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State Equations in Stochastic Process Algebra Models

JIE DING[1,2], (Member, IEEE), XIN-SHAN ZHU[3,4], (Member, IEEE), AND XIAO CHEN[5,6], (Member, IEEE)

[1]China Institute of FTZ Supply Chain, Shanghai Maritime University, Shanghai 201306, China
[2]State Key Laboratory of Digital Publishing Technology, Beijing 100871, China
[3]School of Electrical and Information Engineering, Tianjin University, Tianjin 300072, China
[4]School of Information Engineering, Yangzhou University, Yangzhou 225127, China
[5]State Key Laboratory of Software Development Environment, Beihang University, Beijing 100083, China
[6]School of Computer Science and Communication Engineering, Jiangsu University, Zhenjiang 212013, China

Corresponding author: Xin-Shan Zhu (xszhu126@126.com)

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ABSTRACT State equations are usually used for structural or qualitative analysis, such as deadlock checking, in P/T systems. In this paper, we instead consider timed state equations in stochastic process algebra models, to derive quantified dynamic information on the system modeled in the face of the state space explosion problem. The average of these state equations is demonstrated as the linear combination of the system transitions, with the combination coefficients specified by the bias term of the empirical transition rates to their steady state. The approaches of stochastic simulation and fluid approximation, straightforwardly generated from the quantified state equations, are studied, with the consistency being investigated both theoretically and experimentally.

INDEX TERMS State equation, stochastic process algebra, fluid approximation, stochastic simulation.

I. INTRODUCTION

Stochastic process algebras, such as PEPA [1], EMPA [2] and TIPP [3], are formal performance modelling languages, which compositionally describes a concurrent system as a number of interacting components which undertake actions. The quantified durations associated with activities, usually satisfy exponentially distributions. Therefore, underlying stochastic process algebra models there are usually continuous time Markov chains (CTMCs) upon which quantitative evaluation has relied. The state space of the CTMCs can be generated through the algebraic *state equations* of the models [4]. The state equations are often used as a basis for qualitatively analyzing the systems, such as checking deadlocks [5]. In Place/Transition (P/T) net (or Petri net) modelling there are well-established techniques of qualitative analysis [6]–[9], which can help to provide valuable insight into the behavior of a system. However, the quantitative dynamic information of the system should be incorporated.

In this paper, we will demonstrate how the numerical representation schema of stochastic process algebra models leads to quantified state equations, and further express them by Poisson processes, according to Kurtz's method of Poisson representation of CTMCs. It is easy to be obtained, but never been established before, that the expectation of these state equations are the linear combination of the system transitions, with the combination coefficients specified by the biased term of the empirical transition rates to their steady state. The coefficients can thus be interpreted as the average numbers of all transitions in some sense. Moreover, we also introduce how the state equations straightforwardly leads to several stochastic simulations methods and a fluid approximation approach, to derive the quantified information from the system. In particular, fluid approximation can achieve an acceptable accuracy at a low computational cost, so it is considered as a novel approach to deal with the state space explosion problem encountered in performance derivation of

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large scale systems. This paper justifies, both theoretically and experimentally, that this approach can work very well even for a small scale model with four states.

The remainder of this paper is formed as follows. Section 2 introduces the PEPA language and its numerical representation schema. Section 3 shows how this representation leads to the quantified state equation of every PEPA model, which is represented using Poisson processes by Kurtz's theorem. Stochastic simulation and fluid approximation methods base on the state equations have been discussed in Section 4 and 5 respectively. After presenting some related work in Section 6, we conclude this paper in Section 7.

II. THE PEPA MODELLING FORMALISM

This section will briefly introduce the PEPA language and its numerical representation schema. The numerical representation schema for PEPA, developed by Ding [4], [10], represents a model numerically rather than syntactically supporting the use of mathematical tools and methods to analyze the model.

A. SYNTAX

Prefix: $(\alpha, r).P$: Prefix is the basic mechanism which describes the behavior of the system. Such a component will subsequently behave as P after it carries out the activity (α, r), which has action type α and a duration which satisfies the exponential distribution with parameter r.

Choice: $P + Q$: The component $P + Q$ represents a competition between two components. The system may behave either as P or as Q. The activities of both P and Q are enabled. The choice is resolved by a race policy; the component whose activity is completed first proceeds, the other is discarded.

Cooperation: $P \underset{L}{\bowtie} Q$: The cooperation combinator describes the synchronization of P and Q over the activities in the cooperation set L. In fact, for any activity whose action type is contained in L, P and Q must cooperate to achieve the activity. However, they will proceed independently and concurrently with any activity whose action type is not included in L.

Parallel: $P||Q$: The component $P||Q$ represents two concurrent but completely independent components, meaning the cooperation set is empty. This is simply a shorthand notation for $P \underset{\emptyset}{\bowtie} Q$.

Hiding: P/L: Hiding makes the activities whose action types are in L invisible for an external observer. The component P/L behaves as P except that any activities of types within the set L are hidden.

Constant: $A \stackrel{\text{def}}{=} P$: Constants are components whose meaning is given by a defining equation such as $A \stackrel{\text{def}}{=} P$, which gives the constant A a behavior similar to the behavior of component P.

On the basis of the operational semantic rules (please refer to [1] for details), a PEPA model may be regarded as a labelled multi-transition system

$$\left(\mathcal{C}, \mathcal{A}ct, \left\{\xrightarrow{(\alpha,r)} |(\alpha, r) \in \mathcal{A}ct\right\}\right)$$

where $\mathcal{C}$ is the set of components, $\mathcal{A}ct$ is the set of activities and the multi-relation $\xrightarrow{(\alpha,r)}$ is given by the rules.

The durations of all activities satisfy the exponential distributions, which means that the stochastic process underlying the labelled transition system is a CTMC. The steady state distribution can be obtained by solving the global balance equation associate the CTMC using linear algebra, from which quantitative performance measures such as throughput and utilization can be derived. However, for large scale systems it is difficult to calculating the steady state distribution due to the state space explosion problem. Other two computational approaches have therefore been proposed, which will be discussed later.

