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# $H_-$ Index for Linear Time-Varying Markov Jump Stochastic Systems and Its Application to Fault Detection

**TIANLIANG ZHANG[1], FEIQI DENG[1], (Senior Member, IEEE), AND WEIHAI ZHANG[2], (Senior Member, IEEE)**

[1]School of Automation Science and Engineering, South China University of Technology, Guangzhou 510640, Guangdong, China
[2]College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, Shandong, China

Corresponding author: Weihai Zhang (w_hzhang@163.com)

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**ABSTRACT** In this paper, the $H_-$ index for Markov jump linear time-varying stochastic systems and its application to robust $H_-$ fault detection filter (FDF) are under consideration. First, a set of finite horizon backward generalized differential Riccati equations (GDREs) and a set of matrix inequalities are introduced. Based on the introduced backward GDREs and matrix inequalities, for nominal Markov jump linear time-varying stochastic systems, two necessary and sufficient conditions for the finite horizon $H_-$ index larger than a given prescribed level $\beta > 0$ are given. Second, for norm uncertain Markov jump linear time-varying stochastic systems, sufficient conditions are presented in terms of matrix inequalities. As applications, two equivalent conditions for the existence of robust $H_-$ FDF are obtained for nominal linear stochastic systems. In particular, under the case of norm uncertainties, a robust $H_-$ FDF is designed based on the feasibility of linear matrix inequalities.

**INDEX TERMS** $H_-$ index, fault detection filter, linear Markov jump stochastic systems, generalized differential Riccati equations, linear matrix inequalities.

## I. INTRODUCTION

Along with the development of modern industrial production, higher requirement for safety and reliability has been put forward. As we all know, the fault is one of the biggest threats to the operation of the system. In a broad sense, faults can be defined in many ways. On one hand, faults usually refer to that one or more important variables or performance indexes deviates from the normal range. On the other hand, faults can also be defined as that the dynamic control system shows undesirable characteristics or abnormal phenomena. In order to ensure safety and reliability in industrial process, various techniques for fault detection, fault isolation and fault estimation have appeared [6]. To date, there are various kinds of fault detection techniques such as FDFs, unknown input observers, artificial intelligence techniques and so on. The FDF is, based on the measurement output, to use the estimated value of the system state to generate the residual signal in order to detect the system fault. According to the real time comparison between the designed residual evaluation function and the corresponding threshold, we are in a position to determine whether there is a fault occurring. Some suitable design criteria, such as $H_-$ index, $H_2$ index and $H_\infty$ index have been proposed for FDF [4], [14]–[16], [25], [27], [28], [40], [41]. $H_-$ index was first introduced by [7], which has been extensively studied by many researchers [5], [9], [11]–[13], [27], [38]. $H_-$ FDF is to design the filter such that the $L_2$-gain from the fault signal to the residual signal is larger than the given sensitivity level $\beta > 0$, which measures the sensitivity of the considered system to the fault signal. In [5] and [7], the smallest nonzero singular value of the transfer function from the fault to the residual over a finite frequency range was used to evaluate the worst-case fault sensitivity. By the general Kalman-Yakubovich-Popov lemma, [9] developed a systematic method for designing $H_-$ index of mechanical systems, while [12] studied the minimum input sensitivity analysis problem of linear time-invariant systems in both infinite and finite frequency ranges. There are many results about $H_-$ FDFs for time-invariant

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systems with asymptotic behaviors over the infinite horizon; see, e.g., [4], [15]. As it is well-known, in the real world, almost all engineering systems are of time-varying in nature, i.e., the system parameters are changeable as the environment changes. Therefore, it would be more meaningful to study the $H_-$ FDF for linear Markov jump stochastic systems such as done in [20] and [21].

It is well-known that, in practical modeling, stochastic disturbances often occur, hence, stochastic systems are ideal mathematical models in finance mathematics [30], systems biology [2], [3], benchmark mechanical systems [29]. Markov jump systems often arise in reality with component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances. So the study of stochastic Markov jump systems have attracted many researchers' interest; see the studies of linear-quadratic optimal control [20], output feedback tracking control [1], $p$-th moment stability [31], observability and detectability [8], [19], [26], [34] and so on. Meanwhile, the fault detection problem of stochastic systems has become one of the most important research fields. Reference [25] discussed the FDF of nonlinear switched stochastic systems through T-S fuzzy model approach. While the fault isolation problem for discrete-time fuzzy interconnected systems with unknown interconnections was considered in [41]. Reference [15] studied $H_-/H_\infty$ fault detection observer design for a class of linear parameter-varying descriptor systems in the finite frequency domain . A robust fault detection $H_-/H_\infty$ observer was constructed for a T-S fuzzy model with sensor faults and unknown bounded disturbances via an LMI formulation in [4]. Reference [11] investigated the $H_-$ index of stochastic linear discrete-time systems. Reference [39] proposed an $H_-/H_\infty$ FDF design scheme for nonlinear stochastic systems. However, it can be found that, up to now, there are few work on $H_-$ FDF for linear time-varying Markov jump stochastic systems with uncertain parameters.

Motivated by the aforementioned reason, this paper studies the $H_-$ index for linear time-varying Markov jump stochastic systems with its application to FDF. The main contribution of this paper is as follows:

- Some necessary/sufficient conditions for the finite horizon $H_-$ index large than a prescribed level $\beta > 0$ are given in terms of backward GDREs and matrix inequalities (Theorems 3.1-3.2).
- For norm uncertain stochastic Markov jump systems, sufficient conditions have also been given for the finite horizon $H_-$ index large than a prescribed level $\beta > 0$; see Theorem 3.3 and Corollary 3.2.
- For finite horizon $H_-$ FDF, based on the study of finite horizon $H_-$ index, necessary and sufficient conditions for the existence of the finite horizon $H_-$ FDF of linear uncertain Markov jump stochastic systems are presented (Theorem 4). Moreover, a convenient design method for FDF is also presented via LMIs (Theorem 5).

The organization of this paper is as follows: In Section II, we introduce useful definitions and lemmas and make some preparations. In Section III, some necessary and sufficient conditions about the lower bound of finite horizon $H_-$ index are given. In Section IV, we obtain two necessary and sufficient conditions for the existence of the finite horizon $H_-$ FDF. Moreover, the finite horizon $H_-$ FDF of linear uncertain Markov jump stochastic systems can be designed via solving some LMIs, which is very convenient in practice. Section V presents an example to illustrate the effectiveness of our given results. Section VI concludes this paper with some remarks and perspectives.

For convenience, this paper adopts the following standard notations:

$M'$: the transpose of the matrix $M$ or vector $M$; $M > 0$ ($M < 0$): the matrix $M$ is a positive definite (negative definite) real symmetric matrix; $I_n$: $n \times n$ identity matrix; $\mathcal{R}^n$: the $n$-dimensional real Euclidean vector space with 2-norm $\|x\|$; $\mathcal{R}^{n\times m}$: the $n \times m$ real matrix space; $\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_v})$: the space of nonanticipative stochastic process $v(t) \in \mathcal{R}^{n_v}$ with respect to an increasing $\sigma$-algebra $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying $\|v(t)\|_{\mathcal{L}^2_{[0,T]}} := \{E\int_0^T \|v(t)\|^2\,dt\}^{1/2} < \infty$; $\mathcal{C}^{1,2}([0,T]\times\mathcal{R}^{n_x};\mathcal{R})$: class of $\mathcal{R}$-valued function $V(t,x)$ which are once continuously differential with respect to $t \in [0,T]$, and twice continuously differential with respect to $x \in \mathcal{R}^{n_x}$, except possibly at the point $x = 0$.

## II. PRELIMINARIES

Consider the following Itô stochastic system

$$\begin{cases} dx(t) = \big(A_{e(t)}(t)x(t) + B_{e(t)}(t)f(t)\big)\,dt \\ \qquad\qquad + \big(C_{e(t)}(t)x(t) + D_{e(t)}(t)f(t)\big)\,dw(t), \\ y(t) = H_{e(t)}(t)x(t) + G_{e(t)}(t)f(t), \\ x(0) = x_0,\ t \in [0,T], \end{cases} \tag{1}$$

where $x(t) \in \mathcal{R}^{n_x}$ represents the system state, $y(t) \in \mathcal{R}^{n_y}$ represents the system output, $w(t)$ is a standard one-dimensional Wiener process. Assume $w(t)$ and $e(t)$ are independent of each other defined on the complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathcal{P})$ with the $\sigma$-field $\mathcal{F}_t$ generated by $w(\cdot)$ and $e(\cdot)$ up to time $t$. $e(t)$ is a finite-state $S = \{1,2,\cdots,N\}$ homogeneous Markov jump process with the transition rate matrix $\Pi$ defined by

$$\Pi = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{N1} & \lambda_{N2} & \cdots & \lambda_{NN} \end{bmatrix},$$

where $\lambda_{ij}$ is given by

$$\mathcal{P}\left(e(t+\Delta t)j|e(t)=i\right) = \begin{cases} \lambda_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \lambda_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

$$\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0, \quad \lambda_{ij} \geq 0 (i \neq j),$$

and $\lambda_{ii} = -\sum_{j=1,j\neq i}^N \lambda_{ij}$. $\lambda_i = \mathcal{P}(e(0) = i)$. $f(t) \in \mathcal{R}^{n_f}$ stands for the fault signal with $f(t) \in \mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_f})$.

### A. DEFINITIONS AND LEMMAS

Similar to the perturbed operator defined in [33], a sensitive operator that maps $f(t)$ to $y(t)$ can be given in the following.

*Definition 1:* For stochastic system (1), the finite horizon sensitive operator $\mathcal{L}_{f,y}^{[0,T]}$ and its corresponding $H_-$ index are defined respectively as

$$\mathcal{L}_{f,y}^{[0,T]}: f(t) \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathcal{R}^{n_f}) \mapsto y(t) \in \mathcal{L}_{\mathcal{F}}^2([0,T],\mathcal{R}^{n_y}),$$

and

$$\begin{aligned}\|\mathcal{L}_{f,y}^{[0,T]}\|_- &= \inf_{x_0=0,f(t)\in\mathcal{L}_{\mathcal{F}}^2([0,T],\mathcal{R}^{n_f}),f(t)\neq 0} \frac{\|y(t)\|_{\mathcal{L}^2_{[0,T]}}}{\|f(t)\|_{\mathcal{L}^2_{[0,T]}}} \\ &= \inf_{x_0=0,f(t)\in\mathcal{L}_{\mathcal{F}}^2([0,T],\mathcal{R}^{n_f}),f(t)\neq 0} \frac{\{E\int_0^T \|y(t)\|^2 dt\}^{1/2}}{\{E\int_0^T \|f(t)\|^2 dt\}^{1/2}}.\end{aligned}$$

Set

$$\begin{aligned}&\mathbb{J}_{[0,T],\beta}(\eta_0,f) \\ &\quad = \|y(t)\|^2_{\mathcal{L}^2_{[0,T]}} - \beta^2\|f(t)\|^2_{\mathcal{L}^2_{[0,T]}}, \\ &\mathbb{J}_{[0,T],\beta,i}(\eta_0,f) \\ &\quad = E\left(\int_0^T (\|y(t)\|^2 - \beta^2\|f(t)\|^2)\,dt | e(0)=i\right).\end{aligned}$$

Obviously, we have

$$\mathbb{J}_{[0,T],\beta}(\eta_0,f) = \sum_{i\in S}\lambda_i\mathbb{J}_{[0,T],\beta,i}(\eta_0,f).$$

In order to study finite horizon $\|\mathcal{L}_{f,r}^{[0,T]}\|_-$ for system (1), we introduce the following GDREs

$$\begin{cases}\mathbb{L}_i(t,P) - \dot{P}_{i,T}(t) - \mathbb{K}_i(t,P)'\mathbb{H}_i^{\beta}(t,P)^{-1} \\ \qquad \cdot\mathbb{K}_i(t,P) = 0, \\ \mathbb{H}_i^{\beta}(t,P) > 0, \\ P_{i,T}(T) = 0,\ i\in S,\ t\in[0,T],\end{cases} \tag{2}$$

where

$$\begin{aligned}\mathbb{L}_i(t,P) &= -A_i(t)'P_{i,T}(t) - P_{i,T}(t)A_i(t) - \sum_{j\in S}\lambda_{ij}P_{j,T}(t) \\ &\qquad -C_i(t)'P_{i,T}(t)C_i(t) + H_i(t)'H_i(t), \\ \mathbb{K}_i(t,P) &= -P_{i,T}(t)B_i(t) - C_i(t)'P_{i,T}(t)D_i(t) \\ &\qquad + H_i(t)'G_i(t), \\ \mathbb{H}_i^{\beta}(t,P) &= -D_i(t)'P_{i,T}(t)D_i(t) + G_i(t)'G_i(t) - \beta^2 I.\end{aligned}$$

*Lemma 1 [10]:* For any given matrices $U \in \mathcal{R}^{n\times n}$ with $U = U' < 0$, $V \in \mathcal{R}^{n\times m}$ and $W \in \mathcal{R}^{k\times n}$, if we denote $\mathcal{W} := \{F(t) : F(t)'F(t) \leq I$, $F$ is Lebesgue measurable matrix-valued function, $F(t) \in \mathcal{R}^{m\times k}$, $t\in\mathbb{T}\}$ with $\mathbb{T} := [0,T]$ or $[0,\infty)$, then

$$U + VF(t)W + W'F(t)'V' < 0, \quad \forall F(t)\in\mathcal{W} \tag{3}$$

if and only if there exists $\varepsilon > 0$ such that

$$U + \varepsilon VV' + \varepsilon^{-1}W'W < 0. \tag{4}$$

*Corollary 1:* In Lemma 1, if $\hat{U} = \hat{U}' = -U > 0$, then

$$\hat{U} + VF(t)W + W'F(t)'V' > 0 \tag{5}$$

if and only if there exists $\varepsilon > 0$ such that

$$\hat{U} - \varepsilon VV' - \varepsilon^{-1}W'W > 0. \tag{6}$$

*Proof*: Note that (5) holds if and only if

$$U + V(-F(t))W + W'(-F(t)')V' < 0.$$

Because $-F(t)$ still belongs to $\mathcal{W}$, this corollary is immediately derived. □

Corollary 1 directly yields the following useful lemma.

