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New Quantum Hermite-Hadamard Inequalities Utilizing Harmonic Convexity of the Functions

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ABSTRACT The aim of this paper is to obtain some new Hermite–Hadamard type of inequalities via harmonic convex, strongly harmonic convex, strongly harmonic log-convex functions, and AH-convex in connection with quantum calculus. All the results reduce to ordinary calculus case when $q \to 1^-$.

INDEX TERMS Convex, harmonic convex, strongly harmonic convex, strongly harmonic log-convex, quantum, Hermite-Hadamard, inequalities.

I. INTRODUCTION

A function $\mathcal{F} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function on $\mathcal{I}$, if

$$\mathcal{F}((1-\mathfrak{t})x + \mathfrak{t}y) \leq (1-\mathfrak{t})\mathcal{F}(x) + \mathfrak{t}\mathcal{F}(y), \quad \forall x, \quad y \in \mathcal{I}, \mathfrak{t} \in [0,1].$$

In recent years much attention has been given in studying various aspects of convex functions. As a result this concept has been extended and generalized in different directions, see [1], [2], [4], [11]–[13], [17].

The concept of strong convexity has been introduced by Polyak [13] in connection with solving some extremum problems. Let $[a, b] \subseteq \mathbb{R}$ be an interval and c be a positive number. A function $\mathcal{F} : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is called strongly convex with modulus c if

$$\begin{aligned}&\mathcal{F}((1-\mathfrak{t})x + \mathfrak{t}y)\\ &\quad\leq (1-\mathfrak{t})\mathcal{F}(x) + \mathfrak{t}\mathcal{F}(y) - ct(1-\mathfrak{t})(y-x)^2,\\ &\qquad \forall x, y \in [a, b], \quad \mathfrak{t} \in [0,1].\end{aligned}$$

Note that when $c = 0$, the concept of strong convexity reduces to the concept of classical convexity.

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Harmonic convex sets are defined as:

Definition 1 [17]: A set $\mathcal{H} \subseteq (0, +\infty)$ is said to be a harmonic convex set, if

$$\frac{xy}{\mathfrak{t}x + (1-\mathfrak{t})y} \in \mathcal{H}, \quad \forall x, y \in \mathcal{H}, \ \mathfrak{t} \in [0,1].$$

Işcan [4] introduced and studied the class of harmonic convex functions.

Definition 2 [4]: A function $\mathcal{F} : \mathcal{I} \subseteq (0, +\infty) \to \mathbb{R}$ is said to be harmonic convex function, if

$$\mathcal{F}\left(\frac{xy}{\mathfrak{t}x + (1-\mathfrak{t})y}\right) \leq (1-\mathfrak{t})\mathcal{F}(x) + \mathfrak{t}\mathcal{F}(y), \quad \forall x, y \in \mathcal{I}, \ \mathfrak{t} \in [0,1]. \quad \text{(I.1)}$$

Recently Noor *et al.* [12] defined the class of strongly harmonic convex functions as:

Definition 3: A function $\mathcal{F} : \mathcal{I} \subseteq (0, +\infty) \to \mathbb{R}$ is said to be strongly harmonic convex function with modulus $c > 0$, if

$$\begin{aligned}&\mathcal{F}\left(\frac{xy}{\mathfrak{t}x + (1-\mathfrak{t})y}\right)\\ &\quad\leq (1-\mathfrak{t})\mathcal{F}(x) + \mathfrak{t}\mathcal{F}(y) - ct(1-\mathfrak{t})\left(\frac{x-y}{xy}\right)^2,\\ &\qquad \forall x, y \in \mathcal{I}, \quad \mathfrak{t} \in [0,1].\end{aligned}$$

The function $\mathcal{F}$ is said to be strongly harmonic concave function with modulus $c > 0$, if $-\mathcal{F}$ is strongly harmonic convex function.

Noor *et al.* [12] also introduced the notion of strongly harmonic log-convex functions.

Definition 4 [12]: A function $\mathcal{F}:\mathcal{I}\subseteq(0,+\infty)\to\mathbb{R}_+$ is said to be strongly harmonic log-convex function on $\mathcal{I}$, if

$$\mathcal{F}\left(\frac{xy}{\mathfrak{t}x+(1-\mathfrak{t})y}\right) \leq (\mathcal{F}(x))^{1-\mathfrak{t}}(\mathcal{F}(y))^{\mathfrak{t}} - ct(1-\mathfrak{t})\Big(\frac{x-y}{xy}\Big)^2, \quad \forall x,y\in\mathcal{I},\quad \mathfrak{t}\in(0,1).$$

Remark 5: Strongly harmonic log-convex function implies strongly harmonic convex function.