B. NUMERICAL REPRESENTATION SCHEMA OF PEPA

The above original definition of PEPA is syntactic, making models to be easily understood but not be convenient to employ mathematical tools to analysis models. This subsection presents the numerical representation of PEPA models, which were defined in [4]. For convenience, throughout this paper any such transition $U \xrightarrow{(l,r)} V$ defined in PEPA models may be rewritten as $U \xrightarrow{(l,r_l^{U\to V})} V$, or just $U \xrightarrow{l} V$ if the rate is not being considered, where U and V are two local derivatives. Following [11], hereafter the term *local derivative* refers to the local state of a single sequential component. In the numerical representation schema, the system state is represented by a numerical vector form which is defined below:

Definition 1 (Numerical Vector Form [11]): For an arbitrary PEPA model $\mathcal{M}$ with n component types $C_i, i = 1, 2, \cdots, n$, each with d_i distinct local derivatives, the numerical vector form of $\mathcal{M}$, $\mathbf{m}(\mathcal{M})$, is a vector with $d = \sum_{i=1}^{n} d_i$ entries. The entry $\mathbf{m}[C_{i_j}]$ records how many instances of the jth local derivative of component type C_i are exhibited in the current state.

In the numerical representation schema, the transitions between states of a model are represented by a numerical matrix, called *activity matrix*. In order to introduce activity matrix, we need first give the definitions:

Definition 2 (Pre and Post Local Derivative):

1) *If a local derivative U can enable an activity l, that is $U \xrightarrow{l} \cdot$, then U is called a* pre local derivative *of l. The set of all pre local derivatives of l is denoted by* pre(l), *called the* preset *of l.*
2) *If V is a local derivative obtained by firing an activity l, i.e. $\cdot \xrightarrow{l} V$, then V is called a* post local derivative *of l. The set of all post local derivatives is denoted by* post(l), *called the* post set *of l.*

3) *The set of all the local derivatives derived from U by firing l, i.e.*

$$\text{post}(U, l) = \{V \mid U \xrightarrow{l} V\},$$

is called the post set of l from U.

Definition 3 (Labelled Activity):

1) *For any individual activity l, for each $U \in \text{pre}(l)$, $V \in \text{post}(U, l)$, label l as $l^{U\to V}$.*
2) *For a shared activity l, for each*

$$(V_1, V_2, \cdots, V_k) \in \text{post}(\text{pre}(l)[1], l) \times \text{post}(\text{pre}(l)[2], l) \times \cdots \times \text{post}(\text{pre}(l)[k], l),$$

label l as l^w, where

$$w = (\text{pre}(l)[1] \to V_1, \text{pre}(l)[2] \to V_2, \cdots, \text{pre}(l)[k] \to V_k).$$

Each $l^{U\to V}$ or l^w is called a labelled activity. *The set of all labelled activities is denoted by $\mathcal{A}_{\text{label}}$. For the above labelled activities $l^{U\to V}$ and l^w, their respective pre and post sets are defined as*

$$\text{pre}(l^{U\to V}) = \{U\}, \quad \text{post}(l^{U\to V}) = \{V\},$$
$$\text{pre}(l^w) = \text{pre}(l), \quad \text{post}(l^w) = \{V_1, V_2, \cdots, V_k\}.$$

The impact of labelled activities on local derivatives can be recorded in a matrix form, as defined below.

Definition 4 (Activity Matrix, Pre Activity Matrix, Post Activity Matrix): For a model with $N_{\mathcal{A}_{\text{label}}}$ labelled activities and $N_{\mathcal{D}}$ distinct local derivatives, the activity matrix $\mathbf{C}$ is an $N_{\mathcal{D}} \times N_{\mathcal{A}_{\text{label}}}$ matrix, and the entries are defined as follows

$$\mathbf{C}(U_i, l_j) = \begin{cases} +1 & \text{if } U_i \in \text{post}(l_j) \\ -1 & \text{if } U_i \in \text{pre}(l_j) \\ 0 & \text{otherwise} \end{cases}$$

where l_j is a labelled activity. The pre activity matrix $\mathbf{C}^{\mathbf{pre}}$ and post activity matrix $\mathbf{C}^{\mathbf{post}}$ are defined as

$$\mathbf{C}^{\mathbf{Pre}}(U_i, l_j) = \begin{cases} +1 & \mathbf{C}(U_i, l_j) = -1 \\ 0 & \textit{otherwise}. \end{cases},$$

$$\mathbf{C}^{\mathbf{Post}}(U_i, l_j) = \begin{cases} +1 & \mathbf{C}(U_i, l_j) = +1 \\ 0 & \textit{otherwise}. \end{cases}$$

From Definition 3 and Definition 4, each column of the activity matrix corresponds to a system transition and each transition can be represented by a column of the activity matrix. The activity matrix equals the difference between the pre and post activity matrices, i.e. $\mathbf{C} = \mathbf{C}^{\mathbf{Post}} - \mathbf{C}^{\mathbf{Pre}}$. The rate of the transition between states is specified by a *transition rate function*. We first give the definition of the apparent rate of an activity in a local derivative.

Definition 5 (Apparent Rate of l in U): Suppose l is an activity of a PEPA model and U is a local derivative enabling l (i.e. $U \in \text{pre}(l)$). Let $\text{post}(U, l)$ be the set of all the local derivatives derived from U by firing l, i.e. $\text{post}(U, l) = \{V \mid U \xrightarrow{(l, r_l^{U\to V})} V\}$. Let

$$r_l(U) = \sum_{V \in \text{post}(U,l)} r_l^{U\to V}. \tag{1}$$

The apparent rate *of l in U in state $\mathbf{m}$, denoted by $r_l(\mathbf{m}, U)$, is defined as*

$$r_l(\mathbf{m}, U) = \mathbf{m}[U] r_l(U). \tag{2}$$

The above definition is used to define the following transition rate function.

Definition 6 (Transition Rate Function): Suppose l is an activity of a PEPA model and $\mathbf{m}$ denotes a state vector.