*Lemma 2:* For any given time-varying matrices $U(t) \in \mathcal{R}^{n\times n}$ with $U(t) = U(t)' > 0$, $V(t) \in \mathcal{R}^{n\times m}$ and $W(t) \in \mathcal{R}^{k\times n}$, then for any $F(t)\in\mathcal{W}$, we have

$$U(t) + V(t)F(t)W(t) + W(t)'F(t)'V(t)' > 0,\ t\in\mathbb{T},$$

if and only if there exists $\varepsilon(t) > 0$ such that

$$U(t) - \varepsilon(t)V(t)V(t)' - \varepsilon(t)^{-1}W(t)'W(t) > 0, \quad t\in\mathbb{T}.$$

*Lemma 3 [32]:* For $x, b \in \mathcal{R}^n$, and a real symmetric matrix $A$ with appropriate dimension, if we assume $A^{-1}$ exists, then we have

$$x'Ax + x'b + b'x = (x + A^{-1}b)'A(x + A^{-1}b) - b'A^{-1}b.$$

*Lemma 4 [32] (Generalized Itô formula):* Suppose there exists a function $V_i(t,x) \in \mathcal{C}^{1,2}([0,T]\times\mathcal{R}^{n_x};\mathcal{R})$ for $i\in S$. Then, associated with the following system

$$dx(t) = f_{e(t)}(t,x(t))dt + g_{e(t)}(t,x(t))dw(t),$$

we have

$$\begin{aligned}&E[V_{e(t)}(t,x(t)) - V_{e(0)}(0,x(0))|e(0)=i] \\ &\qquad = E\int_{t_0}^{t}[\mathcal{L}V_{e(s)}(s,x(s))|e(0)=i]\,ds,\end{aligned}$$

where

$$\begin{aligned}&\mathcal{L}V_i(t,x) \\ &\quad = \frac{\partial V_i(t,x)'}{\partial t} + \frac{\partial V_i(t,x)'}{\partial x}f_i(t,x) \\ &\qquad + \frac{1}{2}g_i'(t,x)\frac{\partial^2 V_i(t,x)}{\partial x^2}g_i(t,x) + \sum_{j=1}^{N}\lambda_{ij}V_j(t,x),\end{aligned}$$

## III. FINITE HORIZON $H_-$ INDEX

In this section, we discuss the feasibility of $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ associated with (1). This section is the foundation for designing FDF.

*Theorem 1:* For given $\beta > 0$, the following conditions are equivalent:

(i) $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ associated with system (1), and

$$\min\mathbb{J}_{[0,T],\beta,i}(\eta_0,f) = \eta_0'P_{i,T}(0)\eta_0.$$

(ii) GDRE (2) has a unique real symmetric matrix solution sequence $\{P_{i,T}(t), t\in[0,T]\}_{i\in S}$.

(iii) $\mathbb{M}_i(t) > 0$ has a real symmetric matrix solution sequence $\{P_{i,T}(t), t \in [0,T]\}_{i\in S}$ with $P_{i,T}(T) = 0$, where

$$\mathbb{M}_i(t) = \begin{bmatrix} \mathbb{L}_i(t, P(t)) - \dot{P}_{i,T}(t) & \mathbb{K}_i(t, P(t)) \\ \mathbb{K}_i(t, P(t))' & \mathbb{H}_i^{\beta}(t, P(t)) \end{bmatrix},$$

$\mathbb{L}_i(t, P(t))$, $\mathbb{K}_i(t, P(t))$ and $\mathbb{H}_i^{\beta}(t, P(t))$ are defined in (2).

*Proof:* (*ii*) $\Rightarrow$ (*i*): Consider $V_i(t,x) = x'P_{i,T}(t)x$ associated with (1), $\{P_{i,T}(t), t \in [0,T]\}_{i\in S}$ is a time-varying symmetric matrix solution sequence of GDRE (2), By the generalized Itô's formula-Lemma 4 and Lemma 3, we obtain that

$$\begin{aligned}
&\mathbb{J}_{[0,T],\beta}(0,f) \\
&= \sum_{i\in S} \lambda_i \mathbb{J}_{[0,T],\beta,i}(0,f) \\
&= \sum_{i\in S} \lambda_i E\left(\int_0^T \|y(t)\|^2 - \beta^2\|f(t)\|^2 dt | e(0) = i, x_0 = 0\right) \\
&= \sum_{i\in S} \lambda_i E\Big[\int_0^T \left(\|y(t)\|^2 - \beta^2\|f(t)\|^2 - \mathcal{L}V_{e(t)}(t, x(t))\right)dt \\
&\quad - V_i(0, x_0) + V_{e(T)}(T, x(T)) | e(0) = i, x_0 = 0\Big] \\
&= \sum_{i\in S} \lambda_i E\Big[\int_0^T \left(\begin{bmatrix} x(t) \\ f(t)\end{bmatrix}' \mathbb{M}_{e(t)}(t) \begin{bmatrix} x(t) \\ f(t)\end{bmatrix}\right)dt \\
&\qquad + V_{e(T)}(T, x(T)) | e(0) = i, x_0 = 0\Big], \\
&= \sum_{i\in S} \lambda_i E\Big[\int_0^T \Big(x(t)'(\mathbb{L}_{e(t)}(t,P) - \dot{P}_{e(t)}(t) - \mathbb{K}_{e(t)}(t,P)' \\
&\quad \mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P))x(t) + (f(t) - f^*(t))' \\
&\quad \mathbb{H}_{e(t)}^{\beta}(t,P)(f(t) - f^*(t))\Big)dt + V_{e(T)}(T, x(T)) | e(0) = i, \\
&\quad x_0 = 0\Big],
\end{aligned}$$

where

$$f^*(t) = -\mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P)x(t).$$

In view of GDRE (2), it follows that

$$\begin{aligned}
&\mathbb{J}_{[0,T],\beta}(0,f) \\
&\quad = \sum_{i\in S} \lambda_i E\Big[\int_0^T (f(t) - f^*(t))'\mathbb{H}_{e(t)}^{\beta}(t,P)(f(t) - f^*(t)) \\
&\qquad\qquad | e(0) = i, x_0 = 0\Big]dt. \qquad (7)
\end{aligned}$$

Let $\mathcal{L}_{f,f-f^*}$ be the operator from $f(t) \in \mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^{n_f})$ to $(f(t) - f^*(t)) \in \mathcal{L}^2_{\mathcal{F}}([0,T], \mathcal{R}^{n_f})$, which can be written as

$$\begin{cases}
dx(t) = \left(A_{e(t)}(t)x(t) + B_{e(t)}(t)f(t)\right)dt \\
\qquad\qquad + \left(C_{e(t)}(t)x(t) + D_{e(t)}(t)f(t)\right)dw(t), \\
f(t) - f^*(t) = \mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P)x(t) + f(t), \\
x(0) = x_0,\ t \in [0,T].
\end{cases}$$

We deduce that the inverse operator $\mathcal{L}_{f,f-f^*}^{-1}$ exists given by

$$\begin{cases}
dx(t) = \Big[A_{e(t)}(t)x(t) - B_{e(t)}(t)\mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P) \\
\qquad x(t) + B_{e(t)}(t)(f(t) - f^*(t))\Big]dt + \Big[C_{e(t)}(t)x(t) \\
\qquad - D_{e(t)}(t)\mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P)x(t) + D_{e(t)}(t) \\
\qquad (f(t) - f^*(t))\Big]dw(t), \\
f(t) = \mathbb{H}_{e(t)}^{\beta}(t,P)^{-1}\mathbb{K}_{e(t)}(t,P)x(t) + f(t) - f^*(t), \\
x(0) = x_0,\ t \in [0,T].
\end{cases}$$

Meanwhile, by $\mathbb{H}_i^{\beta}(t,P) > 0$, we have

$$\mathbb{H}_{e(t)}^{\beta}(t,P) \geq \min_{t\in[0,T], i\in S} \lambda_{min}\mathbb{H}_i^{\beta}(t,P)I > 0,$$

so there exists a constant $\alpha > 0$ such that for $f(t) \not\equiv 0$,

$$\begin{aligned}
&\mathbb{J}_{[0,T],\beta}(0,f) \\
&\quad \geq \min_{t\in[0,T], i\in S} \lambda_{min}\mathbb{H}_i^{\beta}(t,P)\|f(t) - f^*(t)\|^2_{\mathcal{L}^2_{[0,T]}} \\
&\quad = \min_{t\in[0,T], i\in S} \lambda_{min}\mathbb{H}_i^{\beta}(t,P)\|\mathcal{L}_{f-f^*,f}f(t)\|^2_{\mathcal{L}^2_{[0,T]}} \\
&\quad \geq \alpha\|f(t)\|^2_{\mathcal{L}^2_{[0,T]}} > 0.
\end{aligned}$$

Thus, (*i*) is derived.

We are now in a position to show (*i*) $\Rightarrow$ (*ii*) by contradiction, that is, if $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ holds, then the finite-time escape of GDRE (2) will lead to a contradiction, i.e., there never exists a solution $P_{i,T}(t)$ backward in an interval $(T_0, T]$ with $T_0 \geq 0$, such that when $t \to T_0$, $P_{i,T}(t)$ never become unbounded.

Take a sufficiently small $\varepsilon > 0$ with $0 < \varepsilon < T - T_0$, and $x_{T_0,\varepsilon} := x(T_0 + \varepsilon) \in \mathcal{R}^{n_x}$. For $t \in [T_0 + \varepsilon, T]$, let

$$\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon}, f) \\
&:= \sum_{i\in S} \lambda_i E\Big(\int_{T_0+\varepsilon}^T \|y(t)\|^2 - \beta^2\|f(t)\|^2 dt | e(T_0+\varepsilon) = i, \\
&\qquad\qquad x(T_0+\varepsilon) = x_{T_0,\varepsilon}\Big).
\end{aligned}$$

By Lemmas 3 and 4, we obtain that

$$\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon}, f) \\
&= \sum_{i\in S} \lambda_i E\Big[\int_{T_0+\varepsilon}^T (\|y(t)\|^2 - \beta^2\|f(t)\|^2 - \mathcal{L}V_{e(t)}(t, x(t)))dt \\
&\qquad - V_{e(T_0+\varepsilon)}(T_0+\varepsilon, x(T_0+\varepsilon)) + V_{e(T)}(T, x(T)) \\
&\qquad | e(T_0+\varepsilon) = i, x(T_0+\varepsilon) = x_{T_0,\varepsilon}\Big] \\
&= \sum_{i\in S} \lambda_i E\Big[\int_{T_0+\varepsilon}^T \Big((f(t) - f^*(t))'\mathbb{H}_{e(t)}^{\beta}(t,P)(f(t) \\
&\quad - f^*(t))\Big)dt - V_{e(T_0+\varepsilon)}(T_0+\varepsilon, x(T_0+\varepsilon)) | e(T_0+\varepsilon) = i, \\
&\qquad x(T_0+\varepsilon) = x_{T_0,\varepsilon}\Big],
\end{aligned}$$

where $f^*(t)$ is defined in (7). So

$$\begin{aligned}
&\min_{f(t)\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_f})} \mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f)\\
&= \mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f^*)\\
&= -V_{e(T_0+\varepsilon)}(T_0+\varepsilon,x(T_0+\varepsilon)) \le \mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},0)\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|y(t)\|^2dt|e(T_0+\varepsilon)=i,\\
&\qquad x(T_0+\varepsilon)=x_{T_0,\varepsilon}\Big]. \qquad (8)
\end{aligned}$$

