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Outer-Independent Italian Domination in Graphs

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ABSTRACT An outer-independent Italian dominating function (OIIDF) on a graph G with vertex set $V(G)$ is defined as a function $f : V(G) \to \{0, 1, 2\}$, such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbors assigned 1 under f or one neighbor w with $f(w) = 2$, and the set $\{u \in V \mid f(u) = 0\}$ is independent. The weight of an OIIDF f is the value $w(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of an OIIDF on a graph G is called the outer-independent Italian domination number $\gamma_{oiI}(G)$ of G. In this paper, we initiate the study of the outer-independent Italian domination number and present the bounds on the outer-independent Italian domination number in terms of the order, diameter, and vertex cover number. In addition, we establish the lower and upper bounds on $\gamma_{oiI}(T)$ when T is a tree and characterize all extremal trees constructively. We also give the Nordhaus–Gaddum-type inequalities.

INDEX TERMS Outer-independent Italian domination, Italian domination, trees.

I. INTRODUCTION

In this paper, we consider finite, undirected and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$, where the *order* of G is $n(G) = |V|$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. A *leaf* is a vertex of degree one, and a *support vertex* is a vertex adjacent to a leaf. We denote the sets of all leaves and all support vertices of G by $L(G)$ and $S(G)$, respectively. Denote also by $S_1(T)$ the set of all support vertices of T that are adjacent to only one leaf and let $S_2(T) = S(G) - S_1(T)$. The *diameter* of a graph G, denoted by diam(G), is the greatest distance between two vertices of G. We write P_n for the *path* of order n, C_n for the *cycle* of length n, $K_{p,q}$ for the complete bipartite graph and $\overline{G}$ for the complement graph of G.

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A set $I \subseteq V(G)$ is *independent* if no two vertices in I are adjacent. The maximum cardinality of an independent set in G equals the *independence number* $\beta_0(G)$. A *vertex cover* of a graph G is a set of vertices that covers all the edges. The minimum cardinality of a vertex cover is denoted by $\alpha_0(G)$. The following result is given in [9].

Theorem 1: Let G be a graph. A subset I of $V(G)$ is independent if and only if $V(G) - I$ is a vertex cover of G. In particular, $\beta_0(G) = |V(G)| - \alpha_0(G)$.

The notion of *Italian domination* in graphs was introduced in [12], where it was called *Roman* $\{2\}$*-domination* and weak $\{2\}$-domination. The concept was studied further in [3] and [4]. An *Italian dominating function* (IDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbors assigned 1 under f or one neighbor w with $f(w) = 2$. The *weight* of an IDF f is the value $w(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* $\gamma_I(G)$ of G. For an IDF f on G, let $V_i^f = \{v \in V(G) : f(v) = i\}$ for i = 0, 1, 2. Since these three sets determine f

uniquely, we can equivalently write $f = (V_0^f, V_1^f, V_2^f)$. If H is a subgraph of G and f an IDF on G, then we denote the restriction of f on H by $f|_H$. The following lower bound on the Italian domination number established in [12].

Theorem 2: If G is a connected graph of order n and maximum degree Δ, then $\gamma_I(G) \geq 2n/(\Delta+2)$.

For Italian domination one can think of each vertex representing a location in the Roman Empire and each edge being a road between two locations. A location is said to be protected if one of the following holds: (a) at least one legion is stationed in it or (b) it is with no legion and has a neighboring location with two legions or at least two neighboring locations with one legion each. A location having no legion is thought of as being vulnerable. In addition, if such a location has one of its neighboring locations with no legion stationed in it, then it is considered to be even more vulnerable. The best protection for a vulnerable location is to be completely surrounded only by neighboring locations with legions. This leads us to seek an Italian dominating function $f = (V_0^f, V_1^f, V_2^f)$ for which V_0^f is independent, that is f is an OIIDF.

In this paper, we initiate the study of outer-independent Italian dominating functions $f = (V_0, V_1, V_2)$ for which V_0 is an independent set. The minimum weight of an OIIDF on a graph G is called the *outer independent Italian domination number* of G and it is denoted by $\gamma_{oiI}(G)$. Clearly $\gamma_I(G) \leq \gamma_{oiI}(G)$. An OIIDF with minimum weight in a graph G will be referred to as a *γ_{oiI}-function* on G. Since any outer-independent Italian dominating function is an Italian dominating function, we have

$$\gamma_{oiI}(G) \geq \gamma_I(G) \tag{1}$$

We establish various bounds on the outer-independent Italian domination number in terms of the order, diameter and vertex cover number. In particular, we give lower and upper bounds on $\gamma_{oiI}(T)$ when T is a tree, and we characterize all extreme trees constructively. Moreover, we provide Nordhaus-Gaddum bounds for $\gamma_{oiI}(G) + \gamma_{oiI}(\overline{G})$, where $\overline{G}$ is the complement graph of G.

In what follows we shall consider only graphs without isolated vertices.

II. PRELIMINARY RESULTS

In this section we present the basic properties of outer-independent Italian domination. We first provide six observations.

