x_{d1}(t), x_{d2}(t), \ldots, x_{dn}(t)]^T$, $v_D(t) = Dv(t) = [v_{d1}(t), v_{d2}(t), \ldots, v_{dn}(t)]^T$, where $D \in \mathcal{D}$ can guarantee DAD to be non-negative. Then, $x(t) = Dx_D(t)$, $v(t) = Dv_D(t)$ because of $D = D^{-1}$. Denote $\zeta_{di}(t) = [x_{di}(t), v_{di}(t)]^T$, $\zeta_D(t) = \left[\zeta_{d1}^T(t), \zeta_{d2}^T(t), \ldots, \zeta_{dn}^T(t)\right]^T$. From (16) and (18), we have

$$\dot{\zeta}_D(t) = (I_n \otimes B)\zeta_D(t) - (L_D \otimes C)\zeta_D(t-\tau(t)), \quad (19)$$

where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & \beta \end{bmatrix}$, $L_D = DLD$. Let $\tilde{\zeta}_i(t) = \zeta_{d1}(t) - \zeta_{di}(t) (i \in \{2, \ldots, n\})$, $\tilde{\zeta}(t) = \left[\tilde{\zeta}_2(t), \ldots, \tilde{\zeta}_n(t)\right]^T$. Then, (19) can be expressed as

$$\dot{\tilde{\zeta}}(t) = (I_{n-1} \otimes B)\,\tilde{\zeta}(t) - \left[\left(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}\right) \otimes C\right] \tilde{\zeta}(t-\tau(t)), \quad (20)$$

where $\bar{L}_D$ is the square matrix of $n-1$ order by deleting first row and first column of L_D, $\mathbf{r} = (sgn(a_{12})a_{12}, sgn(a_{13})a_{13}, \ldots, sgn(a_{1n})a_{1n})$. It can be seen that the bipartite concensus problem is equivalent to the stability problem of error system (20). For notational simplicity, define the following matrix

$$\tilde{e}_j = \Big[\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{j-1}\ I\ \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{3N-j+2}\Big]^T \in R^{(3N+2)(2n-2)\times(2n-2)},$$

where $j \in \{1, \ldots, 3N+2\}$.

Theorem 2: Assume that the topology of (16) has a directed spanning tree and is structurally balanced. Bipartite consensus of multi-agent system (16) using (18) can be asymptotically achieved if there exist positive definite matrices $\tilde{Q}, \tilde{Z}_i, \tilde{Q}_i, \tilde{R}_{2i}, \tilde{R}_{3i} \in R^{(2n-2)\times(2n-2)} (i \in \{1, \ldots, N\})$, $\tilde{P} \in R^{(2n-2)\times(2n-2)}$, $\tilde{S} \in R^{N(2n-2)\times N(2n-2)}$, symmetric matrices $\tilde{W}_1, \tilde{W}_2 \in R^{3(2n-2)\times 3(2n-2)}$, and matrices $\tilde{W}_3 \in R^{3(2n-2)\times 3(2n-2)}$, $\tilde{N}_1, \tilde{N}_2 \in R^{3(2n-2)\times(2n-2)}$, such that the following linear matrix inequalities hold

$$\begin{bmatrix} \tilde{W}_1 & \tilde{W}_3 & \tilde{N}_1 \\ * & \tilde{W}_3 & \tilde{N}_2 \\ * & * & \tilde{Z}_i \end{bmatrix} \geq 0, \quad i \in \{1, \ldots, N\}, \quad (21)$$

$$\tilde{\Psi}_1 + \tilde{\Psi}_2 + \tilde{\Psi}_{3k} + \tilde{\Psi}_4 < 0, \quad k \in \{1, \ldots, N\}, \quad (22)$$

where

$$\tilde{\Psi}_1 = He\left(\tilde{e}_2(I_{n-1} \otimes B)^T \tilde{P}\tilde{e}_2^T\right) - He\left(\tilde{e}_1[(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]^T \tilde{P}\tilde{e}_2^T\right) + He\left(\begin{bmatrix} \tilde{e}_{N+3}^T \\ \vdots \\ \tilde{e}_{2N+2}^T \end{bmatrix}^T S \begin{bmatrix} \tilde{e}_2^T - \tilde{e}_3^T \\ \vdots \\ \tilde{e}_2^T - \tilde{e}_{N+2}^T \end{bmatrix}\right),$$