It is well known that there exists $\alpha_1>0$ such that

$$\begin{aligned}
&\sum_{i\in S}\lambda_i E\left[\int_{T_0+\varepsilon}^{T}\|y(t)\|^2dt|e(T_0+\varepsilon)=i,x(T_0+\varepsilon)=x_{T_0,\varepsilon}\right]\\
&\le \alpha_1\|x(T_0+\varepsilon)\|^2. \qquad (9)
\end{aligned}$$

The above inequalities (8) and (9) lead to

$$-P_{i,T}(T_0+\varepsilon)\le\alpha_1 I \qquad (10)$$

for any $0<\varepsilon<T-T_0$. On the other hand, let $X_{i,T}(t)$ denote the solution of

$$\begin{cases}\mathbb{X}_{i,T}(t) = -\dot{X}_{i,T}(t) - A_i(t)'X_{i,T}(t) - X_{i,T}(t)A_i(t)\\ \qquad -C_i(t)'X_{i,T}(t)C_i(t) + H_i(t)'H_i(t)\\ \qquad -\sum_{j\in S}\lambda_{ij}X_{j,T}(t) = 0,\\ X_{i,T}(T)=0.\end{cases} \qquad (11)$$

For clarity, we denote the solution of system (1) with $x(t_0)=x_{t_0}$ by $x(t,f,x_{t_0},t_0)$. By linearity, the solution $x(t,f,x_{T_0+\varepsilon},T_0+\varepsilon)$ of system (31) satisfies

$$\begin{aligned}
&x(t,f,x_{T_0+\varepsilon},T_0+\varepsilon)\\
&\quad= x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon) + x(t,f,0,T_0+\varepsilon),
\end{aligned}$$

where $x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)$ is the trajectory of fault-free system

$$\begin{cases}dx(t) = A_{e(t)}(t)x(t)dt + C_{e(t)}(t)x(t)dw(t),\\ x(T_0+\varepsilon)=x_{T_0+\varepsilon},\quad t\in[T_0+\varepsilon,T],\end{cases} \qquad (12)$$

and $x(t,f,0,T_0+\varepsilon)$ is the trajectory of the following stochastic system subject to fault with zero initial state:

$$\begin{cases}dx(t) = \big(A_{e(t)}(t)x(t) + B_{e(t)}(t)f(t)\big)\,dt\\ \qquad + \big(C_{e(t)}(t)x(t)+D_{e(t)}(t)f(t)\big)\,dw(t),\\ x(T_0+\varepsilon)=0,\quad t\in[T_0+\varepsilon,T].\end{cases} \qquad (13)$$

It is easy to check that for any $x_{T_0+\varepsilon}$, we can get that

$$\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f) - \mathbb{J}_{[T_0+\varepsilon,T],\beta}(0,f)\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}(\|H_{e(t)}(t)x(t,f,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2\\
&\qquad +\|G_{e(t)}(t)f(t)\|^2)dt|e(T_0+\varepsilon)=i\Big]\\
&\quad -\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}(\|H_{e(t)}(t)x(t,f,0,T_0+\varepsilon)\|^2\\
&\qquad +\|G_{e(t)}(t)f(t)\|^2)dt|e(T_0+\varepsilon)=i\Big]\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}(2x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'H_{e(t)}(t)'H_{e(t)}(t)\\
&\qquad x(t,f,0,T_0+\varepsilon) + \|H_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2)dt\\
&\qquad |e(T_0+\varepsilon)=i\Big]. \qquad (14)
\end{aligned}$$

Considering system (12) and $\mathbb{X}_{i,T}(t)$ in (11), by Lemma 4, it follows that

$$\begin{aligned}
&\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|H_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2dt\\
&\qquad |e(T_0+\varepsilon)=i\Big]\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|H_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2dt\\
&\quad |e(T_0+\varepsilon)=i\Big] - \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}dx(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'\\
&\quad X_{i,T}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)|e(T_0+\varepsilon)=i\Big]\\
&\quad -\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'\mathbb{X}_{i,T}(t)\\
&\qquad x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)dt|e(T_0+\varepsilon)=i\Big]\\
&\qquad -\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}\\
&= -\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}. \qquad (15)
\end{aligned}$$

Similarly, considering system (13) with $x(T_0+\varepsilon,f,0,T_0+\varepsilon)=0$ and $X_{i,T}(T)=0$, we have

$$\begin{aligned}
&\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'H_{e(t)}(t)'\\
&\quad H_{e(t)}(t)x(t,f,0,T_0+\varepsilon)dt|e(T_0+\varepsilon)=i\Big]\\
&= \sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'H_{e(t)}(t)'\\
&\qquad H_{e(t)}(t)x(t,f,0,T_0+\varepsilon)dt|e(T_0+\varepsilon)=i\Big]\\
&\quad -\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}dx(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'X_{i,T}(t)\\
&\qquad x(t,f,0,T_0+\varepsilon)|e(T_0+\varepsilon)=i\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'X_{i,T}(t)B_{e(t)}(t)\\
&\qquad f(t)dt|e(T_0+\varepsilon)=i\Big]\\
&\qquad -\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'\\
&\qquad C_{e(t)}(t)'X_{i,T}(t)D_{e(t)}(t)f(t)dt|e(T_0+\varepsilon)=i\Big].
\end{aligned} \tag{16}
$$

Substituting (15) and (16) into (14) yields that

$$
\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f)-\mathbb{J}_{[T_0+\varepsilon,T],\beta}(0,f)\\
&= -2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'X_{i,T}(t)B_{e(t)}(t)\\
&\quad f(t)dt|e(T_0+\varepsilon)=i\Big]-2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\\
&\quad x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'C_{e(t)}(t)'X_{i,T}(t)D_{e(t)}(t)f(t)dt\\
&\quad |e(T_0+\varepsilon)=i\Big]-\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}.
\end{aligned} \tag{17}
$$

Take $0\le\alpha_2^2\le\|\mathcal{L}_{f,r}\|_-^{[0,T]^2}-\beta^2$, then

$$
\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(0,f)\\
&\quad=\|y(t)\|^2_{\mathcal{L}^2_{[0,T]}}-\beta^2\|\bar f(t)\|^2_{\mathcal{L}^2_{[0,T]}}\\
&\quad\ge\|y(t)\|^2_{\mathcal{L}^2_{[0,T]}}-\beta^2\|\bar f(t)\|^2_{\mathcal{L}^2_{[0,T]}}-\|y(t)\|^2_{\mathcal{L}^2_{[0,T]}}\\
&\qquad+\|\mathcal{L}_{f,r}\|_-^{[0,T]^2}\|\bar f(t)\|^2_{\mathcal{L}^2_{[0,T]}}\\
&\quad\ge\alpha_2^2\|\bar f(t)\|^2_{\mathcal{L}^2_{[0,T]}},
\end{aligned}
$$

where $\bar f$ is the extension of $f$ from $[T_0+\varepsilon,T]$ to $[0,T]$ by setting $\bar f(t)\equiv 0,\ \forall t\in[0,T_0+\varepsilon)$. Therefore, we have

$$
\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f)\\
&\ge -2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'X_{i,T}(t)B_{e(t)}(t)\\
&\quad f(t)dt|e(T_0+\varepsilon)=i\Big]-2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\\
&\quad x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'C_{e(t)}(t)'X_{i,T}(t)D_{e(t)}(t)f(t)dt\\
&\quad |e(T_0+\varepsilon)=i\Big]+E\int_{T_0+\varepsilon}^{T}\alpha_2^2\|\bar f(t)\|^2dt\\
&\quad -\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}.
\end{aligned} \tag{18}
$$

By the inequality $2a'b\le\|a\|^2+\|b\|^2$, we have

$$
\begin{aligned}
&-2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'X_{i,T}(t)B_{e(t)}(t)\\
&\quad f(t)dt|e(T_0+\varepsilon)=i\Big]+E\int_{T_0+\varepsilon}^{T}\alpha_2^2\|\bar f(t)\|^2dt\\
&\quad -2\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T} x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)'C_{e(t)}(t)X_{i,T}(t)\\
&\quad D_{e(t)}(t)f(t)dt|e(T_0+\varepsilon)=i\Big]\\
&\ge\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\Big(-\|X_{i,T}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2\\
&\quad -\|X_{i,T}(t)C_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2\\
&\quad -(\|B_{e(t)}(t)\|^2+\|D_{e(t)}(t)\|^2-\alpha_2^2)\|f(t)\|^2\Big)dt\\
&\quad |e(T_0+\varepsilon)=i\Big].
\end{aligned} \tag{19}
$$

Consequently, substituting the inequality (19) into (18) leads to that

$$
\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f)\\
&\ge-\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\Big(\|X_{i,T}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2\\
&\quad+(\|B_{e(t)}(t)\|^2+\|D_{e(t)}(t)\|^2-\alpha_2^2)\|f(t)\|^2\\
&\quad+\|X_{i,T}(t)C_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2)dt\\
&\quad|e(T_0+\varepsilon)=i\Big]-\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}.
\end{aligned}
$$

It is well known that there exist positive numbers $\alpha_3,\alpha_4,\alpha_5$ and $\alpha_6>0$ such that

$$
\begin{aligned}
&\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|X_{i,T}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2dt\\
&\quad|e(T_0+\varepsilon)=i\Big]\le\alpha_3\|x_{T_0+\varepsilon}\|^2,\\
&\quad\sum_{i\in S}\lambda_i x'_{T_0+\varepsilon}X_{i,T}(T_0+\varepsilon)x_{T_0+\varepsilon}\\
&=\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|H_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2dt\\
&\quad|e(T_0+\varepsilon)=i\Big]\le\alpha_4\|x_{T_0+\varepsilon}\|^2,\\
&\quad\sum_{i\in S}\lambda_i E\Big[\int_{T_0+\varepsilon}^{T}\|X_{i,T}(t)C_{e(t)}(t)x(t,0,x_{T_0+\varepsilon},T_0+\varepsilon)\|^2dt\\
&\quad|e(T_0+\varepsilon)=i\Big]\le\alpha_5\|x_{T_0+\varepsilon}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i\in S}\lambda_i E\Big[\int_0^T(\|B_{e(t)}(t)\|^2+\|D_{e(t)}(t)\|^2-\alpha_2^2)\|f(t)\|^2dt\\
&\qquad\qquad|e(T_0+\varepsilon)=i\Big]\le\alpha_6\|f(t)\|^2_{\mathcal{L}^2_{[0,T]}}.
\end{aligned}
$$

Based on the above discussion, we obtain

$$
\begin{aligned}
&\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f)\\
&\qquad\ge-(\alpha_3+\alpha_4+\alpha_5)\|x_{T_0+\varepsilon}\|^2-\alpha_6\|f(t)\|^2_{\mathcal{L}^2_{[0,T]}}.
\end{aligned}
$$

Since $f(t) \in \mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_f})$, there exists $\alpha_7 > 0$ such that

$$\mathbb{J}_{[T_0+\varepsilon,T],\beta}(x_{T_0,\varepsilon},f) \geq -\alpha_7\|x_{T_0+\varepsilon}\|^2.$$

For any $0<\varepsilon<T-T_0$, the above inequality together with (8) and (10) yields that

$$-\alpha_7 I \leq -P_{i,T}(T_0+\varepsilon) \leq \alpha_1 I. \tag{20}$$

So $P_{i,T}(T_0+\varepsilon)$ cannot tend to $\infty$ as $\varepsilon \to 0$, which shows that (2) has a unique real symmetric matrix solution sequence $\{P_{i,T}(t), t\in[0,T]\}_{i\in S}$.

*(iii)* $\Rightarrow$ *(i)*: As discussed above, we have

$$\mathbb{J}_{[0,T],\beta}(0,f) = \sum_{i\in S}\lambda_i E\Bigg[\int_0^T \begin{bmatrix} x(t)\\ f(t)\end{bmatrix}' \mathbb{M}_{e(t)}(t)\begin{bmatrix} x(t)\\ f(t)\end{bmatrix} dt \,\Big|\, e(0)=i, x_0=0\Bigg].$$

Thus, under the condition $f(t)\in\mathcal{L}^2_{\mathcal{F}}([0,T],\mathcal{R}^{n_f})$ and $f\neq 0$, if $\mathbb{M}_i(t)>0$, we must have (i).

