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Refinement Modal Logic Based on Finite Approximation of Covariant-Contravariant Refinement

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ABSTRACT The notion of covariant–contravariant refinement (CC-refinement) is a generalization of the notions of bisimulation and refinement, which is coinductively defined to describe behavior relation, i.e., two related processes are required to be always able to provide matched transitions each other. The notion of CC-n-refinement, where n is a natural number, presented in this paper, finitely approximates the notion of CC-refinement through weakening the above-mentioned requirement. A process is said to CC-n-refine another process whenever they can, within n-steps, match the transitions each other, and a CC-n-refinement relation may be considered to be a CC-refinement relation if n is big enough. Based on this kind of finite approximation, this paper presents CC-n-refinement modal logic (nCCRML) obtained from the modal system $\boldsymbol{K}$ by adding CC-n-refinement quantifiers, establishes an axiom system for nCCRML, and explores some important properties of this axiom system: soundness, completeness, and decidability. CC-n-refinement quantifiers can be used to formalize some interesting problems in the field of formal method.

INDEX TERMS Approximation, axiom system, covariant-contravariant n-refinement, modal logic.

I. INTRODUCTION

A number of different compatible relations between labeled transitions systems (LTSs) have been presented in the literature (see [1], [2]), which are adopted to capture the behavior relations between processes. Among them, the notion of simulation is often used to describe the refinement between reactive systems. However, for the systems having generative (active) actions (e.g., input/output (I/O) automata), such notion is inappropriate to describe the refinement relations between these systems [2], [3]. To remedy it, the notion of covariant-contravariant refinement (CC-refinement, for short) is presented in [2], which is a behavioral preorder over LTSs and in which all actions are partitioned into three sorts: covariant, contravariant, and bivariant actions. The covariant actions represent the passive actions of a system, whose execution is under the control of the environment. The transitions labeled with these actions in a given specification should be simulated by any correct implementation. The contravariant actions denote the generative actions under the control of a system itself. The transitions labeled with these actions in an implementation must be allowed by its specification. The bivariant actions are treated as in the usual notion of bisimulation. It is easy to see that the notion of CC-refinement generalizes the notions of bisimulation and refinement considered in [1]. More work on it may be found in [2]–[5].

For the coinductively defined notions of behavior relations (e.g., bisimulation, simulation, and CC-refinement, *et al.*), two related processes are required to be always able to provide matched transitions each other. Through weakening such requirement, there is a natural way of finitely approximating these notions. For example, given a natural number n, two processes are said to be n-bisimilar (bisimilar up to n) whenever, within n steps, they can match the transitions each other. These finite approximations are interesting and useful. They may be used to search finite models for a given modal formula, through which the finite model property of a modal logic is established. Moreover, they are useful in formal design and development of dynamic systems. For example, LTSs may be considered as the models of optimization problems in control theory. In this situation, the difference between LTSs occurring in the far future is not as important as the difference occurring in the near future [6], and hence two n-bisimilar LTSs may be considered to be "equivalent" if n is big enough.

It is well known that there exists a deep link between behavior relations and modal logics. A number of modal characterization theorems (e.g., HML theorem) have been established for different behavior relations [7]. Recently, another kind of modal logics related to behavior relations are presented and explored with very different motivation. Bozzelli *et al.* [1], [8] named them refinement modal logic (RML). RML provides a more abstract perspective of future event logic [9], [10] and arbitrary public announcement logic [11]. In addition to usual modal operators, RML contains refinement operator $\exists_B$. Intuitively, the formula $\exists_B\alpha$ says that we can refine the current model so that α is realized. Thus, the problem where a specification, expressed by a LTS M, has an implementation satisfying a given property φ may be formalized as the model checking problem $M \models \exists_B\varphi$.

Inspired by Laura Bozzelli *et al*'s work, a refinement modal logic (CCRML) is established based on the notion of CC-refinement, and a sound and complete axiom system for CCRML is provided [12]. We obtain its language $\mathcal{L}_{CC}$ from the standard modal language $\mathcal{L}_K$ [13] by adding CC-refinement operator $\exists_{(A_1,A_2)}$, where A_1 (A_2) is a set of all covariant (contravariant, resp.) actions. The formula $\exists_{(A_1,A_2)}\alpha$ intuitively represents that there exists a refined model of the current model which realizes α. If a given specification, expressed by a LTS M, involves covariant and contravariant actions, by applying CC-refinement operators, the problem whether this specification has an implementation realizing some given property ψ is boiled down to the model checking problem: $M \models \exists_{(A_1,A_2)}\psi$.

This paper presents the notion of CC-n-refinement, by considering, within n steps, the matching requirement in the notion of CC-refinement, where n is a natural number. In other words, CC-n-refinement weakens the requirement of CC-refinement. This paper considers a refinement modal logic based on this finite approximation and its language $\mathcal{L}^n_{CC}$ is obtained from $\mathcal{L}_{CC}$ by replacing CC-refinement operator $\exists_{(A_1,A_2)}$ with CC-n-refinement operator (or, quantifier) $\exists^n_{(A_1,A_2)}$. This paper also gives an axiom system nCCRML for $\mathcal{L}^n_{CC}$ and explores its important properties: soundness, completeness, and decidability.

We organize this paper as follows. Section 2 gives the notion of CC-n-refinement. Section 3 presents CC-n-refinement modal logic. Section 4 establishes a sound and complete axiomatization for CC-n-refinement modal logic. Finally we end this paper with a brief summary and application discussion of CC-n-refinement in Section 5.

II. CC-*N*-REFINEMENT

Given a finite set A of actions and a countable set P of propositional letters, a model M is a triple $\langle S^M, R^M, V^M\rangle$, where S^M is a non-empty set of states, R^M is an accessibility function from A to $2^{S^M\times S^M}$ which assigns to each action b in A a binary relation $R^M_b \subseteq S^M \times S^M$, and $V^M : P \to 2^{S^M}$ is a valuation function. A pair (M, u) with $u \in S^M$ is said to be a pointed model, often written as M_u. For any binary relation R and u, $R(u) \triangleq \{v \mid uRv\}$, $\pi_1(R) \triangleq \{s \mid \exists t(sRt)\}$ and $\pi_2(R) \triangleq \{t \mid \exists s(sRt)\}$. For any sets B and B' with $B \subseteq B'$, $\mathrm{i}_{B,B'}$ is used to denote the graph of inclusion function from B to B', that is $\mathrm{i}_{B,B'} = \{\langle b, b\rangle \mid b \in B\}$. We use $\circ$ to indicate the composition operator of relations.

As usual, we write $M \uplus N$ to represent the disjoint union of two models M and N such that $S^M \cap S^N = \emptyset$, which is given by $S^{M\uplus N} \triangleq S^M \cup S^N$, $R^{M\uplus N}_b \triangleq R^M_b \cup R^N_b$ for each $b \in A$ and $V^{M\uplus N}(q) \triangleq V^M(q) \cup V^N(q)$ for each $q \in P$.

Recall the notion of CC-refinement, in which, two related models are required to be always able to provide matched transitions each other [2]. Given a natural number n, the notion CC-n-refinement is a finite approximation of the notion of CC-refinement. That is, it is enough that the requirement to match the transitions each other is satisfied within n steps.

Definition 1 (CC-n-refinement): Let $A_1, A_2 \subseteq A$ such that $A_1 \cap A_2 = \emptyset$ and $n < \omega$. Given two models $M = \langle S, R, V\rangle$ and $M' = \langle S', R', V'\rangle$, a sequence of binary relations $\{\mathcal{Z}_i\}_{i\leq n}$ such that $\mathcal{Z}_n \subseteq \cdots \subseteq \mathcal{Z}_0 \subseteq S\times S'$ is an (A_1, A_2)-n-refinement relation between M and M' if, for each $0 < i \leq n$,

(atoms) *$u\mathcal{Z}_0\, u'$ implies $u \in V(q)$ iff $u' \in V'(q)$ for each $q \in P$;*

(forth) *for each $b \in A - A_2$ and $v \in S$, $u\mathcal{Z}_i u'$ and uR_bv imply $u'R'_bv'$ and $v\mathcal{Z}_{i-1}v'$ for some $v' \in S'$;*

(back) *for each $b \in A - A_1$ and $v' \in S'$, $u\mathcal{Z}_i u'$ and $u'R'_bv'$ imply uR_bv and $v\mathcal{Z}_{i-1}v'$ for some $v \in S$.*

Here A_1 and A_2 are said to be ***covariant*** *and* ***contravariant set*** *respectively. We say that $M'_{s'}$ (A_1, A_2)-n-refines M_s (or, M_s (A_1, A_2)-n-simulates $M'_{s'}$), in symbols $M_s \succeq^n_{(A_1,A_2)} M'_{s'}$, if there exists an (A_1, A_2)-n-refinement relation $\{\mathcal{Z}_i\}_{i\leq n}$ between M and M' with $s\mathcal{Z}_ns'$. We also write $\{\mathcal{Z}_i\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} M'_{s'}$ to indicate that $\{\mathcal{Z}_i\}_{i\leq n}$ is an (A_1, A_2)-n-refinement relation such that $s\mathcal{Z}_ns'$.*

As usual, the (A_1, A_2)-ω-refinement relation

$$\succeq^{\omega}_{(A_1,A_2)} \triangleq \bigcap_{n<\omega} \succeq^n_{(A_1,A_2)} .$$

We still write $\mathcal{Z} : M_s \succeq_{(A_1,A_2)} M'_{s'}$ to describe that $\mathcal{Z}$ is an (A_1, A_2)-refinement relation between M_s and $M'_{s'}$ with $s\mathcal{Z}s'$ (see [12] in detail).

