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New Non-Binary Quantum Codes Derived From a Class of Linear Codes

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ABSTRACT In this short paper, we consider the non-binary quantum codes construction from a class of linear codes, which are not self-orthogonal over finite fields. As the computational results, new quantum codes $[[8, 0, \geq 4]]_3$, $[[8, 2, \geq 3]]_3$, $[[11, 1, \geq 5]]_3$, $[[10, 0, \geq 5]]_4$, and $[[16, 6, \geq 4]]_4$ are obtained.

INDEX TERMS Quantum codes, linear codes, Calderbank-Shor-Steane (CSS) construction.

I. INTRODUCTION

Quantum coherence plays an essential role in quantum information theory. Decoherence is a property of quantum, which can destroy quantum coherence. It makes quantum computation easy to make mistakes. Reducing the decoherence or controlling the decoherence to an acceptable level is a key problem that must be solved by scientists. From now on, quantum error-correcting codes is one of the most effective ways to overcome decoherence.

Shor [1] constructed the first quantum error-correcting code $[[9, 1, 3]]$, which can correct 1 bit quantum error with 9 quantum bits. This code space is a 2-dimensional complex space. In 1996, two independent research groups Calderbank and Shor [2] and Steane [3] adopted the idea of classical linear block error-correcting codes, and gave a method to construct quantum error-correcting codes using two classical binary error-correcting codes called Calderbank-Shor-Steane (CSS) construction method. In 1998, Calderbank *et al.* [4] gave the relationship between quantum codes and self-orthogonal codes on the quaternary field. Later, Ketkar *et al.* [5] gave a method to construct quantum codes by using the inclusion relationship of two linear codes over finite fields $\mathbb{F}_q$. This method is called the CSS quantum codes construction. Recently, with the development of quantum information theory, there have been a lot of applications of quantum codes in quantum environment, such as quantum network coding [6], [7], multiparty quantum key management (QKM) protocol [8], quantum cooperative multicast (QCM) [9], etc.

In recent years, a lot of good quantum codes are constructed by self-orthogonal codes over finite fields because the self-orthogonal codes satisfy the conditions of CSS quantum codes construction method, and the self-orthogonal codes can be constructed effectively by algebraic codes such as cyclic codes, constacyclic codes, AG codes and so on. For example, Thangaraj and McLaughlin [10] provided a construction for quantum codes from Hermitian self-orthogonal cyclic codes over $\mathbb{F}_{4^m}$, Guardia [11] constructed some new quantum codes from a class of special cyclic codes whose defining set consists of only one cyclotomic coset containing at least two consecutive integers; Xiaoyan [12] constructed quantum code from some classes of constacyclic codes, Koroglu [13] constructed eight new classes of entanglement-assisted quantum MDS codes by virtue of a decomposition of the defining set of constacyclic codes; Feng *et al.* [14] gave an asymptotic bounds on quantum codes from algebraic geometry codes, Chen [15] constructed asymptotically good family of quantum codes from algebraic geometric codes, Munuera *et al.* [16] studies quantum codes construction from algebraic geometric codes of Castle type.

One question is that if we can give a class of linear codes that are not self-orthogonal to construct quantum codes effectively. In 2018, Hivadi [17] presented a construction method for binary quantum codes from a class of non self-orthogonal binary linear codes. This construction method can be effectively extended to the case of non-binary quantum codes. In this short paper, we will do this issue.

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II. CONSTRUCTION OF QUANTUM CODES

Let $\mathbb{F}_q$ be finite fields, where $q = p^t$, p is a prime, t is a positive integer. If C is a k-dimensional subspace of $\mathbb{F}_q^n$, then C will be called an $[n, k]$ linear code over $\mathbb{F}_q$. The linear code C has q^k codewords. For $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in C$, we define the weight as $\mathrm{wt}(\mathbf{c}) = \#\{i \mid c_i \neq 0,\ 0 \leq i \leq n-1\}$. The minimum distance is $d = \min\{wt(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in C\}$. Further, if the minimum distance d of C is known, it is also sometimes referred to as an $[n, k, d]$ linear code. In coding theory, two frequently-used ways to present a linear code are with either a generator matrix or a parity check matrix.