1) *If l is individual, then for each $U \xrightarrow{(l, r^{U\to V})} V$, the transition rate function of labelled activity $l^{U\to V}$ in state $\mathbf{m}$ is defined as*

$$f(\mathbf{m}, l^{U\to V}) = \mathbf{m}[U] r_l^{U\to V}. \tag{3}$$

2) *If l is synchronized, with $\text{pre}(l) = \{U_1, U_2, \cdots, U_k\}$, then for each*

$$(V_1, V_2, \cdots, V_k) \in \text{post}(U_1, l) \times \text{post}(U_2, l) \times \cdots \times \text{post}(U_k, l),$$

let $w = (U_1 \to V_1, U_2 \to V_2, \cdots, U_k \to V_k)$. Then the transition rate function of labelled activity l^w in state $\mathbf{m}$ is defined as

$$f(\mathbf{m}, l^w) = \left(\prod_{i=1}^{k} \frac{r_l^{U_i\to V_i}}{r_l(U_i)} \right) \min_{i \in \{1,\cdots,k\}} \{r_l(\mathbf{m}, U_i)\},$$

where $r_l(\mathbf{m}, U_i) = \mathbf{x}[U_i] r_l(U_i)$ is the apparent rate of l in U_i in state $\mathbf{m}$. So

$$f(\mathbf{m}, l^w) = \left(\prod_{i=1}^{k} \frac{r_l^{U_i\to V_i}}{r_l(U_i)} \right) \min_{i \in \{1,\cdots,k\}} \{\mathbf{m}[U_i] r_l(U_i)\}. \tag{4}$$

It has been pointed out in [4] that Definition 6 accommodates the passive or unspecified rate $\top$. An algorithm for automatically deriving the numerical representation schema of a PEPA model has been given in [4]. We assume throughout this paper the considered PEPA models satisfy two assumptions. Firstly, there is no cooperation within groups of components of the same type. Secondly, each column of the activity matrix of a model is distinct, i.e. each labelled activity is distinct in terms of pre and post local derivatives.

III. STOCHASTIC STATE EQUATIONS IN PEPA MODELS

This section shows how the numerical representation schema helps to manifest the P/T structure underlying PEPA models. First, the relevant definitions are given below.

Definition 7 (P/T Net, Marking, P/T System, [9]):

1) *A* Place/Transition net *(P/T net) is a structure $\mathcal{N} = (P, T, \mathbf{Pre}, \mathbf{Post})$ where: P and T are the sets of* places *and* transitions *respectively;* **Pre** *and* **Post** *are the $|P| \times |T|$ sized, natural valued,* incidence matrices.

2) *A* marking *is a vector* $\mathbf{m} : P \to \mathbb{N}$ *that assigns to each place of a P/T net a nonnegative integer (number of tokens).*
3) *A* P/T system *is a pair* $\mathcal{S} = \langle \mathcal{N}, \mathbf{m_0} \rangle$*: a net* $\mathcal{N}$ *with an initial marking* $\mathbf{m_0}$.

From Definition 7, it is easy to see that the structure $\mathcal{N} = \left(\mathcal{D}, \mathcal{A}_{\text{label}}, \mathbf{C^{Pre}}, \mathbf{C^{Post}}\right)$ derived from a PEPA model is a P/T net [4], where $\mathcal{D}, \mathcal{A}_{\text{label}}$ are the sets of all local derivatives and all labelled activities of the PEPA model respectively, and $\mathbf{C^{Pre}}$, $\mathbf{C^{Post}}$ are the pre and post activity matrices respectively. Given a starting state $\mathbf{m}_0$, $\mathcal{S} = \langle \mathcal{N}, \mathbf{m}_0 \rangle$ is a P/T system. Clearly, each reachable marking $\mathbf{m}$ from $\mathbf{m}_0$ is a state of the aggregated CTMC underlying the given PEPA model.

A transition l (i.e. a column of the activity matrix $\mathbf{C}$) is *enabled* in a state $\mathbf{m}$ if and only if $\mathbf{m} \geq \mathbf{C^{Pre}}[\cdot, l]$; its firing yields a new state $\mathbf{m}' = \mathbf{m} + \mathbf{C}[\cdot, l]$. This is denoted by $\mathbf{m} \xrightarrow{l} \mathbf{m}'$. In a P/T system an *occurrence sequence* from $\mathbf{m}$ is a sequence of transitions $\sigma = l_1 \cdots l_k \cdots$ such that $\mathbf{m} \xrightarrow{l_1} \mathbf{m_1} \cdots \xrightarrow{l_k} \mathbf{m_k} \cdots$. A state $\mathbf{m}$ is said to be *reachable* from $\mathbf{m}_0$ if there exists a occurrence sequence σ such that $\mathbf{m}_0 \xrightarrow{\sigma} \mathbf{m}$, that is

$$\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}. \tag{5}$$

where $\boldsymbol{\sigma}$ is the *firing count vector* corresponding to σ. The formula (5), called the *state equation* ([9]), reflecting that each state in the state space is related to the starting state through an algebraic equation, can also be written as

$$\mathbf{m} = \mathbf{m_0} + \sum_{l \in \mathcal{T}} l\sigma_l, \tag{6}$$

where $\mathbf{C} \cdot \boldsymbol{\sigma} = \sum_{l \in \mathcal{T}} l\sigma_l$, in which l, each column of $\mathbf{C}$, is a transition vector, σ_l is its occurrence number, and $\mathcal{T}$ is the set of transition vectors. Now we consider a timed state equation or stochastic state equation. Let $\mathbf{m}(t)$ denote the system state at time t, so $\mathbf{m}(0) = \mathbf{m}_0$. Define

$$\sigma_l(t) = \#\{s \leq t : \mathbf{m}(s) - \mathbf{m}(s_-) = l\}, \tag{7}$$

where $\#\{\cdot\}$ denotes the number of elements in a set, then

$$\mathbf{m}(t) = \mathbf{m}(0) + \sum_{l \in \mathcal{T}} l\sigma_l(t). \tag{8}$$

Notice that $\sigma_l(t)$ is the occurrence number of transition l in $[0, t]$. The firing count vector $\boldsymbol{\sigma}(t)$ records $\sigma_l(t)$ for all $l \in \mathcal{T}$. The equation (8) can be written as

$$\mathbf{m}(t) = \mathbf{m}(0) + \mathbf{C} \cdot \boldsymbol{\sigma}(t). \tag{9}$$