Example 1: Consider two models M and N depicted in Fig. 1, where $A = \{a, b, c\}$, $A_1 = \{a\}$, $A_2 = \{b\}$, and $V^M(p) = \emptyset$ and $V^N(p) = \emptyset$ for each $p \in P$. It is not difficult to observe that the relation sequence $\{\mathcal{Z}_i\}_{i\leq 2}$ with

$$\mathcal{Z}_i = \begin{cases} \{\langle s,t\rangle\} & \text{if } i = 2 \\ \{\langle s,t\rangle, \langle u,u_1\rangle, \langle u,u_2\rangle\} & \text{if } i = 1 \\ \{\langle s,t\rangle, \langle u,u_1\rangle, \langle u,u_2\rangle, \langle v,v_1\rangle, \langle v,v_2\rangle\} & \text{if } i = 0, \end{cases}$$

represented by dash arrows, is an (A_1, A_2)-2-refinement relation between M_s and N_t. However, unfortunately, N_t does not (A_1, A_2)-refine M_s as the matching fails in "the third level".

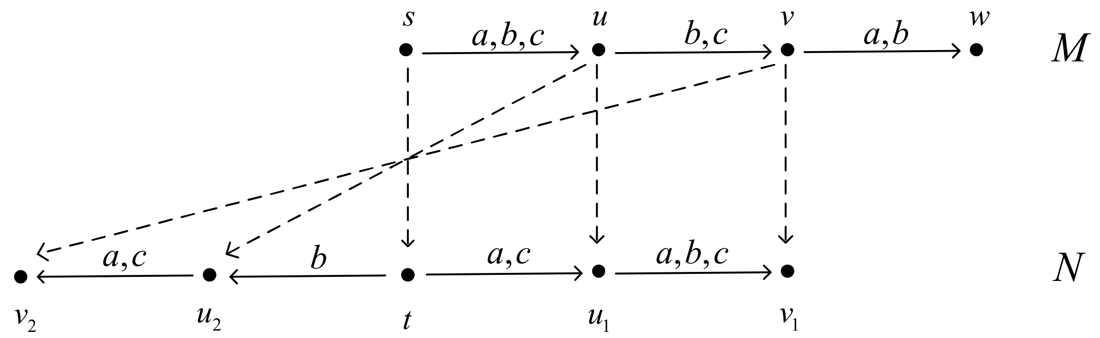

FIGURE 1. $(\{a\}, \{b\})$-2-refinement.

The notion of CC-refinement [12] generalizes the notions of ***bisimulation*** and ***refinement*** considered in [1]. Similarly, an n-***bisimulation*** relation is exactly an $(\emptyset, \emptyset)$-n-refinement, and a B-n-***refinement*** relation an $(\emptyset, B)$-n-refinement. For n-bisimulation, the reader may also refer to [13] and [14]. We write $\mathcal{Z} : M_s \underline{\leftrightarrow} M'_{s'}$ $(\{\mathcal{Z}_i\}_{i\leq n} : M_s \underline{\leftrightarrow}^n M'_{s'})$ to indicate that $\mathcal{Z}$ (resp., $\{\mathcal{Z}_i\}_{i\leq n}$) is a bisimulation (resp., n-bisimulation) which witnesses that M_s is bisimilar (resp., n-bisimilar) to $M'_{s'}$.

Proposition 1: The relation $\succeq^n_{(A_1,A_2)}$ is reflexive and transitive.

Proof: For every pointed model M_s, it is easy to see that $\{\mathrm{i}_{S^M,S^M}\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} M_s$, which implies that the relation $\succeq^n_{(A_1,A_2)}$ has the reflexivity.

Assume $\{\mathcal{Z}_i\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} M'_t$ and $\{\mathcal{Z}'_i\}_{i\leq n} : M'_t \succeq^n_{(A_1,A_2)} M''_u$. For the transitivity, it is enough to show that the sequence of composition relations $\{\mathcal{Z}_i \circ \mathcal{Z}'_i\}_{i\leq n}$ satisfies the conditions (**atoms**), (**forth**) and (**back**) in Definition 1, which is straightforward. □

Proposition 2: $\succeq_{(A_1,A_2)} \subseteq \succeq^\omega_{(A_1,A_2)}$.

Proof: Let $\mathcal{Z} : M_u \succeq_{(A_1,A_2)} N_v$. Then, we have that for all $n < \omega$,

$$\{\mathcal{Z}\}_{i<n} : M_u \succeq^n_{(A_1,A_2)} N_v.$$

It is clear that $\mathcal{Z} \subseteq \bigcap_{n<\omega} \succeq^n_{(A_1,A_2)}$. □

Proposition 3: $M_{s_1} \underline{\leftrightarrow} M'_{s_2} \succeq^n_{(A_1,A_2)} N'_{t_2} \underline{\leftrightarrow} N_{t_1}$ implies that $M_{s_1} \succeq^n_{(A_1,A_2)} N_{t_1}$.

Proof: Straightforward by Definition 1. □

Proposition 4: $M_s \underline{\leftrightarrow}^n M'_{s'} \succeq_{(A_1,A_2)} N_t$ implies that $M_s \succeq^n_{(A_1,A_2)} N_t$.

Proof: Straightforward by Definition 1. □

Recall the notion of tree and tree-like model in the literature [13]. Here, we call a multi-agent model M tree-like if the structure $\langle S^M, \bigcup_{a\in A} R^M_a, V^M\rangle$ is a tree. In detail,

Definition 2 (Tree-like model): We say that a model M is tree-like if the following conditions are satisfied:

(1) *there is a unique $s \in S^M$, called the root, such that*

$$\forall t \in S^M - \{s\} \exists b_1, \cdots, b_n \in A(s \xrightarrow{b_1} v_1 \cdots \xrightarrow{b_n} v_n = t),$$

that is, each $t \in S^M - \{s\}$ is accessible from s;

(2) *for each $t \in S^M - \{s\}$, there is a unique immediate $(\bigcup_{a\in A} R^M_a)$-predecessor, namely, there is a unique $t' \in S^M$ such that $t' R^M_a t$ for some $a \in A$;*

(3) $\forall t \in S^M \neg\exists b_1, \cdots, b_n \in A(t \xrightarrow{b_1} v_1 \cdots \xrightarrow{b_n} v_n = t)$.

Let $M_s \succeq^n_{(A_1,A_2)} N_t$. As every rooted model is bisimilar to a tree-like model [13], by Proposition 3, we may w.l.o.g. assume that M and N are trees rooted at s and t respectively. We use the notation $depth(M_s, u)$ to denote the depth of the state u in the tree M ($depth(M_s, s) = 0$). By removing all the sub-trees of all the states whose depth is n, the model M'_s will be obtained from M_s, and similarly, the model N'_t from N_t. Interestingly, we easily observe that M'_s is n-bisimilar to M_s, N'_t is n-bisimilar to N_t, and N'_t (A_1, A_2)-refines M'_s.

Definition 3 (n-model): Let M be a tree rooted at s. Its n-model $M^{n,s}$ is defined as follows.

$$\begin{aligned} S^{M^{n,s}} &\triangleq S^M - \{u \in S^M \mid depth(M_s, u) > n\} \\ R_b^{M^{n,s}} &\triangleq R_b^M \cap (S^{M^{n,s}} \times S^{M^{n,s}}) \qquad \textit{for each } b \in A \\ V^{M^{n,s}}(q) &\triangleq V^M(q) \cap S^{M^{n,s}} \qquad \textit{for each } q \in P. \end{aligned}$$

Convention. In the remainder of the paper, we always suppose that the model M is a tree rooted at s whenever the model $M^{n,s}$ is used.