Observation 1: If $f = (V_0, V_1, V_2)$ is an γ_{oiI}-function on G, then

i) *each vertex of V_2 (if any) has a private neighbor in V_0.*
ii) *$V_1 \cup V_2$ is an outer-dominating set in G.*
iii) *$V - V_0$ is a vertex cover of G. In particular $\gamma_{oiI}(G) \geq \alpha_0(G)$.*

Observation 2: For any non-complete graph G having at least one edge, there exists an γ_{oiI}-function $f = (V_0, V_1, V_2)$ of G such that $V_0 \neq \emptyset$.

Observation 3: If G is an n-order graph then $\gamma_{oiI}(G) \leq n$. The equality holds if and only if $\Delta(G) \leq 1$.

Observation 4: If H is a complete subgraph of a graph G, then $\gamma_{oiI}(G) \geq |V(H)| - 1$.

Next we determine the exact value of the outer-independent Italian domination number for some classes of graphs. We start with the complete graphs and bipartite complete graphs whose proofs are easy to see.

Observation 5: For $n \geq 1$, $\gamma_{oiI}(K_n) = n-1$.

Observation 6: For integers $p \geq q \geq 1$,

$$\gamma_{oiI}(K_{p,q}) = \begin{cases} 2 & \text{if } q = 1. \\ q & \text{if } q \geq 2. \end{cases}$$

Proposition 1: For $n \geq 3$, $\gamma_{oiI}(C_n) = \lceil \frac{n}{2} \rceil$.

Proof: By Theorem 2 and inequality (1) we have $\gamma_{oiI}(C_n) \geq \lceil \frac{n}{2} \rceil$. To prove the inverse inequality, let $C_n := (v_1 v_2 \ldots v_n)$ and define $f : V(G) \to \{0, 1, 2\}$ by $f(v_{2i-1}) = 1$ and $f(v_{2i}) = 0$ for $1 \leq i \leq n/2$ when n is even, and by $f(v_n) = 1, f(v_{2i-1}) = 1$ and $f(v_{2i}) = 0$ for $1 \leq i \leq (n-1)/2$ when n is odd. Clearly f is an OIIDF of G of weight $\lceil \frac{n}{2} \rceil$ implying that $\gamma_{oiI}(C_n) \leq \lceil \frac{n}{2} \rceil$. Thus $\gamma_{oiI}(C_n) = \lceil \frac{n}{2} \rceil$. □

Next we determine outer-independent Italian domination number of paths. Recall that a *(outer-independent) 2-dominating set* of a graph G is a set D of vertices of G such that every vertex not in S is dominated at least twice (and $V(G) \setminus S$ is independent). The minimum cardinality of a (outer-independent) 2-dominating set of G is the *(outer-independent) 2-domination number* $\gamma_2(G)$ ($\gamma_2^{oi}(G)$). Clearly by assigning a 1 to each vertex of a minimum outer-independent 2-dominating set of a graph G and a 0 to other vertices, we obtain an OIIDF of G and this implies that

$$\gamma_{oiI}(G) \leq \gamma_2^{oi}(G). \tag{2}$$

Fink and Jacobson [13] have established a lower bound on the 2-domination number for every tree in term of its order.

Theorem 3: If T is a tree of order n, then $\gamma_2(T) \geq (n+1)/2$.

Proposition 2: For $n \geq 1$, $\gamma_{oiI}(P_n) = \lceil \frac{n+1}{2} \rceil$.

Proof: First let n is odd. Theorem 2 and inequality (1) imply that $\gamma_{oiI}(P_n) \geq \lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$. To prove the inverse inequality, let $P_n := v_1 v_2 \ldots v_n$ and define $f : V(G) \to \{0, 1, 2\}$ by $f(v_{2i-1}) = 1$ for $1 \leq i \leq (n+1)/2$ and $f(x) = 0$ otherwise. Clearly f is an OIDF of G of weight $\lceil \frac{n+1}{2} \rceil$ yielding $\gamma_{oiI}(P_n) \leq \lceil \frac{n+1}{2} \rceil$. Therefore $\gamma_{oiI}(P_n) = \lceil \frac{n+1}{2} \rceil$ in this case.

Now let n is even. Clearly, the function $f : V(G) \to \{0, 1, 2\}$ defined by $f(v_n) = 1, f(v_{2i-1}) = 1$ for $1 \leq i \leq n/2$ and $f(x) = 0$ otherwise, is an OIDF of G of weight $\lceil \frac{n+1}{2} \rceil$ and so $\gamma_{oiI}(P_n) \leq \lceil \frac{n+1}{2} \rceil$. To prove the inverse inequality, let $P_n := v_1 v_2 \ldots v_n$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{oiI}(P_n)$-function such that $|V_2|$ is as small as possible. If $|V_2| \geq 1$ and $v_i \in V_2$, then by the choice of f we have $f(v_{i-1}) = f(v_{i+1}) = 0$ and the function $g = ((V_0 \setminus \{v_{i-1}, v_{i+1}\}) \cup \{v_i\}, V_1 \cup \{v_{i-1}, v_{i+1}\}, V_2 \setminus \{v_i\})$ is a $\gamma_{oiI}(P_n)$-function which contradicts the choice of f. Hence $V_2 = \emptyset$. Then V_1 is a 2-dominating set of G and we conclude from Theorem 3 that $\gamma_{oiI}(P_n) = \omega(f) = |V_1| \geq \frac{n+1}{2}$. Since $\gamma_{oiI}(P_n)$ is integer, we have $\gamma_{oiI}(P_n) \geq \lceil \frac{n+1}{2} \rceil$.