A $k \times n$ matrix G over $\mathbb{F}_q$ is called a generator matrix of C if the rows of G generates C and no proper subset of the rows of G generates C. Let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$, $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{F}_q^n$, the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is $\mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^{n} x_i y_i$. The dual code of C is defined by $C^{\perp} = \{\mathbf{x} \in \mathbb{F}_q^n \mid \mathbf{x} \cdot \mathbf{c} = 0, \forall\ \mathbf{c} \in C\}$. A parity check matrix H for a linear C is a generator matrix for the dual code $C^{\perp}$. A linear code C of length n is called self-orthogonal if $C \subseteq C^{\perp}$, and self-dual if $C^{\perp} = C$. Clearly, self-dual codes are subclass of self-orthogonal codes.

In the following, we will give the CSS quantum codes construction first.

Lemma 1 [5] (CSS Quantum Codes Construction): Let C_1 and C_2 be two linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively. If $C_2^{\perp} \subseteq C_1$, then there is a quantum code with parameters $[[n, k_1 + k_2 - n, \geq d]]_q$, where $d = \min\{d_1, d_2\}$.

If $C_1 = C_2$, then we have the following Lemma 2. In fact, most of works on quantum codes construction from self-orthogonal linear codes over finite fields based on this fundamental result.

Lemma 2 [5]: Let C be a linear $[n, k, d]_q$ code satisfying $C^{\perp} \subseteq C$, then there is a quantum code with parameters $[[n, 2k - n, \geq d]]_q$.

In the following, we will introduce non-binary quantum codes construction from a class of linear codes, which are not self-orthogonal codes over finite fields.

At the beginning, we need some preparations. The permutation matrix P is a square matrix with exactly one 1 in each row and column and 0s elsewhere. The diagonal matrix D is a square matrix with only non-zero elements on the diagonal line. The monomial matrix is a square matrix with exactly one non-zero entry in each row and column. In other words, the monomial matrix M can be written either in the form DP or PD', where D and D' are diagonal matrices and P is a permutation matrix.

Lemma 3 [18]: Let C_1 and C_2 be linear codes of the same length over $\mathbb{F}_q$, and let G_1 be a generator matrix for C_1. Then

(i) C_1 and C_2 are permutation equivalent if there is a permutation matrix P such that $G_1 P$ is a generator matrix of C_2.

(ii) C_1 and C_2 are monomial equivalent if there is a monomial matrix M such that $G_1 M$ is a generator matrix of C_2.

(iii) C_1 and C_2 are equivalent if there is a monomial matrix M and an automorphism γ of the finite field $\mathbb{F}_q$ such that $G_1 M \gamma$ is a generator matrix of C_2.

Note that two equivalent codes have the same weight distribution.

Let $H(C_1)$ and $H(C_2)$ be parity check matrices of linear codes C_1 and C_2, respectively. If $H(C_2)H(C_1)^{\top} = 0$, then $C_2^{\perp} \subseteq C_1$ which implies that there is a quantum code associated to C_1 and C_2 by CSS quantum codes construction method. Let

$$H = \begin{bmatrix} H(C_2) & 0 \\ 0 & H(C_1) \end{bmatrix}$$

with $H(C_2)H(C_1)^{\top} = 0$. Then we call H is the parity check matrix of the associated quantum code. The problem of quantum codes construction from error-correcting codes over finite fields turns out to be a problem to construct the parity check matrix H.

Now, we will give the main construction results in the following.