Proposition 5: $M_s \underline{\leftrightarrow}^n M^{n,s}_s$.

Proof: For each $0 \leq i \leq n$, put

$$\mathcal{Z}_i \triangleq \{\langle u, u\rangle \mid u \in S^M \text{ and } depth(M_s, u) \leq n - i\}.$$

It is clear that $\{\mathcal{Z}_i\}_{i\leq n} : M_s \underline{\leftrightarrow}^n M^{n,s}_s$. □

Proposition 6: Let $M_s \succeq^n_{(A_1,A_2)} N_t$. Then

$$M^{n,s}_s \succeq_{(A_1,A_2)} N^{n,t}_t.$$

Proof: Let $\{\mathcal{Z}_i\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} N_t$. For each $0 \leq i \leq n$, we set

$$\mathcal{Z}'_i \triangleq \{\langle u, v\rangle \in \mathcal{Z}_i \mid depth(M_s, u) = depth(N_t, v) \leq n - i\}.$$

Clearly, $s\mathcal{Z}'_n t$. Since M_s and N_t be trees rooted at s and t respectively, by the definition of CC-n-refinement, we have that $\{\mathcal{Z}'_i\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} N_t$. Further, it is straightforward to check that

$$\textstyle\bigcup_{i\leq n} \mathcal{Z}'_i : M^{n,s}_s \succeq_{(A_1,A_2)} N^{n,t}_t. \square$$

□

Proposition 7: The relation $\succeq^n_{(A_1,A_2)}$ satisfies Church-Rosser property, that is, if $M_s \succeq^n_{(A_1,A_2)} N_t$ and $M_s \succeq^n_{(A_1,A_2)} N'_{t'}$ then $N_t \succeq^n_{(A_1,A_2)} M'_{s'}$ and $N'_{t'} \succeq^n_{(A_1,A_2)} M'_{s'}$ for some $M'_{s'}$.

Proof: Let $M_s \succeq^n_{(A_1,A_2)} N_t$ and $M_s \succeq^n_{(A_1,A_2)} N'_{t'}$. Then $M^{n,s}_s \succeq_{(A_1,A_2)} N^{n,t}_t$ and $M^{n,s}_s \succeq_{(A_1,A_2)} (N')^{n,t'}_{t'}$ by Proposition 6. As the relation $\succeq_{(A_1,A_2)}$ has Church-Rosser property [12], we have $N^{n,t}_t \succeq_{(A_1,A_2)} M'_{s'}$ and $(N')^{n,t'}_{t'} \succeq_{(A_1,A_2)} M'_{s'}$ for some $M'_{s'}$. Next, by Proposition 5, $N_t \underline{\leftrightarrow}^n N^{n,t}_t$ and $N'_{t'} \underline{\leftrightarrow}^n (N')^{n,t'}_{t'}$, which imply that $N_t \succeq^n_{(A_1,A_2)} M'_{s'}$ and $N'_{t'} \succeq^n_{(A_1,A_2)} M'_{s'}$ follows from Proposition 4, as desired. □

As CC-refinement, through taking compositions, any CC-n-refinement may be captured by CC-n-refinements for singleton covariant and contravariant set.

Proposition 8: Let $A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$. Then, for each $n < \omega$ and A'_1, A''_1, A'_2, A''_2 such that $A'_1 \cup A''_1 = A_1$ and $A'_2 \cup A''_2 = A_2$, it holds that

$$\succeq^n_{(A'_1,A'_2)} \circ \succeq^n_{(A''_1,A''_2)} = \succeq^n_{(A_1,A_2)} .$$

Proof: ($\subseteq$) Let $M_s \succeq^n_{(A'_1,A'_2)} \circ \succeq^n_{(A''_1,A''_2)} N_t$. Then $\{\mathcal{Z}_i\}_{i\leq n} : M_s \succeq^n_{(A'_1,A'_2)} N'_v$ and $\{\mathcal{Z}'_i\}_{i\leq n} : N'_v \succeq^n_{(A''_1,A''_2)} N_t$ for some pointed model N'_v, $\{\mathcal{Z}_i\}_{i\leq n}$ and $\{\mathcal{Z}'_i\}_{i\leq n}$. It is routine to verify that $\{\mathcal{Z}_i \circ \mathcal{Z}'_i\}_{i\leq n} : M_s \succeq^n_{(A_1,A_2)} N_t$.

($\supseteq$) Let $M_s \succeq^n_{(A_1,A_2)} N_t$. By Proposition 6, we get $M^{n,s}_s \succeq_{(A_1,A_2)} N^{n,t}_t$. Recall that, in [12, Proposition 2.6], we have that

$$\succeq_{(A_1,A_2)} \subseteq \succeq_{(A'_1,A'_2)} \circ \succeq_{(A''_1,A''_2)} .$$

Then,

$$M^{n,s}_s \succeq_{(A'_1,A'_2)} \circ \succeq_{(A''_1,A''_2)} N^{n,t}_t .$$

By Proposition 2, this implies that

$$M^{n,s}_s \succeq^n_{(A'_1,A'_2)} \circ \succeq^n_{(A''_1,A''_2)} N^{n,t}_t .$$

Further, due to $M_s \underline{\leftrightarrow}^n M^{n,s}_s$ and $N_t \underline{\leftrightarrow}^n N^{n,t}_t$ by Proposition 5 and the transitivity of CC-n-refinement relation, it follows that

$$M_s \succeq^n_{(A'_1,A'_2)} \circ \succeq^n_{(A''_1,A''_2)} N_t .\square$$

□

Corollary 1: If $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$ then, for each $n < \omega$,

$$\succeq^n_{\theta_1} \circ \succeq^n_{\theta_2} \circ \cdots \circ \succeq^n_{\theta_m} = \succeq^n_{(A_1,A_2)}$$

Here, if $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$ then $\{\theta_i\}_{1\leq i\leq m}$ with $m = |A_1 \times A_2|$ is a permutation of all pairs in $A_1 \times A_2$, otherwise, $\{\theta_i \mid 1 \leq i \leq m\} = A_k$ with $m = |A_k|$ if $A_k \neq \emptyset$ ($k = 1$ or 2).

III. CC-*N*-REFINEMENT MODAL LOGIC

Based on CCRML, this section presents CC-n-refinement modal logic (nCCRML, for short) by replacing CC-refinement operators with CC-n-refinement operators.

Definition 4 (Language $\mathcal{L}^n_{CC}$): Let A be a finite set of actions, P a set of propositional letters and $n < \omega$. The language $\mathcal{L}^n_{CC}$ of CC-n-refinement modal logic is generated by the BNF grammar below, where $\emptyset \neq A_1, A_2 \subseteq A$ with $A_1 \cap A_2 = \emptyset$, $q \in P$, and $b \in A$:

$$\varphi ::= q \mid \neg\varphi \mid (\varphi_1 \wedge \varphi_2) \mid \Box_b\varphi \mid \exists^n_{(A_1,A_2)}\varphi$$

The modal operator $\Diamond_b$ and propositional connectives $\top$, $\bot$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are used in the standard manner. We dually write $\forall^n_{(A_1,A_2)}\varphi$ for $\neg\exists^n_{(A_1,A_2)}\neg\varphi$. $\exists_{(A_1,A_2)}$ and $\forall_{(A_1,A_2)}$ are CC-refinement operators in the language $\mathcal{L}_{CC}$ of CCRML [12].

Here we suppose $A_1, A_2 \neq \emptyset$ for the sake of simplicity, resorting to similar declarations as in [12, Sec. 5]. If both A_1 and A_2 are singleton set, say $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$, we write $\exists^n_{(a_1,a_2)}\varphi$ (or $\forall^n_{(a_1,a_2)}\varphi$) instead of $\exists^n_{(A_1,A_2)}\varphi$ (resp. $\forall^n_{(A_1,A_2)}\varphi$). Furthermore, in order to ease the notation, we shall write $\bigwedge \Diamond_b\exists^n_{(a_1,a_2)}\Phi$ (or $\Box_b \bigvee \exists^n_{(a_1,a_2)}\Phi$) for $\bigwedge_{\varphi\in\Phi} \Diamond_b\exists^n_{(a_1,a_2)}\varphi$ (resp. $\Box_b \bigvee_{\varphi\in\Phi} \exists^n_{(a_1,a_2)}\varphi$).

