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Reliability Assessment of Random Uncertain Multi-State Systems

LINMIN HU[1], DEQUAN YUE[1], AND GUOXI ZHAO[2]
[1]School of Science, Yanshan University, Qinhuangdao 066004, China
[2]School of Mathematics and Information Science, Xinxiang University, Xinxiang 453000, China

Corresponding author: Linmin Hu (linminhu@ysu.edu.cn)

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ABSTRACT The random uncertain multi-state system is defined as a multi-state system consisting of multi-state components whose performance rates and the corresponding state probabilities are presented as uncertain variables. Reliability assessment of random multi-state systems with enough samples based on probability theory has been widely investigated. Nevertheless, in some real-world applications, only a few or even no samples are available to estimate the state probabilities and performance rates of multi-state components or systems. To overcome the problem, by joint employment of probability theory and uncertainty theory, the reliability of a random uncertain multi-state system is analyzed in this paper. The state probabilities and performance rates of multi-state components are considered as uncertain variables. The uncertain universal generating function is introduced to evaluate the state probabilities and performance rates of the random uncertain multi-state systems. The uncertainty distributions, inverse uncertainty distributions, expected values and variances of the state probabilities, and performance rates for the system are discussed. An assessment technique for system reliability is proposed to compute the system reliability under the crisp user demand. A numerical example is presented to illustrate how to assess the state probabilities, performance rates, and reliability of the system.

INDEX TERMS Multi-state system, reliability, uncertainty theory, universal generating function.

I. INTRODUCTION

Reliability assessment is of vital importance in the design phase of a system. Maintaining high reliability is often an essential requisite to achieve desired system functions. Many analysis methods and evaluation techniques [1]–[6] were developed to facilitate the reliability assessment for complex systems based on probability theory. In recent decades, multi-state system reliability has received substantial attention. Based on conventional reliability theory, various reliability models [7]–[9] and optimization problems [10]–[12] have been studied extensively in many multi-state systems. However, using conventional reliability theory in reliability assessment of multi-state systems needs to have two fundamental assumptions [13], [14]: (1) the state probability distributions of multi-state components can be fully characterized by probability measures, and (2) the state performance rates of multi-state components can be precisely determined. When the sample size is large enough, it is possible for us to believe the estimated state probability distribution is close enough to the long-run cumulative frequency, and the estimated state performance rate is close enough to the actual performance behavior. Otherwise, the conventional reliability theory is no longer applicable.

In many real-world applications, it is very difficult to estimate precisely state probabilities and performance rates of some multi-state components. In order to deal with the reliability of multi-state components/systems with imprecise data, Ding and Lisnianski [13] firstly introduced basic concepts of fuzzy multi-state systems where performance rates and corresponding state probabilities are presented as fuzzy values based on fuzzy set theory [16]. Moreover, some key definitions of fuzzy multi-state systems and a general fuzzy multi-state system reliability model were provided by Ding *et al*. [15]. Afterwards, Liu and Huang [14] investigated a dynamic fuzzy reliability assessment problem for fuzzy multi-state systems. Recently, Bamrungsetthapong and Pongpullponsak [17] studied parameter interval estimation of system reliability for a repairable multi-state series-parallel system with fuzzy data. Hu *et al*. [18] provided an assessment

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approach for dynamic fuzzy availability of a discrete time multi-state system under minor failures and repairs.

Each reliability evaluation technique has its advantages and inherent disadvantages [19]. Although fuzzy theory has been applied in reliability analysis of multi-state systems, it was still challenged by some scholars when some kinds of uncertainty was considered. Some surveys showed that human uncertainty does not behave like fuzziness [20]–[22]. The uncertainty theory provides a useful tool to study reliability of multi-state systems with human uncertainty phenomena. The basic uncertainty theory was presented by Liu [23]. Nowadays uncertainty theory has become a branch of mathematics for modeling human uncertainty. Recently, it has received increasing attention and been widely applied in a variety of fields. For example, Dipak *et al*. [24] developed a three-layer supply chain integrated production-inventory model in an uncertain environment. Gao *et al*. [25] proposed an algorithm to determine the distribution function of the diameter of an uncertain graph. Yao [26] analyzed a no-arbitrage determinant theorem for Liu's stock model in uncertain markets. Zhou *et al*. [27] studied the minimum spanning tree problem on a graph with uncertain edge weights which are formulated as uncertain variables. Sheng *et al*. [28] investigated a production-inventory problem in an uncertain environment with bounded production rates and proposed an uncertain optimal control model with Hurwicz criterion. Liu and Ralescu [29] proposed a concept of risk index to quantify the risk of an uncertain random system.

The concept of uncertain system reliability via uncertainty theory was first proposed by Liu [30]. Afterwards, Wen and Kang [31] investigated system reliability based on chance theory which is a generalization of both probability theory and uncertainty theory. Gao *et al*. [32] proposed a new concept of order statistics associated with uncertain random variables and applied it to analyze reliability of k-out-of-n systems with uncertain random lifetimes. Gao and Yao [33] investigated importance index for a component and a group of components in the uncertain random reliability system. Zeng *et al*. [34] developed mathematical foundation of belief reliability for coherent systems based on uncertainty theory. Zu *et al*. [35] proposed an optimal model based on maximum entropy principle to estimate belief reliability distribution. Zhang *et al*. [36] investigated some belief reliability indexes on the basis of the belief reliability metric based on chance theory to measure reliability of uncertain random systems. Liu *et al*. [37] studied the reliability indices redefined by uncertainty measure for general repairable systems with uncertain lifetimes and repair times.

The reliability of random multi-state systems based on probability theory has been discussed widely in many literatures. However, the reliability for a random uncertain multi-state system via probability theory and uncertainty theory has been seldom discussed in previous research. For such system, the state probabilities and performance rates of a component can be represented by uncertain variables when we have no samples but belief degree from the experts. In this paper, we consider a reliability model for a random uncertain multi-state system based on probability theory and uncertainty theory. The uncertain universal generating function is introduced to evaluate the state probabilities and performance rates of the random uncertain multi-state system through aggregating the uncertain behavior of the system components. Then we analyze uncertainty distributions and inverse uncertainty distributions of the state probabilities and the performance rates for the random uncertain multi-state system. Based on obtained uncertainty and inverse uncertainty distributions of these indices, expected values and variances of the state probabilities and the performance rates are calculated. In order to obtain reliability of the random uncertain multi-state system, an assessment technique for system reliability is proposed under the crisp user demand.