Thus $\gamma_{oiI}(P_n) = \lceil \frac{n+1}{2} \rceil$ in this case and the proof is complete. □

A function $f : V(G) \to \{0, 1, 2\}$ is an outer-independent Roman dominating function (OIRDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$ and $\{v \mid f(v) = 0\}$ is an independent set. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. Outer-independent Roman domination was introduced by Abdollahzadeh Ahangar *et al.* [1]. Clearly, any outer-independent Roman dominating function on a graph G is an OIIDF of G and so

$$\gamma_{oiR}(G) \geq \gamma_{oiI}(G). \tag{3}$$

Abdollahzadeh Ahangar et al. proved the following bounds $\gamma_{oiR}(G)$.

Proposition 3: If G is a connected triangle-free graph of order $n \geq 2$ and maximum degree Δ, then $\gamma_{oiR}(G) \leq n - \Delta + 1$.

Proposition 4: Let G be a connected graph of order n. If G has girth $g < \infty$, then $\gamma_{oiR}(G) \leq n + \lceil \frac{g}{2} \rceil - g$.

Next results are immediate consequences of Propositions 3, 4 and inequality (3).

Corollary 1: If G is a connected triangle-free graph of order $n \geq 2$ and maximum degree Δ, then $\gamma_{oiR}(G) \leq n - \Delta + 1$. d This bound is sharp for all stars $K_{1,n-1}$, $n \geq 2$.

Corollary 2: Let G be a connected graph of order n. If G has girth $g < \infty$, then $\gamma_{oiR}(G) \leq n + \lceil \frac{g}{2} \rceil - g$.

III. BOUNDS

In this section we present some sharp bounds on $\gamma_{oiI}(G)$.

Theorem 4: For any connected graph G of order $n \geq 2$ with minimum degree δ and maximum degree Δ,

$$\gamma_{oiI}(G) \geq \lceil n\delta/(\delta + \Delta) \rceil.$$

This bound is sharp for cycles and complete bipartite graphs $K_{n,n}$ ($n \geq 2$).

Proof: Let $f = (V_0, V_1, V_2)$ be an arbitrary $\gamma_{oiI}(G)$-function. Suppose first that $V_0 = \emptyset$. Then $\gamma_{oiI}(G) = |V_1| = n$. Observe that if G contains a vertex y of degree at least two, then we can reduce the weight of f by assigning 0 to y which is a contradiction. Thus $\Delta = 1$ yielding $G = K_2$ and so $\gamma_{oiI}(G) = 2 > \lceil n\delta/(\delta + \Delta) \rceil$. Assume that $V_0 \neq \emptyset$. Since V_0 is independent, we obtain $\delta|V_0| \leq \Delta(|V_2| + |V_1|)$. Using the fact that $n = |V_2| + |V_1| + |V_0|$, we obtain

$$\delta n/(\delta + \Delta) \leq |V_2| + |V_1| \leq 2|V_2| + |V_1| = \gamma_{oiI}(G).$$

Since $\gamma_{oiI}(G)$ is an integer, we deduce that $\gamma_{oiI}(G) \geq \lceil n\delta/(\delta + \Delta) \rceil$. □

Corollary 3: If G is a regular graph of order $n \geq 2$, then $\gamma_{oiI}(G) \geq \lceil n/2 \rceil$.

Corollary 4: Let G be a connected graph of order $n \geq 2$ and $\delta = 1$, then $\gamma_{oiI}(G) \geq \lceil n/(\Delta + 1) \rceil$.

Theorem 5: For a graph G the following hold.

(i) *Each minimum vertex cover of G contains all vertices in $S_2(G)$. There exists a minimum vertex cover of G containing $S(G)$.*

(ii) $\gamma_{oiI}(G) \leq \alpha_0(G) + |S(G)| \leq 2\alpha_0(G)$.

(iii) *If $\delta(G) \geq 2$ then $\gamma_{oiI}(G) = \alpha_0(G) = n(G) - \beta_0(G)$.*

Proof: (i) Obvious.

(ii) By (i) there is a vertex cover F of G such that $S(G) \subseteq F$ and $|F| = \alpha_0(G)$. The set $V(G) - F$ is independent (by Theorem 1) and then each its non-leaf vertex is adjacent to at least 2 vertices of F. Hence the function $f = (V(G) - F; F - S(G); S(G))$ is an OIIDF on G. Thus $\gamma_{oiI}(G) \leq w(f) = |F| + S(G) = \alpha_0(G) + |S(G)|$. It remains to note that clearly $\alpha_0(G) \geq |S(G)|$.

(iii) If $\delta(G) \geq 2$ then $|S(G)| = 0$ and by (ii), $\gamma_{oiI}(G) \leq \alpha_0(G)$. On the other hand, by Observation 1 (Item (iii)) we have $\gamma_{oiI}(G) \geq \alpha_0(G)$. Thus $\gamma_{oiI}(G) = \alpha_0(G)$. The last equality follows by Theorem 1. □

The bounds in Theorem 5(ii) are attainable. Let G be a graph each vertex of which is either a leaf or a support vertex. If each support vertex of G is adjacent to at least 2 leaves, then clearly $S(G)$ is a minimum cover set and $f = (V(G) - S(G); \emptyset; S(G))$ is an OIIRDF on G of minimum weight. Thus $\gamma_{oiI}(G) = \alpha_0(G) + |S(G)| = 2\alpha_0(G)$.