Because the stochastic process underlying a PEPA model is a CTMC, that is, the state function $\mathbf{m}(t)$ of the system is a CTMC. So, for any transition vector $l \in \mathcal{T}$,

$$\begin{aligned} &\Pr\left(\sigma_l(t+h) - \sigma_l(t) = 1 \mid \mathbf{m}(t)\right) \\ &\quad = \Pr\left(\mathbf{m}(t+h) - \mathbf{m}(t) = l \mid \mathbf{m}(t)\right) \\ &\quad = f(\mathbf{m}(t), l)h + o(h), \end{aligned} \tag{10}$$

$$\begin{aligned} &\Pr\left(\sigma_l(t+h) - \sigma_l(t) = 0 \mid \mathbf{m}(t)\right) \\ &\quad = 1 - f(\mathbf{m}(t), l)h + o(h), \end{aligned} \tag{11}$$

and

$$\begin{aligned} &\Pr\left(\sigma_l(t+h) - \sigma_l(t) = 0, \forall l \in \mathcal{T} \mid \mathbf{m}(t)\right) \\ &\quad = \Pr\left(\mathbf{m}(t+h) - \mathbf{m}(t) = 0 \mid \mathbf{m}(t)\right) \\ &\quad = 1 - \sum_{l \in \mathcal{T}} f(\mathbf{m}(t), l)h + o(h). \end{aligned} \tag{12}$$

The following two theorems will demonstrate some properties of the expectation of the system state.

Theorem 1: Let $\mathbf{m}(t)$ *be the state of a PEPA model at time t, then*

$$\frac{\mathrm{dE}\mathbf{m}(t)}{\mathrm{d}t} = \sum_{l \in \mathcal{T}} l\mathrm{E}\left[f(\mathbf{m}(t), l)\right]. \tag{13}$$

Proof: By equations (10) and (11),

$$E\left[\mathbf{m}(t+h) \mid \mathbf{m}(t)\right] = \mathbf{m}(t) + \sum_{l \in \mathcal{T}} lf(\mathbf{m}(t), l)h + o(h).$$

Taking expectations on the both sides leads to

$$E\left[\mathbf{m}(t+h)\right] = E\left[\mathbf{m}(t)\right] + \sum_{l \in \mathcal{T}} lE\left[f(\mathbf{m}(t), l)\right]h + o(h),$$

or

$$\frac{E\left[\mathbf{m}(t+h)\right] - E\left[\mathbf{m}(t)\right]}{h} = \sum_{l \in \mathcal{T}} lE\left[f(\mathbf{m}(t), l)\right] + \frac{o(h)}{h}.$$

Let h tend to zero, then the conclusion is obtained. □

The solution $\mathrm{E}\mathbf{m}(t)$ of the differential equation (13) is specified in the following theorem.

Theorem 2: Let $\mathbf{m}(t)$ *be the state of a PEPA model at time t, then*

$$\mathrm{E}\mathbf{m}(t) = \mathrm{E}\mathbf{m}(0) + \sum_{l \in \mathcal{T}} \alpha_l l + \sum_{l \in \mathcal{T}} o(\exp(-\beta_l t)t)l, \tag{14}$$

where α_l *and* β_l *are some constants.*

Proof: Because $\mathbf{m}(t)$ is the positive recurrent CTMC underline a PEPA model, and $f(\cdot, l)$ is a bounded function defined on the state space S of $\mathbf{m}(t)$, then there exist constants a_l and β_l (where $\beta_l > 0$), satisfying ([12])

$$\mathrm{E}\left(\frac{1}{t}\int_0^t f(\mathbf{m}(s), l)\mathrm{d}s\right) = R_l + \frac{a_l}{t} + o(\exp(-\beta_l t)) \tag{15}$$

as $t \to \infty$, where R_l is the expectation of $f(\mathbf{m}, l)$ with the respect to the steady state probability distribution π_∞, i.e. $R_l = \mathrm{E}_\infty f(\mathbf{m}, l) = \sum_{\mathbf{s} \in S} f(\mathbf{s}, l)\pi_\infty(\mathbf{s})$. Then by Equation (13), we have that

$$\begin{aligned} \mathrm{E}\mathbf{m}(t) &= \mathrm{E}\mathbf{m}(0) + \sum_l l\mathrm{E}\left[\int_0^t f(\mathbf{m}(t), l)ds\right] \\ &= \mathrm{E}\mathbf{m}(0) + \left(\sum_l l\left[R_l t + \alpha_l + o(\exp(-\beta_l t)t)\right]\right) \\ &= \mathrm{E}\mathbf{m}(0) + \left(\sum_l lR_l\right)t + \sum_l l\alpha_l \\ &\quad + \sum_l l(o(\exp(-\beta_l t)t)). \end{aligned}$$

The state space of the CTMC underlying a PEPA model is finite, so $\mathrm{E}\mathbf{m}(t)$ is bounded for any time t, and thus $\sum_l lR_l = 0$. This observation leads to

$$\mathrm{E}\mathbf{m}(t) = \mathrm{E}\mathbf{m}(0) + \sum_l l\alpha_l + \sum_l l(o(\exp(-\beta_l t)t)),$$

which completes the proof. □

In the proof, equation (15) demonstrates that as an estimator of the expectation of $f(\mathbf{m}, l)$, the time average of $f(\mathbf{m}(t), l)$, i.e. $\frac{1}{t}\int_0^t f(\mathbf{m}(s), l)\mathrm{d}s$, is biased. However, the constants α_l and β_l in the biased term in (15) relate to the expectation $\mathrm{E}_\infty\mathbf{m}$, which will be illustrated later.

Contrast to the well-known representation method, i.e. $\mathrm{E}\mathbf{m}(t) = \sum_{\mathbf{s}\in S}\mathbf{s}\pi_t(\mathbf{s})$, this new formulae (14) is only expressed by the transition vectors and their linear combination coefficients α_l and β_l, successfully avoiding directly using the state space S, and thus avoiding the state space explosion in some sense (α_l and β_l still relate to the state space S through equation (15)).