The rest of this paper is structured as follows. In Section 2, some basic concepts and theorems in multi-state system and uncertainty theory are presented. The definitions of random uncertain multi-state component and random uncertain multi-state system are introduced in Section 3. The state probabilities, performance rates and reliability for the random uncertain multi-state system are also discussed in this section. Section 4 presents a numerical example to illustrate the proposed model. Finally, we make concluding remarks in Section 5.

II. PRELIMINARIES

In this section, we will present some basic concepts and properties in multi-state system and uncertainty theory.

A. MULTI-STATE SYSTEM

A system that can have a finite number of performance rates is called a multi-state system. The universal generating function technique [38] is an primary approach for assessing multi-state system reliability.

In order to evaluate multi-state system behavior under crisp value context, one has to recognize the characteristics of its component. Suppose a multi-state system is consisting of m independent components, any component j can have M_j different states corresponding to the performance rates, represented by the ordering set $\mathbf{g}_j = \{g_{j,1}, g_{j,2}, \ldots, g_{j,M_j}\}$, where g_{j,k_j} is the performance rate of the component j in the state k_j, $k_j = 1, 2, \ldots, M_j$. The performance rate G_j of the component j at any time instant is a random variable, taking value from $\mathbf{g}_j$: $G_j \in \mathbf{g}_j$. The probabilities associated with the different states (performance rates) for the component j can be represented by the set $\mathbf{p}_j = \{p_{j,1}, p_{j,2}, \ldots, p_{j,M_j}\}$, that is $p_{j,k_j} = \mathrm{P}\{G_j = g_{j,k_j}\}$. The universal generating function of the component j is defined as [39]:

$$u_j(z) = \sum_{k_j=1}^{M_j} p_{j,k_j} \cdot z^{g_{j,k_j}},$$

where $u_j(z)$ can represent the probability distribution of the performance rates for the component j.

The performance rates and the corresponding state probabilities of the multi-state system are unambiguously determined by the performance rates and the corresponding state probabilities of its components. Without loss of generality, assume that the multi-state system has M possible states, g_k and p_k denote the system performance rate and corresponding state probability in state $k(k = 1, 2, \ldots, M)$, respectively. The system performance rate G is a random variable that takes values from the set $\{g_1, g_2, \ldots, g_M\}$. The universal generating function $U(z)$ of the multi-state system is

$$U(z) = \sum_{k=1}^{M} p_k \cdot z^{g_k},$$

where $M = \prod_{j=1}^{m} M_j$, $p_k = \prod_{j=1}^{m} p_{j,k_j}$ and $g_k = \varphi\left(g_{1,k_1}, g_{2,k_2}, \ldots, g_{m,k_m}\right)$. $\varphi(\cdot)$ is the system structure function.

In order to obtain the output probability distribution for the multi-state system with the arbitrary structure function φ, a general composition operator Θ_φ is used over individual UGF of m components [38]:

$$\begin{aligned} U(z) &= \Theta_\varphi\left(u_1(z), u_2(z), \ldots, u_m(z)\right) \\ &= \Theta_\varphi\left(\sum_{k_1=1}^{M_1} p_{1,k_1} \cdot z^{g_{1,k_1}}, \sum_{k_2=1}^{M_2} p_{2,k_2} \cdot z^{g_{2,k_2}},\right. \\ &\qquad \left. \ldots, \sum_{k_m=1}^{M_m} p_{m,k_m} \cdot z^{g_{m,k_m}}\right) \\ &= \sum_{k_1=1}^{M_1} \sum_{k_2=1}^{M_2} \cdots \sum_{k_m=1}^{M_m} \left(\prod_{j=1}^{m} p_{j,k_j} \cdot z^{\varphi\left(g_{1,k_1}, g_{2,k_2}, \ldots, g_{m,k_m}\right)}\right). \end{aligned}$$

B. UNCERTAINTY THEORY

As a branch of axiomatic mathematics for modeling human uncertainty, uncertainty theory was founded by Liu [23] and subsequently investigated by many researchers. Practically, uncertainty is anything that is described by belief degrees [31].

Definition 1 [23]: Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra on Γ. A set function $\mathcal{M}$: $\mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality) $\mathcal{M}\{\Gamma\} = 1$.

Axiom 2: (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$.

Axiom 3: (Subadditivity) For every sequence of $\{\Lambda_i, i = 1, 2, \ldots, n\} \in \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Then, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Besides, a product axiom was given by Liu [40].

Axiom 4: (Product) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from Γ_k for $k = 1, 2, \ldots$, respectively.

Definition 2 [23]: An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B.

Definition 3 [23]: The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 4 [23]: Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ for $\alpha \in (0, 1)$ is called the inverse uncertainty distribution of ξ.

Definition 5 [41]: The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 1 [41]: Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$, and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an uncertainty distribution

$$\begin{aligned} &\Psi(x) = \\ &\sup_{f(x_1, x_2, \ldots, x_n) = x} \left(\min_{1 \leq i \leq m} \Phi_i(x_i) \bigwedge \min_{m+1 \leq i \leq n} (1 - \Phi_i(x_i))\right), \end{aligned}$$

and an inverse uncertainty distribution

$$\begin{aligned} \Psi^{-1}(\alpha) = f\Big(&\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha) \ldots, \\ &\Phi_n^{-1}(1-\alpha)\Big). \end{aligned}$$

Definition 6 [23]: Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} \mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\} \mathrm{d}x.$$

Definition 7 [23]: Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E\left[(\xi - e)^2\right].$$

Theorem 2 [41]: Let ξ be an uncertain variable with uncertainty distribution Φ. Then

$$E[\xi] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(x),$$

and if $E[\xi]$ is a finite value e, then

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

Theorem 3 [42], [43]: Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions

$\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha)\right. \\ \left. \ldots, \Phi_n^{-1}(1-\alpha)\right) \mathrm{d}\alpha,$$

and a variance

$$V[\xi] = \int_0^1 \left(f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha)\right.\right. \\ \left.\left. \ldots, \Phi_n^{-1}(1-\alpha)\right) - e\right)^2 \mathrm{d}\alpha,$$

where e is the expected value of ξ.

