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Extended Darlington Synthesis of Fractional Order Immittance Function With Two Element Orders

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ABSTRACT Actual circuits have fractional order characteristics essentially. With the widespread application of fractional order circuits and the great strides in the manufacturing of fractional order elements in recent years, passive synthesis of fractional order circuit becomes an important research content. In this paper, classical Darlington's synthesis is extended to the two-variable case. Based on two-variable Darlington's synthesis and variable substitution, synthesis of fractional order immittance function with two element orders is proposed. Finally, an example is given to illustrate the calculation process.

INDEX TERMS Fractional order, passive circuit, network synthesis, circuit theory.

I. INTRODUCTION

Because of inherently fractional order characteristics of materials [1], actual circuits have fractional order characteristics essentially [1]. With the fractional order network widely used in different fields [2], [3], there have been many aspects of research on fractional order network, for example: analysis of frequency response [4], transmission efficiency of energy [5], approximation of the fractional order system by an integer order one [6], etc. In recent years, the great strides in manufacturing of fractional order devices [7], [8] make it possible to establish actual fractional order circuit, and the fractional circuit models [9], have been used, these motivate the need to develop the synthesis methods for fractional order networks. Therefore, it is necessary to study fractional order network synthesis. However, fractional order network synthesis is in its infancy, [10] and [11] can realize integer order RLC two-port terminated one fractional order element, [12] proposes a synthesis method of fractional order two-port. And progress has also been made in the design of fractional order oscillator [13], fractional order filter [14], fractional order devices [15] and analysis of fractional order circuits [16]. Recently, a new synthesis method in [17] is proposed based on multivariable network synthesis theory [18], it provides a new idea for fractional order network synthesis.

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The mathematical basis of one-variable passive realization theory is proposed by Brune [19]. After that, Darlington [20] published his integer order lossy network synthesis method. Based on transducer gain and scattering parameters, he derives a reactance matrix and then establishes a reactance ladder network between the power source and a resistor. This method is suitable to arbitrary one-variable positive real function. Since then, Darlington synthesis is the cornerstone of designing digital filters [21], equalizers [22], matching networks [23], and similar circuits. In 1961, Hazony [24] extended the algorithm of Darlington synthesis on the nonreciprocal structure. Amarit and Sanjit [25] extended Darlington synthesis to the two-variable case and changed an extracted unit resistor into an impedance function in 1975, but this method has restrictions on the function form and only can be used to realize reciprocity network. After that, Dewilde [26] conducted a detailed analysis of Darlington synthesis and Carlin Herbert [27] summarized Darlington Synthesis. Then Belevitch [28] proposed Darlington synthesis to arbitrary integer order $2n$-port in 2011.

After Ozaki and Kasami [29] introduced the concept of multivariable positive real property in 1960, multivariate synthesis theory has made a great progress [30]–[38]. In the case of two-variable, [39] can realize lossless reciprocal network, [40] can realize lossy reciprocal network, and arbitrarily LC n-port can be realized by [41]. Furthermore, there are three-variable [18] reciprocal immittance function $Z\left(p_1, p_2, p_3\right)$ synthesis method, but this method requires $\partial Z\left(p_1, p_2, p_3\right)/\partial p_i$ to be completely squared.

This paper firstly needs a two-variable immittance function synthesis method without restrictions. However, traditional Darlington synthesis [27] and paper [24] only can synthesize one-variable immittance function, and their reactance matrix synthesis method based on residues can't be directly extend to the two-variable case. On the other hand, paper [25], a two-variable Darlington synthesis method, constraints on the form of immittance function. To solve this problem, this paper finds a suitable reactance matrix synthesis method [41], then proposes a Darlington synthesis of two-variable immittance function without restrictions. Based on above mentioned, this paper also gets a conclusion that the necessary and sufficient condition for a two-variable immittance function to be passively realized is that the function is positive real.

This paper is organized as follows. Section 2 mainly presents two-variable Darlington synthesis. And Section 3 briefly introduces two-variable reactance matrix synthesis method of paper [41]. In Section 4, synthesis of two-variable fractional order immittance function with two element orders is proposed, then an application is given to illustrate it. The conclusion of this paper is Section 5.

II. TWO-VARIABLE DARLINGTON SYNTHESIS

As mentioned in the introduction, traditional Darlington synthesis and its expansion method [24] cannot be used directly to synthesize two-variable immittance function, and paper [25] can't solve the problems required by this paper due to its limitation on the immittance function. So, proposing a synthesis method of two-variable immittance function without constraints is necessary.

A. EXTRACTION OF AN UNIT RESISTOR

Definition 1 [37], [41]: A $n \times n$ matrix $\boldsymbol{Z}\left(p_1, p_2\right)$ is said to be a two-variable positive real matrix, if

1) For real p_1 and p_2, the elements of $\boldsymbol{Z}\left(p_1, p_2\right)$ is real;
2) In the domain $Re\left[p_1\right] > 0$ and $Re[p_2] > 0$, the elements of $\boldsymbol{Z}\left(p_1, p_2\right)$ are analytic;
3) $\boldsymbol{Z}\left(p_1, p_2\right) + \boldsymbol{Z}^H\left(p_1, p_2\right)$ is a positive semidefinite matrix in the domain $Re\left[p_1\right] > 0$ and $Re[p_2] > 0$.

where superscript H represents the conjugate transpose of matrices. Since this paper studies finite lumped networks, the functions mentioned in this paper are rational.

Definition 2 [37], [41]: A $n \times n$ matrix $\boldsymbol{X}\left(p_1, p_2\right)$ is said to be a two-variable reactance matrix, if

1) $\boldsymbol{X}\left(p_1, p_2\right)$ is a two-variable positive real matrix;
2) $\boldsymbol{X}\left(p_1, p_2\right) + \boldsymbol{X}^H\left(p_1, p_2\right) \equiv \mathbf{0}_n$.

This paper takes a two-variable impedance function $Z\left(p_1, p_2\right)$ to illustrate the calculation process, admittance functions have similar ideas.

When a unit resistor is extracted from $Z\left(p_1, p_2\right)$, the network of $Z\left(p_1, p_2\right)$ will become Fig. 1, where N expressed by 2×2 matrix $\boldsymbol{X}\left(p_1, p_2\right)$

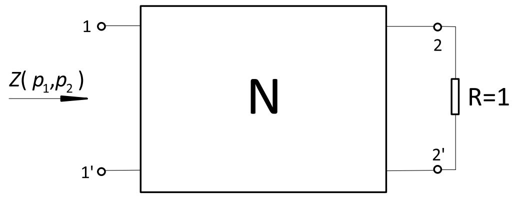

FIGURE 1. The network of impedance function $Z\left(p_1, p_2\right)$.