Now we only need to prove *(i)* $\Rightarrow$ *(iii)*. Consider the following system

$$\begin{cases} d\bar{x}(t) = \big(A_{e(t)}(t)\bar{x}(t)+\bar{B}_{e(t)}(t)\bar{f}(t)\big)\,dt \\ \qquad\quad + \big(C_{e(t)}(t)\bar{x}(t)+\bar{D}_{e(t)}(t)\bar{f}(t)\big)\,dw(t), \\ \bar{y}(t) = \bar{H}_{e(t)}(t)\bar{x}(t)+\bar{G}_{e(t)}(t)\bar{f}(t), \\ \bar{x}(0)=\bar{x}_0,\ t\in[0,T], \end{cases} \tag{21}$$

where $\bar{f}(t)$ is an extension of $f(t)$ with $\bar{f}(t) = \begin{bmatrix} f(t)\\ 0_{n_x}\end{bmatrix}$, $\bar{B}_{e(t)}(t) = \begin{bmatrix} B_{e(t)}(t) & 0_{n_x\times n_x}\end{bmatrix}$, $\bar{D}_{e(t)}(t) = \begin{bmatrix} D'_{e(t)}(t)\\ 0_{n_x\times n_x}\end{bmatrix}'$, $\bar{H}_{e(t)}(t) = \begin{bmatrix} H_{e(t)}(t)\\ \delta I_{n_x\times n_x}\end{bmatrix}$, $\bar{G}_{e(t)}(t) = \begin{bmatrix} G_{e(t)}(t) & 0\\ 0 & \delta I_{n_x\times n_x}\end{bmatrix}$, and $\delta>\beta>0$. If (i) holds, then $\|\mathcal{L}^{[0,T]}_{\bar{f},\bar{y}}\|_- > \beta$. By (ii), it leads to

$$\begin{cases} \mathbb{L}^{\delta}_i(t,P) - \dot{P}_{i,T}(t) - \mathbb{K}^{\delta}_i(t,P)'\mathbb{H}^{\beta,\delta}_i(t,P)^{-1} \\ \mathbb{K}^{\delta}_i(t,P)=0, \\ \mathbb{H}^{\beta,\delta}_i(t,P)>0, \\ P_{i,T}(T)=0, \end{cases} \tag{22}$$

where $i\in S$, $t\in[0,T]$ and

$$\mathbb{L}^{\delta}_i(t,P) = -A_i(t)'P_{i,T}(t) - P_{i,T}(t)A_i(t) - C_i(t)'P_{i,T}(t) C_i(t) + \bar{H}_i(t)'\bar{H}_i(t) - \sum_{j\in S}\lambda_{ij}P_{j,T}(t),$$

$$\mathbb{K}^{\delta}_i(t,P) = -P_{i,T}(t)\bar{B}_i(t) - C_i(t)'P_{i,T}(t)\bar{D}_i(t) + \bar{H}_i(t)'\bar{G}_i(t),$$

$$\mathbb{H}^{\beta,\delta}_i(t,P) = -\bar{D}_i(t)'P_{i,T}(t)\bar{D}_i(t) + \bar{G}_i(t)'\bar{G}_i(t) - \beta^2 I.$$

By a series of calculations, the above inequality is equivalent to that

$$\mathbb{L}_i(t,P) - \dot{P}_{i,T}(t) + \delta^{-2}I_{n_x} - \begin{bmatrix}\mathbb{K}_i(t,P) & \delta^2 I_{n_x}\end{bmatrix} \begin{bmatrix}\mathbb{H}^{-1}_i(t,P) & 0\\ 0 & (-\beta^2+\delta^2)^{-1}I_{n_x}\end{bmatrix}\begin{bmatrix}\mathbb{K}_i(t,P)\\ \delta^2 I_{n_x}\end{bmatrix} = 0.$$

Because $\delta>\beta$, we obtain

$$\mathbb{L}_i(t,P) - \dot{P}_{i,T}(t) + \delta^{-2}I_{n_x} - \begin{bmatrix}\mathbb{K}_i(t,P) & \delta^2 I_{n_x}\end{bmatrix} \begin{bmatrix}\mathbb{H}^{-1}_i(t,P) & 0\\ 0 & \delta^{-2}I_{n_x}\end{bmatrix}\begin{bmatrix}\mathbb{K}_i(t,P)\\ \delta^2 I_{n_x}\end{bmatrix} > 0.$$

Therefore,

$$\mathbb{L}_i(t,P) - \dot{P}_{i,T}(t) - \mathbb{K}_i(t,P)'\mathbb{H}^{\beta}_i(t,P)^{-1}\cdot\mathbb{K}_i(t,P) > 0. \tag{23}$$

By Schur's complement, (23) leads to (iii). The proof is completed. □

*Theorem 2:* For $\beta>0$, if $\|\mathcal{L}^{[0,T]}_{f,y}\|_- > \beta$ associated with system (1), then there exists a real symmetric matrix sequence $\{P_{i,T}(t), t\in[0,T]\}_{i\in S}$ such that

$$\begin{cases} \mathbb{L}_i(t,P) - \dot{P}_{i,T}(t) - \mathbb{K}_i(t,P)'\mathbb{H}^{\beta}_i(t,P)^{-1}\mathbb{K}_i(t,P) < 0, \\ \mathbb{H}^{\beta}_i(t,P) > 0, \\ P_{i,T}(T)=0. \end{cases}$$

*Proof:* The proof is similar to that of Theorem 1 except that the parameter $\delta$ of system (21) is modified as $\beta>\delta>0$. □

If $V_i(t,\eta)$ in Theorem 1 is taken as the time-invariant function $V_i(\eta)=\eta'P_i\eta$ with $i\in S$, Theorem 1 yields the following corollary:

*Corollary 2:* For any given $\beta>0$, if

$$\begin{bmatrix}\bar{\mathbb{L}}_i(P) & \bar{\mathbb{K}}_i(P)\\ \bar{\mathbb{K}}_i(P)' & \bar{\mathbb{H}}^{\beta}_i(P)\end{bmatrix} > 0 \tag{24}$$

has a real symmetric matrix solution sequence $\{P_i\}_{i\in S}$, then $\|\mathcal{L}^{[0,T]}_{f,y}\|_- > \beta$ associated with system (1), where

$$\bar{\mathbb{L}}_i(P) = -A_i(t)'P_i - P_iA_i(t) - C_i(t)'P_iC_i(t) + H_i(t)'H_i(t) - \sum_{j\in S}\lambda_{ij}P_j,$$

$$\bar{\mathbb{K}}_i(P) = -P_iB_i(t) - C_i(t)'P_iD_i(t) + H_i(t)'G_i(t),$$

$$\bar{\mathbb{H}}^{\beta}_i(P) = -D_i(t)'P_iD_i(t) + G_i(t)'G_i(t) - \beta^2 I.$$

*Remark 1:* In finite horizon case, $\|\mathcal{L}^{[0,T]}_{f,y}\|_-$ represents the intensity of the process $y(t)$ generated by $f(t)$ in system (1) under zero initial condition. $H_-$ index of linear time-varying stochastic systems with Markovian jump and multiplicative noise has an important application to the fault detection of system (1). Reference [11] studied the $H_-$ index for linear discrete-time stochastic systems. In [17] and [18], the optimal $H_-/H_\infty$ fault detection problem for linear discrete time-varying systems was considered. In recent years, there is an increasing interest in robust fault detection, isolation and estimation for uncertain systems; see [22]–[24], [35]–[37]. However, the $H_-$ index and FDF of uncertain stochastic Markov jump systems seem not to be considered.

Assume that $A_{e(t)}(t)$, $B_{e(t)}(t)$, $C_{e(t)}(t)$, $D_{e(t)}(t)$, $H_{e(t)}(t)$, and $G_{e(t)}(t)$ are real matrices of suitable dimensions containing the parameter uncertainty affected by Markov jump process

$e(t)$. To describe the uncertainty of system (1), we give the following assumption.

*Assumption 1:* For any $i \in S$, $A_i(t), B_i(t), C_i(t), D_i(t)$, $H_i(t)$, and $G_i(t)$ are norm bounded, i.e.,

$$\begin{aligned} A_i(t) &= A_i + \Delta A_i(t), B_i(t) = B_i + \Delta B_i(t), \\ C_i(t) &= C_i + \Delta C_i(t), D_i(t) = D_i + \Delta D_i(t), \\ H_i(t) &= H_i + \Delta H_i(t), G_i(t) = G_i + \Delta G_i(t), \end{aligned}$$

where $A_i$, $B_i$, $C_i$, $D_i$, $H_i$, and $G_i$ are known constant real matrices. $\Delta A_i(t)$, $\Delta B_i(t)$, $\Delta C_i(t)$, $\Delta D_i(t)$, $\Delta H_i(t)$, and $\Delta G_i(t)$ are uncertain matrices, which satisfy

$$\begin{aligned} &diag\{\Delta A_i(t), \Delta B_i(t), \Delta C_i(t), \Delta D_i(t), \Delta H_i(t), \Delta G_i(t)\} \\ &= diag\{M_{1,i}F(t), M_{2,i}F(t), M_{3,i}F(t), M_{4,i}F(t), M_{5,i}F(t), \\ &\qquad M_{6,i}F(t)\}diag\{N_{1,i}, N_{2,i}, N_{3,i}, N_{4,i}, N_{5,i}, N_{6,i}\}, \end{aligned}$$

where $M_{1,i}, \cdots, M_{6,i}, N_{1,i}, \cdots, N_{6,i}$ are known constant real matrices and $F(t)$ is a time-varying uncertain matrix satisfying

$$F(t)'F(t) \le I, \quad \forall t \in [0, T].$$

Based on Lemma 2, a sufficient condition in terms of matrix inequalities for $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ associated with system (1) under Assumption 1 is given below.

*Theorem 3:* Consider system (1). For the given $\beta > 0$ and any $i \in S$, setting

$$\begin{aligned} \lambda_1 &= \min_{i\in S}\{\lambda(M_{5,i}'M_{5,i})\}, \\ \lambda_2 &= \min_{i\in S}\{\lambda(M_{5,i}'M_{6,i})\}, \\ \lambda_3 &= \min_{i\in S}\{\lambda(M_{6,i}'M_{6,i})\}. \end{aligned} \tag{25}$$

If there exist some positive time-varying scalars $\varepsilon_{1,i}(t) > 0$, $\varepsilon_{2,i}(t) > 0$, $\varepsilon_{3,i}(t) > 0$, $\varepsilon_{4,i}(t) > 0$, and a positive definite matrix sequence $\{P_{i,T}(t), t \in [0, T]\}_{i\in S}$ solving the following matrix inequalities:

$$\Gamma_i(t) = \begin{bmatrix} \Gamma_{1,i}(t) & \bar{\mathbb{M}}_i(t) & \Gamma_{2,i} \\ * & \Gamma_{3,i}(t) & 0 \\ * & * & \Gamma_{4,i}(t) \end{bmatrix} > 0, \quad i \in S, \tag{26}$$

where

$$\Gamma_{1,i}(t) = \begin{bmatrix} \Gamma_{11,i} & \Gamma_{12,i} & C_i'P_{i,T}(t) \\ * & \Gamma_{22,i} & D_i'P_{i,T}(t) \\ * & * & P_{i,T}(t) \end{bmatrix},$$

$$\Gamma_{2,i} = \begin{bmatrix} 0 & N_{5,i}' & 0 \\ N_{6,i}' & 0 & N_{6,i}' \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \Gamma_{3,i}(t) &= diag\{\varepsilon_{1,i}(t)I, \varepsilon_{1,i}(t)I, \varepsilon_{1,i}(t)I, \varepsilon_{1,i}(t)I, \\ &\qquad \varepsilon_{1,i}(t)I, \varepsilon_{1,i}(t)I\}, \\ \Gamma_{4,i}(t) &= diag\{\varepsilon_{2,i}(t)I, \varepsilon_{3,i}(t)I, \varepsilon_{4,i}(t)I\}, \\ \Gamma_{11,i}(t) &= -A_i'P_{i,T}(t) - P_{i,T}(t)A_i - \sum_{j\in S}\lambda_{ij}P_j(t) \\ &\quad + H_i'H_i - \varepsilon_{1,i}(t)\left(N_{1,i}'N_{1,i} + N_{3,i}'N_{3,i} + N_{5,i}'N_{5,i}\right) \\ &\quad - \left(\frac{\lambda_1^2}{4}\varepsilon_{3,i}(t) + \lambda_2^2\varepsilon_{2,i}(t)\right)N_{5,i}'N_{5,i} - \dot{P}_{i,T}(t), \\ \Gamma_{12,i}(t) &= -P_{i,T}(t)B_i + H_i'G_i, \\ \Gamma_{22,i}(t) &= -\beta^2 I + G_i'G_i - \varepsilon_{1,i}(t)(N_{2,i}'N_{2,i} + N_{4,i}'N_{4,i} \\ &\quad + N_{6,i}'N_{6,i}) - \frac{\lambda_3^2}{4}\varepsilon_{4,i}(t)N_{6,i}'N_{6,i}, \end{aligned}$$

and $\bar{\mathbb{M}}_i(t)$ see the top of next page.

Then $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$.