Definition 8 [23]: An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a, \\ (x-a)/(2(b-a)), & \text{if } a \le x \le b, \\ (x+c-2b)/(2(c-b)), & \text{if } b \le x \le c, \\ 1, & \text{if } x \ge c. \end{cases}$$

and denoted by $Z(a, b, c)$ where a,b,c are real numbers with $a < b < c$. The inverse uncertainty distribution of zigzag uncertain variable $Z(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5, \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \ge 0.5. \end{cases}$$

III. RELIABILITY ANALYSIS FOR RANDOM UNCERTAIN MULTI-STATE SYSTEM

A. DEFINITION AND DESCRIPTION

Definition 9: A random uncertain multi-state component is defined as the multi-state component in which the state performance rates and the corresponding state probabilities are represented by uncertain variables.

Definition 10: A random uncertain multi-state system is defined as a multi-state system where one or more of its components are random uncertain multi-state components.

That is to say, the suggested random uncertain multi-state system model are based on the following two assumptions:

(1) state probabilities of a multi-state component can be represented by uncertain variables;

(2) state performance rates of a multi-state component can be presented as uncertain variables.

It is supposed that the random uncertain multi-state system we consider here consists of m components, any component j can have M_j different states with corresponding performance rates, which can be represented by the ordering uncertain variables set $\mathbf{g}_j(\gamma) = \{g_{j,1}(\gamma), g_{j,2}(\gamma), \ldots, g_{j,M_j}(\gamma)\}$, $j = 1, 2, \ldots, m$. The uncertain variable $g_{j,k_j}(\gamma)\,(g_{j,k_j}(\gamma) \ge 0)$ is the performance rate of the component j in the state k_j, $k_j = 1, 2, \ldots, M_j$, which is defined on the uncertainty space $(\Gamma_{1j}, \mathcal{L}_{1j}, \mathcal{M}_{1j})$, $\gamma \in \Gamma_{1j}$. The probabilities associated with different states for the component j can be represented by the ordering uncertain variables set $\mathbf{p}_j(\upsilon) = \{p_{j,1}(\upsilon), p_{j,2}(\upsilon), \ldots, p_{j,M_j}(\upsilon)\}$, and the uncertain variable $p_{j,k_j}(\upsilon)\,(0 \le p_{j,k_j}(\upsilon) \le 1)$ is defined on the uncertainty space $(\Gamma_{2j}, \mathcal{L}_{2j}, \mathcal{M}_{2j})$, $\upsilon \in \Gamma_{2j}$.

Furthermore, other assumptions are given as follows:

(1) The performance rates $g_{j,1}(\gamma), g_{j,2}(\gamma), \ldots, g_{j,M_j}(\gamma)$ of the component j are non-negative uncertain variables with regular uncertainty distributions. We denote the regular uncertainty distribution of $g_{j,k_j}(\gamma)$ by $\Phi_{g_{j,k_j}}(y_{j,k_j})$, $y_{j,k_j} \ge 0$ and the inverse uncertainty distribution of $g_{j,k_j}(\gamma)$ by $\Phi_{g_{j,k_j}}^{-1}(\beta)$, $\beta \in [0, 1]$, $k_j = 1, 2, \ldots, M_j$, respectively.

(2) The state probabilities $p_{j,1}(\upsilon), p_{j,2}(\upsilon), \ldots, p_{j,M_j}(\upsilon)$ of the component j are non-negative uncertain variables $(0 \le p_{j,k_j}(\upsilon) \le 1, k_j = 1, 2, \ldots, M_j)$ with regular uncertainty distributions. We denote the regular uncertainty distribution and the inverse uncertainty distribution of $p_{j,k_j}(\upsilon)$ by $\Phi_{p_{j,k_j}}(x_{j,k_j})$, $0 \le x_{j,k_j} \le 1$ and $\Phi_{p_{j,k_j}}^{-1}(\alpha)$, $\alpha \in [0, 1]$, $k_j = 1, 2, \ldots, M_j$, respectively.

(3) All components are independent of each other.

(4) The state probabilities $p_{1,k_1}(\upsilon), p_{2,k_2}(\upsilon), \ldots, p_{m,k_m}(\upsilon)$ of the m different components are independent uncertain variables.

(5) The state performance rates $g_{1,k_1}(\gamma), g_{2,k_2}(\gamma), \ldots, g_{m,k_m}(\gamma)$ of the m different components are independent uncertain variables.

In conventional multi-state system model, the state probabilities and the performance rates of each component are assumed to be crisp values. The conventional universal generating function can be directly used to analyze the model. However, in the suggested multi-state system model, the state probabilities and the performance rates of each component are presented as uncertain variables. So, combination of the uncertainty theory and universal generating function with uncertain state probabilities and performance rates can be applied to analyze the suggested multi-state system model under the random uncertain context.

Based on Definition 9 and the above assumptions, the universal generating function for the random uncertain component j we consider in this paper can be defined as follows:

$$u_j(z, \upsilon, \gamma) = \sum_{k_j=1}^{M_j} p_{j,k_j}(\upsilon) \cdot z^{g_{j,k_j}(\gamma)}, \tag{1}$$

where the state probability $p_{j,k_j}(\upsilon)$ and the performance rate $g_{j,k_j}(\gamma)$ of the component j are uncertain variables on the uncertainty spaces $(\Gamma_{1j}, \mathcal{L}_{1j}, \mathcal{M}_{1j})$ and $(\Gamma_{2j}, \mathcal{L}_{2j}, \mathcal{M}_{2j})$, respectively. Following the concept of the universal generating function for the random component, Eq. (1) is named as the uncertain universal generating function for the random uncertain component j.