According to the relationship between port-voltage and port-current, equation (1) can be obtained.

$$Z\left(p_1, p_2\right) = x_{11}(p_1, p_2) - \frac{x_{12}(p_1, p_2)x_{21}(p_1, p_2)}{1 + x_{22}(p_1, p_2)} = x_{11}(p_1, p_2)\frac{1 + \frac{x_{11}(p_1,p_2)x_{22}(p_1,p_2) - x_{12}(p_1,p_2)x_{21}(p_1,p_2)}{x_{11}(p_1,p_2)}}{1 + x_{22}(p_1, p_2)} \tag{1}$$

On the other hand, the numerator and denominator of $Z\left(p_1, p_2\right)$ can be divided into odd parts and even parts respectively, then $Z\left(p_1, p_2\right)$ can be written as

$$Z(p_1, p_2) = \frac{o_1(p_1, p_2) + e_1(p_1, p_2)}{o_2(p_1, p_2) + e_2(p_1, p_2)} \tag{2}$$

where polynomials $o_i(p_1, p_2)$ and $e_i(p_1, p_2)(i = 1, 2)$ are odd and even part respectively.

To make $x_{11}(p_1, p_2)$ and $x_{22}(p_1, p_2)$ odd, the calculation of (2) can be divided into two cases.

Case A:

$$Z\left(p_1, p_2\right) = \frac{e_1}{o_2} \cdot \frac{1 + \frac{o_1}{e_1}}{1 + \frac{e_2}{o_2}} \tag{3a}$$

Case B:

$$Z\left(p_1, p_2\right) = \frac{o_1}{e_2} \cdot \frac{1 + \frac{e_1}{0_1}}{1 + \frac{o_2}{e_2}} \tag{3b}$$

By comparing equations (1) and (3), the following equations are obtained.

$$\bar{x}_{11}\left(p_1, p_2\right) = \frac{e_1}{o_2} \tag{4a}$$

$$\bar{x}_{22}\left(p_1, p_2\right) = \frac{e_2}{o_2} \tag{4b}$$

$$\frac{\bar{x}_{11}\left(p_1, p_2\right)\bar{x}_{22}\left(p_1, p_2\right) - \bar{x}_{12}\left(p_1, p_2\right)\bar{x}_{21}\left(p_1, p_2\right)}{\bar{x}_{11}\left(p_1, p_2\right)} = \frac{o_1}{e_1} \tag{4c}$$

$$\bar{\bar{x}}_{11}\left(p_1, p_2\right) = \frac{o_1}{e_2} \tag{5a}$$

$$\bar{\bar{x}}_{22}\left(p_1, p_2\right) = \frac{o_2}{e_2} \tag{5b}$$

$$\frac{\bar{\bar{x}}_{11}\left(p_1, p_2\right)\bar{\bar{x}}_{22}\left(p_1, p_2\right) - \bar{\bar{x}}_{12}\left(p_1, p_2\right)\bar{\bar{x}}_{21}\left(p_1, p_2\right)}{\bar{\bar{x}}_{11}\left(p_1, p_2\right)} = \frac{e_1}{0_1} \tag{5c}$$

where $\bar{x}_{ij}\left(p_1, p_2\right)$ denotes case A; $\bar{\bar{x}}_{ij}\left(p_1, p_2\right)$ denotes case B, $i, j = 1, 2$. Substituting equations (4a) and (4b) into (4c), we can find

$$\bar{x}_{12}(p_1,p_2)\,\bar{x}_{21}(p_1,p_2)=\frac{e_1e_2-o_1o_2}{o_2^2} \tag{6}$$

Substituting equations (4d) and (4d) into (4d), we also can find

$$\bar{\bar{x}}_{12}(p_1,p_2)\,\bar{\bar{x}}_{21}(p_1,p_2)=\frac{o_1o_2-e_1e_2}{e_2^2} \tag{7}$$

However, $e_1e_2-o_1o_2$ and $o_1o_2-e_1e_2$ are not always a simple form to be factorized.

To ensure the feasibility of the synthesis process, this paper introduces two-variable positive real polynomial $L(p_1,p_2)=o_0(p_1,p_2)+e_0(p_1,p_2)$ and makes

$$e_0^2-o_0^2=e_1e_2-o_1o_2 \tag{8}$$

where $o_0(p_1,p_2)$ is odd, $e_0(p_1,p_2)$ is even. According to paper [29], $(e_1e_2-o_1o_2)|_{p_1=j\omega_1,\,p_2=j\omega_2}\geq 0$.

In one-variable case, paper [24] does not give a clear calculation method for equation (8), this paper solves this problem by using two-variable spectral factorization.

The calculation method of $L(p_1,p_2)=o_0+e_0$ is as follows:

The property $(e_1e_2-o_1o_2)|_{p_1=j\omega_1,\,p_2=j\omega_2}\geq 0$ ensures that two-variable spectral factorization [42], [43] can be performed. Then, polynomial $e_1e_2-o_1o_2$ is performed spectral factorization and equation (9) is obtained.

$$e_1e_2-o_1o_2=L(-p_1,-p_2)\,L(p_1,p_2) \tag{9}$$

The calculation of two-variable spectral factorization is in Appendix A. Hereafter, dividing polynomial $L(p_1,p_2)$ into odd part and even part, that is,

$$L(p_1,p_2)=o_0+e_0 \tag{10}$$

Obviously, when $Re[p_1]=0$ and $Re[p_2]=0$,

$$L(-p_1,-p_2)=-o_0+e_0 \tag{11}$$

The next calculation process is divided into reciprocity and nonreciprocity two situations.

B. EXPRESSIONS OF REACTANCE MATRIX IN NONRECIPROCAL SITUATION

According to equation (8), equation (6) and (7) becomes

$$\bar{x}_{\alpha,12}(p_1,p_2)\,\bar{x}_{\alpha,21}(p_1,p_2)=\frac{e_0^2-o_0^2}{o_2^2} \tag{12a}$$

$$\bar{\bar{x}}_{\alpha,12}(p_1,p_2)\,\bar{\bar{x}}_{\alpha,21}(p_1,p_2)=\frac{o_0^2-e_0^2}{e_2^2} \tag{12b}$$

where, the right subscript α denotes nonreciprocal situation. It is easy to factorize equation (12a) and get equation (13).

$$\bar{x}_{\alpha,12}(p_1,p_2)=\pm\frac{e_0+o_o}{o_2}\bar{x}_{\alpha,21}(p_1,p_2)=\pm\frac{e_0-o_o}{o_2} \tag{13}$$

Similarly, it is easy to factorize equation (12b) get equation (14).

$$\bar{\bar{x}}_{\alpha,12}(p_1,p_2)=\pm\frac{o_0+e_o}{e_2}\bar{\bar{x}}_{\alpha,21}(p_1,p_2)=\pm\frac{o_0-e_o}{e_2} \tag{14}$$

Remark: equation (13) is also feasible to take $\bar{x}_{\alpha,12}(p_1,p_2)=\pm\frac{e_0-o_o}{o_2}$ and $\bar{x}_{\alpha,21}(p_1,p_2)=\pm\frac{e_0+o_o}{o_2}$, but there is no essential difference between their network. Hence, this paper just takes the case of equation (13). The case of equation (14) is similar.