As time t tends to infinity, the transient probability distribution π_t of the CTMC will converge to the steady state distribution π_∞, leading to

$$\mathrm{E}\mathbf{m}(t) \longrightarrow \mathrm{E}_\infty\mathbf{m} = \mathrm{E}\mathbf{m}(0) + \sum_l l\alpha_l,$$

where $\mathrm{E}_\infty\mathbf{m} = \sum_{\mathbf{s}\in S}\mathbf{s}\pi_\infty(\mathbf{s})$, is the expectation of the CTMC with respect to π_∞, and $\mathrm{E}\mathbf{m}(0) = \sum_{\mathbf{s}\in S}\mathbf{s}\pi_0(\mathbf{s})$ where π_0 is the initial probability distribution. So we have that

Corollary 1: Denote $\mathbf{F}(\mathbf{m}) = \sum_{l\in\mathcal{T}} lf(\mathbf{m}, l)$, *then*

$$\mathrm{E}_\infty\mathbf{m} = \mathrm{E}\mathbf{m}(0) + \sum_{l\in\mathcal{T}} l\alpha_l, \tag{16}$$

and

$$\mathbf{C}\mathrm{E}_\infty\mathbf{F}(\mathbf{m}) = \sum_{l\in\mathcal{T}} l\mathrm{E}_\infty f(\mathbf{m}, l) = 0, \tag{17}$$

These results reveal the relationship among the transition vectors, the transition rate functions, and the expectation of the underlying CTMC. The rate of the CTMC convergence to its expectation is also being specified. By equation (16), α_l can be interpreted as the "average" number of the occurrence of l in some sense, which also explains its meaning in (15). To the best of our knowledge, it is the first time to obtain these conclusions such as equation (14) and (16) for finite CTMCs.

IV. STOCHASTIC SIMULATION OF THE STATE EQUATION

The stochastic process underlying a PEPA model is a CTMC, so the number of transition l of this CTMC in $[0, t]$, i.e. $\sigma_l(t)$, is a counting process with random intensity function $f(\mathbf{m}(t), l)$, shown in Equation (10) and (11). That is,

$$\Pr(\sigma_l(t+h) - \sigma_l(t) = 1 \mid \mathbf{m}(t)) = f(\mathbf{m}(t), l)h + o(h),$$
$$\Pr(\sigma_l(t+h) - \sigma_l(t) = 0 \mid \mathbf{m}(t)) = 1 - f(\mathbf{m}(t), l)h + o(h).$$

In the following, we will show that $\sigma_l(t)$ can be represented using a Poisson process.

Let $\mathcal{F}_t$ be the filtration generated by $\mathbf{m}(t)$, so σ_l and $f(\mathbf{m}(t), l)$ are adapted to $\mathcal{F}_t$. In addition, if we denote ξ_k by the k-th jump time of $\sigma_l(t)$, then for each l and k, $\sigma_l(t\wedge\xi_k) - \int_0^{t\wedge\xi_k} f(\mathbf{m}(t), l)$ is a martingale, by noticing that $E\left[\sigma_l(t\wedge\xi_k) - \int_0^{t\wedge\xi_k} f(\mathbf{m}(t), l) \mid \mathcal{F}_t\right] = 0$. According to the Meyer's result [13], $\sigma_l(t)$ can be written as

$$\sigma_l(t) = Y_l\left(\int_0^t f(\mathbf{m}(s), l)\mathrm{d}s\right), \tag{18}$$

where $Y_l(t)$ are independent Poisson processes with intensity 1, corresponding to transition $l \in \mathcal{T}$. Therefore,

$$\mathbf{m}(t) = \mathbf{m}(0) + \sum_l l\sigma_l(t) \tag{19}$$

$$= \mathbf{m}(0) + \sum_l lY_l\left(\int_0^t f(\mathbf{m}(s), l)\mathrm{d}s\right). \tag{20}$$

That is to say, the CTMC $\mathbf{m}(t)$ underlying a PEPA model (or the marking of a stochastic P/T system), can be represented using Poisson processes.

Theorem 3: Let $\mathbf{m}(t)$ *be the state of a PEPA model at time* t, $f(\cdot, l)(l \in \mathcal{T})$ *be the transition rate functions, where* $\mathcal{T}$ *is the transition set. Then the state equation can be represented by*

$$\mathbf{m}(t) = \mathbf{m}(0) + \sum_{l\in\mathcal{T}} lY_l\left(\int_0^t f(\mathbf{m}(s), l)\mathrm{d}s\right). \tag{21}$$

This kind of representation method has also been given by Kurtz in [14]. As pointed out in his another paper [15], the stochastic state equation (21) can straightforwardly lead to several simulation methods of CTMCs, such as next reaction (next jump) method given by Gibson and Bruck [16], Gillespie's direct method or the stochastic simulation algorithm [17], as well as the following Gillespie's τ-leap method [18],

$$\widehat{\mathbf{m}}(\tau_n) = \mathbf{m}(0) + \sum_{l\in\mathcal{T}} lY_l\left(\sum_{k=0}^{n-1} f\left(\widehat{\mathbf{m}}(\tau_k), l\right)(\tau_{k+1} - \tau_k)\right), \tag{22}$$

where $0 = \tau_0 < \tau_1 < \cdots$, which is the Euler-type approximation for (21).

However, it is easy to directly simulate the state equation $\mathbf{m}(t)$. As mentioned before, a transition between states, namely from $\mathbf{m}$ to $\mathbf{m} + l$, is represented by a transition vector l, with the rate $f(\mathbf{m}, l)$. That is, $\mathbf{m} \overset{(l, f(\mathbf{m}, l))}{\longrightarrow} \mathbf{m} + l$. Given a starting state $\mathbf{m}_0$, the transition chain corresponding to a firing sequence $l_0, l_1, \ldots, l, \ldots$ is

$$\mathbf{m}_0 \overset{(l_0, f(\mathbf{m}_0, l_0))}{\longrightarrow} \mathbf{m}_0 + l_0 \overset{(l_1, f(\mathbf{m}_0 + l_0, l_1))}{\longrightarrow} (\mathbf{m}_0 + l_0) + l_1 \overset{\cdots}{\longrightarrow} \cdots \overset{\cdots}{\longrightarrow} \mathbf{m} \overset{(l, f(\mathbf{m}, l))}{\longrightarrow} \mathbf{m} + l \overset{\cdots}{\longrightarrow} \cdots.$$

The above sequence can be considered to be one path or realization of the CTMC, if the enabled activity at each state is chosen stochastically, i.e. is chosen through sampling. That is, if the current state is $\mathbf{m}$, then the next state is chosen as $\mathbf{m} + l_u$ where l_u is the transition vector to make that $t_u = \min_{l\in\mathcal{T}}\{t_l\}$, where t_l are the numbers sampled from the

exponential distributions with the rates $f(\mathbf{m}, l), l \in \mathcal{T}$. The time duration of the chain stayed in $\mathbf{m}$ is t_u.