Remark 6: . Let $\mathcal{I}=[a,b]\subseteq\mathbb{R}\setminus\{0\}$ and consider the function $\mathcal{G}:[\frac{1}{b},\frac{1}{a}]\to\mathbb{R}$ defined by $\mathcal{G}(\mathfrak{t})=\mathcal{F}(\frac{1}{\mathfrak{t}})$. Then $\mathcal{F}$ is strongly harmonic convex function on $[a,b]$ with modulus $c>0$, if and only if, $\mathcal{G}$ is strongly convex function on $[\frac{1}{b},\frac{1}{a}]$ with modulus $c>0$.

In order to avoid any confusion with the class of AH-convex functions, the functions satisfying the condition

$$\mathcal{F}(\mathfrak{t}x+(1-\mathfrak{t})y) \leq \frac{\mathcal{F}(x)\mathcal{F}(y)}{(1-\mathfrak{t})\mathcal{F}(x)+\mathfrak{t}\mathcal{F}(y)}, \tag{I.2}$$

we call the class of functions satisfying (I.1) as HA-convex functions (see [1], [4]).

The following result will be helpful in obtaining our main results.

Proposition 7 [2]: If $[a,b]\subset\mathcal{I}\subset(0,\infty)$ and if we consider the function $\mathcal{G}:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, defined by $\mathcal{G}(\mathfrak{t})=\mathcal{F}(\frac{1}{\mathfrak{t}})$, then $\mathcal{F}$ is harmonic convex function on $[a,b]$ if and only if $\mathcal{G}$ is convex in classical sense on $\left[\frac{1}{b},\frac{1}{a}\right]$.

Convexity in connection with inequalities has also attracted many researchers from all over the world. Many mathematical inequalities have been obtained using the concept of convex functions, for example, see [3]. For a convex function $\mathcal{F}:[a,b]\to\mathbb{R}$, the following inequality is well known in the literature as the Hermite-Hadamard inequality:

$$\mathcal{F}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(u)du \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{I.3}$$

This famous result of Hermite and Hadamard which provides us an equivalent condition to convexity has extensively been studied. This inequality plays significant role in theory of means and in numerical analysis. Numerous extensions for the result have been obtained in the literature, for example see [3].

We now recall some basic concepts of quantum calculus. Let $0<q<1$, the q-Jackson integral from 0 to b is defined by [5] as:

$$\int_0^b \mathcal{F}(x)\mathrm{d}_qx = (1-q)b\sum_{n=0}^{\infty}\mathcal{F}(bq^n)q^n$$

provided the sum converge absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by [5]

$$\int_a^b \mathcal{F}(x)\mathrm{d}_qx = \int_0^b \mathcal{F}(x)\mathrm{d}_qx - \int_0^a \mathcal{F}(x)\mathrm{d}_qx.$$

Rajkovic *et al.* [14] presented a Riemann-type q-integral by:

$$R_q(\mathcal{F};a,b) = (b-a)(1-q)\sum_{k=0}^{\infty}\mathcal{F}(a+(b-a)q^k)q^k.$$

We can get another definition from the Riemman-type q-integral:

$$\frac{2}{b-a}\int_a^b \mathcal{F}(x)\mathrm{d}_q^Rx = (1-q)\sum_{k=0}^{\infty}\left(\mathcal{F}\left(\frac{a+b}{2}+q^k\left(\frac{b-a}{2}\right)\right) + \mathcal{F}\left(\frac{a+b}{2}-q^k\left(\frac{b-a}{2}\right)\right)\right)q^k.$$

From the q-Jackson integral we can write:

$$\frac{2}{b-a}\int_a^b \mathcal{F}(x)\mathrm{d}_q^Rx = \int_{-1}^1 \mathcal{F}\left(\frac{1-\mathfrak{t}}{2}a+\frac{1+\mathfrak{t}}{2}b\right)\mathrm{d}_q\mathfrak{t} = \int_{-1}^1 \mathcal{F}\left(\frac{1+\mathfrak{t}}{2}a+\frac{1-\mathfrak{t}}{2}b\right)\mathrm{d}_q\mathfrak{t}. \tag{I.4}$$

Contrary to the q-Jackson integral, if

$$\mathcal{F}(x)\leq\mathcal{G}(x),\quad x\in[a,b]$$

then

$$\int_a^b \mathcal{F}(x)\mathrm{d}_q^Rx \leq \int_a^b \mathcal{G}(x)\mathrm{d}_q^Rx.$$

Taf *et al.* [16] established the q-analogue of Hermite-Hadamard's inequalities for convex function

Theorem 8: Let $\mathcal{F}:[a,b]\to\mathbb{R}$ be a q-integrable convex function, then

$$\mathcal{F}\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b \mathcal{F}(x)\mathrm{d}_q^Rx \leq \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}. \tag{I.5}$$

For some recent studies on quantum integral inequalities, see [8]–[10].