Next result is an immediate consequence of Theorem 5(iii).

Corollary 5: For any graph G of order n with $\delta(G) \geq 2$,

$$\gamma_{oiI}(G) \leq 2\alpha'(G)$$

where $\alpha'(G)$ is the matching number of G.

We will say that a graph G is a *vertex cover outer independent Italian graph*, a *VCOI-Italian graph* for short, if $\gamma_{oiI}(G) = 2\alpha_0(G)$.

Theorem 6: A graph G is VCOI-Italian if and only if the function $f = (V(G) - S(G), \emptyset, S(G))$ is a γ_{oiI}-function on G.

Proof: Suppose that G is a VCOI-Italian graph. By Theorem 5, $S(G)$ is a minimum vertex cover of G. Hence $f = (V(G) - S(G), \emptyset, S(G))$ is a γ_{oiI}-function on G.

Assume now that $f = (V(G) - S(G), \emptyset, S(G))$ is a γ_{oiI}-function on G. Then $\gamma_{oiI}(G) = 2|S(G)|$ and $S(G)$ is a vertex cover of G. Now by Theorem 5, $S(G)$ is a minimum vertex cover of G and so G is VCOI-Italian. □

Proposition 5: Let H be an induced subgraph of a graph G. Then $\gamma_{oiI}(G) \leq \gamma_{oiI}(H) + |V(G)| - |V(H)|$.

Proof: Let f be a γ_{oiI}-function on H. Define an OIIDF h on G as follows: $h(x) = f(x)$ when $x \in V(H)$ and $h(x) = 1$ otherwise. Since $w(h) = w(f) + |V(G)| - |V(H)|$, we obtained the desired inequality. □

Corollary 6: Let G be a connected graph of order n. If $\mathrm{diam}(G) = d \geq 2$, then $\gamma_{oiI}(G) \leq n - \lfloor \frac{d}{2} \rfloor$.

Proof: Let P_{d+1} be a diametral path in G. By Propositions 2 and 5 we have

$$\begin{aligned}\gamma_{oiI}(G) &\leq \gamma_{oiI}(P_{d+1}) + |V(G)| - |V(P_{d+1})| \\ &= \left\lceil \frac{d+2}{2} \right\rceil + n - (d+1) \\ &= n - \left\lfloor \frac{d}{2} \right\rfloor.\end{aligned}$$

□

Applying Corollary 6, we can characterize all graphs G of order n with $\gamma_{oiI}(G) = n-1$.

Theorem 7: Let G be a connected graph of order $n \geq 3$. Then $\gamma_{oiI}(G) = n-1$ if and only if $G \in \{P_3, P_4, K_n\}$ or G is obtained from a complete graph K_t ($t \geq 3$) by adding at most one pendant edge at each vertex of K_t.

Proof: If $G \in \{P_3, P_4, K_n\}$ or G is obtained from a complete graph K_t ($t \geq 3$) by adding at most one pendant edge at each vertex of K_t, then it is easy too see that $\gamma_{oiI}(G) = n-1$.

Conversely, let $\gamma_{oiI}(G) = n-1$. By Corollary 6, we have $diam(G) \leq 3$. If $diam(G) = 1$, then G is a complete graph and we are done. Assume that $2 \leq diam(G) \leq 3$. If G has an induced subgraph isomorphic to $K_{1,3}$ centered at x and with leaves x_1, x_2, x_3, then assigning a 2 to x, a 0 to x_1, x_2, x_3 and a 1 to other vertices introduces an OIIDF of G of weight $n-2$, a contradiction. Hence G is $K_{1,3}$-free graph. We consider two cases.

Case 1. $diam(G) = 3$.

Let $P := v_1v_2v_3v_4$ be a diametrical path in G. If $n = 4$, then $G = P_4$ and we are done. Assume that $n \geq 5$. If v_1 is adjacent to a vertex $w \in V(G) - V(P)$, then the function $(\{v_1, v_3\}, V(G) - \{v_1, v_3\}, \emptyset)$ is an OIIDF of G of weight $n-2$ which is a contradiction. This implies that $d(v_1) = 1$. Similarly, we have $d(v_4) = 1$. Since G is $K_{1,3}$-free, v_1 is the unique leaf adjacent to v_2 and v_4 is the unique leaf adjacent to v_3. Also since G is $K_{1,3}$-free, we conclude that $N[v_2] - \{v_1\} = N[v_3] - \{v_4\}$ and that $N[v_2] - \{v_1\}$ induces a complete subgraph of G. Since $diam(G) = 3$, each vertex in $V(G) - (N[x_2] \cup \{v_4\})$ must be adjacent to a vertex in $N(v_2) \cap N(v_3)$ and must be an end vertex of a diametrical path. Thus each vertex in $V(G) - (N[x_2] \cup \{v_4\})$ has degree 1. Using above argument, we deduce that any vertex in $N(v_2) \cap N(v_3)$ is adjacent to at most one leaf and so G is obtained from a complete graph K_t ($t \geq 3$) by adding at most one pendant edge at each vertex of K_t.