*Proof:* From Theorem 1, we know that $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ if $\mathbb{M}_i(t) > 0$ has a symmetric matrix solution sequence $\{P_{i,T}(t), t \in [0, T]\}_{i\in S}$ with $P_{i,T}(T) = 0$ for any $i \in S$. If $P_{i,T}(t) > 0$, by Schur's complement, $\mathbb{M}_i(t) > 0$ yields that $\Pi(t) > 0$, where

$$\Pi(t) = \begin{bmatrix} \Pi_{11}(t) & -P_{i,T}(t)B_i(t)+H_i(t)'G_i(t) & C_i(t)'P_{i,T}(t) \\ -B_i(t)'P_{i,T}(t) + G_i(t)'H_i(t) & -\beta^2 I + G_i(t)'G_i(t) & D_i(t)'P_{i,T}(t) \\ P_{i,T}(t)C_i(t) & P_{i,T}(t)D_i(t) & P_{i,T}(t) \end{bmatrix}$$

and

$$\Pi_{11}(t) = -\dot{P}_{i,T}(t) - A_i(t)'P_{i,T}(t) - P_{i,T}(t)A_i(t) - \sum_{j\in S}\lambda_{ij}(t)P_j(t) + H_i(t)'H_i(t).$$

According to whether there are uncertain parameters, $\Pi(t)$ can be divided into two parts:

$$\Pi(t) = \Pi_1(t) + \Pi_2(t),$$

where

$$\Pi_1(t) = \begin{bmatrix} \psi_1(t) & \begin{matrix}-P_{i,T}(t)B_i \\ +H_i'G_i\end{matrix} & C_i'P_{i,T}(t) \\ -B_i'P_{i,T}(t) + G_i'H_i & -\beta^2 I + G_i'G_i & D_i'P_{i,T}(t) \\ P_{i,T}(t)C_i & P_{i,T}(t)D_i & P_{i,T}(t) \end{bmatrix},$$

$$\Pi_2(t) = \begin{bmatrix} \psi_2 + \Delta H_i(t)'\Delta H_i(t) & \psi_3 + \Delta H_i(t)'\Delta G_i(t) & \Delta C_i(t)'P_{i,T}(t) \\ \psi_3' + \Delta G_i(t)'\Delta H_i(t) & \psi_4 + \Delta G_i(t)'\Delta G_i(t) & \Delta D_i(t)'P_{i,T}(t) \\ P_{i,T}(t)\Delta C_i(t) & P_{i,T}(t)\Delta D_i(t) & 0 \end{bmatrix},$$

$$\begin{aligned} \psi_1(t) &= -\dot{P}_{i,T}(t) - A_i'P_{i,T}(t) - P_{i,T}(t)A_i \\ &\quad - \sum_{j\in S}\lambda_{ij}P_j(t) + H_i'H_i, \\ \psi_2(t) &= -\Delta A_i(t)'P_{i,T}(t) - P_{i,T}(t)\Delta A_i(t) \\ &\quad + \Delta H_i(t)'H_i + H_i'\Delta H_i(t), \\ \psi_3(t) &= -P_{i,T}(t)\Delta B_i(t) + \Delta H_i(t)'G_i + H_i'\Delta G_i(t) \end{aligned}$$

and

$$\psi_4(t) = \Delta G_i(t)'G_i + G_i'\Delta G_i(t).$$

$$\bar{\mathbb{M}}_i(t) = \begin{bmatrix} -P_i(t)M_{1,i} & -P_i(t)M_{2,i} & 0 & 0 & H_i'M_{5,i} & H_i'M_{6,i} \\ 0 & 0 & 0 & 0 & G_i'M_{5,i} & G_i'M_{6,i} \\ 0 & 0 & P_{i,T}(t)M_{3,i} & P_{i,T}(t)M_{4,i} & 0 & 0 \end{bmatrix}.$$

According to (25), we have

$$\begin{aligned} & diag\{\Delta G_i(t)'\Delta G_i(t), \Delta H_i(t)'\Delta G_i(t), \Delta H_i(t)'\Delta H_i(t)\} \\ & \geq diag\{\lambda_3 N_{6,i}'F(t)'F(t)N_{6,i}, \lambda_2 N_{5,i}'F(t)'F(t)N_{6,i}, \\ & \lambda_1 N_{5,i}'F(t)'F(t)N_{5,i}\}. \end{aligned}$$

Calculate $\Pi_2$ together with considering the above inequality, we have

$$\begin{aligned} \Pi_2 = & \begin{bmatrix} -P_{i,T}(t)\Delta A_i(t) + H_i(t)'\Delta H_i(t) & -P_{i,T}(t)\Delta B_i(t) + H_i'\Delta G_i(t) & 0 \\ G_i'\Delta H_i(t) & G_i'\Delta G_i(t) & 0 \\ P_{i,T}(t)\Delta C_i(t) & P_{i,T}(t)\Delta D_i(t) & 0 \end{bmatrix} \\ & + \begin{bmatrix} -P_{i,T}(t)\Delta A_i(t) + H_i(t)'\Delta H_i(t) & -P_{i,T}(t)\Delta B_i(t) + H_i'\Delta G_i(t) & 0 \\ G_i'\Delta H_i(t) & G_i'\Delta G_i(t) & 0 \\ P_{i,T}(t)\Delta C_i(t) & P_{i,T}(t)\Delta D_i(t) & 0 \end{bmatrix}' \\ & + \begin{bmatrix} \Delta H_i(t)'\Delta H_i(t) & \Delta H_i(t)'\Delta G_i(t) & 0 \\ \Delta G_i(t)'\Delta H_i(t) & \Delta G_i(t)'\Delta G_i(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & \geq \bar{\mathbb{M}}_i(t) diag\{F(t), F(t), F(t), F(t), F(t), F(t)\} \\ & \bar{\mathbb{N}}_i(t) + \bar{\mathbb{N}}_i(t)' diag\{F(t), F(t), F(t), F(t), F(t), F(t)\}' \\ & \bar{\mathbb{M}}_i(t)' + \lambda_2 \begin{bmatrix} N_{5,i} & 0 & 0 \end{bmatrix}' F(t)'F(t) \begin{bmatrix} 0 & N_{6,i} & 0 \end{bmatrix} + \lambda_2 \\ & \begin{bmatrix} 0 & N_{6,i} & 0 \end{bmatrix}' F(t)'F(t) \begin{bmatrix} N_{5,i} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda_1 N_{5,i} & 0 & 0 \\ 0 & \lambda_3 N_{6,i} & 0 \end{bmatrix}' \\ & \begin{bmatrix} F(t)'F(t) & 0 \\ 0 & F(t)'F(t) \end{bmatrix} \begin{bmatrix} N_{5,i} & 0 & 0 \\ 0 & N_{6,i} & 0 \end{bmatrix}, \end{aligned}$$

where $\bar{\mathbb{M}}_i(t)$ is defined in (26) and

$$\bar{\mathbb{N}}_i(t) = \begin{bmatrix} N_{1,i}' & 0 & N_{3,i}' & 0 & N_{5,i}' & 0 \\ 0 & N_{2,i}' & 0 & N_{4,i}' & 0 & N_{6,i}' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}'.$$

Note that $(F(t)'F(t))'(F(t)'F(t)) \leq I$. Consider $\Pi(t) > 0$, by Lemma 2, its sufficient condition is that there are real time-varying scalars $\varepsilon_{1,i}(t) > 0$, $\varepsilon_{2,i}(t) > 0$, $\varepsilon_{3,i}(t) > 0$, and $\varepsilon_{4,i}(t) > 0$, such that

$$\begin{aligned} & \Pi_1(t) - \varepsilon_{1,i}(t)^{-1}\bar{\mathbb{M}}_i(t)\bar{\mathbb{M}}_i(t)' - \varepsilon_{1,i}(t)\bar{\mathbb{N}}_i(t)'\bar{\mathbb{N}}_i(t) \\ & - \left(\lambda_2^2\varepsilon_{2,i}(t) + \frac{\lambda_1^2}{4}\varepsilon_{3,i}(t) + \varepsilon_{3,i}(t)^{-1}\right) \begin{bmatrix} N_{5,i} & 0 & 0 \end{bmatrix}' \\ & \begin{bmatrix} N_{5,i} & 0 & 0 \end{bmatrix} - \left(\varepsilon_{2,i}(t)^{-1} + \frac{\lambda_3^2}{4}\varepsilon_{4,i}(t) + \varepsilon_{4,i}(t)^{-1}\right) \\ & \begin{bmatrix} 0 & N_{6,i} & 0 \end{bmatrix}' \begin{bmatrix} 0 & N_{6,i} & 0 \end{bmatrix} > 0. \end{aligned}$$

Then, we can get Theorem 3 via Schur's complement. □

*Remark 2:* In Theorem 3, if the condition is slightly strengthened as $M_{5,i} = M_{6,i}$, then $\lambda_1 = \lambda_2 = \lambda_3$. Furthermore, (26) turns into a necessary and sufficient condition.

*Remark 3:* Based on the analysis of Theorem 3, it is not difficult to find that if we choose a time-invariant Lyapunov function as in Corollary 2. By using a similar method to that of Theorem 3, we can get a time-invariant version of Theorem 3.

For the following state-measurement system

$$\begin{cases} dx(t) = \left(A_{e(t)}(t)x(t) + B_{e(t)}(t)f(t)\right) dt \\ \qquad + \left(C_{e(t)}(t)x(t) + D_{e(t)}(t)f(t)\right) dw(t), \\ y(t) = \begin{bmatrix} H_{e(t)}(t)x(t) \\ G_{e(t)}(t)f(t) \end{bmatrix}, \\ x(0) = x_0, \end{cases} \tag{27}$$

using the method as in Theorem 3, we can get the following sufficient condition for $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$ in the form of matrix inequalities .

*Corollary 3:* Set

$$\lambda_1 = \min_{i\in S}\{\lambda(M_{5,i}'M_{5,i})\}, \ \lambda_3 = \min_{i\in S}\{\lambda(M_{6,i}'M_{6,i})\}. \tag{28}$$

For $\beta > 0$ and system (27), if there exist some positive time-varying scalars $\varepsilon_{1,i}(t) > 0$, $\varepsilon_{2,i}(t) > 0$, $\varepsilon_{3,i}(t) > 0$, $\varepsilon_{4,i}(t) > 0$, and a positive definite matrix sequence $\{P_{i,T}(t), t \in [0, T]\}_{i\in S}$ such that the following matrix inequalities

$$\Gamma_i(t) = \begin{bmatrix} \Gamma_{1,i}(t) & \bar{\mathbb{M}}_i(t) & \Gamma_{2,i} \\ * & \Gamma_{3,i}(t) & 0 \\ * & * & \Gamma_{4,i}(t) \end{bmatrix} > 0, \quad i \in S \tag{29}$$

hold, then $\|\mathcal{L}_{f,y}^{[0,T]}\|_- > \beta$, where

$$\Gamma_{1,i}(t) = \begin{bmatrix} \Gamma_{11,i} & \Gamma_{12,i} & C_i'P_{i,T}(t) \\ * & \Gamma_{22,i} & D_i'P_{i,T}(t) \\ * & * & P_{i,T}(t) \end{bmatrix},$$

$$\Gamma_{2,i} = \begin{bmatrix} 0 & N_{5,i}' & 0 \\ N_{6,i}' & 0 & N_{6,i}' \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} \Gamma_{3,i}(t) &= diag\{\varepsilon_{1,i}(t), \varepsilon_{1,i}(t), \varepsilon_{1,i}(t), \varepsilon_{1,i}(t), \varepsilon_{1,i}(t), \\ & \qquad \varepsilon_{1,i}(t)\} \otimes I, \\ \Gamma_{4,i}(t) &= diag\{\varepsilon_{2,i}(t), \varepsilon_{3,i}(t), \varepsilon_{4,i}(t)\} \otimes I, \\ \Gamma_{11,i}(t) &= -A_i'P_{i,T}(t) - P_{i,T}(t)A_i - \sum_{j\in S}\lambda_{ij}P_j(t) \\ & \quad + H_i'H_i - \varepsilon_{1,i}(t)(N_{1,i}'N_{1,i} + N_{3,i}'N_{3,i} \\ & \quad + N_{5,i}'N_{5,i}) - \left(\frac{\lambda_1^2}{4}\varepsilon_{3,i}(t) + \lambda_2^2\varepsilon_{2,i}(t)\right) \\ & \quad N_{5,i}'N_{5,i} - \dot{P}_{i,T}(t), \end{aligned}$$

$$\Gamma_{12,i}(t) = -P_{i,T}(t)B_i + H_i'G_i,$$
$$\Gamma_{22,i}(t) = -\beta^2 I + G_i'G_i - \varepsilon_{1,i}(t)(N_{2,i}'N_{2,i} + N_{4,i}'N_{4,i} + N_{6,i}'N_{6,i}) - \frac{\lambda_3^2}{4}\varepsilon_{4,i}(t)N_{6,i}'N_{6,i}$$

and

$$\bar{\mathbb{M}}_i(t) = \begin{bmatrix} -P_i(t)M_{1,i} & -P_i(t)M_{2,i} & 0 & 0 & H_i'M_{5,i} & H_i'M_{6,i} & \\ 0 & 0 & 0 & 0 & G_i'M_{5,i} & G_i'M_{6,i} & \\ 0 & 0 & P_{i,T}(t)M_{3,i} & P_{i,T}(t)M_{4,i} & 0 & 0 & \end{bmatrix}.$$

## IV. $H_-$ FDF

The above section presents some necessary and sufficient conditions of the $H_-$ index problem in finite time horizon for linear time-varying Markov jump systems. It is shown that the $H_-$ index greater than a prescribed value is equivalent to the feasibility of a set of finite horizon backward time-varying GDREs or certain matrix inequalities. Moreover, under Assumption 1, a sufficient condition is given via time-invariant LMIs as said in Remark 3, which is easily tested. In this section, we will study the $H_-$ FDF of system (1) and achieve a practical design method.