In summary, when $e_0(p_1,p_2)\neq 0$ and $o_0(p_1,p_2)\neq 0$, the expressions of $\boldsymbol{X}(p_1,p_2)$ in nonreciprocal situation are

$$\bar{\boldsymbol{X}}_\alpha(p_1,p_2)=\begin{bmatrix}\frac{e_1}{o_2} & \pm\frac{e_0+o_o}{o_2}\\ \pm\frac{e_0-o_o}{o_2} & \frac{e_2}{o_2}\end{bmatrix} \tag{15}$$

$$\bar{\bar{\boldsymbol{X}}}_\alpha(p_1,p_2)=\begin{bmatrix}\frac{o_1}{e_2} & \pm\frac{o_0+e_o}{e_2}\\ \pm\frac{o_0-e_o}{e_2} & \frac{o_2}{e_2}\end{bmatrix} \tag{16}$$

Theorem 1: $\bar{\boldsymbol{X}}_\alpha(p_1,p_2)$ and $\bar{\bar{\boldsymbol{X}}}_\alpha(p_1,p_2)$ are reactance matrix.

The proof of Theorem 1 is in Appendix B.

C. EXPRESSIONS OF REACTANCE MATRIX IN RECIPROCAL AND ANTIRECIPROCAL SITUATION

If the numerator and denominator of equation (3) is multiplied by factor $L(p_1,p_2)=o_0+e_0$, $Z(p_1,p_2)$ will become equation (17).

$$\begin{aligned}Z(p_1,p_2)&=\frac{o_1+e_1}{o_2+e_2}\cdot\frac{o_0+e_0}{o_0+e_0}\\&=\frac{(o_1e_0+e_1o_0)+(o_1o_0+e_1e_0)}{(o_2e_0+e_2o_0)+(o_2o_0+e_2e_0)}\end{aligned} \tag{17}$$

Similar to the idea of Section A, we can get

$$\bar{x}_{\beta,11}(p_1,p_2)=\frac{o_1o_0+e_1e_0}{o_2e_0+e_2o_0}\bar{x}_{\beta,22}(p_1,p_2)=\frac{o_2o_0+e_2e_0}{o_2e_0+e_2o_0} \tag{18a}$$

$$\bar{\bar{x}}_{\beta,11}(p_1,p_2)=\frac{o_1e_0+e_1o_0}{o_2o_0+e_2e_0}\bar{\bar{x}}_{\beta,22}(p_1,p_2)=\frac{o_2e_0+e_2o_0}{o_2o_0+e_2e_0} \tag{18b}$$

and

$$\bar{x}_{\beta,12}(p_1,p_2)\,\bar{x}_{\beta,21}(p_1,p_2)=\frac{(e_0^2-o_0^2)^2}{(o_2e_0+e_2o_0)^2} \tag{19}$$

$$\bar{\bar{x}}_{\beta,12}(p_1,p_2)\,\bar{\bar{x}}_{\beta,21}(p_1,p_2)=-\frac{(e_0^2-o_0^2)^2}{(o_2o_0+e_2e_0)^2} \tag{20}$$

Using reciprocal property $x_{\beta,12}(p_1,p_2)=x_{\beta,21}(p_1,p_2)$ for (19). Using antireciprocal property $-x_{\beta,12}(p_1,p_2)=x_{\beta,21}(p_1,p_2)$ for equation (20). The results are

$$\bar{x}_{\beta,12}(p_1,p_2)=\bar{x}_{\beta,21}(p_1,p_2)=\pm\frac{e_0^2-o_0^2}{o_2e_0+e_2o_0} \tag{21}$$

$$\bar{\bar{x}}_{\beta,12}(p_1,p_2) = -\bar{\bar{x}}_{\beta,21}(p_1,p_2) = \pm\frac{e_0^2 - o_0^2}{o_2 o_0 + e_2 e_0} \tag{22}$$

The expressions of $\boldsymbol{X}(p_1,p_2)$ in reciprocal and antireciprocal situation are

$$\bar{\boldsymbol{X}}_\beta(p_1,p_2) = \begin{bmatrix} \frac{o_1 o_0 + e_1 e_0}{o_2 e_0 + e_2 o_0} & \pm\frac{e_0^2 - o_0^2}{o_2 e_0 + e_2 o_0} \\ \pm\frac{e_0^2 - o_0^2}{o_2 e_0 + e_2 o_0} & \frac{o_2 o_0 + e_2 e_0}{o_2 e_0 + e_2 o_0} \end{bmatrix} \tag{23}$$

$$\bar{\bar{\boldsymbol{X}}}_\beta(p_1,p_2) = \begin{bmatrix} \frac{o_1 e_0 + e_1 o_0}{o_2 o_0 + e_2 e_0} & \pm\frac{e_0^2 - o_0^2}{o_2 o_0 + e_2 e_0} \\ \mp\frac{e_0^2 - o_0^2}{o_2 o_0 + e_2 e_0} & \frac{o_2 e_0 + e_2 o_0}{o_2 o_0 + e_2 e_0} \end{bmatrix} \tag{24}$$

When $e_0(p_1,p_2) = 0$ and $o_0(p_1,p_2) = 0$, four reactance matrices in formula (15) and (16) are

$$\bar{\boldsymbol{X}}_\alpha(p_1,p_2) = \begin{bmatrix} \frac{e_1}{o_2} & 0 \\ 0 & \frac{e_2}{o_2} \end{bmatrix} \tag{25}$$

$$\bar{\bar{\boldsymbol{X}}}_\alpha(p_1,p_2) = \begin{bmatrix} \frac{o_1}{e_2} & 0 \\ 0 & \frac{o_2}{e_2} \end{bmatrix} \tag{26}$$

And the four reactance matrices in formula (23) and (24) are $\frac{0}{0}$ type, so they will not be discussed.

When $e_0(p_1,p_2) = 0$ or $o_0(p_1,p_2) = 0$, four reactance matrices in formula (15) and (16) can be transformed into reciprocal and antireciprocal situation, and it is also possible to simplify four reactance matrices in formula (23) and (24), the simplified results are shown in Table 1.

TABLE 1. Simplified results of reactance matrices.

	$\bar{\boldsymbol{X}}_\alpha(p_1,p_2)$	$\bar{\bar{\boldsymbol{X}}}_\alpha(p_1,p_2)$	$\bar{\boldsymbol{X}}_\beta(p_1,p_2)$	$\bar{\bar{\boldsymbol{X}}}_\beta(p_1,p_2)$
$e_0 = 0$	$\begin{bmatrix} \frac{e_1}{o_2} & \pm\frac{o_0}{o_2} \\ \mp\frac{o_0}{o_2} & \frac{e_2}{o_2} \end{bmatrix}$	$\begin{bmatrix} \frac{o_1}{e_2} & \pm\frac{o_0}{e_2} \\ \pm\frac{o_0}{e_2} & \frac{o_2}{e_2} \end{bmatrix}$	$\begin{bmatrix} \frac{o_1}{e_2} & \pm\frac{o_0}{e_2} \\ \pm\frac{o_0}{e_2} & \frac{o_2}{e_2} \end{bmatrix}$	$\begin{bmatrix} \frac{e_1}{o_2} & \mp\frac{o_0}{o_2} \\ \pm\frac{o_0}{o_2} & \frac{e_2}{o_2} \end{bmatrix}$
$o_0 = 0$	$\begin{bmatrix} \frac{e_1}{o_2} & \pm\frac{e_0}{o_2} \\ \pm\frac{e_0}{o_2} & \frac{e_2}{o_2} \end{bmatrix}$	$\begin{bmatrix} \frac{o_1}{e_2} & \pm\frac{e_0}{e_2} \\ \mp\frac{e_0}{e_2} & \frac{o_2}{e_2} \end{bmatrix}$	$\begin{bmatrix} \frac{e_1}{o_2} & \pm\frac{e_0}{o_2} \\ \pm\frac{e_0}{o_2} & \frac{e_2}{o_2} \end{bmatrix}$	$\begin{bmatrix} \frac{e_1}{o_2} & \pm\frac{e_0}{o_2} \\ \mp\frac{e_0}{o_2} & \frac{e_2}{o_2} \end{bmatrix}$

From Table I, the results in formula (15) and (16) are equivalent to those in formula (23) and (24). Hence, formula (23) and (24) also meet Theorem 1.