In this paper, we also adopt cover operator ∇_b [15], [16]. As usual, for each $b \in A$, $\nabla_b\Phi$ is defined by $\Box_b \bigvee_{\varphi\in\Phi} \varphi \wedge \bigwedge_{\varphi\in\Phi} \Diamond_b\varphi$, where Φ is a finite set of formulas. By the way, the operators $\Box_b$ and $\Diamond_b$ may be defined in terms of ∇_b. Formally, $\Box_b\psi$ and $\Diamond_b\psi$ are captured by $\nabla_b\emptyset \vee \nabla_b\{\psi\}$ and $\nabla_b\{\psi, \top\}$ respectively.

Given a model M, the notion of a formula $\varphi \in \mathcal{L}^n_{CC}$ being satisfied in M at a state u is defined inductively by:

$M_u \models q$	iff	$u \in V^M(q)$, where $q \in P$
$M_u \models \neg\varphi$	iff	$M_u \not\models \varphi$
$M_u \models \varphi_1 \wedge \varphi_2$	iff	$M_u \models \varphi_1$ and $M_u \models \varphi_2$
$M_u \models \Box_b\varphi$	iff	for all $v \in R^M_b(u)$, $M_v \models \varphi$
$M_u \models \exists^n_{(A_1,A_2)}\varphi$	iff	$M_u \succeq^n_{(A_1,A_2)} N_v$ and $N_v \models \varphi$ for some N_v

As usual, for every $\psi \in \mathcal{L}^n_{CC}$, we say ψ is valid, written as $\models \psi$, whenever $M_u \models \psi$ for each pointed model M_u. It is not difficult to see that $\mathcal{L}^n_{CC}$-satisfaction is invariant under bisimilarity. This is a known result:

Proposition 9: If $M_u \underline{\leftrightarrow} N_v$ then

$$M_u \models \psi \text{ iff } N_v \models \psi \quad \text{for each } \psi \in \mathcal{L}^n_{CC}.$$

Proof: Prove inductively by the structure of formulas. □

Proposition 10: $M_u \underline{\leftrightarrow}^n N_v \models \exists_{(a_1,a_2)}\psi$ implies that $M_u \models \exists^n_{(a_1,a_2)}\psi$.

Proof: Let $M_u \underline{\leftrightarrow}^n N_v \models \exists_{(a_1,a_2)}\psi$. Then $N_v \succeq_{(a_1,a_2)} N'_{v'} \models \psi$ for some $N'_{v'}$. By Proposition 2, we have $N_v \succeq^n_{(a_1,a_2)} N'_{v'}$. Next, $M_u \succeq^n_{(a_1,a_2)} N'_{v'}$ due to the transitivity of $\succeq^n_{(a_1,a_2)}$. Hence it holds that $M_u \models \exists^n_{(a_1,a_2)}\psi$. □

Proposition 11: For each $n < \omega$ and $A'_1, A'_2, A''_1, A''_2 \subseteq A$ such that $(A'_1 \cup A''_1) \cap (A'_2 \cup A''_2) = \emptyset$, we have that

$$\models \exists^n_{(A'_1,A'_2)}\exists^n_{(A''_1,A''_2)}\psi \leftrightarrow \exists^n_{(A''_1,A''_2)}\exists^n_{(A'_1,A'_2)}\psi.$$

Proof: By Proposition 8, immediately. □

IV. AXIOM SYSTEM

A sound and complete axiom system for nCCRML will be presented in this section. Analogous to the axiom system CCRML [12] and **RML** [1], the uniform substitution rule is

TABLE 1. Axiom system of *n*CCRML.

Axiom schemes

Here $n < \omega, a_1, a_2, b \in A, A_1, A_2, B \subseteq A, A_1 \cap A_2 = \emptyset, q \in P, \Gamma, \Gamma_b \subseteq_f \mathcal{L}_K$, and $\Phi, \Phi_b \subseteq_f \mathcal{L}^n_{CC}$.

Prop	*All propositional tautologies*	
K	$\Box_b(\varphi \to \psi) \to (\Box_b\varphi \to \Box_a\psi)$	
nCCR	$\forall^n_{(a_1,a_2)}(\varphi \to \psi) \to (\forall^n_{(a_1,a_2)}\varphi \to \forall^n_{(a_1,a_2)}\psi)$	
nCCRp1	$\forall^n_{(a_1,a_2)}q \leftrightarrow q$	
nCCRp2	$\forall^n_{(a_1,a_2)}\neg q \leftrightarrow \neg q$	
nCCRD	$\exists^n_{(A_1,A_2)}\varphi \leftrightarrow (\exists^n_{\theta_1}\cdots\exists^n_{\theta_{\|A_1\times A_2\|}})\varphi$	where all pairs in $A_1 \times A_2$ are arranged in a permutation $\{\theta_j\}_{1\le j\le \|A_1\times A_2\|}$
0CCRK	$\exists^0_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Gamma_b$	if $\nvdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for all $\beta \in \bigcup_{b\in B}\Gamma_b$
nCCRKco1	$\exists^n_{(a_1,a_2)}\nabla_{a_1}\Gamma \leftrightarrow \bot$	if $\vdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for some $\beta \in \Gamma$
nCCRKco2	$\exists^{n+1}_{(a_1,a_2)}\nabla_{a_1}\Gamma \leftrightarrow \Box_{a_1}\bigvee\exists^n_{(a_1,a_2)}\Gamma$	if $\nvdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for all $\beta \in \Gamma$
nCCRKcontra	$\exists^{n+1}_{(a_1,a_2)}\nabla_{a_2}\Phi \leftrightarrow \bigwedge\Diamond_{a_2}\exists^n_{(a_1,a_2)}\Phi$	
nCCRKbis	$\exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi \leftrightarrow \nabla_b\exists^n_{(a_1,a_2)}\Phi$	where $b \neq a_1, a_2$
nCCRKconj	$\exists^n_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bigwedge_{b\in B}\exists^n_{(a_1,a_2)}\nabla_b\Phi_b$	

Rules

MP $\dfrac{\varphi \to \psi, \varphi}{\psi}$ **NK** $\dfrac{\varphi}{\Box_b\varphi}$ **nNCCR** $\dfrac{\varphi}{\forall^n_{(a_1,a_2)}\varphi}$

also not sound in nCCRML. For instance, $q \to \forall^n_{(a_1,a_2)}q$ is valid for each $q \in P$ and $n < \omega$, however, it is clear that

$$\Diamond_{a_2}\top \to \forall^0_{(a_1,a_2)}\Diamond_{a_2}\top$$

is not valid. Hence nCCRML is not a normal modal logic.

We use $\mathcal{L}_K$ to denote the set of all $\mathcal{L}^n_{CC}$ formulas containing no CC-n-refinement operators, and $\mathcal{L}_p$ the set of all propositional formulas in $\mathcal{L}^n_{CC}$. Obviously, $\mathcal{L}_K$ is indeed the multi-agent modal language, which may be axiomatized by the system $\boldsymbol{K}$ [13].

We list the axiom schemes and rules for nCCRML in Table 1. The axiom schemes **nCCR**, **nCCRp1**, **nCCRp2**, **nCCRD** and **nCCRKco1** depict the similar properties as the axiom schemes **CCR**, **CCRp1**, **CCRp2**, **CCRD** and **CCRKco1** of CCRML [12], respectively. The axiom scheme **0CCRK** reveals that $\exists^0_{(a_1,a_2)}\nabla_b$ preserves the consistency of $\mathcal{L}_K$-formulas. **nCCRKco2**, **nCCRKcontra** and **nCCRKbis** allow us to transform $\exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi$ into a formula of the form $F_b(\exists^n_{(a_1,a_2)}\Phi)$, where the format F_b depends on the sort of the action b. **nCCRKconj** is the $\exists^n_{(a_1,a_2)}$-over-$\wedge$ distribution.

As usual, $\vdash \beta$ ($\vdash_{\boldsymbol{K}} \beta$) means that β is a theorem in nCCRML (resp., $\boldsymbol{K}$ [13]).

A. TECHNICAL PRELIMINARY

In this subsection, we will make some technical preparations in order to establish the soundness of the axiom system.

In verifying Lemma 3, 4, 5 and 6, we intend to apply the validity of the axiom schemes **CCRKco2**, **CCRKcontra**, **CCRKbis** and **CCRKconj** of CCRML [12], which are listed in Table 2, where $B \subseteq A$, $\Gamma \subseteq_f \mathcal{L}_K$, and $\Phi, \Phi_b \subseteq_f \mathcal{L}_{CC}$.

To realize this, given $\{\mathcal{Z}_i\}_{i\le n} : M_s \succeq^n_{(a_1,a_2)} N_t$, we intend to obtain a model $M'_{s'}$ such that $M_s \underline{\leftrightarrow}^n M'_{s'} \succeq_{(a_1,a_2)} N_t$.