Theorem 1: Let C be an $[n, k, d]$ linear code over $\mathbb{F}_p$ with parity check matrix H, where p is an odd prime. If there is a monomial matrix M such that $H(HM)^{\top} = 0$, then there is a non-binary $[[n, 2k - n, \geq d]]_p$ quantum code with parity check matrix

$$\begin{bmatrix} H & 0 \\ 0 & HM \end{bmatrix}.$$

Proof: Let C be an $[n, k, d]$ linear code over $\mathbb{F}_p$ with parity check matrix H, M be an $(n-k) \times (n-k)$ monomial matrix over $\mathbb{F}_p$. Let C' be a linear code with parity check matrix HM. Then $C^{\perp}$ and $C'^{\perp}$ are monomial equivalent. Since $H(HM)^{\top} = 0$, it follows that $C^{\perp} \subseteq C'$. By Lemma 1, there is a quantum code associated to C and C'. Further, the matrix $\begin{bmatrix} H & 0 \\ 0 & HM \end{bmatrix}$ is the parity check matrix of the associated quantum code.

Now, we will show that the parameters of this quantum code are $[[n, 2k - n, \geq d]]_p$. The number of $\mathbb{F}_p$-linearity independent rows in the matrix $\begin{bmatrix} H & 0 \\ 0 & HM \end{bmatrix}$ is $2(n-k)$, so the quantum code has the dimension $n - 2(n-k) = 2k - n$. The linear code C has the minimum distance d. That is, the parity check matrix H of C has a set of d linear dependent columns but no set of $d-1$ linearly dependent columns. Clearly, H and HM have the same linearly correlation. Thus, the parameters of the quantum code are $[[n, 2k - n, \geq d]]_p$. ■

Theorem 2: Let C be an $[n, k, d]$ linear code over $\mathbb{F}_q$ with parity check matrix H, where $q = p^t$, p is an odd prime and $t \geq 2$ is a positive integer. If there is a monomial matrix M and an automorphism γ of the finite field $\mathbb{F}_q$ such that $H(HM\gamma)^{\top} = 0$, then there is a non-binary $[[n, 2k - n, \geq d]]_q$ quantum code with parity check matrix

$$\begin{bmatrix} H & 0 \\ 0 & HM\gamma \end{bmatrix}.$$

Proof: The proof process of this theorem is similar to Theorem 1. We omit it here. ■

III. SOME NEW NON-BINARY QUANTUM CODES

In this section, we will give some computational examples to illustrate that our construction method can produce good non-binary quantum codes. More importantly, some of these quantum codes can not be obtained by self-orthogonal codes.

Example 1: Let C be an $[8, 4, 4]$ linear code over $\mathbb{F}_3$. The parity check matrix H of this linear code is given as follows

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

This linear code is not self-orthogonal. Consider the monomial matrix M on the matrix H as follows

$$HM = \begin{bmatrix} 2 & 2 & 0 & 0 & 2 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 1 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

We have $H(HM)^{\top} = 0$. By Theorem 1, there is an $[[8, 0, \geq 4]]_3$ quantum code. This quantum code has the same length and dimension as the known quantum code $[[8, 0, 3]]_3$ appeared in [19], but our code has the larger minimum distance than that code. Further, since the highest minimum distance of the self-dual code of length 8 over $\mathbb{F}_3$ is 3, then we can not get the quantum code $[[8, 0, \geq 4]]_3$ by self-dual codes.

Example 2: Let C be an $[8, 5, 3]$ linear code over $\mathbb{F}_3$. The parity check matrix H of this linear code is given as follows

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 & 2 & 0 & 1 \end{bmatrix}.$$

This linear code is not self-orthogonal. Consider the monomial matrix M on the matrix H as follows

$$HM = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \end{bmatrix}.$$

Then $H(HM)^{\top} = 0$. By Theorem 1, there is an $[[8, 2, \geq 3]]_3$ quantum code. This quantum code has the same length and minimum distance as the known quantum code $[[8, 0, 3]]_3$ appeared in [19], but our code has the larger dimension than that code.