$$\tilde{\Psi}_2 = \tilde{e}_2\tilde{Q}\tilde{e}_2^T - (1-\mu)\tilde{e}_1\tilde{Q}\tilde{e}_1^T + \sum_{i=1}^{N} \tilde{e}_{i+1}\tilde{Q}_i\tilde{e}_{i+1}^T - \sum_{i=1}^{N} \tilde{e}_{i+2}\tilde{Q}_i\tilde{e}_{i+2}^T,$$

$$\begin{aligned}
\tilde{\Psi}_{3k} = &\sum_{i=1, i\neq k}^{N} \begin{bmatrix} \tilde{e}_{i+1}^T \\ \tilde{e}_{i+2}^T \\ \tilde{e}_{2N+2+i}^T \end{bmatrix}^T \Omega_{3i} \begin{bmatrix} \tilde{e}_{i+1}^T \\ \tilde{e}_{i+2}^T \\ \tilde{e}_{2N+2+i}^T \end{bmatrix} + \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_{k+1}^T \\ \tilde{e}_{k+2}^T \end{bmatrix}^T \begin{bmatrix} -2\tilde{Z}_k & \tilde{Z}_k & \tilde{Z}_k \\ * & -\tilde{Z}_k & 0 \\ * & * & \tilde{Z}_k \end{bmatrix} \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_{k+1}^T \\ \tilde{e}_{k+2}^T \end{bmatrix} \\
&+ \sum_{i=1}^{N} \tilde{e}_2\delta_i^2(I_{n-1} \otimes B)^T \tilde{Z}_i(I_{n-1} \otimes B)\tilde{e}_2^T - He\left(\sum_{i=1}^{N} \delta_i^2\tilde{e}_2(I_{n-1} \otimes B)^T \tilde{Z}_i[(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]\tilde{e}_1^T\right) \\
&+ \sum_{i=1}^{N} \delta_i^2\tilde{e}_1[(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]^T \tilde{Z}_i \times [(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]\tilde{e}_1^T
\end{aligned}$$

$$\tilde{\Omega}_{3i} = \delta_i^2(\tilde{W}_1 + \frac{1}{3}\tilde{W}_2) + \delta_i He(\tilde{N}_1\Lambda_1 + \tilde{N}_2\Lambda_2),$$

$$\begin{aligned}
\tilde{\Psi}_4 = &\sum_{i=1}^{N} \frac{1}{4}\tilde{e}_2(I_{n-1} \otimes B)^T[\tau_i^2\tilde{R}_{2i} + (\tau_i^2 - \tau_{i-1}^2)\tilde{R}_{3i}] \times (I_{n-1} \otimes B)\tilde{e}_2^T \\
&- He\left(\sum_{i=1}^{N} \frac{1}{4}\tilde{e}_2(I_{n-1} \otimes B)^T[\tau_i^2\tilde{R}_{2i} + (\tau_i^2 - \tau_{i-1}^2)\tilde{R}_{3i}][(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]\tilde{e}_1^T\right) \\
&+ \sum_{i=1}^{N} \frac{1}{4}\tilde{e}_1[(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]^T[\tau_i^2\tilde{R}_{2i} + (\tau_i^2 - \tau_{i-1}^2)\tilde{R}_{3i}][(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \otimes C]\tilde{e}_1^T \\
&+ \sum_{i=1}^{N} \begin{bmatrix} \tau_i\tilde{e}_2^T \\ \tilde{e}_{N+2+i}^T \end{bmatrix}^T \begin{bmatrix} -\tilde{R}_{2i} & \tilde{R}_{2i} \\ * & -\tilde{R}_{2i} \end{bmatrix} \begin{bmatrix} \tau_i\tilde{e}_2^T \\ \tilde{e}_{N+2+i}^T \end{bmatrix} \\
&+ \sum_{i=1}^{N} \delta_i^2 \begin{bmatrix} \tilde{e}_2^T \\ \tilde{e}_{2N+2+i}^T \end{bmatrix}^T \begin{bmatrix} -\tilde{R}_{3i} & \tilde{R}_{3i} \\ * & -\tilde{R}_{3i} \end{bmatrix} \times \begin{bmatrix} \tilde{e}_2^T \\ \tilde{e}_{2N+2+i}^T \end{bmatrix}.
\end{aligned}$$

Moreover, to ensure that (22) holds, it can be chosen that

$$\beta > \max_{2\leq i\leq n} \frac{|Im(\mu_i)|}{\sqrt{Re(\mu_i)}|\mu_i|}, \tag{23}$$

where $\mu_i (2 \leq i \leq n)$ are the non-zero eigenvalues of L.