Definition 5 gives such a model, denoted by $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$. In the following, we will explain its construction.

TABLE 2. Some axiom schemes of CCRML.

CCRKco2	$\exists_{(a_1,a_2)}\nabla_{a_1}\Gamma \leftrightarrow \Box_{a_1}\bigvee\exists_{(a_1,a_2)}\Gamma$	if $\nvdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for all $\beta \in \Gamma$
CCRKcontra	$\exists_{(a_1,a_2)}\nabla_{a_2}\Phi \leftrightarrow \bigwedge\Diamond_{a_2}\exists_{(a_1,a_2)}\Phi$	
CCRKbis	$\exists_{(a_1,a_2)}\nabla_b\Phi \leftrightarrow \nabla_b\exists_{(a_1,a_2)}\Phi$	where $b \neq a_1, a_2$
CCRKconj	$\exists_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bigwedge_{b\in B}\exists_{(a_1,a_2)}\nabla_b\Phi_b$	

At first glance, since $M_s\underline{\leftrightarrow}^n M^{n,s}_s$, we may obtain the desired model by modifying $M^{n,s}_s$. If $u\mathcal{Z}_0\ v$ with $depth(M_s, u) = depth(N_t, v) = n$, we will append the v-generated submodel of N to u and prescribe the behaviors between u and this submodel according to the behaviors of v. Unfortunately, there exists the possibility that $u\mathcal{Z}_0\ v$ and $u\mathcal{Z}_0\ w$ such that the depth of u in M and v, w in N is n. To remedy it, we will intend to replace u by the pairs $\langle u, v\rangle$ and $\langle u, w\rangle$. Moreover, the transitions from these new states in $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ are prescribed according to the ones entering u in $M^{n,s}$ and the ones outgoing from v or w in N. In Definition 5, the set of such u is denoted by $\mathcal{Z}^{\blacktriangle}_0$, which is defined as

$$\mathcal{Z}^{\blacktriangle}_0 \triangleq \{\langle u, v\rangle \in \mathcal{Z}_0 \mid depth(M_s, u) = depth(N_t, v) = n\}.$$

Example 2: Consider two tree-like models M and N rooted at s and t respectively, in which the submodels consisting of the states whose depth is not less than n are depicted in Fig. 2. Here $A = \{a, b, c\}$, the depth of u_1, u_2, v_1, and v_2 is n, and $\{\mathcal{Z}_i\}_{i\le n} : M_s \succeq^n_{(a,b)} N_t$ for some $\{\mathcal{Z}_i\}_{i\le n}$ with $\langle u_1, v_1\rangle, \langle u_1, v_2\rangle \in \mathcal{Z}_0$. The submodel of the model $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ obtained by the construction mentioned in the preceding paragraph, consisting of the states whose depth is not less than n, may be depicted as in Fig. 3.

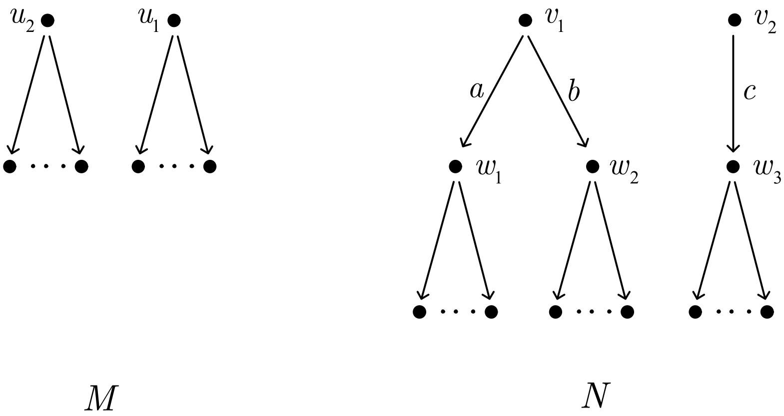

FIGURE 2. The submodels of *M* and *N* consisting of the states whose depth is not less than *n*.

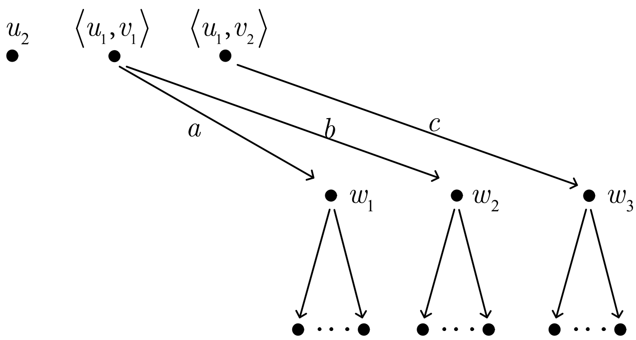

FIGURE 3. The submodels of $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ consisting of the states whose depth is not less than *n*.

Definition 5: Given two disjoint tree-like models M and N rooted at s and t respectively such that $\{\mathcal{Z}_i\}_{i\le n} : M_s \succeq^n_{(a_1,a_2)} N_t$*, the model* M' *is defined as follows.*

(M_1') $S^{M'} \triangleq (S^{M^{n,s}} - \pi_1(\mathcal{Z}_0^{\blacktriangle})) \cup S^N \cup \mathcal{Z}_0^{\blacktriangle}$

(M_2') *For each* $b \in A$, $R_b^{M'} \subseteq S^{M'} \times S^{M'}$ *is given by*

$$R_b^{M'} \triangleq R_b^N \cup \{\langle w, \langle u, v\rangle\rangle \mid wR_b^M u \text{ and } u\mathcal{Z}_0^{\blacktriangle} v\} \cup$$
$$(R_b^M \cap (S^{M'})^2) \cup \{\langle\langle u, v\rangle, w\rangle \mid u\mathcal{Z}_0^{\blacktriangle} v \text{ and } vR_b^N w\}$$

(M_3') *For each* $q \in P$,

$$V^{M'}(q) \triangleq (V^M(q) \cap S^{M'}) \cup V^N(q)$$
$$\cup\{\langle u, v\rangle \mid u\mathcal{Z}_0^{\blacktriangle} v \text{ and } u \in V^M(q)\}$$

Here

$$\mathcal{Z}_0^{\blacktriangle} \triangleq \{\langle u, v\rangle \in \mathcal{Z}_0 \mid depth(M_s, u) = depth(N_t, v) = n\}.$$

The model $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ *is the s-generated submodel of* M'*, which is indeed a tree rooted at s.*

Convention. In the remainder of the paper, we always suppose that the models *M* and *N* are trees rooted at *s* and *t* respectively and they are disjoint whenever the model $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ is used.

The model $M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$ enjoys the desired properties:

Proposition 12: With the notations as in Definition 5, we have

$$M_s \underline{\leftrightarrow}^n (M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t)_s \succeq_{(a_1,a_2)} N_t.$$

Proof: Let $M' \triangleq M_s +_{\{\mathcal{Z}_i\}_{i\le n}} N_t$. Put

$$\mathcal{Z}_i' \triangleq \begin{cases} \mathcal{Z}_i^{\blacktriangledown} & \text{if } 1 \le i \le n \\ \mathcal{Z}_i^{\blacktriangledown} \cup \{\langle u, \langle u, v\rangle\rangle \mid u\mathcal{Z}_0^{\blacktriangle} v\} & \text{if } i = 0 \end{cases}$$

where

$$\mathcal{Z}_i^{\blacktriangledown} \triangleq \{\langle u, u\rangle \mid u \in S^{M'} \cap S^M \text{ and } depth(M_s, u) \le n - i\}.$$

Clearly, $\{\mathcal{Z}_i'\}_{i\le n} : M_s \underline{\leftrightarrow}^n M_s'$. For each $1 \le i \le n$, set

$$\mathcal{Z}_i'' \triangleq \{\langle u, v\rangle \in \mathcal{Z}_i \mid depth(M_s, u) = depth(N_t, v) = n - i\}$$

and then

$$\mathcal{Z} \triangleq \bigcup_{1\le i\le n} \mathcal{Z}_i'' \cup \{\langle\langle u, v\rangle, v\rangle \mid u\mathcal{Z}_0^{\blacktriangle} v\} \cup \mathrm{i}_{S^{M'}\cap S^N, S^N}.$$

It is routine to check that $\mathcal{Z} : M_s' \succeq_{(a_1,a_2)} N_t$. □

B. SOUNDNESS

In this subsection, we devote ourselves to establish the soundness of the axiom system, beginning with giving some validities.