To obtain the state probabilities and the state performance rates of the random uncertain multi-state system with the arbitrary structure, a general composition operator Θ_φ is introduced as

$$\begin{aligned}
&\Theta_\varphi\left(u_1(z,\upsilon,\gamma),u_2(z,\upsilon,\gamma),\ldots,u_m(z,\upsilon,\gamma)\right)\\
&=\Theta_\varphi\left(\sum_{k_1=1}^{M_1}p_{1,k_1}(\upsilon)\cdot z^{g_{1,k_1}(\gamma)},\sum_{k_2=1}^{M_2}p_{2,k_2}(\upsilon)\cdot z^{g_{2,k_2}(\gamma)},\right.\\
&\qquad\left.\ldots,\sum_{k_m=1}^{M_m}p_{m,k_m}(\upsilon)\cdot z^{g_{m,k_m}(\gamma)}\right)\\
&=\sum_{k_1=1}^{M_1}\sum_{k_2=1}^{M_2}\cdots\sum_{k_m=1}^{M_m}\left(\prod_{j=1}^{m}p_{j,k_j}(\upsilon)\cdot z^{\varphi\left(g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)\right)}\right),
\end{aligned}\tag{2}$$

where $\varphi(\cdot)$ is a measurable system structure function. According to (2), the uncertain universal generating function of the system can be written as

$$U(z,\gamma,\upsilon)=\sum_{k=1}^{M}p_{s,k}(\upsilon)\cdot z^{g_{s,k}(\gamma)},\tag{3}$$

where M is the highest possible state for the multi-state system. $p_{s,k}(\upsilon)=\prod_{j=1}^{m}p_{j,k_j}(\upsilon)$ and $g_{s,k}(\gamma)=\varphi\left(g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)\right)$ denote the probability and performance rate in the system state k, respectively. When the performance rate of the system is equal to the sum of that of components, $g_{s,k}(\gamma)=g_{1,k_1}(\gamma)+g_{2,k_2}(\gamma)+\cdots+g_{m,k_m}(\gamma)$, the operator Θ_φ is denoted as $\Theta_{\varphi P}$. When the performance rate of the system is equal to the minimum of that of components, $g_{s,k}(\gamma)=\min\left\{g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)\right\}$, the operator Θ_φ is denoted as $\Theta_{\varphi S}$.

Since the state probability $p_{j,k_j}(\upsilon)$ and performance rate $g_{j,k_j}(\gamma)$ of the component j are uncertain variables, the state probability $p_{s,k}(\upsilon)=\prod_{j=1}^{m}p_{j,k_j}(\upsilon)$ and performance rate $g_{s,k}(\gamma)=\varphi\left(g_{1,k_1}(\gamma),\ldots,g_{m,k_m}(\gamma)\right)$ of the random uncertain multi-state system are also uncertain variables. By using the operational law of uncertain variables, we can obtain the uncertainty distributions and inverse uncertainty distributions of $p_{s,k}(\upsilon)$ and $g_{s,k}(\gamma)$.

B. SYSTEM STATE PROBABILITY ANALYSIS

Since the uncertain variables $p_{1,k_1}(\upsilon)$, $p_{2,k_2}(\upsilon),\ldots,p_{m,k_m}(\upsilon)$ are independent, by Theorem 1, the uncertainty distribution function of the state probability $p_{s,k}(\upsilon)=\prod_{j=1}^{m}p_{j,k_j}(\upsilon)$ can be determined by

$$\Psi_{p_{s,k}}(x)=\sup_{\prod_{j=1}^{m}x_{j,k_j}=x}\ \min_{1\le k_j\le M_j,1\le j\le m}\Phi_{p_{j,k_j}}\left(x_{j,k_j}\right).\tag{4}$$

The inverse uncertainty distribution of $p_{s,k}(\upsilon)=\prod_{j=1}^{m}p_{j,k_j}(\upsilon)$ can be obtained as

$$\Psi^{-1}_{p_{s,k}}(\alpha)=\prod_{j=1}^{m}\Phi^{-1}_{p_{j,k_j}}(\alpha).\tag{5}$$

According to the theory of expected value and variance for uncertain variable (See Theorems 2 and 3), the expected value $\overline{p}_{s,k}$ and variance $\widehat{p}_{s,k}$ of the probability $p_{s,k}(\upsilon)$ in the system state k can be determined respectively by

$$\overline{p}_{s,k}=E\left[p_{s,k}(\upsilon)\right]=\int_0^1 x\mathrm{d}\Psi_{p_{s,k}}(x),\tag{6}$$

and

$$\widehat{p}_{s,k}=V\left[p_{s,k}(\upsilon)\right]=\int_0^1\left(x-\overline{p}_{s,k}\right)^2\mathrm{d}\Psi_{p_{s,k}}(x).\tag{7}$$

Moreover, the expected value $\overline{p}_{s,k}$ and variance $\widehat{p}_{s,k}$ of the state probability $p_{s,k}(\upsilon)$ can also be calculated by the inverse uncertainty distribution $\Psi^{-1}_{p_{s,k}}(\alpha)$ of $p_{s,k}(\upsilon)$. We have

$$\begin{aligned}
\overline{p}_{s,k}&=E\left[p_{s,k}(\upsilon)\right]\\
&=\int_0^1\Psi^{-1}_{p_{s,k}}(\alpha)\,\mathrm{d}\alpha\\
&=\int_0^1\prod_{j=1}^{m}\Phi^{-1}_{p_{j,k_j}}(\alpha)\mathrm{d}\alpha,
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
\widehat{p}_{s,k}&=V\left[p_{s,k}(\upsilon)\right]\\
&=\int_0^1\left(\Psi^{-1}_{p_{s,k}}(\alpha)-\overline{p}_{s,k}\right)^2\mathrm{d}\alpha\\
&=\int_0^1\left(\prod_{j=1}^{m}\Phi^{-1}_{p_{j,k_j}}(\alpha)-\overline{p}_{s,k}\right)^2\mathrm{d}\alpha.
\end{aligned}\tag{9}$$