After a long time, the steady-state of the system is assumed to be achieved. Hence the average performance $R = \sum_{\mathbf{s}\in S} \rho(\mathbf{s})\pi_\infty(\mathbf{s})$ can be calculated, where ρ is a reward function defined on the state space. Performance metrics, such as throughput of an activity and capacity utilization of a local derivative, which are discussed in [19], can be derived through this algorithm by choosing appropriate reward functions. For details, please refer to [4]. The empirical performance $\frac{1}{t}\int_0^t \rho(\mathbf{m}(s))\mathrm{d}s$ will converge to its expectation $\mathrm{E}[\rho(\mathbf{m})] = \sum_{\mathbf{s}\in S} \rho(\mathbf{s})\pi_\infty(\mathbf{s})$, as the following ergodic theorem shows.

Theorem 4: The time average of the state reward in a PEPA model converges to its statistical average, as time tends to infinite. That is,

$$\Pr\left(\frac{1}{t}\int_0^t \rho(\mathbf{m}(s))\mathrm{d}s \to \mathrm{E}[\rho(\mathbf{m})] \text{ as } t \to \infty\right) = 1.$$

where $\mathrm{E}[\rho(\mathbf{m})] = \sum_{\mathbf{s}\in S} \rho(\mathbf{s})\pi_\infty(\mathbf{s})$. *Here* $\rho : S \to \mathbb{R}$ *is any bounded reward function, and S is the state space of the underlying irreducible and positive recurrent CTMC with the unique steady state distribution* π_∞.

Proof: The result is a consequence of Theorem 3.8.1 in [20]. □

This ergodic theorem states the convergence of the time average or empirical performance to its expectation, but in a biased manner as shown in Equation (15).

V. APPROXIMATION OF THE STATE EQUATION

Stochastic simulations have to rely on more computational costs, particularly longer running time, to derive more accurate performance measures from PEPA models. This becomes unacceptable for realtime requirement, particularly in the case of large scale models. However, fluid approximation, as a novel approach to obtain an acceptable accuracy at a low cost, has attracted lots of interests [4], [11], [19], [21], [22]. This section will illustrate the important role of the state equation in investigating fluid approximation methods.

A. APPROXIMATING THE STATE EQUATION

Let $\mathbf{m}(t)$ be the CTMC underlying a PEPA model, then by equation (13),

$$\mathrm{E}\mathbf{m}(t+\Delta t) - \mathrm{E}\mathbf{m}(t) = \sum_l l \int_t^{t+\Delta t} \mathrm{E}f(\mathbf{m}(s), l)\mathrm{d}s. \quad (23)$$

Approximating $\mathrm{E}[f(\mathbf{m}(s), l)]$ by $f(\mathrm{E}\mathbf{m}(s), l)$, as discussed by Hayden [21], we have that

$$\mathrm{E}\mathbf{m}(t+\Delta t) - \mathrm{E}\mathbf{m}(t) \approx \sum_l l \int_t^{t+\Delta t} f(\mathrm{E}\mathbf{m}(s), l)\mathrm{d}s. \quad (24)$$

This leads to the following ODEs, by considering $\mathrm{E}\mathbf{m}(t)$ as $\mathbf{x}(t)$ in (24),

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}), \quad (25)$$

where

$$\mathbf{F}(\mathbf{x}) = \sum_{l\in\mathcal{T}} l f(\mathbf{x}, l). \quad (26)$$

The solution $\mathbf{x}(t)$ of the ODEs (25) can be considered as an approximation of $\mathrm{E}\mathbf{m}(t)$, in contrast to the following direct differentiation of equation (9):

$$\dot{\mathbf{m}}(t) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(t),$$

which is considered in [23].

Let U be a local derivative. For any transition vector l, $l[U]$ is either ± 1 or 0. If $l[U] = -1$ then U is in the pre set of l, i.e. $U \in \mathrm{pre}(l)$, while $l[U] = 1$ implies $U \in \mathrm{post}(l)$. According to (25) and (26),

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}(U,t)}{\mathrm{d}t} &= \sum_l l[U] f(\mathbf{x}, l) \\ &= -\sum_{l:l[U]=-1} f(\mathbf{x}, l) + \sum_{l:l[U]=1} f(\mathbf{x}, l) \\ &= -\sum_{\{l|U\in\mathrm{pre}(l)\}} f(\mathbf{x}, l) + \sum_{\{l|U\in\mathrm{post}(l)\}} f(\mathbf{x}, l). \quad (27)\end{aligned}$$

The term $\sum_{\{l|U\in\mathrm{pre}(l)\}} f(\mathbf{x}, l)$ represents the "exit rates" in local derivative U, while the term $\sum_{\{l|U\in\mathrm{post}(l)\}} f(\mathbf{x}, l)$ reflects the "entry rates" in U. The formulae (25) and (26) are activity centric while (27) is local derivative centric. It has been proved that

Theorem 5 [4]: If $f(\mathbf{x}, l)$ *is Lipschtz, then the approximated ODEs of this model have a unique solution in time interval* $[0, \infty)$. *Moreover, the solution is bounded and nonnegative, given the initial condition is nonnegative.*

For an arbitrary CTMC, the evolution of probabilities distributed on each state can be described by a set of linear ODEs ([24], page 52). For example, for the CTMC underlying a PEPA model, the corresponding differential equations describing the evolution of the probability distributions are

$$\frac{\mathrm{d}\pi_t}{\mathrm{d}t} = Q^T \pi_t, \quad (28)$$

where each entry of π_t represents the probability of the system being in each state at time t, and Q is an infinitesimal generator matrix corresponding to the CTMC with the entries $Q_{\mathbf{s},\mathbf{s}+l} = f(\mathbf{s}, l)$. Clearly, the dimension of the coefficient matrix Q is the square of the size of the state space, which increases with the number of components.