Lemma 1: If all the pairs in $A_1 \times A_2$ *are arranged in a permutation* $\{\theta_i\}_{1\le i\le |A_1\times A_2|}$ *then*

$$\models \exists^n_{(A_1,A_2)}\psi \leftrightarrow (\exists^n_{\theta_1} \cdots \exists^n_{\theta_{|A_1\times A_2|}})\psi.$$

Proof: By Corollary 1 and Proposition 11, immediately. □

Actually, for every sequence $\{\theta_i\}_{1\le i\le m}$ of the pairs in $A_1 \times A_2$ such that every action in $A_1 \cup A_2$ occurs in θ_i for some $1 \le i \le m$, we always have

$$\models \exists^n_{(A_1,A_2)}\psi \leftrightarrow (\exists^n_{\theta_1} \cdots \exists^n_{\theta_m})\psi.$$

Lemma 2: Let $\Phi_b \subseteq_f \mathcal{L}^n_{CC}$ for each $b \in B(\subseteq A)$. Then

$$\models \exists^0_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b$$

whenever each ψ in $\bigcup_{b\in B} \Phi_b$ is satisfiable.

Proof: Let M_s be a pointed model. For each $b \in B$, since each ψ in Φ_b is satisfiable, we may choose arbitrarily and fix a pointed model $N^{b\psi}_{u^{b\psi}}$ such that $N^{b\psi}_{u^{b\psi}} \models \psi$ for each $\psi \in \Phi_b$. W.l.o.g., we assume that all these $N^{b\psi}_{u^{b\psi}}$ ($b \in B$, $\psi \in \Phi_b$) are pairwise disjoint. The model N is obtained from $\biguplus_{b\in B,\psi\in\Phi_b} N^{b\psi}_{u^{b\psi}}$ by adding a new state s' and imposing the following clauses:

(i) for each $b \in B$, $s' R^N_b w$ iff $w = u^{b\psi}$ for some $\psi \in \Phi_b$, and

(ii) for each $q \in P$, $s' \in V^N(q)$ iff $s \in V^M(q)$.

Obviously,

$$\{\langle s, s'\rangle\} : M_s \succeq^0_{(a_1,a_2)} N_{s'}.$$

Moreover, for each $b \in B$, on the one hand, it is not difficult to see that for each $t \in R^N_b(s') = \{u^{b\psi} \mid \psi \in \Phi_b\}$, $N_t \models \psi$ for some $\psi \in \Phi_b$, hence $N_{s'} \models \Box_b \bigvee \Phi_b$; on the other hand, for each $\psi \in \Phi_b$, there exists a corresponding $u^{b\psi} \in R^N_b(s')$ such that $N_{u^{b\psi}} \models \psi$, which implies $N_{s'} \models \bigwedge \Diamond_b \Phi_b$. Together, $N_{s'} \models \bigwedge_{b\in B} \nabla_b \Phi_b$. Thus, $M_s \models \exists^0_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b$ due to $M_s \succeq^0_{(a_1,a_2)} N_{s'}$. □

If $|B| = 1$ then we get that for each $b \in A$,

$$\models \exists^0_{(a_1,a_2)} \nabla_b \Phi \quad \text{whenever each } \psi \text{ in } \Phi \text{ is satisfiable.}$$

Lemma 3: Let $\Phi \subseteq_f \mathcal{L}^n_{CC}$. Then

(1) $\models \exists^{n+1}_{(a_1,a_2)} \nabla_{a_1} \Phi \to \Box_{a_1} \bigvee \exists^n_{(a_1,a_2)} \Phi$,

(2) $\models \Box_{a_1} \bigvee \exists^n_{(a_1,a_2)} \Phi \to \exists^{n+1}_{(a_1,a_2)} \nabla_{a_1} \Phi$ *whenever each ψ in Φ is satisfiable.*

Proof: **(1)** Suppose that $M_s \models \exists^{n+1}_{(a_1,a_2)} \nabla_{a_1} \Phi$. Then for some N_t,

$$M_s \succeq^{n+1}_{(a_1,a_2)} N_t \models \nabla_{a_1} \Phi.$$

So $N_t \models \Box_{a_1} \bigvee \Phi$, namely, for each $v \in R^N_{a_1}(t)$, $N_v \models \psi$ for some $\psi \in \Phi$. From $M_s \succeq^{n+1}_{(a_1,a_2)} N_t$, it follows that for each $u \in R^M_{a_1}(s)$, $M_u \succeq^n_{(a_1,a_2)} N_{v'}$ for some $v' \in R^N_{a_1}(t)$. Hence, for each $u \in R^M_{a_1}(s)$, $M_u \models \exists^n_{(a_1,a_2)} \psi$ for some $\psi \in \Phi$, which implies that $M_u \models \bigvee \exists^n_{(a_1,a_2)} \Phi$. Therefore $M_s \models \Box_{a_1} \bigvee \exists^n_{(a_1,a_2)} \Phi$.

(2) Let $M_s \models \Box_{a_1} \bigvee \exists^n_{(a_1,a_2)} \Phi$. Then, for each $u \in R^M_{a_1}(s)$, $M_u \models \exists^n_{(a_1,a_2)} \psi$ for some $\psi \in \Phi$. So we have $\{\mathcal{Z}^u_i\}_{i\le n} : M_u \succeq^n_{(a_1,a_2)} N^u_{v^u} \models \psi$ for some fixed $N^u_{v^u}$ and $\{\mathcal{Z}^u_i\}_{i\le n}$. W.l.o.g., suppose that M and all these N^u ($u \in R^M_{a_1}(s)$) are pairwise disjoint. By Definition 5 and Proposition 12, we have

$$M_u \underline{\leftrightarrow}^n (M_u +_{\{\mathcal{Z}^u_i\}_{i\le n}} N^u_{v^u})_u \succeq_{(a_1,a_2)} N^u_{v^u} \models \psi.$$

Let $M_u +_{\{\mathcal{Z}^u_i\}_{i\le n}} N^u_{v^u} = \langle S^u, R^u, V^u\rangle$. By renaming each state $w \in S^u$ to $\langle u, w\rangle$ (assume, $(\{u\} \times S^u) \cap S^M = \emptyset$), we obtain the model M^u from $M_u +_{\{\mathcal{Z}^u_i\}_{i\le n}} N^u_{v^u}$. The purpose is to make all the models in $\{M\} \cup \bigcup_{u\in R^M_{a_1}(s)} M^u$ pairwise disjoint. Next, we construct the model M' from $M \uplus \biguplus_{u\in R^M_{a_1}(s)} M^u$ by removing all the a_1-labeled transitions outgoing from s in M and adding the a_1-labeled transition $s \xrightarrow{a_1} \langle u, u\rangle$ for each $u \in R^M_{a_1}(s)$, which results in $R^{M'}_{a_1}(s) = \{\langle u, u\rangle \mid u \in R^M_{a_1}(s)\}$. It is not difficult to see that for each $u \in R^M_{a_1}(s)$, we still have

$$M_u \underline{\leftrightarrow}^n M'_{\langle u,u\rangle} \succeq_{(a_1,a_2)} N^u_{v^u} \models \psi$$

Then $M'_s \models \Box_{a_1} \bigvee \exists_{(a_1,a_2)} \Phi$. Fortunately, based on the validity of the axiom scheme **CCRKco2** (in Table 2) of CCRML [12], we get

$$M'_s \models \exists_{(a_1,a_2)} \nabla_{a_1} \Phi.$$

Further, $M_s \underline{\leftrightarrow}^{n+1} M'_s$ follows easily from $M_u \underline{\leftrightarrow}^n M'_{\langle u,u\rangle}$. So $M_s \models \exists^{n+1}_{(a_1,a_2)} \nabla_{a_1} \Phi$ by Proposition 10. □

Lemma 2 implies the validity of the axiom scheme **0CCRK**. However, we can not apply the following formula

$$\exists^0_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b \quad \text{whenever each } \psi \text{ in} \bigcup_{b\in B} \Phi_b \text{ is satisfiable}$$

as an axiom scheme. The reason is that its side condition involves a semantical concept. Similarly, Lemma 3 implies the validity of the axiom scheme **nCCRKco2**, however the formula

$$\exists^{n+1}_{(a_1,a_2)} \nabla_{a_1} \Phi \leftrightarrow \Box_{a_1} \bigvee \exists^n_{(a_1,a_2)} \Phi,$$

where each ψ in Φ is satisfiable, is also unavailable as an axiom scheme.