D. SUMMARY OF TWO-VARIABLE DARLINGTON SYNTHESIS

The process of two-variable Darlington synthesis is as follows.

1) Dividing $Z(p_1,p_2)$ into odd parts and even parts,

$$Z(p_1,p_2) = \frac{o_1(p_1,p_2) + e_1(p_1,p_2)}{o_2(p_1,p_2) + e_2(p_1,p_2)} \tag{27}$$

where, polynomials $o_i(p_1,p_2)$ and $e_i(p_1,p_2)(i = 1,2)$ are odd and even part, respectively.

2) Polynomial $e_1e_2 - o_1o_2$ is performed spectral factorization [42], [43] and we can get

$$e_1e_2 - o_1o_2 = L(-p_1,-p_2)\,L(p_1,p_2) = e_0^2 - o_0^2 \tag{28}$$

3) When $e_0(p_1,p_2) \neq 0$ and $o_0(p_1,p_2) \neq 0$, the result is formula (15) and (16);
When $e_0(p_1,p_2) = 0$ or $o_0(p_1,p_2) = 0$, the result is formula (23) and (24);
When $e_0(p_1,p_2) = 0$ and $o_0(p_1,p_2) = 0$, the result is formula (25) and (26).

4) Combining with Section 3, the final two-variabl network is obtained.

III. TWO-VARIABLE REACTANCE MATRIX SYNTHESIS

Paper [25], a method based on extracting immittance function, uses traditional one-variable reactance matrix synthesis method, which led to limitation of synthesized immittance function form. This paper uses suitable synthesis method [41] without effect on synthesized immittance function form. The synthesis steps of $\boldsymbol{X}(p_1,p_2)$ is as follows.

1) Assuming that reactance matrix $\boldsymbol{X}(p_1,p_2)$ has no independent pole, the concept of independent poles and how to extract them are referenced to [41].

2) Writing $\boldsymbol{X}(p_1,p_2)$ as the following form.

$$\boldsymbol{X}(p_1,p_2) = \frac{\boldsymbol{B}_0(p_1)\,p_2^r + \boldsymbol{B}_1(p_1)\,p_2^{r-1} + \cdots + \boldsymbol{B}_r(p_1)}{a_0(p_1)\,p_2^r + a_1(p_1)\,p_2^{r-1} + \cdots + a_r(p_1)} \tag{29}$$

where scalar

$$g(p_1,p_2) = a_0(p_1)\,p_2^r + a_1(p_1)\,p_2^{r-1} + \cdots + a_r(p_1) \tag{30}$$

is the least common denominator of matrix $\boldsymbol{X}(p_1,p_2)$.

3) Calculating $\boldsymbol{A}_l(p_1)$, the expression is as follows.

$$\boldsymbol{A}_l(p_1) = \frac{(-1)^{l+1}}{a_0^{l+2}} \begin{vmatrix} \boldsymbol{B}_0 & a_0 & 0 & \cdots & 0 \\ \boldsymbol{B}_1 & a_1 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{B}_l & a_l & a_{l-1} & \cdots & a_0 \\ \boldsymbol{B}_{l+1} & a_{l+1} & a_l & \cdots & a_1 \end{vmatrix} \tag{31}$$

where, $l = -1, 0, 1, \cdots$. When $l \geq r$, $\boldsymbol{B}_l(p_1) = \boldsymbol{0}_n$ and $a_l(p_1) = 0$.

4) Set

$$\boldsymbol{N}_{r-1}(p_1) = \begin{bmatrix} \boldsymbol{A}_0 & \cdots & \boldsymbol{A}_{r-1} \\ -\boldsymbol{A}_1 & \cdots & -\boldsymbol{A}_r \\ \vdots & \ddots & \vdots \\ (-1)^{r-1}\boldsymbol{A}_{r-1} & \cdots & (-1)^{r-1}\boldsymbol{A}_{2r-2} \end{bmatrix} \tag{32}$$

and then calculating $\hat{\boldsymbol{N}}_{r-1}(p_1) = a_0^{2r}(p_1)\,\boldsymbol{N}_{r-1}(p_1)$.

5) Matrix $\hat{\boldsymbol{N}}_{r-1}(p_1)$ can be factorized [44], that is, when r is odd and $a_0(p_1) = -a_0(-p_1)$,

$$-\hat{\boldsymbol{N}}_{r-1}(p_1) = \boldsymbol{M}(p_1)\boldsymbol{M}^T(-p_1) \tag{33a}$$

other case,

$$\hat{\boldsymbol{N}}_{r-1}(p_1) = \boldsymbol{M}(p_1)\boldsymbol{M}^T(-p_1) \tag{33b}$$

6) $\boldsymbol{M}(p_1)$ is divided into r submatrices $\boldsymbol{M}_i(p_1)$ yields $\boldsymbol{M}(p_1) = \begin{bmatrix} \boldsymbol{M}_0(p_1) & \boldsymbol{M}_1(p_1) & \cdots \boldsymbol{M}_{r-1}(p_1) \end{bmatrix}^T$.

7) Set

$$\boldsymbol{\Omega}(p_1) = \begin{bmatrix} 0_n & \boldsymbol{E}_n & \cdots & 0_n \\ 0_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & \cdots & \boldsymbol{E}_n \\ -\frac{a_r}{a_0}\boldsymbol{E}_n & -\frac{a_{r-1}}{a_0}\boldsymbol{E}_n & \cdots & -\frac{a_1}{a_0}\boldsymbol{E}_n \end{bmatrix} \tag{34}$$

8) The expression of one-variable reactance matrix $\hat{\boldsymbol{X}}(p_1)$ is

$$\hat{\boldsymbol{X}}(p_1) = \begin{bmatrix} \boldsymbol{X}(p_1, \infty) & \frac{\boldsymbol{M}_0(p_1)}{a_0^r(p_1)} \\ -\frac{\boldsymbol{M}_0^T(-p_1)}{a_0^r(-p_1)} & \boldsymbol{M}^{-1}(p_1)\boldsymbol{\Omega}(-p_1)\boldsymbol{M}(p_1) \end{bmatrix} \tag{35}$$

where $\boldsymbol{M}^{-1}(p_1)$ is the left inverse [45] of $\boldsymbol{M}(p_1)$.