Remark 1: If the system degenerates to a random multi-state system with crisp state probabilities $p_{1,k_1},p_{2,k_2},\ldots,p_{m,k_m}$, then

$$\overline{p}_{s,k}=\prod_{j=1}^{m}p_{j,k_j},\quad\widehat{p}_{s,k}=0.$$

C. SYSTEM STATE PERFORMANCE RATE ANALYSIS

Since $g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)$ are independent uncertain variables, then system structure function $g_{s,k}(\gamma)=\varphi\left(g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)\right)$ is an uncertain variable, too. If the function $\varphi(y_{1,k_1},y_{2,k_2},\ldots,y_{m,k_m})$ is strictly increasing with respect to $y_{1,k_1},y_{2,k_2},\ldots,y_{r,k_r}$ and strictly decreasing with respect to $y_{r+1,k_{r+1}},y_{r+2,k_{r+2}},\ldots,y_{m,k_m}$, the uncertainty distribution function of $g_{s,k}(\gamma)=\varphi\left(g_{1,k_1}(\gamma),g_{2,k_2}(\gamma),\ldots,g_{m,k_m}(\gamma)\right)$ can be determined by

$$\begin{aligned}
&\Psi_{g_{s,k}}(y)\\
&=\sup_{\varphi(y_{1,k_1},y_{2,k_2},\ldots,y_{m,k_m})=y}\left(\min_{1\le k_j\le M_j,1\le j\le r}\Phi_{g_{j,k_j}}(y_{j,k_j})\bigwedge\right.\\
&\qquad\left.\min_{1\le k_j\le M_j,r+1\le j\le m}\left(1-\Phi_{g_{j,k_j}}(y_{j,k_j})\right)\right).
\end{aligned}\tag{10}$$

The inverse uncertainty distribution of $g_{s,k}(\gamma)$ can be obtained as

$$\begin{aligned}
\Psi^{-1}_{g_{s,k}}(\beta)=\varphi&\left(\Phi^{-1}_{g_{1,k_1}}(\beta),\ldots,\Phi^{-1}_{g_{r,k_r}}(\beta),\right.\\
&\left.\Phi^{-1}_{g_{r+1,k_{r+1}}}(1-\beta),\ldots,\Phi^{-1}_{g_{m,k_m}}(1-\beta)\right).
\end{aligned}\tag{11}$$

Two cases (Liu [14]) are given in the random uncertain context as follows:

Case 1: Random uncertain flow transmission system with two components

Assume that the independent uncertain variables $g_{1,k_1}(\gamma)$ and $g_{2,k_2}(\gamma)$ denote the transmission capacities of the two components at their states k_1 and k_2, respectively.

(1) If any two components are connected in parallel, we have

$$g_{s,k}(\gamma) = \varphi\left(g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right) = g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma), \quad k_1 = 1, 2, \ldots, M_1, \quad k_2 = 1, 2, \ldots, M_2. \tag{12}$$

The uncertainty distribution function of $g_{s,k}(\gamma) = g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma)$ is

$$\Psi_{g_{s,k}}(y) = \sup_{y_{1,k_1}+y_{2,k_2}=y} \Phi_{g_{1,k_1}}\left(y_{1,k_1}\right) \bigwedge \Phi_{g_{2,k_2}}\left(y_{2,k_2}\right). \tag{13}$$

The inverse uncertainty distribution of $g_{s,k}(\gamma) = g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma)$ is

$$\Psi^{-1}_{g_{s,k}}(\beta) = \Phi^{-1}_{g_{1,k_1}}(\beta) + \Phi^{-1}_{g_{2,k_2}}(\beta). \tag{14}$$

(2) If any two components are connected in series, we have

$$g_{s,k}(\gamma) = \varphi\left(g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right) = \min\{g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\}, \quad k_1 = 1, 2, \ldots, M_1, \quad k_2 = 1, 2, \ldots, M_2. \tag{15}$$

The uncertainty distribution function of $g_{s,k}(\gamma) = \min\left\{g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right\}$ is

$$\Psi_{g_{s,k}}(y) = \Phi_{g_{1,k_1}}(y) \bigvee \Phi_{g_{2,k_2}}(y). \tag{16}$$

The inverse uncertainty distribution of $g_{s,k}(\gamma) = \min\left\{g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right\}$ is

$$\Psi^{-1}_{g_{s,k}}(\beta) = \Phi^{-1}_{g_{1,k_1}}(\beta) \bigwedge \Phi^{-1}_{g_{2,k_2}}(\beta). \tag{17}$$

Case 2: Random uncertain task processing system with two components

Assume that the independent uncertain variables $g_{1,k_1}(\gamma)$ and $g_{2,k_2}(\gamma)$ denote the task processing speeds of the two components at their states k_1 and k_2, respectively.

(1) If any two components are connected in parallel, we have

$$g_{s,k}(\gamma) = \varphi\left(g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right) = g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma), \quad k_1 = 1, 2, \ldots, M_1, \quad k_2 = 1, 2, \ldots, M_2. \tag{18}$$

Eqs. (18) and (12) are the same. Thus, the uncertainty distribution function and the inverse uncertainty distribution of $g_{s,k}(\gamma) = g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma)$ can be determined in the same manners as Eqs. (13) and (14), respectively.