Proof: When $\tau(t) \equiv 0$, system (19) can be written as

$$\dot{\zeta}_D(t) = (I_n \otimes B - L_D \otimes C)\zeta_D(t).$$

The transformation $L \to DLD$ is a similarity transformation. So the eigenvalues of L are same as the ones of DLD. Let $\mu_i (i \in \{2, \ldots, n\})$ be the ith non-zero eigenvalue of L and DLD. According to [27, Th. 3], the LMI (22) is feasible when (23) is satisfied.

When $\tau(t) \not\equiv 0$ for $t \geq 0$, there should exist an integer $k \in \{1, \ldots, N\}$ such that $\tau(t) \in H_k$. Define an augmented vector as

$$\tilde{\xi}(t) = \left[\tilde{\zeta}^T(t-\tau(t)), \tilde{\zeta}^T(t), \tilde{\eta}^T(t), \tilde{\eta}_1^T(t), \tilde{\eta}_2^T(t)\right]^T,$$

where $\tilde{\xi}(t) \in R^{(3N+2)(2n-2)}$, $\tilde{\eta}(t) = \left[\tilde{\zeta}^T(t-\tau_1), \ldots, \tilde{\zeta}^T(t-\tau_N)\right]^T$, $\tilde{\eta}_1(t) = \left[\int_{t-\tau_1}^t \tilde{\zeta}^T(s)ds, \ldots, \int_{t-\tau_N}^t \tilde{\zeta}^T(s)ds\right]^T$, $\tilde{\eta}_2(t) = \left[\frac{1}{\delta_1}\int_{t-\tau_1}^{t-\tau_0} \tilde{\zeta}^T(s)ds, \ldots, \frac{1}{\delta_N}\int_{t-\tau_N}^{t-\tau_{N-1}} \tilde{\zeta}^T(s)ds\right]^T$.

A $Lyapunov-Krasovskii$ functional is chosen as

$$\tilde{V}(t) = \tilde{V}_1(t) + \tilde{V}_2(t) + \tilde{V}_3(t) + \tilde{V}_4(t),$$

where

$$\begin{aligned}
\tilde{V}_1(t &= \tilde{\zeta}^T(t)\tilde{P}\tilde{\zeta}t) + \tilde{\eta}_1^T(t)\tilde{S}\tilde{\eta}_1(t),\\
\tilde{V}_2(t) &= \int_{t-\tau(t)}^t \tilde{\zeta}^T(s)\tilde{Q}\tilde{\zeta}(s)ds\\
&\quad + \sum_{i=1}^N \int_{t-\tau_i}^{t-\tau_{i-1}} \tilde{\zeta}^T(s)\tilde{Q}_i\tilde{\zeta}(s)ds,\\
\tilde{V}_3(t) &= \sum_{i=1}^N \delta_i \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\beta}^t \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)dsd\beta,\\
\tilde{V}_4(t) &= \sum_{i=1}^N \frac{\tau_i^2}{2} \int_{-\tau_i}^0 \int_\theta^0 \int_{t+\beta}^t \dot{\tilde{\zeta}}^T(s)\tilde{R}_{2i}\dot{\tilde{\zeta}}(s)dsd\beta d\theta\\
&\quad + \sum_{i=1}^N \frac{\tau_i^2 - \tau_{i-1}^2}{2} \int_{-\tau_i}^{-\tau_{i-1}} \int_\theta^0 \int_{t+\beta}^t \dot{\tilde{\zeta}}^T(s)\tilde{R}_{3i}\\
&\quad \times \dot{\tilde{\zeta}}(s)dsd\beta d\theta.
\end{aligned}$$

The time derivative of $\tilde{V}(t)$ is given as

$$\dot{\tilde{V}}(t) = \dot{\tilde{V}}_1(t) + \dot{\tilde{V}}_2(t) + \dot{\tilde{V}}_3(t) + \dot{\tilde{V}}_4(t).$$