II. MAIN RESULT

In this section, we derive our main results. In this context the first result is related to strongly harmonic convex function.

Theorem 9: Let $\mathcal{F}:\mathcal{I}\subseteq(0,+\infty)\to\mathbb{R}$ be a strongly harmonic convex function with modulus $c>0$, and $a,b\in\mathcal{I}$, with $0<a<b$, we have

$$2\mathcal{F}\left(\frac{2ab}{a+b}\right) \leq \frac{2ab}{q(b-a)}\int_a^b \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^Ru - \frac{c}{2[3]_q}\left(\frac{b-a}{ab}\right)^2 \leq \mathcal{F}(b)+\mathcal{F}(a) - \frac{c}{2}\left(\frac{b-a}{ab}\right)^2.$$

Proof: By Remark 6 and the fact that $\mathcal{G}$ is strongly convex function on $[\frac{1}{b}, \frac{1}{a}]$, defined by $\mathcal{G}(\mathfrak{t}) = \mathcal{F}(\frac{1}{\mathfrak{t}})$, we obtain

$$\begin{aligned}
\mathcal{G}\left(\frac{a+b}{2ab}\right) &= \mathcal{G}\left(\frac{1}{2}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right) + \frac{1}{2}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{b}+\frac{1-\mathfrak{t}}{2}\frac{1}{a}\right)\right)\\
&\le \frac{1}{2}\mathcal{G}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right) + \frac{1}{2}\mathcal{G}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{b}+\frac{1-\mathfrak{t}}{2}\frac{1}{a}\right) - \frac{ct^2}{4}\left(\frac{b-a}{ab}\right)^2.
\end{aligned}$$

q-integrating with respect to $\mathfrak{t}$ on $[-1, 1]$, we have

$$2\mathcal{G}\left(\frac{a+b}{2ab}\right) \le \frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{G}(u)\mathrm{d}_q^R u - \frac{c}{2[3]_q}\left(\frac{b-a}{ab}\right)^2,$$

with this first inequality is proved.

The proof of second inequality is given as

$$\mathcal{G}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right) \le \frac{1-\mathfrak{t}}{2}\mathcal{G}(\frac{1}{b}) + \frac{1+\mathfrak{t}}{2}\mathcal{G}(\frac{1}{a}) - \frac{c(1-\mathfrak{t}^2)}{4}\left(\frac{b-a}{ab}\right)^2.$$

q-integrating with respect to $\mathfrak{t}$ on $[-1, 1]$, we get

$$\frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{G}(u)\mathrm{d}_q^R u \le \mathcal{G}(\frac{1}{b}) + \mathcal{G}(\frac{1}{a}) - \frac{c}{2}\left(1-\frac{1}{[3]_q}\right)\left(\frac{b-a}{ab}\right)^2.$$

Thus

$$\begin{aligned}
2\mathcal{F}\left(\frac{2ab}{a+b}\right) &\le \frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{F}\left(\frac{1}{u}\right)\mathrm{d}_q^R u - \frac{c}{2[3]_q}\left(\frac{b-a}{ab}\right)^2\\
&\le \mathcal{F}(b) + \mathcal{F}(a) - \frac{c}{2}\left(\frac{b-a}{ab}\right)^2.
\end{aligned}$$

This implies

$$\begin{aligned}
2\mathcal{F}\left(\frac{2ab}{a+b}\right) &\le \frac{2ab}{q(b-a)}\int_a^b \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^R u - \frac{c}{2[3]_q}\left(\frac{b-a}{ab}\right)^2\\
&\le \mathcal{F}(b) + \mathcal{F}(a) - \frac{c}{2}\left(\frac{b-a}{ab}\right)^2.
\end{aligned}$$

This completes the proof. □

We now give the proof of q-analogue of Hermite-Hadamard's inequality using HA-convex functions.