Example 3: Let C be an $[11, 6, 5]$ linear code over $\mathbb{F}_3$. The parity check matrix H of this linear code is given as follows

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 1 \end{bmatrix}.$$

This linear code is not self-orthogonal. Consider the monomial matrix M on the matrix H as follows

$$HM = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then $H(HM)^{\top} = 0$. By Theorem 1, there is an $[[11, 1, \geq 5]]_3$ quantum code. This quantum code has the same length and dimension as the known quantum code $[[11, 1, 4]]_3$ appeared in [19], but our code has the larger minimum distance than that code.

Example 4: Let C be an $[10, 5, 5]$ linear code over $\mathbb{F}_4$. The parity check matrix H of this linear code is given as follows

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & w^2 & w^2 & 0 & w \\ 0 & 1 & 0 & 0 & 0 & w & 0 & w & w^2 & w^2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & w^2 & w & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & w & w & w^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & w^2 & w^2 & 0 & w & w \end{bmatrix},$$

where w is a primitive element of $\mathbb{F}_4$. This linear code is not self-orthogonal. Consider the monomial matrix M and the automorphism γ over $\mathbb{F}_4$ on the matrix H as follows

$$HM\gamma = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & w & w & 0 & w^2 \\ 0 & w^2 & 0 & 0 & w & 1 & w^2 & 0 & 0 & w \\ 0 & 1 & 0 & 1 & w^2 & 0 & w & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & w & 0 & w^2 & w^2 & 0 & 0 \\ 0 & w & 0 & 0 & w^2 & 0 & 0 & w & 1 & w^2 \end{bmatrix}.$$

Then $H(HM\gamma)^{\top} = 0$. By Theorem 2, there is an $[[10, 0, \geq 5]]_4$ quantum code. Since the highest minimum distance of the self-dual code of length 10 over $\mathbb{F}_4$ is 4, then we can not get the quantum code $[[10, 0, \geq 5]]_4$ by self-dual codes.

Example 5: Let C be an $[14, 10, 4]$ linear code over $\mathbb{F}_4$. The parity check matrix H of this linear code is given as follows

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & w & w & w^2 & 1 & 0 & w & w & 0 & 1 \\ 0 & 1 & 0 & 0 & w & w & 1 & w^2 & 1 & 1 & w^2 & 1 & w & w \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & w & w & 1 & w^2 & 1 & 1 & w^2 \\ 0 & 0 & 0 & 1 & w & w & w^2 & 1 & 0 & w & w & 0 & 1 & w^2 \end{bmatrix},$$

where w is a primitive element over $\mathbb{F}_4$. This linear code is not self-orthogonal. Consider the monomial matrix M and the automorphism γ over $\mathbb{F}_4$ on the matrix H as follows

$$HM\gamma = \begin{bmatrix} w & 0 & 0 & 1 & 0 & w^2 & 1 & 1 & 0 & 1 & w^2 & 0 & w^2 & w^2 \\ w & 0 & 1 & 1 & 1 & w & w^2 & 0 & 0 & w^2 & w^2 & w^2 & 1 & 1 \\ w^2 & 1 & 0 & w^2 & 1 & w & w & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & w^2 & w^2 & w & 0 & 1 & w^2 & w^2 & 1 & 0 & w \end{bmatrix}.$$

Then $H(HM\gamma)^{\top} = 0$. By Theorem 2, there is an $[[14, 6, \geq 4]]_4$ quantum code. This quantum code has the same length and minimum distance as the known quantum code $[[14, 0, 4]]_4$ appeared in [19], but our code has the larger dimension than that code. □

IV. CONCLUSION

In this paper, we studied the non-binary quantum codes construction from a class of linear codes, which are not self-orthogonal over finite fields. We can construct some quantum codes with special parameters $[[n, 0, \geq d]]$, which can not be constructed by self-orthogonal codes over finite fields.

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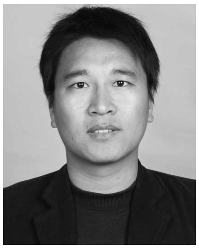

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