Invoking (20), we have

$$\begin{aligned}
\dot{\tilde{V}}_1(t) &= \dot{\zeta}^T(t)\tilde{P}\tilde{\zeta}(t) + \tilde{\zeta}(t)^T\tilde{P}\dot{\zeta}^T(t) + \dot{\tilde{\eta}}_1^T(t)\tilde{S}\tilde{\eta}_1(t)\\
&\quad + \tilde{\eta}_1(t)^T\tilde{S}\dot{\tilde{\eta}}_1\\
&= \tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\tilde{P}\tilde{\zeta}(t) - \tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D\\
&\quad + 1_{n-1}\mathbf{r})\otimes C]^T\tilde{P}\tilde{\zeta}(t) + \tilde{\zeta}^T(t)\tilde{P}(I_{n-1}\otimes B)\tilde{\zeta}(t)\\
&\quad - \tilde{\zeta}^T(t)\tilde{P}[(\bar{L}_D + 1_{n-1}\mathbf{r})\otimes C]\tilde{\zeta}(t-\tau(t))\\
&\quad + \begin{bmatrix} \tilde{\zeta}^T(t) - \tilde{\zeta}^T(t-\tau_1)\\ \vdots \\ \tilde{\zeta}^T(t) - \tilde{\zeta}^T(t-\tau_N)\end{bmatrix}^T \tilde{S}\tilde{\eta}_1(t)\\
&\quad + \tilde{\eta}_1^T(t)\tilde{S}\begin{bmatrix} \tilde{\zeta}^T(t) - \tilde{\zeta}^T(t-\tau_1)\\ \vdots \\ \tilde{\zeta}^T(t) - \tilde{\zeta}^T(t-\tau_N)\end{bmatrix}\\
&= \tilde{\xi}^T(t)\tilde{\Psi}_1\tilde{\xi}(t),\\
\dot{\tilde{V}}_2(t) &\leq \sum_{i=1}^N \Big[\tilde{\zeta}^T(t-\tau_{i-1})\tilde{Q}_i\tilde{\zeta}(t-\tau_{i-1}) - \tilde{\zeta}^T(t-\tau_i)\tilde{Q}_i\\
&\quad \times \tilde{\zeta}(t-\tau_i)\Big] + \tilde{\zeta}^T(t)\tilde{Q}\tilde{\zeta}(t) - (1-\mu)\tilde{\zeta}^T(t-\tau(t))\\
&\quad \times \tilde{Q}\zeta(t-\tau(t))\\
&= \tilde{\xi}^T(t)\tilde{\Psi}_2\tilde{\xi}(t).
\end{aligned}$$

The time derivative of $\tilde{V}_3(t)$ is expressed as

$$\begin{aligned}
\dot{\tilde{V}}_3(t) &= \sum_{i=1}^N \delta_i^2\dot{\tilde{\zeta}}^T(t)\tilde{Z}_i\dot{\tilde{\zeta}}(t) - \sum_{i=1}^T \delta_i \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)ds\\
&= -\sum_{i=1}^N \delta_i^2\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\tilde{Z}_i[(\bar{L}_D + \mathbf{1}_{n-1}\cdot r)\\
&\quad \otimes C]\tilde{\zeta}(t-\tau(t)) - \sum_{i=1}^N \delta_i^2\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D\\
&\quad + \mathbf{1}_{n-1}\cdot r)\otimes C]^T\tilde{Z}_i(I_{n-1}\otimes B)\tilde{\zeta}(t)\\
&\quad + \sum_{i=1}^N \delta_i^2\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D + \mathbf{1}_{n-1}\cdot r)\otimes C]^T\tilde{Z}_i\\
&\quad \times [(\bar{L}_D + \mathbf{1}_{n-1}\cdot r)\otimes C]\tilde{\zeta}^T(t-\tau(t))\\
&\quad + \sum_{i=1}^N \delta_i^2\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\tilde{Z}_i(I_{n-1}\otimes B)\tilde{\zeta}(t)\\
&\quad - \sum_{i=1}^N \delta_i \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)ds.
\end{aligned} \tag{24}$$

For the case of $\tau(t) \in H_k (1 \leq k \leq N)$, the last term in (24) can be deduced as follows

$$\begin{aligned}
&-\sum_{i=1}^N \delta_i \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)ds\\
&\quad = -\sum_{i=1, i\neq k}^N \delta_i \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)ds\\
&\qquad - \delta_k \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)Z_i\dot{\tilde{\zeta}}(s)ds.
\end{aligned} \tag{25}$$