(2) If any two components are connected in series, we have

$$g_{s,k}(\gamma) = \varphi\left(g_{1,k_1}(\gamma), g_{2,k_2}(\gamma)\right) = \frac{g_{1,k_1}(\gamma) \cdot g_{2,k_2}(\gamma)}{g_{1,k_1}(\gamma) + g_{2,k_2}(\gamma)}, \quad k_1 = 1, 2, \ldots, M_1, \quad k_2 = 1, 2, \ldots, M_2. \tag{19}$$

The function $y = \frac{y_{1,k_1} \cdot y_{2,k_2}}{y_{1,k_1}+y_{2,k_2}}$ is strictly increasing with respect to y_{1,k_1} and y_{2,k_2}, So the uncertainty distribution function of $g_{s,k}(\gamma) = \frac{g_{1,k_1}(\gamma) \cdot g_{2,k_2}(\gamma)}{g_{1,k_1}(\gamma)+g_{2,k_2}(\gamma)}$ can be given as

$$\Psi_{g_{s,k}}(y) = \sup_{\frac{y_{1,k_1} \cdot y_{2,k_2}}{y_{1,k_1}+y_{2,k_2}}=y} \Phi_{g_{1,k_1}}\left(y_{1,k_1}\right) \bigwedge \Phi_{g_{2,k_2}}\left(y_{2,k_2}\right). \tag{20}$$

The inverse uncertainty distribution of $g_{s,k}(\gamma) = \frac{g_{1,k_1}(\gamma) \cdot g_{2,k_2}(\gamma)}{g_{1,k_1}(\gamma)+g_{2,k_2}(\gamma)}$ can be given as

$$\Psi^{-1}_{g_{s,k}}(\beta) = \frac{\Phi^{-1}_{g_{1,k_1}}(\beta) \cdot \Phi^{-1}_{g_{2,k_2}}(\beta)}{\Phi^{-1}_{g_{1,k_1}}(\beta) + \Phi^{-1}_{g_{2,k_2}}(\beta)}. \tag{21}$$

According to Theorems 2 and 3, the expected value $\overline{g}_{s,k}$ and variance $\widehat{g}_{s,k}$ of the performance rate $g_{s,k}(\gamma)$ in the system state k can be determined by

$$\overline{g}_{s,k} = E\left[g_{s,k}(\gamma)\right] = \int_0^{+\infty} y \mathrm{d}\Psi_{g_{s,k}}(y), \tag{22}$$

and

$$\widehat{g}_{s,k} = V\left[g_{s,k}(\gamma)\right] = \int_0^{+\infty} \left(y - \overline{g}_{s,k}\right)^2 \mathrm{d}\Psi_{g_{s,k}}(y), \tag{23}$$

respectively.

Moreover, the expected value $\overline{g}_{s,k}$ and variance $\widehat{g}_{s,k}$ of the performance rate $g_{s,k}(\gamma)$ can also be calculated by the inverse uncertainty distribution $\Psi^{-1}_{g_{s,k}}(\beta)$ of $g_{s,k}(\gamma)$. We have

$$\begin{aligned}
\overline{g}_{s,k} &= E\left[g_{s,k}(\gamma)\right] \\
&= \int_0^1 \Psi^{-1}_{g_{s,k}}(\beta)\, \mathrm{d}\beta \\
&= \int_0^1 \varphi\Big(\Phi^{-1}_{g_{1,k_1}}(\beta), \ldots, \Phi^{-1}_{g_{r,k_r}}(\beta), \\
&\qquad \Phi^{-1}_{g_{r+1,k_{r+1}}}(1-\beta), \ldots, \Phi^{-1}_{g_{m,k_m}}(1-\beta)\Big)\, \mathrm{d}\beta,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\widehat{g}_{s,k} &= V\left[g_{s,k}(\gamma)\right] \\
&= \int_0^1 \left(\Psi^{-1}_{g_{s,k}}(\beta) - \overline{g}_{s,k}\right)^2 \mathrm{d}\beta \\
&= \int_0^1 \Big(\varphi\Big(\Phi^{-1}_{g_{1,k_1}}(\beta), \ldots, \Phi^{-1}_{g_{r,k_r}}(\beta), \\
&\qquad \Phi^{-1}_{g_{r+1,k_{r+1}}}(1-\beta), \ldots, \Phi^{-1}_{g_{m,k_m}}(1-\beta)\Big) - \overline{g}_{s,k}\Big)^2 \mathrm{d}\beta.
\end{aligned} \tag{25}$$

Remark 2: If the system degenerates to a random multi-state system with crisp state performance rates $g_{1,k_1}, g_{2,k_2}, \ldots, g_{m,k_m}$, then

$$\overline{g}_{s,k} = \varphi(g_{1,k_1}, g_{2,k_2}, \ldots, g_{m,k_m}), \quad \widehat{g}_{s,k} = 0.$$

D. RELIABILITY ASSESSMENT

To evaluate the reliability of the random uncertain multi-state system for the required crisp performance rate ω, we introduce the following δ operator over the uncertain universal

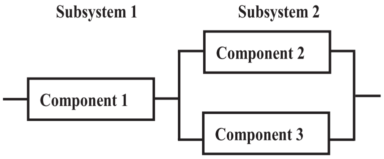

FIGURE 1. Flow transmission multi-state system structure.

generating function $U(z, \gamma, \upsilon)$ of the system

$$\overline{\delta}\left(U(z,\gamma,\upsilon),\omega\right)=\overline{\delta}\left(\sum_{k=0}^{M}p_{s,k}\left(\upsilon\right)\cdot z^{g_{s,k}(\gamma)},\omega\right)=\sum_{k=0}^{M}\overline{\delta}\left(p_{s,k}\left(\upsilon\right)\cdot z^{g_{s,k}(\gamma)},\omega\right),\quad(26)$$

where

$$\overline{\delta}\left(p_{s,k}\left(\upsilon\right)\cdot z^{g_{s,k}(\gamma)},\omega\right)=\begin{cases}\overline{p}_{s,k}, & \overline{g}_{s,k}\geq\omega,\\ 0, & \overline{g}_{s,k}<\omega.\end{cases}$$

$\overline{p}_{s,k}$ can be obtained by Eqs. (6) or (8), and $\overline{g}_{s,k}$ can be obtained by Eqs. (22) or (24).