Fortunately, in order to show the completeness of nCCRML in Section IV-C, by relying on the completeness of ***K***, it is enough to require $\Phi \subseteq_f \mathcal{L}_K$ and to express the side condition in terms of ***K***-derivability (see, **0CCRK** and **nCCRKco2** in Table 1).

Lemma 4: $\models \exists^{n+1}_{(a_1,a_2)} \nabla_{a_2} \Phi \leftrightarrow \bigwedge \Diamond_{a_2} \exists^n_{(a_1,a_2)} \Phi$ for every $\Phi \subseteq_f \mathcal{L}^n_{CC}$.

Proof: Suppose that $M_s \models \exists^{n+1}_{(a_1,a_2)} \nabla_{a_2} \Phi$. Then there exists N_t such that

$$M_s \succeq^{n+1}_{(a_1,a_2)} N_t \models \nabla_{a_2} \Phi.$$

We intend to check $M_s \models \Diamond_{a_2} \exists^n_{(a_1,a_2)} \psi$ for every $\psi \in \Phi$. Let ψ be any formula in Φ. From $N_t \models \nabla_{a_2} \Phi$, it follows that $N_v \models \psi$ for some $v \in R^N_{a_2}(t)$. Next, since $M_s \succeq^{n+1}_{(a_1,a_2)} N_t$, we have that $M_u \succeq^n_{(a_1,a_2)} N_v$ for some $u \in R^M_{a_2}(s)$. Then $M_u \models \exists^n_{(a_1,a_2)} \psi$. Hence $M_s \models \Diamond_{a_2} \exists^n_{(a_1,a_2)} \psi$ follows from $u \in R^M_{a_2}(s)$.

Let $M_s \models \bigwedge \Diamond_{a_2} \exists^n_{(a_1,a_2)} \Phi$ and $\Phi = \{\varphi_1, \cdots, \varphi_m\}$. Then, for each $1 \le j \le m$, $M_{u^j} \models \exists^n_{(a_1,a_2)} \varphi^j$ for some $u^j \in R^M_{a_1}(s)$. Next, we choose arbitrarily and fixed a pointed model $N^j_{v^j}$ and a sequence of binary relations $\{\mathcal{Z}^j_i\}_{i\le n}$ such that $\{\mathcal{Z}^j_i\}_{i\le n} : M_{u^j} \succeq^n_{(a_1,a_2)} N^j_{v^j} \models \varphi^j$. W.l.o.g., we assume that M and all these N^j ($1 \le j \le m$) are pairwise disjoint. By Definition 5 and Proposition 12, it follows that

$$M_{u^j} \underline{\leftrightarrow}^n (M_{u^j} +_{\{\mathcal{Z}^j_i\}_{i\le n}} N^j_{v^j})_{u^j} \succeq_{(a_1,a_2)} N^j_{v^j} \models \varphi^j.$$

Let $M_{u^j} +_{\{\mathcal{Z}_i^j\}_{i\leq n}} N_{v^j}^j = \langle S^j, R^j, V^j\rangle$. By renaming each state $w \in S^j$ to $\langle j, w\rangle$ (suppose, $(\{j\} \times S^j) \cap S^M = \emptyset$), the model M^j is obtained from $M_{u^j} +_{\{\mathcal{Z}_i^j\}_{i\leq n}} N_{v^j}^j$. Immediately, all the models in $\{M\} \cup \bigcup_{1\leq j\leq m}\{M^j\}$ are pairwise disjoint. Further, the model M' is constructed from $M \uplus \biguplus_{1\leq j\leq m} M^j$ by adding the a_2-labeled transition $s \xrightarrow{a_2} \langle j, u^j\rangle$ for every $1 \leq j \leq m$. Obviously, it still holds that

$$M_{u^j} \underline{\leftrightarrow}^n M'_{\langle j,u^j\rangle} \succeq_{(a_1,a_2)} N_{v^j}^j \models \varphi^j,$$

and then we get $M'_s \models \bigwedge \Diamond_{a_2}\exists_{(a_1,a_2)}\Phi$. From the validity of the axiom scheme **CCRKcontra** (in Table 2) of CCRML [12], it follows that

$$M'_s \models \exists_{(a_1,a_2)}\nabla_{a_2}\Phi.$$

Moreover, it is easy to see that $M_s \underline{\leftrightarrow}^{n+1} M'_s$ due to $M_{u^j}\underline{\leftrightarrow}^n M'_{\langle j,u^j\rangle}$. Together, by Proposition 10, it holds that $M_s \models \exists^{n+1}_{(a_1,a_2)}\nabla_{a_2}\Phi$ as desired. □

Lemma 5: Let $b \in A$ and $\Phi \subseteq_f \mathcal{L}^n_{CC}$. Then

(1) $\models \exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi \to \nabla_b\exists^n_{(a_1,a_2)}\Phi$ whenever $b \neq a_1, a_2$,

(2) $\models \nabla_b\exists^n_{(a_1,a_2)}\Phi \to \exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi$.

Proof: **(1)** Let $M_s \models \exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi$. By the analysis similar to that in the proof of Lemma 4 and Lemma 3 (1), we get that

$$M_s \models \bigwedge \Diamond_b\exists^n_{(a_1,a_2)}\Phi \text{ and } M_s \models \Box_b \bigvee \exists^n_{(a_1,a_2)}\Phi.$$

Immediately, $M_s \models \nabla_b\exists^n_{(a_1,a_2)}\Phi$.

(2) Let $M_s \models \nabla_b\exists^n_{(a_1,a_2)}\Phi$ and $\Phi = \{\varphi_1, \cdots, \varphi_m\}$. Then $M_s \models \bigwedge\Diamond_b\exists^n_{(a_1,a_2)}\Phi$ and $M_s \models \Box_b \bigvee \exists^n_{(a_1,a_2)}\Phi$. The analyses similar to those in the proof of Lemma 4 and Lemma 3 (2) give us the desired model M' here. Namely, the model M' is obtained from $M \uplus \biguplus_{u\in R_b^M(s)} M^u \uplus \biguplus_{1\leq j\leq m} M^j$ by removing all the b-labeled transitions outgoing from s in M and adding the b-labeled transition $s \xrightarrow{b} \langle u, u\rangle$ for each $u \in R_b^M(s)$ and $s \xrightarrow{b} \langle j, u^j\rangle$ for each $1 \leq j \leq m$ ($\Phi = \{\varphi_1, \cdots, \varphi_m\}$). Also, it is not difficult to see that $M'_s \models \nabla_b\exists_{(a_1,a_2)}\Phi$ and $M_s\underline{\leftrightarrow}^{n+1}M'_s$. Then, it follows from the validity of the axiom scheme **CCRKbis** (in Table 2) of CCRML [12] that

$$M'_s \models \exists_{(a_1,a_2)}\nabla_b\Phi.$$

Hence, by Proposition 10, $M_s \models \exists^{n+1}_{(a_1,a_2)}\nabla_b\Phi$ holds. □

Note that Lemma 5 (1) would be invalid without the assumption that $b \neq a_1, a_2$. We will give the following counterexamples. Consider the models depicted in Fig. 4 (1). We have that

$$\{\mathcal{Z}_i\}_{i\leq 1} : M_s \succeq^1_{(a_1,a_2)} N_t$$

with

$$\mathcal{Z}_i \triangleq \begin{cases}\{\langle s, t\rangle\} & \text{if } i = 1\\ \{\langle s, t\rangle, \langle u, v\rangle\} & \text{if } i = 0\end{cases}$$

and $N_t \models \nabla_{a_1}\Phi$ with $\Phi = \{p, q\} \subseteq P$. It is clear that $M_s \models \exists^1_{(a_1,a_2)}\nabla_{a_1}\Phi$, but it is easy to see $M_s \not\models \bigwedge\Diamond_{a_1}\exists^0_{(a_1,a_2)}\Phi$, hence $M_s \not\models \nabla_{a_1}\exists^0_{(a_1,a_2)}\Phi$.

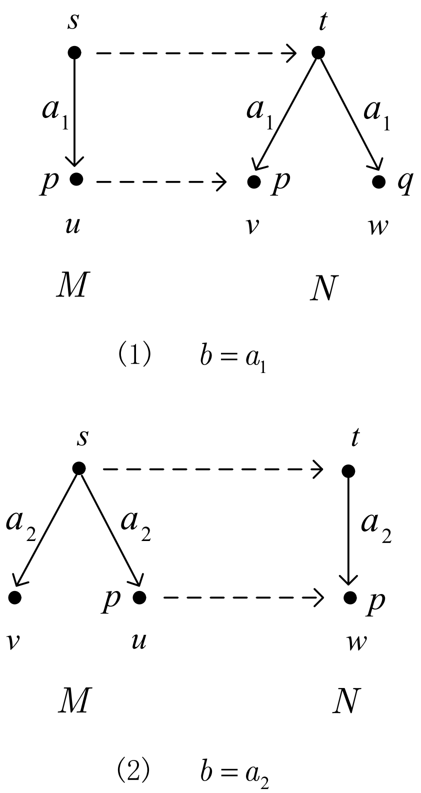

FIGURE 4. Counterexamples used in Lemma 5 (1).