9) The network of $\hat{\boldsymbol{X}}(p_1)$ can be realized by traditional synthesis method [46], [47].

10) Terminating rank $[\boldsymbol{N}_{r-1}(p_1)]$ p_2-type unit inductors on the network of $\hat{\boldsymbol{X}}(p_1)$, then this paper gets the network of $\boldsymbol{X}(p_1, p_2)$.

Theorem 2: The necessary and sufficient condition for a two-variable immittance function to be passively realized is that the function is positive real.

The proof of Theorem 2 is in Appendix C.

IV. SYNTHESIS OF FRACTIONAL ORDER IMMITTANCE FUNCTION WITH TWO ELEMENT ORDERS

A. SYNTHESIS STEPS

This paper takes impedance function as an example, and admittance function is the same.

There is a fractional order impedance function with two element orders $Z(s)$ and it can be written as

$$Z(s) = \frac{\sum_{k_1=0}^{l}\sum_{k_2=0}^{t} b_{k_1k_2} s^{k_1\alpha + k_2\beta}}{\sum_{k_1=0}^{l}\sum_{k_2=0}^{t} c_{k_1k_2} s^{k_1\alpha + k_2\beta}} + d_1 s^\alpha + d_2 s^\beta \tag{36}$$

The synthesis steps are as follows.

1) Set $s^\alpha = p_1, s^\beta = p_2, Z(s)$ is transformed into two-variable impedance function $Z(p_1, p_2)$, i.e. equation (37).

$$Z(p_1, p_2) = \frac{\sum_{k_1=0}^{l}\sum_{k_2=0}^{t} p_1^{k_1} p_2^{k_2}}{\sum_{k_1=0}^{l}\sum_{k_2=0}^{t} p_1^{k_1} p_2^{k_2}} + d_1 p_1 + d_2 p_2 \tag{37}$$

2) Realizing the network of $Z(p_1, p_2)$ by two-variable Darlington synthesis method.

3) p_1-type and p_2-type elements are transformed into fractional order elements by $p_1 = s^\alpha, p_2 = s^\beta$.

Immittance matrix of a network composed of fractional order elements with their orders ranging from 0 and 1, and other passive elements are multivariable positive real matrix through appropriate variable substitutions $s^\alpha = p_1, s^\beta = p_2$ [48].

The sign of fractional order LC elements is shown in Fig. 2, where, C_α and L_β is the value of fractional order capacitor and fractional order inductor respectively; α and β is the order of fractional order capacitor and fractional order inductor respectively.

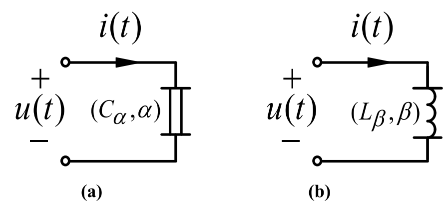

FIGURE 2. The sign of fractional order elements. (a) Fractional order capacitor. (b) Fractional order inductor.

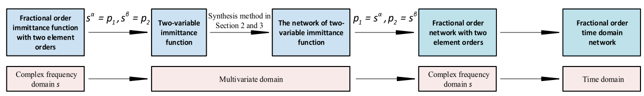

FIGURE 3. Synthesis of fractional order immittance functions with two element orders.

The whole process of synthesizing fractional order immittance function with two element orders is shown in Fig. 3.

B. APPLICATION EXAMPLE

This section illustrates above synthesis process, then gives two resultant fractional order networks and their simulation.

There is a fractional order impedance function with two element orders,

$$Z(s) = \frac{s^{1.1} + s^{0.95}}{2s^{0.95} + s^{0.8} + s^{0.3} + s^{0.15}}$$

Set $s^\alpha = s^{0.15} = p_1, s^\beta = s^{0.8} = p_2$, then $Z(p_1, p_2)$ is obtained, i.e.,

$$Z(p_1, p_2) = \frac{p_1^2 p_2 + p_1 p_2}{p_1^2 + 2p_1 p_2 + p_1 + p_2}$$

Odd part and even part in $Z(p_1, p_2)$ are

$$o_1 = p_1^2 p_2 \quad e_1 = p_1 p_2$$
$$o_2 = p_1 + p_2 \quad e_2 = p_1^2 + 2p_1 p_2$$

Using equation (8), we find that

$$e_0^2 - o_0^2 = e_1 e_2 - o_1 o_2$$

$$= p_1 p_2 \left(p_1^2 + 2p_1 p_2\right) - p_1^2 p_2 (p_1 + p_2) = p_1^2 p_2^2$$

Obviously, $e_0 = p_1 p_2$; $o_0 = 0$.

Since $o_0 = 0$, the resultant reactance matrix is reciprocal or antireciprocal. This paper takes two results.

1) When all the parameters $e_1, e_2, e_0, o_1, o_2, o_0$ be gotten, reactance matrix $\bar{X}_\beta(p_1, p_2)$ can be obtained from formula (23) directly.

$$\bar{X}_\beta(p_1, p_2) = \begin{bmatrix} \dfrac{p_1 p_2}{p_1 + p_2} & \dfrac{p_1 p_2}{p_1 + p_2} \\ \dfrac{p_1 p_2}{p_1 + p_2} & \dfrac{p_1^2 + 2p_1 p_2}{p_1 + p_2} \end{bmatrix}$$

Synthesis method in Section 3 can realize its network, this network as shown in Fig. 4(a).

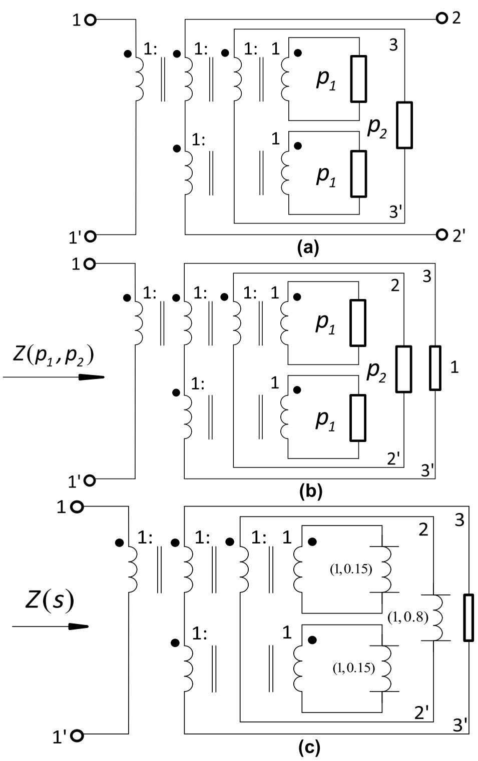

FIGURE 4. (a) The network of $\bar{X}_\beta(p_1, p_2)$. (b) The network of $Z(p_1, p_2)$. (c) The network of $Z(s)$.

The network of $\bar{X}_\beta(p_1, p_2)$ terminating a unit resistor, then we can obtain the network of $Z(p_1, p_2)$, as shown in Fig. 4(b).