The reliability of the random uncertain multi-state system, denoted by R_ω, is defined as the expected value of the probability that the expected value of the system performance rate greater than the user demand ω, and is written as

$$R_\omega=\overline{\delta}\left(U(z,\gamma,\upsilon),\omega\right)=\sum_{\overline{g}_{s,k}\geq\omega}\overline{p}_{s,k}.\quad(27)$$

IV. ILLUSTRATIVE EXAMPLE

In this section, a numerical example is provided to illustrate how to compute the state probabilities, performance rates and reliability of random uncertain multi-state systems. Consider a random uncertain flow transmission multi-state series-parallel system with two parallel subsystems connected in series, as shown in Fig. 1. For subsystem 1, there is only one component, and the component has two possible states 1 and 2. For subsystem 2, there are two different components, and each component has three possible states 1, 2 and 3. For each component j ($j = 1, 2, 3$), the state probability $p_{j,k_j}(\upsilon)$ and performance rate $g_{j,k_j}(\gamma)$ are treated as uncertain variables with zigzag uncertainty distributions from the uncertainty space $\left(R_{1j},\mathcal{L}_{1j},\mathcal{M}_{1j}\right)$ to R_{1j} $\left(R_{1j}=\{x\,|x\in[0,1]\}\right)$ and from the uncertainty space $\left(R_{2j},\mathcal{L}_{2j},\mathcal{M}_{2j}\right)$ to R_{2j} $\left(R_{2j}=\{x\,|x\geq 0\}\right)$, respectively. They are listed in Table 1.

According to Eq. (1), the uncertain universal generating functions of the system components are given as

$$\begin{aligned}u_1(z,\upsilon,\gamma)&=p_{1,1}(\upsilon)\cdot z^{g_{1,1}(\gamma)}+p_{1,2}(\upsilon)\cdot z^{g_{1,2}(\gamma)},\\ u_2(z,\upsilon,\gamma)&=p_{2,1}(\upsilon)\cdot z^{g_{2,1}(\gamma)}+p_{2,2}(\upsilon)\cdot z^{g_{2,2}(\gamma)}\\&\quad+p_{2,3}(\upsilon)\cdot z^{g_{2,3}(\gamma)},\\ u_3(z,\upsilon,\gamma)&=p_{3,1}(\upsilon)\cdot z^{g_{3,1}(\gamma)}+p_{3,2}(\upsilon)\cdot z^{g_{3,2}(\gamma)}\\&\quad+p_{3,3}(\upsilon)\cdot z^{g_{3,3}(\gamma)}.\end{aligned}$$

According to Eqs. (2) and (3), we can obtain the uncertain universal generating function of the random uncertain flow transmission multi-state series-parallel system. There are eight different states corresponding to performance rates greater than zero in the system. The performance rates and state probabilities corresponding to these states are

State 1: $\min\{g_{1,2}(\gamma), g_{2,2}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,2}(\upsilon)p_{3,1}(\upsilon)$.

State 2: $\min\{g_{1,2}(\gamma), g_{2,3}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,3}(\upsilon)p_{3,1}(\upsilon)$.

State 3: $\min\{g_{1,2}(\gamma), g_{3,2}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,1}(\upsilon)p_{3,2}(\upsilon)$.

State 4: $\min\{g_{1,2}(\gamma), g_{3,3}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,1}(\upsilon)p_{3,3}(\upsilon)$.

State 5: $\min\{g_{1,2}(\gamma), g_{2,2}(\gamma)+g_{3,2}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,2}(\upsilon)p_{3,2}(\upsilon)$.

State 6: $\min\{g_{1,2}(\gamma), g_{2,3}(\gamma)+g_{3,2}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,3}(\upsilon)p_{3,2}(\upsilon)$.

State 7: $\min\{g_{1,2}(\gamma), g_{2,2}(\gamma)+g_{3,3}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,2}(\upsilon)p_{3,3}(\upsilon)$.

State 8: $\min\{g_{1,2}(\gamma), g_{2,3}(\gamma)+g_{3,3}(\gamma)\}$ and $p_{1,2}(\upsilon)p_{2,3}(\upsilon)p_{3,3}(\upsilon)$.

According to Eqs. (5) and (11), the inverse uncertainty distributions of the performance rates and corresponding state probabilities of the system when the performance rates are greater than zero can be given as:

State 1: $\Psi_{g_{s1}}^{-1}(\beta)=1.2+0.6\beta,\ 0<\beta<1,$

$$\Psi_{p_{s1}}^{-1}(\alpha)=\begin{cases}(0.946+0.008\alpha)\cdot(0.145+0.01\alpha)\cdot(0.047+0.006\alpha), & 0<\alpha<0.5,\\ (0.947+0.006\alpha)\cdot(0.145+0.01\alpha)\cdot(0.047+0.006\alpha), & 0.5\leq\alpha<1.\end{cases}$$

State 2: $\Psi_{g_{s2}}^{-1}(\beta)=1.6+0.8\beta,\ 0<\beta<1,$

$$\Psi_{p_{s2}}^{-1}(\alpha)=\begin{cases}(0.946+0.008\alpha)\cdot(0.745+0.01\alpha)\cdot(0.047+0.006\alpha), & 0<\alpha<0.5,\\ (0.947+0.006\alpha)\cdot(0.745+0.01\alpha)\cdot(0.047+0.006\alpha), & 0.5\leq\alpha<1.\end{cases}$$

TABLE 1. Characteristics of the components for the random uncertain multi-state system.

component	State	State probability	Performance rate
1	1	$p_{1,1}(\upsilon)=Z(0.048, 0.050, 0.053)$	$g_{1,1}(\gamma)=0$
	2	$p_{1,2}(\upsilon)=Z(0.946, 0.950, 0.953)$	$g_{1,2}(\gamma)=Z(4.0, 5.2, 6.0)$
	1	$p_{2,1}(\upsilon)=Z(0.095, 0.100, 0.105)$	$g_{2,1}(\gamma)=0$
2	2	$p_{2,2}(\upsilon)=Z(0.145, 0.150, 0.155)$	$g_{2,2}(\gamma)=Z(1.2, 1.5, 1.8)$
	3	$p_{2,3}(\upsilon)=Z(0.745, 0.750, 0.755)$	$g_{2,3}(\gamma)=Z(1.6, 2.0, 2.4)$
	1	$p_{3,1}(\upsilon)=Z(0.047, 0.050, 0.053)$	$g_{3,1}(\gamma)=0$
3	2	$p_{3,2}(\upsilon)=Z(0.094, 0.100, 0.106)$	$g_{3,2}(\gamma)=Z(1.8, 2.0, 2.2)$
	3	$p_{3,3}(\upsilon)=Z(0.846, 0.850, 0.854)$	$g_{3,3}(\gamma)=Z(2.5, 3.0, 3.5)$

TABLE 2. Expected values and variances of the uncertain performance rates and state probabilities for the random uncertain multi-state system.