Applying Lemma 2 to deal with the first term of (25), it follows that

$$\begin{aligned}
&-\sum_{i=1, i\neq k}^N \delta_i \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{\tilde{\zeta}}^T(s)\tilde{Z}_i\dot{\tilde{\zeta}}(s)ds\\
&\quad \leq \sum_{i=1, i\neq k}^N \tilde{\varpi}_{3i}^T(t)\tilde{\Omega}_{3i}\tilde{\varpi}_{3i}(t),
\end{aligned} \tag{26}$$

where $\tilde{\varpi}_{3i}(t) = \left[\tilde{\zeta}^T(t-\tau_{i-1}), \tilde{\zeta}^T(t-\tau_i), \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}\tilde{\zeta}^T(s)ds\right]^T$. Applying Lemma 1 to deal with the second term of (25), it follows that

$$
\begin{aligned}
&-\delta_k\int_{t-\tau_k}^{t-\tau_{k-1}}\dot{\tilde{\zeta}}^T(s)\tilde{Z}_k\dot{\tilde{\zeta}}(s)ds\\
&\le -(\tau_k-\tau(t))\int_{t-\tau k}^{t-\tau(t)}\dot{\tilde{\zeta}}^T(s)\tilde{Z}_k\dot{\tilde{\zeta}}(s)ds\\
&\quad -(\tau(t)-\tau_{k-1})\int_{t-\tau(t)}^{t-\tau_{k-1}}\dot{\tilde{\zeta}}^T(s)\tilde{Z}_k\dot{\tilde{\zeta}}(s)ds\\
&\le \begin{bmatrix}\tilde{\zeta}(t-\tau_k)\\ \tilde{\zeta}(t-\tau(t))\end{bmatrix}^T\begin{bmatrix}-\tilde{Z}_k & \tilde{Z}_k\\ * & -\tilde{Z}_k\end{bmatrix}\begin{bmatrix}\tilde{\zeta}(t-\tau_k)\\ \tilde{\zeta}(t-\tau(t))\end{bmatrix}\\
&\quad+\begin{bmatrix}\tilde{\zeta}(t-\tau(t))\\ \tilde{\zeta}(t-\tau_{k-1})\end{bmatrix}^T\begin{bmatrix}-\tilde{Z}_k & \tilde{Z}_k\\ * & -\tilde{Z}_k\end{bmatrix}\begin{bmatrix}\tilde{\zeta}(t-\tau(t))\\ \tilde{\zeta}(t-\tau_{k-1})\end{bmatrix}\\
&= \tilde{\eta}_0^T(t)\begin{bmatrix}-2\tilde{Z}_k & \tilde{Z}_k & \tilde{Z}_k\\ * & -\tilde{Z}_k & 0\\ * & * & -\tilde{Z}_k\end{bmatrix}\tilde{\eta}_0(t),
\end{aligned}
\tag{27}
$$

where $\tilde{\eta}_0(t) = \left[\tilde{\zeta}^T(t-\tau(t)), \tilde{\zeta}^T(t-\tau_{k-1}), \tilde{\zeta}^T(t-\tau_k)\right]^T$. Then, it follows from (24)-(27) that

$$
\begin{aligned}
\dot{\tilde{V}}_3(t) \le &-\sum_{i=1}^N\delta_i^2\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\tilde{Z}_i[(\bar{L}_D+\mathbf{1}_{n-1}\cdot r)\otimes C]\\
&\times\tilde{\zeta}(t-\tau(t))-\sum_{i=1}^N\delta_i^2\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D+\mathbf{1}_{n-1}\cdot r)\\
&\otimes C]^T\tilde{Z}_i(I_{n-1}\otimes B)\tilde{\zeta}(t)+\sum_{i=1}^N\delta_i^2\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D\\
&+\mathbf{1}_{n-1}\cdot r)\otimes C]^T\tilde{Z}_i[(\bar{L}_D+\mathbf{1}_{n-1}\cdot r)\otimes C]\\
&\times\tilde{\zeta}^T(t-\tau(t))+\sum_{i=1}^N\delta_i^2\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\\
&\times\tilde{Z}_i(I_{n-1}\otimes B)\tilde{\zeta}(t)+\sum_{i=1,i\ne k}^N\tilde{\varpi}_{3i}^T(t)\tilde{\Omega}_{3i}\tilde{\varpi}_{3i}(t)\\
&+\tilde{\eta}_0^T(t)\begin{bmatrix}-2\tilde{Z}_k & \tilde{Z}_k & \tilde{Z}_k\\ * & -\tilde{Z}_k & 0\\ * & * & -\tilde{Z}_k\end{bmatrix}\tilde{\eta}_0(t)\\
=&\ \tilde{\xi}^T(t)\tilde{\Psi}_{3k}\tilde{\xi}(t).
\end{aligned}
\tag{28}
$$