To get the network of $Z(s)$, this paper replaces two-variable elements with fractional order elements in Fig. 4(b) by using $p_1 = s^{0.15}$ and $p_2 = s^{0.8}$. The result as shown in Fig. 4(c).

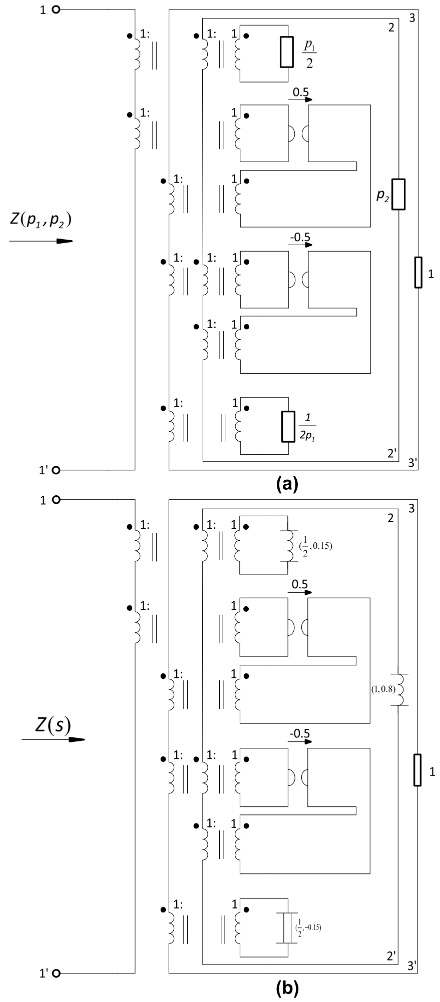

FIGURE 5. (a) The network of $Z(p_1, p_2)$. (b) The network of $Z(s)$.

2) According to formula (24), the other reactance matrix $\bar{\bar{X}}_\beta(p_1, p_2)$ is

$$\bar{\bar{X}}_\beta(p_1, p_2) = \begin{bmatrix} \dfrac{p_1 p_2}{p_1 + 2p_2} & \dfrac{p_1 p_2}{p_1^2 + 2p_1 p_2} \\ -\dfrac{p_1 p_2}{p_1^2 + 2p_1 p_2} & \dfrac{p_1 + p_2}{p_1^2 + 2p_1 p_2} \end{bmatrix}$$

The synthesis process is similar to those of $\bar{X}_\beta(p_1, p_2)$. The network of $Z(p_1, p_2)$ as shown in Fig. 5(a).

By using variable substitution $p_1 = s^{0.15}$ and $p_2 = s^{0.8}$, the resultant fractional order network as shown in Fig. 5(b).

In order to verify the correctness of resultant networks, two current excitations shown in Fig. 6 are applied to fractional order impedance function and fractional order network respectively, then this paper gets two voltages by frequency domain analysis.

The voltage responses are shown in Fig. 7.

Since the zero-impedance branch may appear in the circuits, the modified nodal approach (MNA) [49] is used to establish the circuit equation in simulation. In Figure 7,

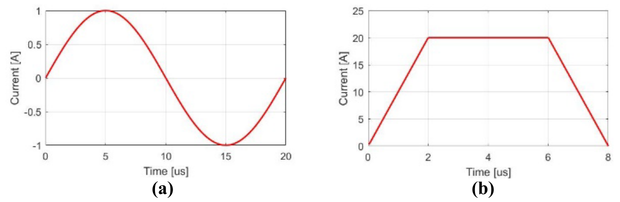

FIGURE 6. **Current excitation. (a) Sinusoidal steady-state current. (b) Step transient current.**

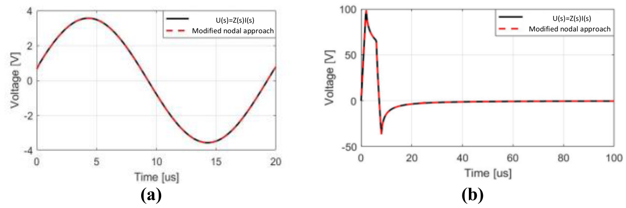

FIGURE 7. **Port-voltage response. (a) Sinusoidal steady-state current. (b) Step transient current.**

"$U(s) = Z(s)I(s)$" and "Modified nodal approach" are consistent, hence the resultant networks are correct.

V. CONCLUSION

The necessary and sufficient condition for a two-variable immittance function to be passively realized is that the function is positive real. This paper extends classical Darlington synthesis to two-variable case, then above method is applied to synthesize fractional order immittance function with two element orders.

APPENDIX A

SPECTRAL FACTRICATION OF TWO-VARIABLE POLYNOMIALS

There is a two-variable polynomial $H(p_1, p_2)$. If $H(p_1, p_2)|_{\substack{p_1=j\omega_1 \\ p_2=j\omega_2}} \geq 0$, then $H(p_1, p_2)$ can be factored into

$$H(p_1, p_2) = F(-p_1, -p_2)\, F(p_1, p_2) \tag{A-1}$$

The following is a brief description of the calculation process.

1) Writing $H(p_1, p_2)$ as the following form.

$$H(p_1, p_2) = \sum_{k=0}^{M_1} \sum_{l=0}^{M_2} h_{(k,l)} p_1^k p_2^l \tag{A-2}$$

where M_1 and M_2 are the highest order of p_1 and p_2, respectively.

2) Set matrix $\boldsymbol{\Phi}_i$,

$$\boldsymbol{\Phi}_i = \begin{bmatrix} h_{(0,i)} & h_{(1,i)} & \cdots & h_{(\frac{M_1}{2},i)} \\ 0 & 0 & \cdots & -h_{(\frac{M_1}{2}+1,i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^{\frac{M_1}{2}} h_{(M_1,i)} \end{bmatrix} \tag{A-3}$$

where $i = 0, 1, \cdots, M_2$. If M_1 and M_2 are odd, then replace them with $M_1 + 1$ and $M_2 + 1$ respectively.

3) Set $(\frac{M_1}{2}+1)(\frac{M_2}{2}+1)-$dimensional matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{\psi}_{(0,0)} & \boldsymbol{\psi}_{(0,1)} & \cdots & \boldsymbol{\psi}_{(0,\frac{M_2}{2})} \\ \boldsymbol{\psi}_{(1,0)} & \mathbf{0} & \cdots & \boldsymbol{\psi}_{(1,\frac{M_2}{2})} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\psi}_{(\frac{M_2}{2}-1,0)} & \mathbf{0} & \cdots & \boldsymbol{\psi}_{(\frac{M_2}{2}-1,\frac{M_2}{2})} \\ \boldsymbol{\psi}_{(\frac{M_2}{2},0)} & \boldsymbol{\psi}_{(\frac{M_2}{2},1)} & \cdots & \boldsymbol{\psi}_{(\frac{M_2}{2},\frac{M_2}{2})} \end{bmatrix} \tag{A-4}$$

where

$$\boldsymbol{\psi}_{(0,0)} = \frac{1}{2}(\boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_0^{\mathrm{T}}) \tag{A-5a}$$