Theorem 10: Let $\mathcal{F} : \mathcal{I} \subseteq (0, +\infty) \to \mathbb{R}$ be an HA-convex function and $a, b \in \mathcal{I}$, with $a < b$, we have

$$\mathcal{F}\left(\frac{2ab}{a+b}\right) \le \frac{ab}{q(b-a)}\int_a^b \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^R u \le \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}.$$

Proof: Consider the function $\mathcal{G} : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$, $\mathcal{G}(u) = \mathcal{F}(\frac{1}{u})$. Since $\mathcal{G}$ is convex on $[\frac{1}{b}, \frac{1}{a}]$, then by (I.5), we have

$$\mathcal{G}\left(\frac{a+b}{2ab}\right) \le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{G}(u)\mathrm{d}_q^R u \le \frac{\mathcal{G}(\frac{1}{b})+\mathcal{G}(\frac{1}{a})}{2},$$

then

$$\mathcal{F}\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{F}(\frac{1}{u})\mathrm{d}_q^R u \le \frac{\mathcal{F}(b)+\mathcal{F}(a)}{2},$$

by using change of variable, we get

$$\mathcal{F}\left(\frac{2ab}{a+b}\right) \le \frac{ab}{q(b-a)}\int_a^b \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^R u \le \frac{\mathcal{F}(a)+\mathcal{F}(b)}{2}.$$

The proof is completed. □

Note that, when $c = 0$ Theorem 9 reduces to Theorem 10.

Our next result is pertaining to strongly harmonic log-convex functions.

Theorem 11: Let $\mathcal{F} : [a, b] \subseteq (0, +\infty) \to \mathbb{R}_+$ be a strongly harmonic log*-convex function, then we have*

$$\begin{aligned}
\mathcal{F}\left(\frac{2ab}{a+b}\right) &\le \frac{ab}{q(b-a)}\int_a^b \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^R u - \frac{c}{4}\left(1-\frac{1}{[3]_q}\right)(b-a)^2\\
&\le \frac{1}{2}(\mathcal{F}(a)\mathcal{F}(b))^{\frac{1}{2}}\int_{-1}^{1}\left(\frac{\mathcal{F}(b)}{\mathcal{F}(a)}\right)^{\frac{\mathfrak{t}}{2}}\mathrm{d}_q\mathfrak{t}.
\end{aligned}$$

Proof: Consider the function $\mathcal{G}(u) = \mathcal{F}(\frac{1}{u}), u \in [\frac{1}{b}, \frac{1}{a}]$. Since $\mathcal{F}$ is strongly harmonic log-convex, for all $\mathfrak{t} \in [-1, 1]$, $x, y \in [a, b]$, we have

$$\mathcal{G}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{y}+\frac{1-\mathfrak{t}}{2}\frac{1}{x}\right) \le \left(\mathcal{G}\left(\frac{1}{x}\right)\right)^{\frac{1+\mathfrak{t}}{2}}\left(\mathcal{G}\left(\frac{1}{y}\right)\right)^{\frac{1-\mathfrak{t}}{2}} - c\frac{1-\mathfrak{t}^2}{4}(y-x)^2,$$

Then

$$\begin{aligned}
\mathcal{F}\left(\frac{2ab}{a+b}\right) &= \mathcal{G}\left(\frac{a+b}{2ab}\right)\\
&= \mathcal{G}\left(\frac{1}{2}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right) + \frac{1}{2}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{b}+\frac{1-\mathfrak{t}}{2}\frac{1}{a}\right)\right)\\
&\le \left(\mathcal{G}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right)\right)^{\frac{1}{2}} \times \left(\mathcal{G}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{b}+\frac{1-\mathfrak{t}}{2}\frac{1}{a}\right)\right)^{\frac{1}{2}} - c\frac{(1-\mathfrak{t}^2)}{4}(b-a)^2\\
&\le \frac{1}{2}\mathcal{G}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right) + \frac{1}{2}\mathcal{G}\left(\frac{1+\mathfrak{t}}{2}\frac{1}{b}+\frac{1-\mathfrak{t}}{2}\frac{1}{a}\right) - c\frac{(1-\mathfrak{t}^2)}{4}(b-a)^2.
\end{aligned}$$

q-integrating with respect $\mathfrak{t}$ over $[-1, 1]$, we get

$$\begin{aligned}
2\mathcal{F}\left(\frac{2ab}{a+b}\right) &= 2\mathcal{G}\left(\frac{a+b}{2ab}\right) \\
&\leq \frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{G}(u)\mathrm{d}_q^R u - \frac{c}{2}(1-\frac{1}{[3]_q})(b-a)^2 \\
&= \frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{F}(\frac{1}{u})\mathrm{d}_q^R u - \frac{c}{2}(1-\frac{1}{[3]_q})(b-a)^2. \quad \text{(II.1)}
\end{aligned}$$