Case 2. $diam(G) = 2$.

If $n = 3$, then we have $G \in \{P_3, K_3\}$ and we are done. Let $n \geq 4$ and v a vertex of G with maximum degree. Since $diam(G) = 2$, v has 2 nonadjacent neighbors, say v_1 and v_2. If $d(v_1), d(v_2) \geq 2$, then the function $(\{v_1, v_2\}, V(G) - \{v_1, v_2\}, \emptyset)$ is an OIIDF of G of weight $n-2$ which is a contradiction. Assume that $d(v_1) = 1$. It follows from $diam(G) = 2$ that v_2 is adjacent to all neighbors of v but v_1. Since G is a $K_{1,3}$-free graph, $G - v_1$ is a complete graph. Thus G is obtained from the complete graph K_{n-1} by adding a pendant edge at a vertex and this completes the proof. □

Nordhaus and Gaddum [10] found sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters; see for example [11]. Jafari Rad and Krzywkowski [6] proved the following Nordhaus-Gaddum type result for outer-independent 2-domination number.

Theorem 8: For any graph G on n vertices,

$$\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) \leq 2n,$$

with equality if and only if $G \in \{K_1, K_2, \overline{K_2}\}$. Moreover, $\gamma_2^{oi}(G) + \gamma_2^{oi}(\overline{G}) = 2n-1$ if and only if G or $\overline{G}$ is a complete graph or a path P_3.

Here we provide similar inequalities for the outer independent Italian domination number.

Theorem 9: For any graph G on n vertices,

$$n-1 \leq \gamma_{oiI}(G) + \gamma_{oiI}(\overline{G}) \leq 2n.$$

Both bound are attainable. Moreover, (a) $\gamma_{oiI}(G) + \gamma_{oiI}(\overline{G}) = 2n$ if and only if $G \in \{K_1, K_2, \overline{K_2}\}$, and (b) $\gamma_{oiI}(G) + \gamma_{oiI}(\overline{G}) = 2n-1$ if and only if $n \geq 3$ and $G \in \{K_n, \overline{K_n}, P_3, \overline{P_3}\}$.

Proof: The right inequality follows from Theorem 8 and inequality (2). Now we prove the left equality. If $\gamma_{oiI}(G) = n$ or/and $\gamma_{oiI}(\overline{G}) = n$ then we are done. So, let $\gamma_{oiI}(G) \leq n-1$ and $\gamma_{oiI}(\overline{G}) \leq n-1$. Suppose $f = (V_0, V_1, V_2)$ is a $\gamma_{oiI}(G)$-function. Since $\gamma_{oiI}(G) \leq n-1$, we have $V_0 \neq \emptyset$. By definition, V_0 is a clique in $\overline{G}$ and we deduce from Observation 4 that

$$\begin{aligned}\gamma_{oiI}(G) + \gamma_{oiI}(\overline{G}) &= w(f) + |V_0| - 1 \\ &= |V_1| + 2|V_2| + |V_0| - 1 \\ &= |V_1| + |V_2| + |V_0| - 1 \\ &\geq n-1, \qquad (4)\end{aligned}$$

as required. If G is the graph obtained from the complete bipartite graph $K_{5,5}$ with bipartition (X, Y) by deleting three perfect matchings from $K_{5,5}$ and adding all edges between the vertices of X, then clearly $\gamma_{oiI}(G) + \gamma_{oiI}(\overline{G}) = n-1$. □

IV. TREES

A. AN UPPER BOUND IN TERMS OF ORDER

Denote by $F^v_{r,t}$ the tree obtained from a star $K_{1,r+t}$, $r+t \geq 1$, with a central vertex v, by subdividing exactly t edges once. Clearly $\gamma_{oiI}(F^v_{r,t}) \leq 3|V(F^v_{r,t})|/4$ whenever $(r, t) \neq (1, 0)$ and the equality holds if and only if $(r, t) = (1, 1)$, i.e. for $F^v_{1,1} = P_4$. Our first result in this section shows that $\gamma_{oiI}(T) \leq \frac{3n}{4}$ for any tree of order $n \geq 3$.

Theorem 10: Let T be a tree of order $n \geq 3$. Then $\gamma_{oiI}(T) \leq \frac{3n}{4}$.

Proof: It is easy to verify that the theorem holds for all trees with diameter at most three. Suppose, to the contrary, that there exists a tree T on n vertices such that $\gamma_{oiI}(T) > \frac{3n}{4}$. In addition choose T so that n is as small as possible. Then $diam(T) \geq 4$. Let $P := u_1u_2 \ldots u_r (k \geq 5)$ be a diametrical path in T with the property that $d(u_2)$ is as large as possible. We distinguish the following three cases depending on the degrees of u_2 and u_3.

Case 1: $d(u_2) \geq 3$.