The derived ODEs (25) describe the evolution of the population of the components in *each local derivative*, while (28) reflects the the probability evolution at *each state*. Since the scale of (25), i.e. the number of the ODEs, is only determined by the number of local derivatives and is unaffected by the size of the state space, so it avoids the state space explosion problem. In contrast, the scale of (28) depends on the size of the state space, so it suffers from the explosion problem. The price paid is that the ODEs (25) are generally nonlinear due to synchronizations, while (28) is linear.

However, if there is no synchronization then the ODEs (25) become linear and $\mathrm{E}[f(\mathbf{m}(s), l)] = f(\mathrm{E}\mathbf{m}(s), l)$, resulting

in that $\mathbf{x}(t) = \mathrm{E}\mathbf{m}(t)$ and consequently, $\lim_{t\to\infty}\mathbf{x}(t) = \lim_{t\to\infty}\mathrm{E}\mathbf{m}(t) = \mathrm{E}_\infty\mathbf{m}$. This justifies the consistence of the fluid approximation. In a general case with synchronization, since the formulae (24) is an approximation, so $\mathbf{x}(t) \neq \mathrm{E}\mathbf{m}(t)$ in general. But $\mathbf{x}(t)$ is very close to $\mathrm{E}\mathbf{m}(t)$, which will be illustrated later by an example.

B. STEADY STATE EQUATIONS

If the solution $\mathbf{x}(t)$ of the ODEs (25) converges to a limit $\mathbf{x}(\infty)$, as time tends to infinity, then

$$\sum_{l\in\mathcal{T}} lf(\mathbf{x}(\infty), l) = \lim_{t\to\infty}\dot{\mathbf{x}}(t) = 0.$$

Notice that $\sum_{l\in\mathcal{T}} l\mathrm{E}_\infty f(\mathbf{m}, l) = 0$, shown in Corollary 1. So, both $f(\mathbf{x}(\infty), l)$ and $\mathrm{E}_\infty f(\mathbf{m}, l)$ are the solutions of the following algebra equation

$$\sum_{l\in\mathcal{T}} lz_l = 0, \tag{29}$$

with the conservation condition

$$\sum_{l\in\mathcal{T}} z_l = \text{constant}, \tag{30}$$

which origins from the conservation law satisfied by PEPA models, and the "constant" equals the number of processes in a PEPA model. See [4] for details. Equation (29) with (30) is in fact a steady state equation, compared with the global balance equation of (28), i.e. $Q^T\pi_\infty = 0$, with the "conservation" condition $\mathbf{1}^T\pi_\infty = 1$.

According to the theory of linear algebra, if the rank of the activity matrix $\mathbf{C}$ (consists of all transition vectors l) of a given PEPA model is $\#\mathcal{T} - 1$, where $\#\mathcal{T}$ is the number of the elements in set $\mathcal{T}$, i.e. the number of transition vectors, then $\mathrm{E}_\infty f(\mathbf{m}, l)$ is proportional to $f(\mathbf{x}(\infty), l)$, for all l. This relationship is revealed in the following theorem.

Theorem 6: If rank $\mathbf{C} = \#\mathcal{T} - 1$, *then there exists a constant* k *such that for any* $l \in \mathcal{T}$,

$$\mathrm{E}_\infty f(\mathbf{m}, l) = kf(\mathbf{x}(\infty), l). \tag{31}$$

In realistic scenarios, the proportion factor k is very closed to one, even for some small scale models. We use the following PEPA model to illustrate this conclusion:

$$\begin{aligned}
User_1 &\overset{\text{def}}{=} (task_1, a).User_2\\
User_2 &\overset{\text{def}}{=} (task_2, b).User_1\\
Provider_1 &\overset{\text{def}}{=} (task_1, a).Provider_2\\
Provider_2 &\overset{\text{def}}{=} (reset, d).Provider_1\\
(User_1[1]) &\underset{\{task1\}}{\bowtie} (Provider_1[1]).
\end{aligned}$$

The activity matrix and transition rate functions have been specified in Table 1. In this table, U_i, P_i $(i = 1, 2)$ are the local derivatives representing $User_i$ and $Provider_i$ respectively. For convenience, the labelled activities or transition vectors $task_1^{(U_1\to U_2, P_1\to P_2)}$, $task_2^{U_2\to U_1}$, $reset^{P_2\to P_1}$ will subsequently be denoted by l_1, l_2, l_3 respectively.

TABLE 1. Activity matrix and transition rate function.

	$task_1^{(U_1\to U_2, P_1\to P_2)}$	$task_2^{U_2\to U_1}$	$reset^{P_2\to P_1}$
U_1	-1	1	0
U_2	1	-1	0
P_1	-1	0	1
P_2	1	0	-1
$f(\mathbf{m}, l)$	$a\min(\mathbf{m}[U_1], \mathbf{m}[P_1])$	$b\mathbf{m}[U_2]$	$d\mathbf{m}[P_2]$

Let $\mathbf{m}(t)$ be the CTMC underlying the model with initial state $\mathbf{s}_1 = (1, 0, 1, 0)^T$. Then the state space of S is composed of

$$\begin{aligned}
\mathbf{s}_1 &= (1, 0, 1, 0)^T, \quad \mathbf{s}_2 = (0, 1, 0, 1)^T,\\
\mathbf{s}_3 &= (1, 0, 0, 1)^T, \quad \mathbf{s}_4 = (0, 1, 1, 0)^T.
\end{aligned} \tag{32}$$

The generator matrix Q of the CTMC has the following form:

$$Q = \begin{pmatrix} -a & a & 0 & 0\\ 0 & -(b+d) & b & d\\ d & 0 & -d & 0\\ b & 0 & 0 & -b \end{pmatrix}.$$