State k	Expected value $\overline{p}_{sk}$	Variance $\widehat{p}_{sk}$	Expected value $\overline{g}_{sk}$	Variance $\widehat{g}_{sk}$
1	0.0071	0.16×10^{-6}	1.50	0.0300
2	0.0356	2.09×10^{-6}	2.00	0.0533
3	0.0095	0.39×10^{-6}	2.00	0.0133
4	0.0807	7.41×10^{-6}	3.00	0.0833
5	0.0143	0.64×10^{-6}	3.50	0.0833
6	0.0713	8.37×10^{-6}	4.00	0.1200
7	0.1211	8.51×10^{-6}	4.50	0.2133
8	0.6055	27.7×10^{-6}	4.99	0.2838

State 3: $\Psi_{g_{s3}}^{-1}(\beta) = 1.8 + 0.4\beta, 0 < \beta < 1,$

$$\Psi_{p_{s3}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.095+0.01\alpha)\cdot(0.094+0.012\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.095+0.01\alpha)\cdot(0.094+0.012\alpha), & 0.5\le\alpha<1. \end{cases}$$

State 4: $\Psi_{g_{s4}}^{-1}(\beta) = 2.5 + \beta, 0 < \beta < 1,$

$$\Psi_{p_{s4}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.095+0.01\alpha)\cdot(0.846+0.008\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.095+0.01\alpha)\cdot(0.846+0.008\alpha), & 0.5\le\alpha<1. \end{cases}$$

State 5: $\Psi_{g_{s5}}^{-1}(\beta) = 3 + \beta, 0 < \beta < 1,$

$$\Psi_{p_{s5}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.145+0.01\alpha)\cdot(0.094+0.012\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.145+0.01\alpha)\cdot(0.094+0.012\alpha), & 0.5\le\alpha<1. \end{cases}$$

State 6: $\Psi_{g_{s6}}^{-1}(\beta) = 3.4 + 1.2\beta, 0 < \beta < 1,$

$$\Psi_{p_{s6}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.745+0.01\alpha)\cdot(0.094+0.012\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.745+0.01\alpha)\cdot(0.094+0.012\alpha), & 0.5\le\alpha<1. \end{cases}$$

State 7: $\Psi_{g_{s7}}^{-1}(\beta) = 3.7 + 1.6\beta, 0 < \beta < 1,$

$$\Psi_{p_{s7}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.145+0.01\alpha)\cdot(0.846+0.008\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.145+0.01\alpha)\cdot(0.846+0.008\alpha), & 0.5\le\alpha<1. \end{cases}$$

TABLE 3. Reliability for the random uncertain multi-state system at four distinct ω values.

User demand ω	3	3.5	4	4.5
Reliability	0.8929	0.8122	0.7979	0.7266

$$\text{State 8: } \Psi_{g_{s8}}^{-1}(\beta) = \begin{cases} 4.0 + 2.4\beta, & 0 < \beta < 1/6, \\ 4.1 + 1.8\beta, & 1/6 < \beta < 1, \end{cases}$$

$$\Psi_{p_{s8}}^{-1}(\alpha) = \begin{cases} (0.946+0.008\alpha)\cdot(0.745+0.01\alpha)\cdot(0.846+0.008\alpha), & 0<\alpha<0.5, \\ (0.947+0.006\alpha)\cdot(0.745+0.01\alpha)\cdot(0.846+0.008\alpha), & 0.5\le\alpha<1. \end{cases}$$

Table 2 shows the expected values and variances of the performance rates and state probabilities of the system when the performance rates are greater than zero by using Eqs.(8), (9), (24) and (25). According to Eq. (27), the reliability for the random uncertain multi-state system can be obtained. Table 3 shows that the obtained results at four distinct ω values: 3.0, 3.5, 4.0 and 4.5.

V. CONCLUDING REMARKS

This paper proposed the definition of random uncertain multi-state system model based on probability theory and uncertainty theory. The state probabilities and performance rates of the system components are presented as uncertain variables. The uncertain universal generating function was introduced to evaluate the reliability of random uncertain multi-state system. The uncertainty distributions and inverse uncertainty distributions of some indices for the system were analyzed. The expected values and variances of these uncertain indices were calculated based on obtained uncertainty and inverse uncertainty distributions. To illustrate how to compute the expected values and variances of these uncertain indices, a numerical example was given in the end. The proposed model is applicable for the reliability and performance evaluation of multi-state systems when we have no samples but belief degree from the experts.

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LINMIN HU received the Ph.D. degree in operations research and management science from the School of Economics and Management, Yanshan University, China, in 2014, where he is currently an Associate Professor with the Department of Applied Mathematics. He has published more than 30 papers in journals, including *Applied Mathematical Modelling*, the *Journal of Intelligent and Fuzzy Systems*, the *Journal of Systems Science and Complexity*, *Eksploatacja i Niezawodnosc-Maintenance and Reliability*, and other journals. His research interests include system reliability, operations research, and stochastic models.

DEQUAN YUE received the Ph.D. degree in operations research and cybernetics from the Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China, in 1998. He is currently a Professor with the Department of Statistics, School of Science, Yanshan University, Qinhuangdao, China. He has published more than 100 papers in journals, including *Naval Research Logistics*, *Operations Research Letters*, *Computer Communications*, *Applied Mathematical Modelling*, the *Journal of Industrial and Management Optimization*, *Optimization and Engineering*, and other journals. His main research areas include performance analysis of queuing systems, optimization and inventory control of queuing-inventory systems, analysis of reliability models, stochastic order, and its applications.

GUOXI ZHAO received the Ph.D. degree in operations research and management science from the School of Economics and Management, Yanshan University, China, in 2017. He is currently a Professor with the School of Mathematics and Information Science, Xinxiang University, China. His research interests include equilibrium strategies of queuing systems and stochastic models.