Denote by T' the component of $T - u_2u_3$ containing u_3. By the choice of T there is an OIIDF f' on T' with $w(f') \leq \frac{3n(T)}{4}$. Define an OIIDF f on T with $f(x) = f'(x)$ for $x \in V(T')$, $f(u_2) = 2$ and $f(y) = 0$ for any leaf y adjacent to u_2. Then $\gamma_{oiI}(T) \leq w(f) = w(f') + 2 \leq \frac{3(n-3)}{4} + 2 < \frac{3n}{4}$, which is a contradiction.

Case 2: $d(u_2) = d(u_3) = 2$.

Let $T' = T - N[u_2]$, and f' an OIIDF on T' with $w(f') \le \frac{3n(T)}{4}$. Define an OIIDF f on T so that $f(x) = f'(x)$ for all $x \in V(T')$, $f(u_2) = 0$ and $f(u_1) = f(u_3) = 1$. This implies that $\gamma_{oiI}(T) \le w(f) = w(f') + 2 \le \frac{3(n-3)}{4} + 2 < \frac{3n}{4}$, a contradiction again.

Case 3: $d(u_2) = 2$ and $d(u_3) \ge 3$.

Denote by T' and T'' the components of $T - u_3u_4$, where $u_4 \in V(T')$. Let f' be an OIIDF on T' with $w(f') \le \frac{3n(T')}{4}$. Note that $T'' = F^{v_3}_{l-t,t}$, where $l = d_T(u_3) - 1 \ge t \ge 1$.

Assume first that u_3 is adjacent to a leaf. Define an OIIDF f'' on T'' so that $f''(u_3) = 2$, $f''(y) = 0$ for any neighbor y of u_3, and $f''(z) = 1$ for any other vertex z of T''. Now the function f on G with $f|_{T'} = f'$ and $f|_{T''} = f''$ is an OIIDF on T with $\gamma_{oiI}(T) \le w(f) = w(f') + w(f'') \le \frac{3(n-l-t-1)}{4} + 2 + t = \frac{3n}{4} + \frac{5+t-3l}{4} \le \frac{3n}{4}$, because of $l > t \ge 1$, and this leads to a contradiction.

It remains the case when $t = l$. Define now an OIIDF g on T as follows: $g|_{T'} = f'$, $g(u_3) = 1$, $g(y) = 0$ if y is a neighbor of u_3 in T'', and $g(z) = 1$ for each leaf of T''. But then $\gamma_{oiI}(T) \le w(f) = w(f')+1+t \le \frac{3(n-2t-1)}{4}+1+t < \frac{3n}{4}$, because of $t \ge 2$; a contradiction again. □

In the next theorem we give a constructive characterization of all trees T with $\gamma_{oiI}(T) = \frac{3n}{4}$. We need the following definition. Let $\mathcal{T}$ be the family of all trees T that can be obtained from a sequence of trees $T_1, T_2, \ldots, T_k$ for some $k \ge 1$, where T_1 is P_4 and $T = T_k$. If $k \ge 2$ then T_{i+1} is obtained from T_i by the following Operation $\mathcal{O}_1$.

Operation $\mathcal{O}$. If $u \in V(T_i)$ is a non-leaf vertex, then T_{i+1} is obtained from T_i by adding a path P_4 and joining u to a non-leaf vertex of P_4.

Theorem 11: For an n-order tree T, $\gamma_{oiI}(T) = \frac{3n}{4}$ if and only if $T \in \mathcal{T}$.

Proof: Necessity: Let T be a tree of order $n \ge 3$ with $\gamma_{oiI}(T) = \frac{3n}{4}$. We will prove that the following holds:

($\mathcal{P}$) $T \in \mathcal{T}$ and for each leaf x of T there is a γ_{oiI}-function $f_x = (V_0, V_1, V_2)$ on T such that $N[x] \subseteq V_1$.

If $n \le 4$ then $T = P_4$ and we are done. We proceed by induction on n. Let $n \ge 5$ and ($\mathcal{P}$) hold for all trees of order less than n. Let T be a tree of order n with $\gamma_{oiI}(T) = \frac{3n}{4}$. Clearly $T \ne F^{v}_{r,t}$. Let $P := v_1v_2, \ldots, v_k$ be a diametrical path in T.

Claim 1. $d(v_2) = 2$.

Proof: Suppose, to the contrary, that $d(v_2) \ge 3$. Let $x_1, x_2, \ldots, x_k$ ($k \ge 2$) be the leaves adjacent to v_2 and let $T' = T - \{v_2, x_1, x_2, \ldots, x_k\}$. Let f be any γ_{oiI}-function on T'. By Theorem 10, $\gamma_{oiI}(T') \le \frac{3(n-k-1)}{4}$. Define now an *OIIDF* g on T with $g|_{T'} = f$, $g(v_2) = 2$ and $g(x_i) = 0$ for each i. Now $w(g) = w(f)+2 \le \frac{3(n-k-1)}{4}+2 = \frac{3n-3k+5}{4} < \frac{3n}{4}$ which is a contradiction. □