Suppose $a = 2$, $b = 2$, $d = 8$, then the steady state probability distribution π_∞ can be calculated:

$$\begin{aligned}
\pi_\infty(\mathbf{s}_1) &= \frac{20}{41}, \quad \pi_\infty(\mathbf{s}_2) = \frac{4}{41},\\
\pi_\infty(\mathbf{s}_3) &= \frac{1}{41}, \quad \pi_\infty(\mathbf{s}_4) = \frac{16}{41}.
\end{aligned}$$

It is easy to obtain that $\mathrm{E}_\infty\mathbf{m} = \left(\frac{21}{41}, \frac{20}{41}, \frac{36}{41}, \frac{5}{41}\right)^T$, and

$$\mathrm{E}_\infty f_{l_1}(\mathbf{m}) = \frac{40}{41}, \quad \mathrm{E}_\infty f_{l_2}(\mathbf{m}, l_2) = \frac{40}{41}, \quad \mathrm{E}_\infty f_{l_3}(\mathbf{m}) = \frac{40}{41}.$$

The ODEs derived from the model through fluid approximation are

$$\begin{cases}
\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -a\min\{x_1, x_3\} + bx_2,\\
\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a\min\{x_1, x_3\} - bx_2,\\
\dfrac{\mathrm{d}x_3}{\mathrm{d}t} = -a\min\{x_1, x_3\} + dx_4,\\
\dfrac{\mathrm{d}x_4}{\mathrm{d}t} = a\min\{x_1, x_3\} - dx_4,
\end{cases} \tag{33}$$

with the initial condition $\mathbf{x}(0) = (x_1(0), x_2(0), x_3(0), x_4(0))^T = (1, 0, 1, 0)^T$. Let $a = 2$, $b = 2$, $d = 8$. The solution of (33), $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$, will converge to the limit (see [4, Sec. 6.4])

$$\begin{aligned}
\mathbf{x}(\infty) &= (x_1(\infty), x_2(\infty), x_3(\infty), x_4(\infty))^T\\
&= \left(\frac{1}{2}, \frac{1}{2}, \frac{7}{8}, \frac{1}{8}\right)^T\\
&= \left(\frac{20}{40}, \frac{20}{40}, \frac{35}{40}, \frac{5}{40}\right)^T.
\end{aligned}$$

So we have

$$f_{l_1}(\mathbf{x}(\infty)) = \frac{40}{40}, \quad f_{l_2}(\mathbf{x}(\infty)) = \frac{40}{40}, \quad f_{l_3}(\mathbf{x}(\infty)) = \frac{40}{40}.$$

It is clearly to see that $\mathbf{x}(\infty)$ tightly approximates $\mathrm{E}_\infty\mathbf{m}$, and for any $l \in \mathcal{T}$,

$$\mathrm{E}_\infty f_l(\mathbf{m}) = \frac{21}{20} f_l(\mathbf{x}(\infty)).$$

This example illustrates that the approximation works very well even for a small scale PEPA model. For the discussions in large scale cases, particularly on the topic of consistence of this approach, please see [4], [21], [22], and [25].

VI. RELATED WORK

The numerical representation schema of stochastic process algebra models, which leads to discover the underlying P/T structure, is established in [4] and [10]. Algorithms to automatically generated a PEPA model from an activity matrix is presented in [26]. Qualitative analysis of PEPA models based on this structure is presented in [5]. In the circumstance of Petri net or P/T system, state equations and timed state equations are common topics, see [6]–[9], [23], [27], [28]. However, we have not seen the quantified state equations are expressed using Poisson processes, although Kurtz's related theory has established in the early of 1980s [14]. The advantages of this kind of Poisson representation for CTMCs, such as straightforwardly leading to several simulation methods, have also been presented in [15]. The fluid approximation of PEPA models have been intensively discussed in [4], [11], [19], [21], and [22], employed a moment approximation method, similarly to our discussion of equation (13) that originated from timed state equations. In [4] and [22], the derived ODEs are the same to the ones here and in [21], which are considered as the approximation of a family of density dependent CTMCs underlying the model. As mentioned before, Silva *et al.* [23] instead directly differentiate the state equation to derive a set of different ODEs. A recent detailed comparison of the fluid approximation and stochastic simulation approaches, in terms of both accuracy and computational costs, have been presented and discussed in [25].

VII. CONCLUSIONS

In this paper, we have demonstrated how the numerical representation schema of stochastic process algebra models leads to quantified state equations, which can be expressed by Poisson processes according to Kurtz's theorem. The expectation of these state equations have been presented as the linear combination of the system transitions, with the combination coefficients specified by the deviated term of the empirical transition rates to their steady state. This conclusion is not difficult to be obtained, but never been revealed before. Moreover, the state equations are shown to derive quantified dynamic information about the system, through stochastic simulation and fluid approximation approaches. The consistence of the latter has also been discusses both theoretically and experimentally.

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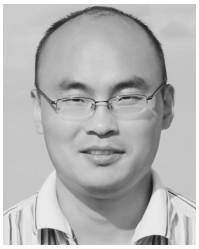

JIE DING received the B.S. degree in mathematical education from Yangzhou University, Yangzhou, China, in 2001, the M.S. degree in mathematical statistics from Southeast University, Nanjing, China, in 2004, and the Ph.D. degree in computer science from Edinburgh University, Edinburgh, U.K., in 2010. He is currently a Professor with the China Institute of FTZ Supply Chain, Shanghai Maritime University, China. His research interests include performance modeling and evaluation for computer, and communication systems.

XIN-SHAN ZHU was born in Xinmin, China, in 1977. He received the B.E. and M.E. degrees in automation control from the Harbin Institute of Technology, Harbin, China, in 2000 and 2002, respectively, and the Ph.D. degree in pattern recognition and intelligent systems from the Institute of Automation, Chinese Academic Sciences, Beijing, China, in 2005. He is currently an Associate Professor with the School of Electrical and Information Engineering, Tianjin University, Tianjin, China. His research interests include signal processing, information hiding, and multimedia security.

XIAO CHEN received the M.Sc. and Ph.D. degrees in computing science from Newcastle University, in 2009 and 2013, respectively. His research interests include performance modelling and analysis for large-scale systems, e.g., smart systems, cloud/fog systems, and blockchain-based applications and systems.