The time derivative of $\tilde{V}_4(t)$ is presented as

$$
\begin{aligned}
&\dot{\tilde{V}}_4(t)\\
&=\sum_{i=1}^N\frac{1}{4}\tau_i^4\dot{\tilde{\zeta}}^T(t)\tilde{R}_{2i}\dot{\tilde{\zeta}}(t)\\
&\quad+\sum_{i=1}^N\frac{1}{4}(\tau_i^2-\tau_{i-1}^2)^2\dot{\tilde{\zeta}}^T(t)\tilde{R}_{3i}\dot{\tilde{\zeta}}(t)\\
&\quad-\sum_{i=1}^N\frac{\tau_i^2}{2}\int_{-\tau_i}^0\int_{t+\theta}^t\dot{\tilde{\zeta}}^T(\beta)\tilde{R}_{2i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta\\
&\quad-\sum_{i=1}^N\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau-i}^{-\tau_{i-1}}\int_{t+\theta}^t\dot{\tilde{\zeta}}^T(\beta)\tilde{R}_{3i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta\\
&=\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right]\\
&\quad\times(I_{n-1}\otimes B)\tilde{\zeta}(t)-\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\\
&\quad\times\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right][(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\otimes B]\\
&\quad\times\tilde{\zeta}(t-\tau(t))-\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\\
&\quad\otimes B]^T\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right](I_{n-1}\otimes B)\tilde{\zeta}(t)\\
&\quad+\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\otimes B]^T\\
&\quad\times\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right][(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\otimes B]\\
&\quad\times\tilde{\zeta}(t-\tau(t))-\sum_{i=1}^N\frac{\tau_i^2}{2}\int_{-\tau_i}^0\int_{t+\theta}^t\dot{\tilde{\zeta}}^T(\beta)\tilde{R}_{2i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta\\
&\quad-\sum_{i=1}^N\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^t\dot{\tilde{\zeta}}^T(\beta)R_{3i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta.
\end{aligned}
\tag{29}
$$