In order to detect the fault of system (1) and ensure the system to run normally, a FDF is considered as

$$\begin{cases} d\hat{x}(t) = \hat{A}_{e(t)}\hat{x}(t)dt + \hat{B}_{e(t)}y(t)dt, \\ \hat{y}(t) = \hat{C}_{e(t)}\hat{x}(t), \\ r(t) = \hat{U}_{e(t)}(y(t) - \hat{y}(t)), \\ \hat{x}(0) = 0, \end{cases} \tag{30}$$

where $\hat{x}(t)$ is the estimated values of $x(t)$. $r(t)$ is the residual signal, and $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{U}_i$ are the filter parameters to be designed. Because all parameters are uncertain, which leads to a time-varying stochastic system (1). In this case, it is advisable to design a definite filter to resist the worst fluctuation of the parameter. Therefore, the filter (30) is a robust filter.

Set $\eta(t) = \begin{bmatrix} x(t)' & \hat{x}(t)' \end{bmatrix}'$, then we get the following augmented system:

$$\begin{cases} \eta(t) = \left(\tilde{A}_{e(t)}(t)\eta(t) + \tilde{B}_{e(t)}(t)f(t)\right)dt \\ \qquad + \left(\tilde{C}_{e(t)}(t)\eta(t) + \tilde{D}_{e(t)}(t)f(t)\right)dw(t), \\ r(t) = \tilde{H}_{e(t)}(t)\eta(t) + \tilde{G}_{e(t)}(t)f(t), \\ \eta(0) = \eta_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \end{cases} \tag{31}$$

where

$$\tilde{A}_{e(t)}(t) = \begin{bmatrix} A_{e(t)}(t) & 0 \\ \hat{B}_{e(t)}H_{e(t)}(t) & \hat{A}_{e(t)} \end{bmatrix},$$
$$\tilde{B}_{e(t)}(t) = \begin{bmatrix} B_{e(t)}(t) \\ \hat{B}_{e(t)}G_{e(t)}(t) \end{bmatrix},$$
$$\tilde{C}_{e(t)}(t) = \begin{bmatrix} C_{e(t)}(t) & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{D}_{e(t)}(t) = \begin{bmatrix} D_{e(t)}(t) \\ 0 \end{bmatrix},$$
$$\tilde{H}_{e(t)}(t) = \begin{bmatrix} \hat{U}_{e(t)}H_{e(t)}(t) & -\hat{U}_{e(t)}\hat{C}_{e(t)} \end{bmatrix},$$
$$\tilde{G}_{e(t)}(t) = \hat{U}_{e(t)}G_{e(t)}(t).$$

*Definition 2:* For stochastic system (31), its finite horizon sensitive operator $\mathcal{L}_{f,r}^{[0,T]}$, and the corresponding stochastic $H_-$ index $H_-^{[0,T]}$ are defined respectively as

$$\mathcal{L}_{f,r}^{[0,T]} : f \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathcal{R}^{n_f}) \mapsto r \in \mathcal{L}_{\mathcal{F}}^2([0,T], \mathcal{R}^{n_r})$$

and

$$\|\mathcal{L}_{f,r}^{[0,T]}\|_- = \inf_{x_0=0, f(t)\in\mathcal{L}_{\mathcal{F}}^2([0,T],\mathcal{R}^{n_f}), f(t)\not\equiv 0} \frac{\|r(t)\|_{\mathcal{L}^2_{[0,T]}}}{\|f(t)\|_{\mathcal{L}^2_{[0,T]}}}.$$

*Definition 3:* For any given $\beta > 0$, the filter (30) is called the finite horizon $H_-$ FDF for system (1), if for system (31), we have

$$\|\mathcal{L}_{f,r}^{[0,T]}\|_- > \beta.$$

After obtaining the gain matrices of the filter, we are in a position to discuss the residual estimation. For the purpose of evaluating the residual signal, one needs to adopt a threshold $J_{th} > 0$, and the $J_{th}$ conforms to the following decision logic:

$$\begin{cases} J_r(t) > J_{th} \Rightarrow \text{fault} \Rightarrow \text{alarm}, \\ J_r(t) \le J_{th} \Rightarrow \text{fault-free}, \end{cases} \tag{32}$$

where the residual evaluation function $J_r(t)$ is

$$J_r(t) = \left(\int_0^t r'(s)r(s)ds\right)^{\frac{1}{2}}, \quad J_r(0) = 0, \tag{33}$$

and the threshold is determined as

$$J_{th} = \sup_{f(t)\equiv 0, t\in[T_1,T_2]} EJ_r(t), \tag{34}$$

where $[T_1, T_2]$ is the evaluation interval.

Summarizing the above discussion, we can immediately get the following theorem about the $H_-$ FDF for time-varying Markov jump stochastic system (1).

*Theorem 4:* For any given $\beta > 0$, the following conditions are equivalent:

(i) The filter (30) is a finite horizon $H_-$ FDF for system (1).

(ii) The following coupled backward GDREs

$$\begin{cases} \tilde{\mathbb{L}}_i(t,P) - \dot{P}_{i,T}(t) - \tilde{\mathbb{K}}_i(t,P)'\tilde{\mathbb{H}}_i^{\beta}(t,P)^{-1} \\ \tilde{\mathbb{K}}_i(t,P) = 0, \\ \tilde{\mathbb{H}}_i^{\beta}(t,P) > 0, \\ P_{i,T}(T) = 0, \ i \in S \end{cases} \tag{35}$$

have a unique real symmetric matrix solution sequence $\{P_{i,T}(t), t \in [0,T]\}_{i\in S}$, where

$$\begin{cases} \tilde{\mathbb{L}}_i(t,P) = -\tilde{A}_i(t)'P_{i,T}(t) - P_{i,T}(t)\tilde{A}_i(t) - \tilde{C}_i(t)' \\ \cdot P_{i,T}(t)\tilde{C}_i(t) + \tilde{H}_i(t)'\tilde{H}_i(t) - \sum_{j\in S}\lambda_{ij}P_{j,T}(t), \\ \tilde{\mathbb{K}}_i(t,P) = -P_{i,T}(t)\tilde{B}_i(t) - \tilde{C}_i(t)'P_{i,T}(t)\tilde{D}_i(t) \\ +\tilde{H}_i(t)'\tilde{G}_i(t), \\ \tilde{\mathbb{H}}_i^{\beta}(t,P) = -\tilde{D}_i(t)'P_{i,T}(t)\tilde{D}_i(t) - \beta^2 I + \tilde{G}_i(t)'\tilde{G}_i(t). \end{cases}$$

(iii) $\tilde{\mathbb{M}}_i(t) > 0$, $i \in S$, have a real symmetric matrix solution sequence $\{P_{i,T}(t), t \in [0,T]\}_{i\in S}$ with $P_{i,T}(T) = 0$, where

$$\tilde{\mathbb{M}}_i(t) = \begin{bmatrix} \tilde{\mathbb{L}}_i(t, P(t)) - \dot{P}_{i,T}(t) & \tilde{\mathbb{K}}_i(t, P(t)) \\ \tilde{\mathbb{K}}_i(t, P(t))' & \tilde{\mathbb{H}}_i^{\beta}(t, P(t)) \end{bmatrix}.$$

By Assumption 1, the parameter matrices of (31) can be rewritten as

$$\tilde{A}_{e(t)}(t) = \tilde{A}_{e(t)} + \Delta\tilde{A}_{e(t)}(t), \tilde{B}_{e(t)}(t) = \tilde{B}_{e(t)} + \Delta\tilde{B}_{e(t)}(t),$$
$$\tilde{C}_{e(t)}(t) = \tilde{C}_{e(t)} + \Delta\tilde{C}_{e(t)}(t), \tilde{D}_{e(t)}(t) = \tilde{D}_{e(t)} + \Delta\tilde{D}_{e(t)}(t),$$
$$\tilde{H}_{e(t)}(t) = \tilde{H}_{e(t)} + \Delta\tilde{H}_{e(t)}(t), \tilde{G}_{e(t)}(t) = \tilde{G}_{e(t)} + \Delta\tilde{G}_{e(t)}(t),$$

where

$$\tilde{A}_{e(t)} = \begin{bmatrix} A_{e(t)} & 0 \\ \hat{B}_{e(t)}H_{e(t)} & \hat{A}_{e(t)} \end{bmatrix}, \quad \tilde{B}_{e(t)} = \begin{bmatrix} B_{e(t)} \\ \hat{B}_{e(t)}G_{e(t)} \end{bmatrix},$$
$$\Delta\tilde{A}_{e(t)}(t) = \begin{bmatrix} \Delta A_{e(t)}(t) & 0 \\ \hat{B}_{e(t)}\Delta H_{e(t)}(t) & 0 \end{bmatrix}, \quad \tilde{C}_{e(t)} = \begin{bmatrix} C_{e(t)} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\Delta\tilde{B}_{e(t)}(t) = \begin{bmatrix} \Delta B_{e(t)}(t) \\ \hat{B}_{e(t)}\Delta G_{e(t)}(t) \end{bmatrix}, \quad \tilde{D}_{e(t)} = \begin{bmatrix} D_{e(t)} \\ 0 \end{bmatrix},$$
$$\Delta\tilde{C}_{e(t)}(t) = \begin{bmatrix} \Delta C_{e(t)}(t) & 0 \\ 0 & 0 \end{bmatrix}, \ \Delta\tilde{D}_{e(t)}(t) = \begin{bmatrix} \Delta D_{e(t)}(t) \\ 0 \end{bmatrix},$$
$$\tilde{H}_{e(t)} = \begin{bmatrix} \hat{U}_{e(t)}H_{e(t)} & -\hat{U}_{e(t)}\hat{C}_{e(t)} \end{bmatrix},$$
$$\Delta\tilde{H}_{e(t)}(t) = \begin{bmatrix} \hat{U}_{e(t)}\Delta H_{e(t)}(t) & 0 \end{bmatrix},$$

and

$$\tilde{G}_{e(t)} = \hat{U}_{e(t)}G_{e(t)}, \quad \Delta\tilde{G}_{e(t)}(t) = \hat{U}_{e(t)}\Delta G_{e(t)}(t).$$

The following theorem gives a practical design method for the filter (30) based on LMIs.

*Theorem 5:* For the given $\beta > 0$ and any $i \in S$, if there exist scalars $\alpha_i$, $i \in S$, such that the following LMIs

$$\Psi_i = \begin{bmatrix} \Psi_{1,i} & \Psi_{2,i} & \Psi_{3,i} \\ * & \Psi_{4,i} & 0 \\ * & * & \Psi_{5,i} \end{bmatrix} > 0, \quad i \in S \tag{36}$$

have solutions $P_{1,i} > 0$, $P_{2,i} > 0$, $\hat{P}_{A,i}$, $\hat{P}_{B,i}$, $\hat{C}_i$, $\varepsilon_{1,i} > 0$, $\varepsilon_{2,i} > 0$, $\varepsilon_{3,i} > 0$, and $\varepsilon_{4,i} > 0$, then (30) is a finite horizon $H_-$ FDF for system (1), where in (36),

$$\Psi_{1,i} = \begin{bmatrix} \Psi_{111,i} & \Psi_{112,i} & \Psi_{113,i} \\ * & \Psi_{122,i} & \Psi_{123,i} \\ * & * & \Psi_{133,i} \end{bmatrix},$$

$$\Psi_{111,i} = \begin{bmatrix} -P_{1,i}A_i - A_i'P_{1,i} - \sum_{j\in S}\lambda_{ij}P_{1,j}(t) + \alpha_i^2 H_i'H_i' & -\alpha_i^2 H_i'\hat{C}_i - H_i'\hat{P}_{B,i}' \\ * & -\sum_{j\in S}\lambda_{ij}P_{2,j}(t) - \hat{P}_{A,i} - \hat{P}_{A,i}' \end{bmatrix}$$
$$-\varepsilon_{1,i}\begin{bmatrix} N_{1,i}'N_{1,i} + N_{3,i}'N_{3,i} + 2N_{5,i}'N_{5,i} & 0 \\ 0 & 0 \end{bmatrix}$$
$$-\begin{bmatrix} \alpha_i^2(\frac{\lambda_1^2}{4}\varepsilon_{3,i} + \lambda_2^2\varepsilon_{2,i})N_{5,i}'N_{5,i} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Psi_{112,i} = \begin{bmatrix} -P_{1,i}B_i + \alpha_i^2 H_i'G_i \\ -\hat{P}_{B,i}G_i - \alpha_i^2\hat{C}_i'G_i \end{bmatrix},$$