$$\boldsymbol{\psi}_{(0,i)} = \frac{1}{2}\boldsymbol{\Phi}_i \tag{A-5b}$$

$$\boldsymbol{\psi}_{(i,0)} = \frac{1}{2}\boldsymbol{\Phi}_i^{\mathrm{T}} \tag{A-5c}$$

$$\boldsymbol{\psi}_{(i,\frac{M_2}{2})} = (-1)^i \boldsymbol{\Phi}_{\frac{M_2}{2}+i} \tag{A-5d}$$

$$\boldsymbol{\psi}_{(\frac{M_2}{2},i)} = (-1)^i \boldsymbol{\Phi}_{\frac{M_2}{2}+i}^{\mathrm{T}} \tag{A-5e}$$

$$\boldsymbol{\psi}_{(\frac{M_2}{2},\frac{M_2}{2})} = (-1)^{\frac{M_2}{2}} \frac{1}{2}(\boldsymbol{\Phi}_{M_2} + \boldsymbol{\Phi}_{M_2}^{\mathrm{T}}) \tag{A-5f}$$

and $i < \frac{M_1}{2}, \frac{M_2}{2}$.

4) After that, set matrices $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{\sigma}_4$, we can obtain

$$\boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & a & \cdots & b \\ 0 & \vdots & 0 & \vdots \\ \vdots & b & \vdots & c \\ 0 & 0 & \cdots & 0 \end{bmatrix} \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a & \cdots & b & 0 \\ \vdots & 0 & \vdots & \vdots \\ b & \cdots & c & 0 \end{bmatrix} \tag{A-6a}$$

$$\boldsymbol{\sigma}_3 = \begin{bmatrix} \mathbf{0} & d & e \\ \vdots & e & f \\ \mathbf{0}_{M_2+1} & \cdots & \mathbf{0} \end{bmatrix} \boldsymbol{\sigma}_4 = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0}_{M_2+1} \\ d & e & \vdots \\ e & f & \mathbf{0} \end{bmatrix} \tag{A-6b}$$

5) Solving linear matrix inequality (LMI) [50],

$$\boldsymbol{B} = \boldsymbol{A} - \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_3 - \boldsymbol{\sigma}_4 \geq \mathbf{0} \tag{A-7}$$

Then we can get a, b, c, d, e, f.

6) Constant matrix $\boldsymbol{B}$ is factorized [51], $1 \times (\frac{M_1}{2}+1)(\frac{M_2}{2}+1)$ matrix $\bar{\boldsymbol{F}}$ is obtained.

$$\boldsymbol{B} = \bar{\boldsymbol{F}}^{\mathrm{T}} \bar{\boldsymbol{F}} \tag{A-8}$$

7) The resultant polynomial $F(p_1, p_2)$ is

$$F(p_1, p_2) = \bar{\boldsymbol{F}} \cdot \begin{bmatrix} 1 & p_1 & \cdots & p_1^{\frac{M_1}{2}} & p_2 & p_1 p_2 & \cdots & p_1^{\frac{M_1}{2}} p_2^{\frac{M_1}{2}} \end{bmatrix}^{\mathrm{T}} \tag{A-9}$$

APPENDIX B
THE PROOF OF THEOREM 1

Lemma 1 [37]: The necessary and sufficient condition for a $n \times n$ matrix $\boldsymbol{X}\left(p_1, p_2\right)$ to be a two-variable positive matrix is that

$$\boldsymbol{S}\left(p_1, p_2\right) = [\boldsymbol{X} - \boldsymbol{E}]\,[\boldsymbol{X} + \boldsymbol{E}]^{-1} \geq \boldsymbol{0} \qquad \text{(B-1)}$$

is analytic in the$Re\,[p_1] > 0$, $Re\,[p_2] > 0$ and

$$\boldsymbol{E} - \boldsymbol{S}^H\left(p_1, p_2\right)\boldsymbol{S}\left(p_1, p_2\right) \geq \boldsymbol{0}_n \qquad \text{(B-2)}$$

for $Re\,[p_1] = 0 Re\,[p_2] = 0$. Except at singularities of indeterminacy, if they exist. Where $\boldsymbol{E}$ is unit matrix.

Since $\bar{\boldsymbol{X}}_\alpha\left(p_1, p_2\right)$ and $\bar{\bar{\boldsymbol{X}}}_\alpha\left(p_1, p_2\right)$ are real rational function matrix, so the real property is holds.

Subscript "+" and "−" presents that (15) and (16) take upper and lower sign, respectively. Based on equation (8), the scattering matrices of $\bar{\boldsymbol{X}}_{\alpha,+}\left(p_1, p_2\right)$ and $\bar{\bar{\boldsymbol{X}}}_{\alpha,+}\left(p_1, p_2\right)$ are

$$\bar{\boldsymbol{S}}\left(p_1, p_2\right) = \begin{bmatrix} \frac{m}{n} & \frac{L}{n} \\ \frac{L^*}{n} & -\frac{m^*}{n} \end{bmatrix} \qquad \text{(B-3a)}$$

$$\bar{\bar{\boldsymbol{S}}}\left(p_1, p_2\right) = \begin{bmatrix} \frac{m}{n} & \frac{L}{n} \\ -\frac{L^*}{n} & \frac{m^*}{n} \end{bmatrix} \qquad \text{(B-3b)}$$

where

$$mm^* + LL^* = nn^* \qquad \text{(B-4)}$$

And even parts $n_e\left(p_1, p_2\right) = \frac{e_1+e_2}{2}$, $m_e\left(p_1, p_2\right) = \frac{e_1-e_2}{2}$; odd parts $n_o\left(p_1, p_2\right) = \frac{o_1+o_2}{2}$, $m_o\left(p_1, p_2\right) = \frac{o_1-o_2}{2}$. This paper takes

$$\boldsymbol{S}\left(p_1, p_2\right) = \begin{bmatrix} \frac{m}{n} & \frac{L}{n} \\ \pm\frac{L^*}{n} & \mp\frac{m^*}{n} \end{bmatrix} \qquad \text{(B-5)}$$

The combination of equation (B-4) and (B-5) can get

$$\boldsymbol{E} - \boldsymbol{S}\left(p_1, p_2\right)\boldsymbol{S}^H\left(p_1, p_2\right) = \boldsymbol{0} \qquad \text{(B-6)}$$

Based on Lemma 1, Definition 1 and Definition 2, $\bar{\boldsymbol{X}}_{\alpha,+}\left(p_1, p_2\right)$ and $\bar{\bar{\boldsymbol{X}}}_{\alpha,+}\left(p_1, p_2\right)$ are reactance matrix.