By using Lemma 1, the last two terms of (29) are deduced as

$$
\begin{aligned}
&-\sum_{i=1}^N\frac{\tau_i^2}{2}\int_{-\tau_i}^0\int_{t+\theta}^t\dot{\tilde{\zeta}}^T(\beta)\tilde{R}_{2i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta\\
&\le\sum_{i=1}^N\begin{bmatrix}\tau_i\tilde{\zeta}(t)\\ \int_{t-\tau_i}^t\tilde{\zeta}(\beta)d\beta\end{bmatrix}^T\begin{bmatrix}-\tilde{R}_{2i} & \tilde{R}_{2i}\\ * & -\tilde{R}_{2i}\end{bmatrix}\begin{bmatrix}\tau_i\tilde{\zeta}(t)\\ \int_{t-\tau_i}^t\tilde{\zeta}(\beta)d\beta\end{bmatrix},\\
&\quad-\sum_{i=1}^N\frac{\tau_i^2-\tau_{i-1}^2}{2}\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^t\dot{\tilde{\zeta}}^T\tilde{R}_{3i}\dot{\tilde{\zeta}}(\beta)d\beta d\theta\\
&\le\sum_{i=1}^N\begin{bmatrix}(\tau_i-\tau_{i-1})\tilde{\zeta}(t)\\ \int_{t-\tau_i}^{t-\tau_{i-1}}\tilde{\zeta}(\beta)d\beta\end{bmatrix}^T\begin{bmatrix}-\tilde{R}_{3i} & \tilde{R}_{3i}\\ * & -\tilde{R}_{3i}\end{bmatrix}\begin{bmatrix}(\tau_i-\tau_{i-1})\tilde{\zeta}(t)\\ \int_{t-\tau_i}^{t-\tau_{i-1}}\tilde{\zeta}(\beta)d\beta\end{bmatrix}\\
&\le\sum_{i=1}^N(\tau_i-\tau_{i-1})^2\begin{bmatrix}\tilde{\zeta}(t)\\ \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}\tilde{\zeta}(\beta)d\beta\end{bmatrix}^T\begin{bmatrix}-\tilde{R}_{3i} & \tilde{R}_{3i}\\ * & -\tilde{R}_{3i}\end{bmatrix}\\
&\quad\times\begin{bmatrix}\tilde{\zeta}(t)\\ \frac{1}{\delta_i}\int_{t-\tau_i}^{t-\tau_{i-1}}\tilde{\zeta}(\beta)d\beta\end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\dot{\tilde{V}}_4(t)\le&\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right]\\
&\times(I_{n-1}\otimes B)\tilde{\zeta}(t)-\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t)(I_{n-1}\otimes B)^T\\
&\times\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right][(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\otimes B]\\
&\times\tilde{\zeta}(t-\tau(t))-\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\\
&\otimes B]^T\left[\tau_i^2\tilde{R}_{2i}+(\tau_i^2-\tau_{i-1}^2)\tilde{R}_{3i}\right](I_{n-1}\otimes B)\tilde{\zeta}(t)\\
&+\sum_{i=1}^N\frac{1}{4}\tilde{\zeta}^T(t-\tau(t))[(\bar{L}_D+\mathbf{1}_{n-1}\cdot\mathbf{r})\otimes B]^T
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\tau_i^2 \tilde{R}_{2i} + (\tau_i^2 - \tau_{i-1}^2)\tilde{R}_{3i}\right] [(\bar{L}_D + \mathbf{1}_{n-1} \cdot \mathbf{r}) \\
&\otimes B]\tilde{\xi}(t - \tau(t)) + \sum_{i=1}^{N} \begin{bmatrix} \tau_i \tilde{\xi}(t) \\ \int_{t-\tau_i}^{t} \tilde{\xi}(\beta) d\beta \end{bmatrix}^T \\
&\times \begin{bmatrix} -\tilde{R}_{2i} & \tilde{R}_{2i} \\ * & -\tilde{R}_{2i} \end{bmatrix} \begin{bmatrix} \tau_i \tilde{\xi}(t) \\ \int_{t-\tau_i}^{t} \tilde{\xi}(\beta) d\beta \end{bmatrix} \\
&+ \sum_{i=1}^{N} (\tau_i - \tau_{i-1})^2 \begin{bmatrix} \tilde{\xi}(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} \tilde{\xi}(\beta) d\beta \end{bmatrix}^T \\
&\times \begin{bmatrix} -\tilde{R}_{3i} & \tilde{R}_{3i} \\ * & -\tilde{R}_{3i} \end{bmatrix} \begin{bmatrix} \tilde{\xi}(t) \\ \frac{1}{\delta_i} \int_{t-\tau_i}^{t-\tau_{i-1}} \tilde{\xi}(\beta) d\beta \end{bmatrix} \\
&= \tilde{\xi}^T(t) \tilde{\Psi}_4 \tilde{\xi}(t).
\end{aligned}
$$

Therefore, we can get that

$$\dot{\tilde{V}}(t) \leq \tilde{\xi}^T(t) \left(\tilde{\Psi}_1 + \tilde{\Psi}_2 + \tilde{\Psi}_{3k} + \tilde{\Psi}_4 \right) \tilde{\xi}(t).$$

Obviously, $\dot{\tilde{V}}(t) < 0$ if (21) and (22) hold, which means that the error system (20) is asymptotically stable, i.e., the system (16) using (18) can achieve bipartite consensus. □

Remark 1: *Consensus for multi-agent systems under double integrator dynamics with time-varying communication delays was considered in [21] and [27]. Here, when N=1, our results include the aforementioned works. In [13], the method of equally partitioning is employed in dealing with second-order consensus of multi-agent systems with constant communication delay. Here, when $\alpha = 1$, the delays $\tau(t)$ are equally partitioned into N parts. So, our partitioning method is more general compared with the method of equally partitioning in [13].*

V. SIMULATIONS

In this section, we give some numerical simulations to demonstrate the effectiveness of the obtained theoretical results.