$$\Psi_{122,i} = -\beta^2 I + \alpha_i^2 G_i'G_i - \alpha_i^2\varepsilon_{1,i}(2N_{2,i}'N_{2,i} + N_{4,i}'N_{4,i} + 2N_{6,i}'N_{6,i}) - \alpha_i^2\frac{\lambda_3^2}{4}\varepsilon_{4,i}N_{6,i}'N_{6,i},$$

$$\Psi_{113,i} = \begin{bmatrix} C_i'P_{1,i} & 0 \\ 0 & 0 \end{bmatrix}, \ \Psi_{123,i} = \begin{bmatrix} D_i'P_{1,i} & 0 \end{bmatrix},$$

$$\Psi_{133,i} = \begin{bmatrix} P_{1,i} & 0 \\ 0 & P_{2,i} \end{bmatrix},$$

$$\Psi_{2,i} = \begin{bmatrix} -P_{1,i}M_{1,i} & 0 & P_{1,i}M_{2,i} & 0 & 0 & 0 & 0 & \alpha_i^2 H_i'M_{5,i} & 0 & \alpha_i^2 H_i'M_{6,i} \\ -P_{2,i}M_{5,i} & 0 & \hat{P}_{B,i}M_{6,i} & 0 & 0 & 0 & 0 & -\alpha_i^2\hat{C}_i'M_{5,i} & 0 & -\alpha_i^2\hat{C}_i'M_{6,i} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_i^2 G_i'M_{5,i} & 0 & -\alpha_i^2 G_i'M_{6,i} \\ 0 & 0 & 0 & P_{1,i}M_{3,i} & 0 & P_{1,i}M_{4,i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{3,i} = \begin{bmatrix} 0 & N_{5,i}' & 0 \\ 0 & 0 & 0 \\ N_{6,i}' & 0 & N_{6,i}' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{5,i} = \begin{bmatrix} \varepsilon_{2,i}I & 0 & 0 \\ 0 & \varepsilon_{3,i}I & 0 \\ 0 & 0 & \varepsilon_{4,i}I \end{bmatrix},$$

and

$$\Psi_{4,i} = \varepsilon_{1,i}I.$$

Moreover,

$$\hat{A}_i = P_{2,i}^{-1}\hat{P}_{A,i},$$
$$\hat{B}_i = P_{2,i}^{-1}\hat{P}_{B,i},$$
$$\hat{U}_i = \alpha_i I.$$

*Proof:* Similar to the proof of Theorem 3, a sufficient condition for $\|\mathcal{L}_{f,r}^{[0,T]}\|_- > \beta$ associated with system (31) is

$$\tilde{\Pi} = \tilde{\Pi}_1 + \tilde{\Pi}_2(t) > 0$$

for some $P_i > 0$, where

$$\tilde{\Pi}_1 = \begin{bmatrix} -\tilde{A}_i'P_i - P_i\tilde{A}_i - \sum_{j\in S}\lambda_{ij}P_j(t) + \tilde{H}_i'\tilde{H}_i & -P_i\tilde{B}_i + \tilde{H}_i'\tilde{G}_i & \tilde{C}_i'P_i \\ -\tilde{B}_i'P_i + \tilde{G}_i'\tilde{H}_i & -\beta^2 I + \tilde{G}_i'\tilde{G}_i & \tilde{D}_i'P_i \\ P_i\tilde{C}_i & P_i\tilde{D}_i & P_i \end{bmatrix},$$

$$\tilde{\Pi}_2(t) = \begin{bmatrix} \psi_{11} + \Delta\tilde{H}_i(t)'\Delta\tilde{H}_i(t) & \psi_{12} + \Delta\tilde{H}_i(t)'\Delta\tilde{G}_i(t) & \Delta\tilde{C}_i(t)'P_i \\ \psi_{12}' + \Delta\tilde{G}_i(t)'\Delta\tilde{H}_i(t) & \psi_{22} + \Delta\tilde{G}_i(t)'\Delta\tilde{G}_i(t) & \Delta\tilde{D}_i(t)'P_i \\ P_i\Delta\tilde{C}_i(t) & P_i\Delta\tilde{D}_i(t) & 0 \end{bmatrix},$$

$$\psi_{11}(t) = -\Delta\tilde{A}_i(t)'P_i - P_i\Delta\tilde{A}_i(t) + \Delta\tilde{H}_i(t)'\tilde{H}_i + \tilde{H}_i'\Delta\tilde{H}_i(t),$$
$$\psi_{12}(t) = -P_i\Delta\tilde{B}_i(t) + \Delta\tilde{H}_i(t)'\tilde{G}_i + \tilde{H}_i'\Delta\tilde{G}_i(t)$$

and

$$\psi_{22}(t) = \Delta\tilde{G}_i(t)'\tilde{G}_i + \tilde{G}_i'\Delta\tilde{G}_i(t).$$

Assuming $\hat{U}_i = \alpha_i I$, by Corollary 1 and the condition (25), it follows immediately that there exist scalars $\varepsilon_{1,i} > 0$, $\varepsilon_{2,i} > 0$, $\varepsilon_{3,i} > 0$ and $\varepsilon_{4,i} > 0$ such that

$$\tilde{\Pi}_1 - \varepsilon_{1,i}^{-1}\tilde{\mathbb{M}}_i\tilde{\mathbb{M}}_i' - \varepsilon_{1,i}\bar{\mathbb{N}}_i'\bar{\mathbb{N}}_i - \alpha^2\left(\lambda_2^2\varepsilon_{2,i} + \frac{\lambda_1^2}{4}\varepsilon_{3,i} + \varepsilon_{3,i}^{-1}\right)$$
$$\begin{bmatrix} N_{5,i} & 0 & 0\end{bmatrix}'\begin{bmatrix} N_{5,i} & 0 & 0\end{bmatrix} - \alpha^2\left(\varepsilon_{2,i}^{-1} + \frac{\lambda_3^2}{4}\varepsilon_{4,i} + \varepsilon_{4,i}^{-1}\right)$$
$$\begin{bmatrix} 0 & N_{6,i} & 0\end{bmatrix}'\begin{bmatrix} 0 & N_{6,i} & 0\end{bmatrix} > 0,$$

where

$$\tilde{\mathbb{M}}_i = \begin{bmatrix} -P_i\tilde{M}_{1,i} & -P_i\tilde{M}_{2,i} & 0 & 0 & \tilde{H}_i'\tilde{M}_{5,i} & H_i'\tilde{M}_{6,i} \\ 0 & 0 & 0 & 0 & G_i'\tilde{M}_{5,i} & G_i'\tilde{M}_{6,i} \\ 0 & 0 & P_i\tilde{M}_{3,i} & P_i\tilde{M}_{4,i} & 0 & 0 \end{bmatrix},$$

$$\bar{\mathbb{N}}_i(t) = \begin{bmatrix} \tilde{N}_{1,i}' & 0 & \tilde{N}_{3,i}' & 0 & \tilde{N}_{5,i}' & 0 \\ 0 & \tilde{N}_{2,i}' & 0 & \tilde{N}_{4,i}' & 0 & \tilde{N}_{6,i}' \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}',$$

$$\tilde{M}_{1,i} = \begin{bmatrix} M_{1,i} & 0 \\ \hat{B}_i M_{5,i} & 0 \end{bmatrix}, \quad \tilde{M}_{2,i} = \begin{bmatrix} M_{2,i} \\ \hat{B}_i M_{6,i} \end{bmatrix},$$
$$\tilde{M}_{3,i} = \begin{bmatrix} M_{3,i} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{M}_{4,i} = \begin{bmatrix} M_{4,i} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\tilde{M}_{5,i} = \begin{bmatrix} \alpha M_{5,i} & 0 \end{bmatrix}, \quad \tilde{M}_{6,i} = \alpha M_{6,i},$$
$$\tilde{N}_{1,i} = \begin{bmatrix} N_{1,i} & 0 \\ N_{5,i} & 0 \end{bmatrix}, \quad \tilde{N}_{2,i} = \begin{bmatrix} N_{2,i} \\ N_{6,i} \end{bmatrix}$$

and

$$\tilde{N}_{3,i} = \begin{bmatrix} N_{3,i} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{N}_{4,i} = \begin{bmatrix} N_{4,i} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\tilde{N}_{5,i} = \begin{bmatrix} N_{5,i} & 0 \end{bmatrix},$$
$$\tilde{N}_{6,i} = N_{6,i}.$$

For convenience, we set $P_i$ as $P_i = \begin{bmatrix} P_{1,i} & 0 \\ 0 & P_{3,i} \end{bmatrix} > 0$. Then, by a trivial computation, it shows LMI (36) with $\hat{P}_{A,i} = P_{2,i}A_i$ and $\hat{P}_{B,i} = P_{2,i}B_i$.

*Remark 4:* Theorem 5 is only a sufficient condition about (30) is a finite horizon $H_-$ FDF for system (1). Generally speaking, it is not difficult to analyze the feasibility of the matrix inequalities (36) in theory. The difficulty in solving LMIs (36) lies in the choosing of scalars $\alpha_i$.

## V. NUMERICAL EXAMPLE

In this section, one numerical example is provided to illustrate the effectiveness of Theorem 5.

*Example 1:* In system (1), we set $T = 30$. Consider a two-mode nonlinear stochastic system (1) with the following parameters:

$$A_1 = \begin{bmatrix} -0.52 & 0.01 \\ 0.01 & -1.09 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.38 & -0.01 \\ 0.01 & -1.82 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.68 \\ 0.69 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.22 \\ 0.55 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.46 & 0.11 \\ 2.12 & -0.92 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1.32 & -0.41 \\ 0.1 & -1.35 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} -0.47 \\ -1.63 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.68 \\ 1.94 \end{bmatrix},$$
$$H_1 = \begin{bmatrix} 0.74 & 1.36 \\ -0.32 & 0.29 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -2.22 & 0.08 \\ -0.14 & 0.56 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} -0.16 \\ 1.17 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1.33 \\ 2.46 \end{bmatrix}, \quad M_{j,i} = \begin{bmatrix} -0.02 \\ -0.1 \end{bmatrix},$$
$$N_{1,i} = N_{3,i} = N_{5,i} = \begin{bmatrix} -0.5 \\ 0.2 \end{bmatrix},$$
$$N_{2,i} = N_{4,i} = N_{6,i} = -0.5, \quad \forall j \in \{1, 2, \cdots, 6\}, i \in \{1, 2\}.$$

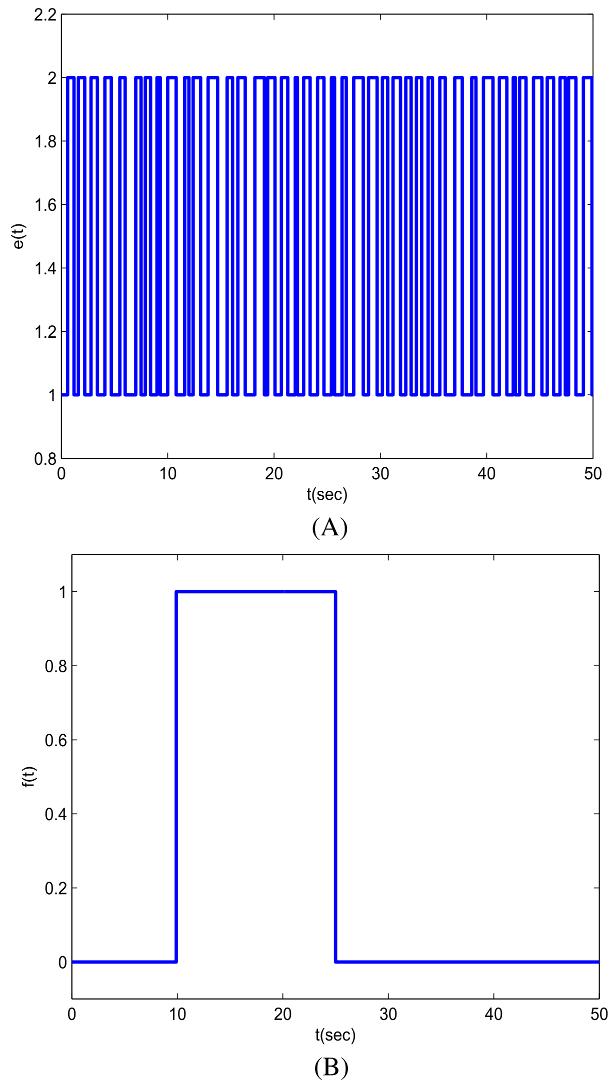
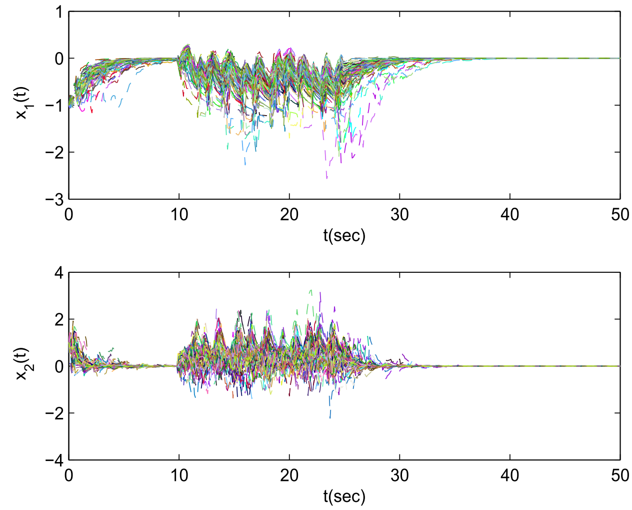
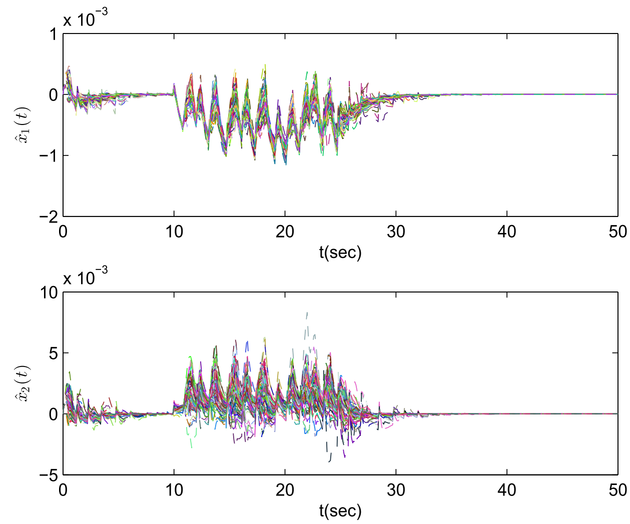