Denote by T'' the component of $T - v_3v_4$ containing v_4. By the choice of P and Claim 1, the other component of $T - v_3v_4$ is $F^{v_3}_{r,t}$, where $t \ge 1$. Let h_1 be a γ_{oiI}-function on T'' and h_2 a γ_{oiI}-function on $F^{v_3}_{r,t}$. It is easy to see that we can choose h_2 so that $h_2(v_3) \ne 0$. But then the function h defined on T as $h|_{T''} = h_1$ and $h|_{F^{v_3}_{r,t}} = h_2$ is an OIIDF on T and $3n/4 = \gamma_{oiI}(T) \le w(h) = w(h_1) + w(h_2) = \gamma_{oiI}(T'') + \gamma_{oiI}(F^{v_3}_{r,t}) \le 3|V(T'')|/4+3|V(F^{v_3}_{r,t})|/4 = 3n/4$. This immediately implies $\gamma_{oiI}(T'') = 3|V(T'')|/4$ and $F^{v_3}_{r,t} = F^{v_3}_{1,1} = P_4$. It follows from the induction hypothesis that $T \in \mathcal{T}$ and for each leaf x of T there is a γ_{oiI}-function $f_x = (V_0, V_1, V_2)$ on T such that $N[x] \subseteq V_1$. In particular, each vertex of T'' is a leaf or a support vertex. We claim that v_4 is a support vertex. If v_4 is a leaf, then the function $h : V(T) \to \{0, 1, 2\}$ defined by $h(v_4) = h(v_2) = 0$, $h(v_3) = h(v_1) = h(w) = 1$ and $h(x) = f_{v_4}(x)$ otherwise, where w is the leaf adjacent to v_3, is an OIIDF on T of weight less than $3n/4$ which is a contradiction. Thus v_4 is a support vertex. Now T can be obtained from T'' by operation $\mathcal{O}$ and so $T \in \mathcal{T}$. Since any $\gamma_{oiI}(T'')$-function can be extended to a $\gamma_{oiI}(T)$-function by assigning a 1 to v_1, v_3, w and a o to v_2 or by assigning a 1 to v_1, v_2, w and a o to v_3, we conclude that ($\mathcal{P}$) is valid and the necessity is proved.

Sufficiency: Let T be a tree in $\mathcal{T}$. Then there is a sequence $T_1 = P_4, T_2, \ldots, T_k = T$ of trees in $\mathcal{T}$, where if $k \ge 2$, then T_{i+1} is obtained from T_i by Operation $\mathcal{O}$. We proceed by induction on the number of operations performed to construct T. If $k = 1$ then we are done. So let $k \ge 2$. Assume that the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, $\gamma_{oiI}(T') = \frac{3(n-4)}{4}$. Now we show that $\gamma_{oiI}(T) = \frac{3n}{4}$. Let $T = T_k$ be obtained from $T' = T_{k-1}$ and a path $P_4 : wv_3v_2v_1$ by adding an edge v_3v_4, where v_4 is a non-leaf vertex of T_{k-1}. Clearly, any $\gamma_{oiI}(T')$-function can be extended to an OIIDF of T by assigning a 1 to v_1, v_3, w and a 0 to v_2 implying that $\gamma_{oiI}(T) \le \gamma_{oiI}(T') + 3 = \frac{3n}{4}$. To prove the inverse inequality, let f be an arbitrary γ_{oiI}-function on T. Clearly $f(w)+f(v_3)+f(v_2)+f(v_1) \ge 3$. If $f(v_4) \ge 1$, then $f|_{T'}$ is an OIIDF of T' and so $\gamma_{oiI}(T) \ge \gamma_{oiI}(T')+3 = \frac{3n}{4}$. If $f(v_4) = 0$, then f must assign a positive weight to each neighbor of v_4 implying that $f|_{T'}$ is an OIIDF of T' and as above we have $\gamma_{oiI}(T) \ge \frac{3n}{4}$. Thus $\gamma_{oiI}(T) = \frac{3n}{4}$ and the proof is complete. □

B. LOWER BOUNDS

First we provide a lower bound on outer-independent Italian domination number of a tree in terms of the order and the number of leaves.

Theorem 12: For any tree T of order $n \ge 2$,

$$\gamma_{oiI}(T) \ge \frac{n + 3 - \ell(T)}{2}$$

where $\ell(T)$ is the number of leaves of T. This bound is sharp for stars and paths.

Proof: We proceed by induction on n. The result is immediate for $n = 2, 3$. Let $n \ge 4$ and the statement hold for all trees of order less than n. Let T be a tree of order n. If $diam(T) = 2$, then T is a star and we have $\gamma_{oiI}(T) = 2 = \frac{n+3-\ell(T)}{2}$ and if $diam(T) = 3$, then T is a double star and we have $\gamma_{oiI}(T) \ge 3 > \frac{n+3-\ell(T)}{2}$. Henceforth, we assume that $diam(T) \geq 4$. Let $v_1 v_2 \ldots v_k$ $(k \geq 5)$ be a diametrical path in T and let f be a $\gamma_{oiI}(T)$-function such that $f(v_2)$ is as large as possible. If $d(v_2) \geq 3$, then we may assume that $f(v_2) = 2$ and the function f, restricted to $T - v_1$ is an OIIDF and it follows from the induction hypothesis that $\gamma_{oiI}(T) = \omega(f) \geq \frac{n+3-\ell(T-v_1)}{2} = \frac{n+3-\ell(T)}{2}$. Assume that $d(v_2) = 2$. If $f(v_2) = 2$, then as above we can see that $\gamma_{oiI}(T) \geq \frac{n+3-\ell(T)}{2}$. Let $f(v_2) \leq 1$. We conclude from the choice of f that $f(v_2) = 0, f(v_1) = 1$ and $f(v_3) \geq 1$. Now the function f, restricted to $T' = T - \{v_1, v_2, v_3\}$ is an OIIDF and by the induction hypothesis we have

$$\gamma_{oiI}(T) = \omega(f) \geq \frac{n+3-\ell(T')}{2} \geq \frac{n+3-\ell(T)}{2}.$$

This completes the proof. □

Next we establish a lower bound in terms of the diameter.