Example 1: *Consider a first-order multi-agent system composed of four agents under a direct topology shown in FIGURE 2. Obviously, G_1 is structurally balanced with $V_1 = \{1, 3\}, V_2 = \{2, 4\}$. According to Theorem 1, $h = 1.46 (N = 3)$ when $\mu = 0.9$. Based on this, we assume that $\tau(t) = 0.56 + 0.9|\sin \frac{t}{10^3}|$ and the initial values of agents*

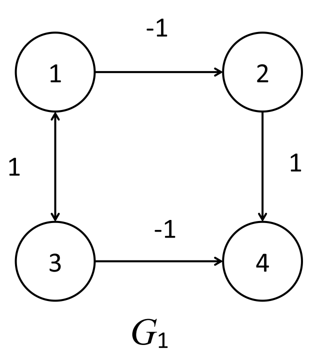

FIGURE 2. **Directed topology G_1.**

are $[x_1(0), x_2(0), x_3(0), x_4(0)] = [1, 2, -3, -4]$. FIGURE 3 shows the positions of system (1) using (3) under topology G_1.

Example 2: *Consider a second-order multi-agent system composed of four agents under a same direct topology shown in FIGURE 2. According to Theorem 2, $h = 1.83 (N = 3)$ when $\mu = 1.2$ and $\beta = 0.5$. According to [21, Th. 2], the upper bound of time-varying delays is 0.31 when we choose $\mu = 1.2$ and $\beta = 0.5$(i.e.,$\gamma_1 = 0.5$ and $\gamma_2 = 1$ in [21]), which implies that our conclusion is superior to the one in [21]. Here, we assume that $\tau(t) = 0.63 + 1.2|\sin \frac{t}{10^3}|$ and the initial values of agents are $[x_1(0), x_2(0), x_3(0), x_4(0), v_1(0), v_2(0), v_3(0), v_4(0)] = [1, 2, 3, 4, 1, 2, -3, -4]$. FIGURE 4 and FIGURE 5 show the positions and velocities of system (16) using (18) under topology G_1, respectively.*

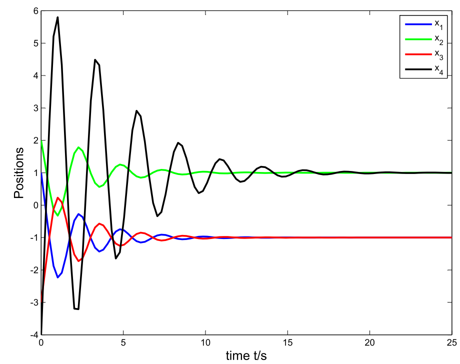

FIGURE 3. **Positions of first-order system (1).**

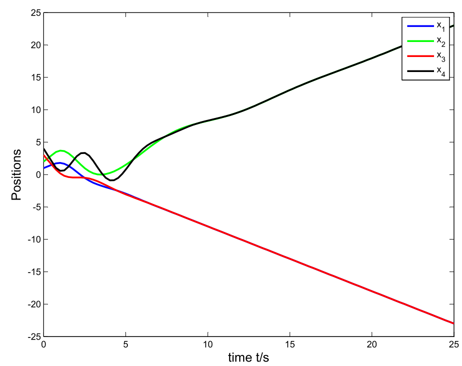

FIGURE 4. **Positions of second-order system (16).**

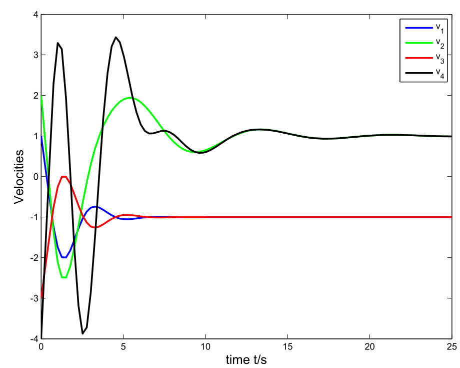

FIGURE 5. **Velocities of second-order system (16).**

VI. CONCLUSION

This paper investigates consensus problems for first-order and second-order multi-agent systems with antagonistic interactions and time-varying delays under directed topology, respectively. By using the method of delay partitioning, Lyapunov-Krasovskii functional is constructed and the linear matrix inequality theory is used to obtain sufficient conditions for bipartite consensus of multi-agent systems. Compared with the results which did not use the method of delay partitioning, the permitted upper bound of time-varying delays of corresponding system to guarantee the achievement of consensus becomes larger. Bipartite consensus of multi-agent systems with time-varying delays under switching topologies is full of challenge and will be investigated in the future work.

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