**FIGURE 1. Switching signal and fault signal. (A) Switching signal e(t). (B) Fault signal f(t).**

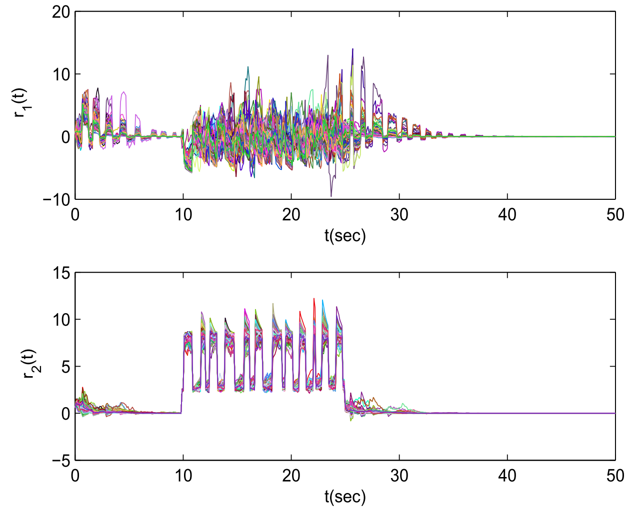



(A)

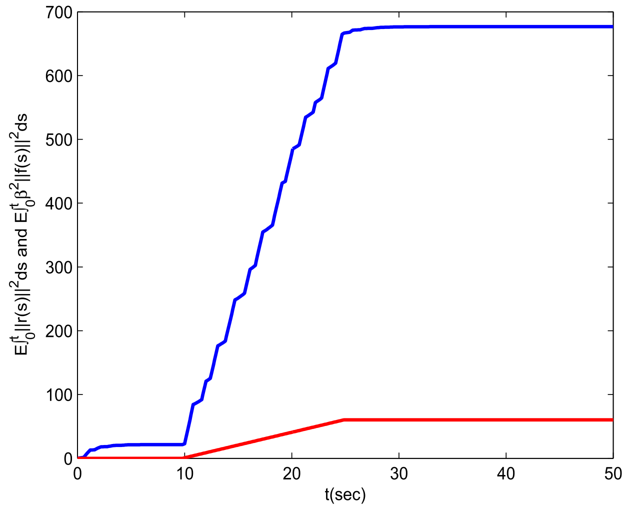



(B)

**FIGURE 2. State trajectories $x(t)$ and $\hat{x}(t)$. (A) State trajectory $x(t)$ of the system (1). (B) State trajectory $\hat{x}(t)$ of the system (30).**




(A)




(B)

**FIGURE 3. Residual signal $r(t)$ and $\|\mathcal{L}_{f,r}^{[0,30]}\|_- > \beta = 2$. (A) Residual signal $r(t)$. (B) $E\int_0^{30}\|r(s)\|ds$ and $E\int_0^{30}\beta^2\|f(s)\|ds$.**

The switching between the two modes is described by the following transition rate matrix:

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} -0.1 & 0.1 \\ 0.3 & -0.3 \end{bmatrix}.$$

The random switching signal $e(t)$ is shown in Figure 1-(A). If we choose the $H_-$ performance level $\beta = 2$ and set $\alpha_1 = 1$, $\alpha_2 = 2$, $T_1 = 25$ and $T_2 = 30$, then we obtain the solutions of LMIs (36) as

$$P_{11} = \begin{bmatrix} 7.11 & -0.44 \\ -0.44 & 0.05 \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} 6.79 & -4.65 \\ -4.65 & 5.24 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 3342.09 & -515 \\ -515 & 866.75 \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} 3380.93 & -320.5 \\ -320.5 & 1840.69 \end{bmatrix},$$

$$\hat{P}_{A,1} = \begin{bmatrix} -1943.22 & -240.1 \\ -240.3 & -3098.21 \end{bmatrix},$$

$$\hat{P}_{A,2} = \begin{bmatrix} -2922.87 & 28.04 \\ 28.04 & -2787.77 \end{bmatrix},$$

$$\hat{P}_{B,1} = \begin{bmatrix} -0.11 & 0.95 \\ 5.80 & 3.58 \end{bmatrix}, \quad \hat{P}_{B,2} = \begin{bmatrix} -0.54 & -1.62 \\ -2.37 & -0.53 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} -0.24 & -0.31 \\ 0.29 & -0.38 \end{bmatrix}, \quad \hat{C}_2 = \begin{bmatrix} 0.1 & 0.08 \\ -0.31 & -0.01 \end{bmatrix},$$

$$\varepsilon_{11} = 2.28, \quad \varepsilon_{12} = 20.16, \quad \varepsilon_{21} = 1809.44,$$

$$\varepsilon_{22} = 3201.7, \quad \varepsilon_{31} = 2574.22,$$

$$\varepsilon_{32} = 3349.55, \quad \varepsilon_{41} = 3416.86, \quad \varepsilon_{42} = 3443.84.$$

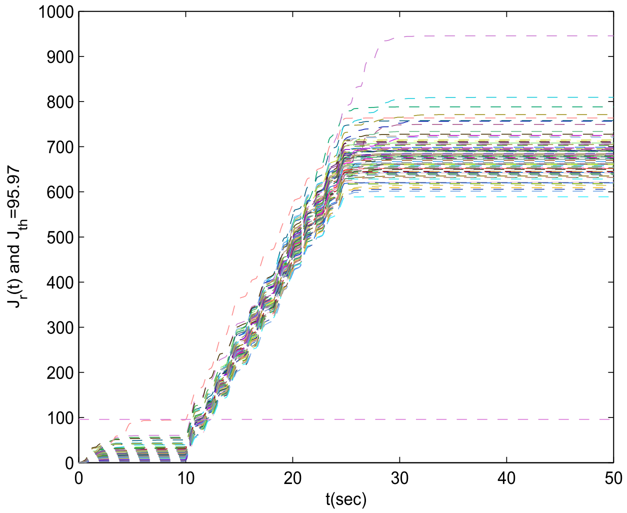



**FIGURE 4.** $J_r(t)$ and $J_{th}$.

Thus, we adopt the linear filters as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \begin{bmatrix} -0.69 & -0.69 \\ -0.69 & -3.98 \end{bmatrix} \hat{x}_1(t) + \begin{bmatrix} 0.001 & 0.001 \\ 0.007 & 0.005 \end{bmatrix} y_1(t), \\ \hat{y}_1(t) = \begin{bmatrix} -0.24 & -0.31 \\ 0.29 & -0.38 \end{bmatrix} \hat{x}_1(t), \\ r_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y_1(t) - \hat{y}_1(t)), \end{cases}$$

$$\begin{cases} \dot{\hat{x}}_2(t) = \begin{bmatrix} -0.88 & -0.14 \\ -0.14 & -1.5385 \end{bmatrix} \hat{x}_2(t) \\ \qquad + \begin{bmatrix} -0.0003 & -0.0005 \\ -0.0013 & -0.0004 \end{bmatrix} y_2(t), \\ \hat{y}_2(t) = \begin{bmatrix} 0.099 & 0.078 \\ -0.314 & -0.006 \end{bmatrix} \hat{x}_2(t), \\ r_2(t) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} (y_2(t) - \hat{y}_2(t)), \end{cases}$$

Assume $f(t) = \begin{cases} 1 & t \in [10, 25], \\ 0 & else \end{cases} \in \mathcal{L}^2_{\mathcal{F}}([0, 30], \mathcal{R})$ shown in Figure 1-(B). We use Matlab to simulate the state trajectories of systems (1) and (30) under $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in Figure 2. Meanwhile, we simulate the system state trajectories 1000 times to obtain the approximate value of $\|\mathcal{L}^{[0,30]}_{f,r}\|_-$. The residual signal $r(t)$ and $\|\mathcal{L}^{[0,30]}_{f,r}\|_- > \beta$ are described in Figure 3. After the Monte Carlo simulation, $J_{th} = 95.97$ with the evaluation window $T = 30$. To show the effectiveness of the designed filter, the residual evaluation function $J_r(t)$ and the fault signal $J_{th} = 95.97$ are depicted in Figure 4.

## VI. CONCLUSION

In this paper, the finite horizon $H_-$ index and $H_-$ FDF for linear time-varying Markov jump stochastic systems have been discussed, and some necessary and sufficient conditions for the $H_-$ index larger than a given $\beta > 0$ and for the existence of the desired FDF filter have also been given, respectively. Moreover, a convenient design method of the FDF is presented via LMIs. Lemma 2 is a time-variant modified version of Lemma 1, which has a direct application to the FDF design of linear uncertain Markov jump stochastic systems. We believe that both Corollary 1 and Lemma 2 will play important roles as Lemma 1 has done in robust control theory. How to generalize the results of this paper to the infinite horizon case is a valuable work, which merits further study in our future work.

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**TIANLIANG ZHANG** received the B.S. degree from the Shandong University of Science and Technology, in 2014, and the M.S. degree from the China University of Petroleum, Qingdao, China, in 2017. He is currently pursuing the Ph.D. degree with the South China University of Technology, Guangzhou, China. His research interests include robust control, fault diagnosis, mean-field systems, and stochastic systems.

**FEIQI DENG** (SM'18) received the Ph.D. degree in control theory and control engineering from the South China University of Technology, Guangzhou, in 1997, where he has been a Professor and the Director of the Systems Engineering Institute, since 1999. His main research interests include stability, stabilization, and the robust control theory of complex systems, including time-delay systems, nonlinear systems, and stochastic systems. He is currently a member of the Technical Committee on Control Theory (TCCT), Chinese Association of Automation. He is a member of the Editorial Boards of the *Journal of Control Theory and Applications*, the *Journal of Systems Engineering and Electronics*, and the *Journal of Systems Engineering*. He is also serving as the Chair for the IEEE SMC Guangzhou Chapter, the Director of the Sub-Committee on Stochastic Systems Control of TCCT, a Vice Editor-in-Chief of the *Journal of South China University of Technology*.

**WEIHAI ZHANG** (SM'16) received the M.S. and Ph.D. degrees from Zhejiang University, Hangzhou, China, in 1994 and 1998, respectively.

He was a Postdoctoral Researcher, from 2001 to 2003, and a Visiting Professor, from 2010 to 2011 and from 2015 to 2016, with National Tsing Hua University, Hsinchu, Taiwan. He is currently a Professor with the Shandong University of Science and Technology, Qingdao, China. He has authored or co-authored more than 110 peer-reviewed journal papers and one monograph *Stochastic $H_2/H_\infty$ Control: A Nash Game Approach* (Boca Raton, FL, USA: CRC Press, 2017). His research interests include stochastic control theory and its applications, linear and nonlinear stochastic optimal controls, mean-field systems, robust $H_\infty$ control, stochastic stability and stabilization, multiobjective optimization, and fuzzy adaptive control.

Dr. Zhang is a member of the Technical Committee on Control Theory of the Chinese Association of Automation. He received the Second Prize of the Ministry of Education of the People's Republic of China twice. He is a Taishan Scholar of Shandong Province of China and serves as an Associate Editor for the *Asian Journal of Control* and the *Journal of the Franklin Institute*.