Now, we have

$$\begin{aligned}
\frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{F}(\frac{1}{u})\mathrm{d}_q^R u &= \int_{-1}^{1} \mathcal{G}\left(\frac{1-\mathfrak{t}}{2}\frac{1}{b}+\frac{1+\mathfrak{t}}{2}\frac{1}{a}\right)\mathrm{d}_q\mathfrak{t} \\
&\leq \int_{-1}^{1} (\mathcal{G}(\frac{1}{b}))^{\frac{1+\mathfrak{t}}{2}}(\mathcal{G}(\frac{1}{a}))^{\frac{1-\mathfrak{t}}{2}}\mathrm{d}_q\mathfrak{t} \\
&\quad - \frac{c(b-a)^2}{4}\int_{-1}^{1}(1-\mathfrak{t}^2)\mathrm{d}_q\mathfrak{t}. \quad \text{(II.2)}
\end{aligned}$$

By (II.1) and (II.2), we get

$$\begin{aligned}
2\mathcal{F}\left(\frac{2ab}{a+b}\right) &\leq \frac{2ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} \mathcal{F}(\frac{1}{u})\mathrm{d}_q^R u - \frac{c}{2}(1-\frac{1}{[3]_q})(b-a)^2 \\
&\leq \int_{-1}^{1} (\mathcal{F}(b))^{\frac{1+\mathfrak{t}}{2}}(\mathcal{F}(a))^{\frac{1-\mathfrak{t}}{2}}\mathrm{d}_q\mathfrak{t} \\
&= (\mathcal{F}(a)\mathcal{F}(b))^{\frac{1}{2}}\int_{-1}^{1}\left(\frac{\mathcal{F}(b)}{\mathcal{F}(a)}\right)^{\frac{\mathfrak{t}}{2}}\mathrm{d}_q\mathfrak{t},
\end{aligned}$$

by using change of variable, we have

$$\begin{aligned}
2\mathcal{F}\left(\frac{2ab}{a+b}\right) &\leq \frac{ab}{q(b-a)}\int_{a}^{b} \frac{\mathcal{F}(u)}{u^2}\mathrm{d}_q^R u - \frac{c}{4}(1-\frac{1}{[3]_q})(b-a)^2 \\
&\leq \frac{1}{2}(\mathcal{F}(a)\mathcal{F}(b))^{\frac{1}{2}}\int_{-1}^{1}\left(\frac{\mathcal{F}(b)}{\mathcal{F}(a)}\right)^{\frac{\mathfrak{t}}{2}}\mathrm{d}_q\mathfrak{t}.
\end{aligned}$$

The proof is completed. □

Theorem 12: Let $\mathcal{F} : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ be an AH-convex function, then we have

$$\frac{2}{b-a}\int_{a}^{b} \mathcal{F}(u)\mathrm{d}_q^R u \leq \frac{q-1}{\log q}\left(\log \mathcal{F}(b) - \log \mathcal{F}(a)\right).$$

Proof: Since $\mathcal{F}$ is an AH-convex function, for all $\mathfrak{t} \in [-1, 1]$, we obtain

$$\mathcal{F}\left(\frac{1-\mathfrak{t}}{2}a+\frac{1+\mathfrak{t}}{2}b\right) \leq \frac{2\mathcal{F}(a)\mathcal{F}(b)}{(1+\mathfrak{t})\mathcal{F}(a)+(1-\mathfrak{t})\mathcal{F}(b)}$$

q-integrating with respect $\mathfrak{t}$ over $[-1, 1]$, we get

$$\begin{aligned}
\frac{2}{b-a}\int_{a}^{b} \mathcal{F}(u)\mathrm{d}_q^R u &\leq \frac{\mathcal{F}(a)\mathcal{F}(b)}{\mathcal{F}(b)-\mathcal{F}(a)}\int_{\frac{1}{\mathcal{F}(b)}}^{\frac{1}{\mathcal{F}(a)}} \frac{\mathrm{d}_q u}{u} \\
&= \frac{q-1}{\log q}\left(\log \mathcal{F}(b) - \log \mathcal{F}(a)\right).
\end{aligned}$$

The proof is completed. □

III. CONCLUSION

To the best of our knowledge this is the first pervasive note on q-analogues of the Hermite-Hadamard type of integral inequalities using different classes of harmonic convexity such as strongly harmonic convex functions, HA-convex functions, strongly harmonic log-convex functions AH-convex functions.

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