Lemma 1: If v is a leaf of a graph G, then $\gamma_{oiI}(G - v) \leq \gamma_{oiI}(G)$.

Proof: Let $f = (V_0^f, V_1^f, V_2^f)$ be a γ_{oiI}-function on G and u the neighbor of v. If $f(v) = 0$ then f is an OIIDF on $G - v$. If $f(v) \neq 0$ then the function g defined on $G - v$ by $g(u) = 1$ and $g(x) = f(x)$ for $x \in V(G) - \{u, v\}$ is an OIIDF of $G - v$ with $w(g) \leq w(f)$. □

By Proposition 2 and Lemma 1, we immediately obtain the following result.

Corollary 7: For any tree T with $diam(T) = d$, $\gamma_{oiI}(T) \geq \lceil \frac{d}{2} \rceil + 1$.

Theorem 13: For any n-order tree T the following are equivalent:

(i) $\gamma_{oiI}(T) = \lceil \frac{diam(T)}{2} \rceil + 1$.

(ii) *T is a path or T is a star or T is a tree obtained from a path of even order by adding some pendant edges at one of its support vertices.*

Proof: The theorem is clearly true when $d = diam(T) \leq 2$ or T is a path. So, let $d \geq 3$, $\Delta(T) \geq 3$ and P_{d+1} : $v_1, v_2, \ldots, v_{d+1}$ a diametral path of T. $(ii) \Leftarrow (i)$ is obvious. Hence we prove $(i) \Rightarrow (ii)$. Let f be any γ_{oiI}-function on T. Note first that the function f, restricted to P_{d+1} is an OIIDF on P_{d+1} and so

$$\lceil d/2 \rceil + 1 = \gamma_{oiI}(T) = w(f) \geq w(f|_{P_{d+1}}) \geq \gamma_{oiI}(P_{d+1})$$
$$= \lceil (d+2)/2 \rceil.$$

But then $f|_{P_{d+1}}$ is a γ_{oiI}-function on P_{d+1} and $f(x) = 0$ for all $x \in V(T) - V(P_{d+1})$. It follows that $f(v_i) = 2$ if $d(v_i) \geq 3$. If $f|_{P_{d+1}}(v_i) = 2$ for some $3 \leq i \leq d-1$, then the function $g : V(P_{d+1}) \rightarrow \{0, 1, 2\}$ defined by $g(v_i) = 1$ and $g(x) = f(x)$ otherwise, is an OIIDF on P_{d+1} of weight less that $\omega(f)$ which leads to a contradiction. Thus $f|_{P_{d+1}}(v_i) \leq 1$ for each $3 \leq i \leq d-1$ implying that $d(v_i) = 2$ for each $3 \leq i \leq d-1$. Since $\Delta(G) \geq 3$, we have $d(v_2) \geq 3$ or $d(v_d) \geq 3$. Thus $f|_{P_{d+1}}(v_2) = 2$ or $f|_{P_{d+1}}(v_d) = 2$. If $f|_{P_{d+1}}(v_2) = 2$ or $f|_{P_{d+1}}(v_d) = 2$, then we can easily define an OIIDF on P_{d+1} with weight less that $\omega(f)$ which leads to a contradiction. Hence either $f|_{P_{d+1}}(v_2) = 2$ or $f|_{P_{d+1}}(v_d) = 2$. Assume without loss of generality that $f(v_2) = 2$ and $f(v_d) = 0$. This implies that $d(v_d) = 2$. If $d+1$ is odd, then we can easily define an OIIDF on P_{d+1} with weight less that $\omega(f)$ which leads to a contradiction. Hence $d+1$ is even. Thus T is obtained from the path P_{d+1} of even order by adding some pendant edges at v_2 and the proof is complete. □

V. CONCLUSION

As a variation of domination, the outer-independent domination was introduced and studied [5], [6]. More recently, known as Roman-{2} domination [12], Italian domination was proposed in 2016 and its study was continued by some authors [3], [4]. This paper considers the combination of the properties of the outer-independent domination and Italian domination. We show bounds relating the outer-independent Italian domination number to the vertex cover number, order and diameter. Moreover, lower and upper bounds on $\gamma_{oiI}(T)$ of a tree T, characterization of extremal graphs, and Nordhaus-Gaddum type inequalities are given.

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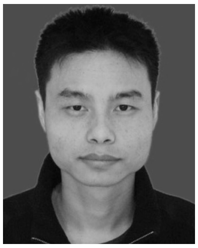

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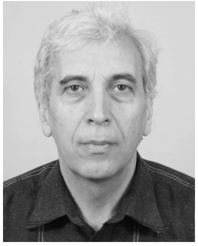

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