The relationship between $\bar{\boldsymbol{X}}_{\alpha,+}\left(p_1, p_2\right)$ and $\bar{\boldsymbol{X}}_{\alpha,-}\left(p_1, p_2\right)$ is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \bar{\boldsymbol{X}}_{\alpha,+}\left(p_1, p_2\right) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \bar{\boldsymbol{X}}_{\alpha,-}\left(p_1, p_2\right) \qquad \text{(B-7)}$$

Hence $\bar{\boldsymbol{X}}_{\alpha,+}\left(p_1, p_2\right)$ and $\bar{\boldsymbol{X}}_{\alpha,-}\left(p_1, p_2\right)$ are similar, $\bar{\boldsymbol{X}}_{\alpha,-}\left(p_1, p_2\right)$ is also reactance matrix. Similarly, $\bar{\bar{\boldsymbol{X}}}_{\alpha,-}\left(p_1, p_2\right)$ is reactance matrix.

In summary, $\bar{\boldsymbol{X}}_\alpha\left(p_1, p_2\right)$ and $\bar{\bar{\boldsymbol{X}}}_\alpha\left(p_1, p_2\right)$ are reactance matrix. ■

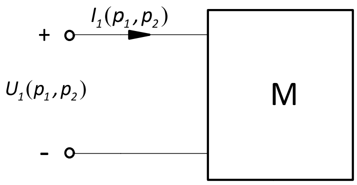

FIGURE 8. Two-variable passive network M.

APPENDIX C
THE PROOF OF THEOREM 2

***Necessary*:** This paper takes the impedance function $Z\left(p_1, p_2\right)$ an example, the proof of admittance function is similar.

Supposing that the two-variable passive network M shown in Fig. 8 is composed of p_1−type and p_2−type passive inductors, p_1−type and p_2−type passive capacitors, passive resistors, ideal transformers and ideal gyrators. The input impedance of network M is $Z\left(p_1, p_2\right)$.

According to generalized Tellegen's theorem [52], equation (C-1) is obtained.

$$-U_1\left(p_1, p_2\right) I_1^*\left(p_1, p_2\right) + \sum_{k=2}^{b} U_k\left(p_1, p_2\right) I_k^*\left(p_1, p_2\right) = 0 \qquad \text{(C-1)}$$

where, $U_k\left(p_1, p_2\right)$ and $I_k\left(p_1, p_2\right)$ $(k = 2, \cdots, b)$are internal branch voltage and current in the network M, respectively. Equation (C-1) can be changed into equation (C-2).

$$U_1\left(p_1, p_2\right) I_1^*\left(p_1, p_2\right) = \sum_{k=2}^{b} U_k\left(p_1, p_2\right) I_k^*\left(p_1, p_2\right) \qquad \text{(C-2)}$$

The expression of $Z\left(p_1, p_2\right)$ can be written as equation (C-3).

$$Z\left(p_1, p_2\right) = \frac{U_1\left(p_1, p_2\right)}{I_1\left(p_1, p_2\right)} = \frac{U_1\left(p_1, p_2\right) I_1^*\left(p_1, p_2\right)}{I_1\left(p_1, p_2\right) I_1^*\left(p_1, p_2\right)} = \frac{1}{\left|I_1\left(p_1, p_2\right)\right|^2} \sum_{k=2}^{b} U_k\left(p_1, p_2\right) I_k^*\left(p_1, p_2\right) \qquad \text{(C-3)}$$

Polynomial $\sum_{k=2}^{b} U_k\left(p_1, p_2\right) I_k^*\left(p_1, p_2\right)$ can be decomposed into equation (C-4).

$$\sum_{k=2}^{b} U_k\left(p_1, p_2\right) I_k^*\left(p_1, p_2\right) = F_0\left(p_1, p_2\right) + \left[p_1 T_{01}\left(p_1, p_2\right) + p_2 T_{02}\left(p_1, p_2\right)\right] + [\frac{1}{p_1} V_{01}\left(p_1, p_2\right) + \frac{1}{p_2} V_{02}\left(p_1, p_2\right)] \qquad \text{(C-4)}$$

And each part of (C-4) is

$$F_0(p_1,p_2) = \sum_{k=2}^{b} R_k \left|I_k(p_1,p_2)\right|^2$$

$$T_{01}(p_1,p_2) = \sum_{k=2}^{b} L_{k1} \left|I_k(p_1,p_2)\right|^2 + \sum_{\substack{k=2\\k\neq l}}^{b}\sum_{l=2}^{b} M_{kl1} I_k^*(p_1,p_2) I_l(p_1,p_2) \tag{C-5a, C-5b}$$

$$T_{02}(p_1,p_2) = \sum_{k=2}^{b} L_{k2} \left|I_k(p_1,p_2)\right|^2 + \sum_{\substack{k=2\\k\neq l}}^{b}\sum_{l=2}^{b} M_{kl2} I_k^*(p_1,p_2) I_l(p_1,p_2) \tag{C-5c}$$

$$V_{01}(p_1,p_2) = \sum_{k=2}^{b} C_{k1} \left|I_k(p_1,p_2)\right|^2 \tag{C-5d}$$

$$V_{02}(p_1,p_2) = \sum_{k=2}^{b} C_{k2} \left|I_k(p_1,p_2)\right|^2 \tag{C-5e}$$

where, $F_0(p_1,p_2)$ is the power consumed by all the branch resistors; $T_{01}(p_1,p_2)$ and $T_{02}(p_1,p_2)$ are average magnetic energy; $V_{01}(p_1,p_2)$ and $V_{02}(p_1,p_2)$ are average electric field energy. For sinusoidal steady-state signals, the average magnetic energy stored on the coupled inductors is

$$\xi_{T_{01}}(p_1,p_2) = \frac{1}{4}\sum_{k=2}^{b} L_{k1} I_k(p_1,p_2) I_k^*(p_1,p_2) + \frac{1}{4}\sum_{\substack{k=2\\k\neq l}}^{b}\sum_{l=2}^{b} M_{kl1} I_k^*(p_1,p_2) I_l(p_1,p_2) = \frac{1}{4}T_{01}(p_1,p_2) \geq 0 \tag{C-6a}$$

$$\xi_{T_{02}}(p_1,p_2) = \frac{1}{4}\sum_{k=2}^{b} L_{k2} I_k(p_1,p_2) I_k^*(p_1,p_2) + \frac{1}{4}\sum_{\substack{k=2\\k\neq l}}^{b}\sum_{l=2}^{b} M_{kl2} I_k^*(p_1,p_2) I_l(p_1,p_2) = \frac{1}{4}T_{02}(p_1,p_2) \geq 0 \tag{C-6b}$$

Since all the elements in network M are passive, the values of R_k, C_{k1} and C_{k2} are all equal to or greater than 0. Obviously, all the parts in (C-4) are greater than or equal to 0, we further know that $Z(p_1,p_2) \geq 0$. In addition, the values of $R_k, L_{k1}, L_{k2}, M_{kl1}, M_{kl2}, C_{k1}$ and C_{k2} are real, so all the coefficients of $Z(p_1,p_2)$ are real.

From above mentions, $Z(p_1,p_2)$ is positive real function.

Sufficiency: The two-variable Darlington synthesis presented in this paper is a passive synthesis method. This method does not make any special requirement for the positive real function in calculation process, so it is suitable to arbitrary two-variable real positive function. ■

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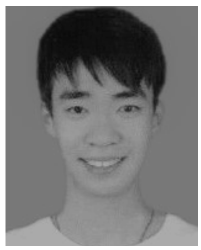

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