Analogously, for the models in Fig. 4 (2), we have that $\{\mathcal{Z}'_i\}_{i\leq 1} : M_s \succeq^1_{(a_1,a_2)} N_t$ with

$$\mathcal{Z}'_i \triangleq \begin{cases}\{\langle s, t\rangle\} & \text{if } i = 1\\ \{\langle s, t\rangle, \langle u, w\rangle\} & \text{if } i = 0\end{cases}$$

and $N_t \models \nabla_{a_2}\Phi$ with $\Phi = \{p\} \subseteq P$. Thus $M_s \models \exists^1_{(a_1,a_2)}\nabla_{a_2}\Phi$, however, obviously, $M_s \not\models \Box_{a_2}\bigvee\exists^0_{(a_1,a_2)}\Phi$, so $M_s \not\models \nabla_{a_2}\exists^0_{(a_1,a_2)}\Phi$.

Lemma 6: Let $\Phi_b \subseteq_f \mathcal{L}^n_{CC}$ for every $b \in B(\subseteq A)$. Then

$$\models \exists^n_{(a_1,a_2)} \textstyle\bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bigwedge_{b\in B}\exists^n_{(a_1,a_2)}\nabla_b\Phi_b.$$

Proof: If β is unsatisfiable for some $\beta \in \bigcup_{b\in B}\Phi_b$, then

$$\models \bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bot.$$

It is easy to see that

$$\models \exists^n_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bot \text{ and } \models \bigwedge_{b\in B}\exists^n_{(a_1,a_2)}\nabla_b\Phi_b \leftrightarrow \bot.$$

Immediately, we get that

$$\models \exists^n_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Phi_b \leftrightarrow \bigwedge_{b\in B}\exists^n_{(a_1,a_2)}\nabla_b\Phi_b.$$

Below, we consider the nontrivial case where each β in $\bigcup_{b\in B}\Phi_b$ is satisfiable. It is not difficult to observe that

$$\models \exists^n_{(a_1,a_2)}\bigwedge_{b\in B}\nabla_b\Phi_b \to \bigwedge_{b\in B}\exists^n_{(a_1,a_2)}\nabla_b\Phi_b.$$

Next we prove the converse implication. Suppose that $M_s \models \bigwedge_{a\in B} \exists^n_{(a_1,a_2)} \nabla_a \Phi_a$. Consider two cases based on n.

(I) $n = 0$

By Lemma 2, $\exists^0_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b$ and $\exists^0_{(a_1,a_2)} \nabla_b \Phi_b$ for each $b \in B$ are valid, hence this implication holds trivially.

(II) $n > 0$

Let $b \in B$ and $\Phi_b = \{\varphi_1, \cdots, \varphi_{m_b}\}$. For $b = a_1$, by Lemma 3 (1), it holds that $M_s \models \Box_{a_1} \bigvee \exists^{n-1}_{(a_1,a_2)} \Phi_{a_1}$. For $b = a_2$, by Lemma 4, we get that $M_s \models \bigwedge \Diamond_{a_2} \exists^{n-1}_{(a_1,a_2)} \Phi_{a_2}$. For $b \neq a_1, a_2$, by Lemma 5 (1), $M_s \models \nabla_b \exists^{n-1}_{(a_1,a_2)} \Phi_b$ follows from $M_s \models \exists^n_{(a_1,a_2)} \nabla_b \Phi_b$. Through the analyses analogous as those in the proof of Lemmas 3 (2), 4 and 5 (2), the desired model M' here is obtained from $M \uplus \biguplus_{b\in B, u\in R^M_b(s)} M^{bu} \uplus \biguplus_{b\in B, 1\le j\le m_b} M^{bj}$ by modifying correspondingly the a_1-labeled, a_2-labeled and b-labeled transitions respectively. Thus, we may get that $M'_s \models \Box_{a_1} \bigvee \exists_{(a_1,a_2)} \Phi_{a_1}$, $M'_s \models \bigwedge \Diamond_{a_2} \exists_{(a_1,a_2)} \Phi_{a_2}$, $M'_s \models \nabla_b \exists_{(a_1,a_2)} \Phi_b$ $(b \neq a_1, a_2)$, and $M_s \underline{\leftrightarrow}^n M'_s$. From the validity of the axiom schemes **CCRKco2**, **CCRKcontra** and **CCRKbis** (listed in Table 2) of CCRML [12], it follows that

$$M'_s \models \bigwedge_{b\in B} \exists_{(a_1,a_2)} \nabla_b \Phi_b,$$

which implies

$$M'_s \models \exists_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b$$

due to the validity of the axiom scheme **CCRK-conj** (in Table 2) of CCRML [12]. Finally, $M_s \models \exists^n_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b$ follows from $M_s \underline{\leftrightarrow}^n M'_s$ and Proposition 10. □

We now arrive at the soundness of the axiom system for nCCRML.

Theorem 1 (Soundness): For each $\psi \in \mathcal{L}^n_{CC}$, $\vdash \psi$ implies that $\models \psi$.

Proof: As usual, it suffices to check that all the axiom schemes are valid, and the rules **MP**, **NK** and **nNCCR** are sound. It is trivial to prove that the axiom schemes **Prop**, **K**, **nCCR**, **nCCRp1**, **nCCRp2** and **nCCRKco1** are valid, and the rules **MP**, **NK** and **nNCCR** are sound. Furthermore, it follows from Lemma 1, 2, 3, 4, 5 and 6 that the axiom schemes **nCCRD**, **0CCRK**, **nCCRKco2**, **nCCRKcontra**, **nCCRKbis** and **nCCRKconj** are valid. □

C. COMPLETENESS

In this subsection, we intend to establish the completeness of the axiom system nCCRML, by the same method as in [1] and [12]. We will check that every nCCRML-formula is provably equivalent to a $\boldsymbol{K}$-formula. Based on the completeness of $\boldsymbol{K}$, this brings the completeness of nCCRML,.

Firstly, some general statements will be given as the preparations for the reduction argument.

Proposition 13: Let φ_1, φ_2, $\psi \in \mathcal{L}^n_{CC}$, and $q \in P$. Then

$$\vdash \varphi_1 \leftrightarrow \varphi_2 \text{ implies } \vdash \psi[\varphi_1 \backslash q] \leftrightarrow \psi[\varphi_2 \backslash q]$$

Proof: Proceed by the induction on the formula ψ. □

Proposition 14:

(1) $\vdash \forall^n_{(a_1,a_2)}(\varphi \wedge \psi) \leftrightarrow \forall^n_{(a_1,a_2)}\varphi \wedge \forall^n_{(a_1,a_2)}\psi$.
(2) $\vdash \exists^n_{(a_1,a_2)}(\varphi \vee \psi) \leftrightarrow \exists^n_{(a_1,a_2)}\varphi \vee \exists^n_{(a_1,a_2)}\psi$.
(3) $\vdash \forall^n_{(a_1,a_2)}\varphi \vee \forall^n_{(a_1,a_2)}\psi \to \forall^n_{(a_1,a_2)}(\varphi \vee \psi)$.
(4) $\vdash \exists^n_{(a_1,a_2)}(\varphi \wedge \psi) \to \exists^n_{(a_1,a_2)}\varphi \wedge \exists^n_{(a_1,a_2)}\psi$.

Proof: Trivially. □

Proposition 15: For each $\alpha \in \mathcal{L}_p$, we have

(1) $\vdash \forall^n_{(a_1,a_2)}\alpha \leftrightarrow \alpha$
(2) $\vdash \exists^n_{(a_1,a_2)}\alpha \leftrightarrow \alpha$

Proof: By the same strategy as that applied in the proof of [12, Proposition 4.12]. □

The above proposition generalizes the axiom schemes **nCCRp1** and **nCCRp2**, which guarantees that the CC-n-refinement quantifiers over any propositional formula may be eliminated by proof-theoretical method. However, $\exists^n_{(a_1,a_2)}\beta$ with $\beta \in \mathcal{L}_K$ is not always logical equivalent to β. Then we have $\nvdash \exists^n_{(a_1,a_2)}\beta \leftrightarrow \beta$ due to the soundness. Fortunately, $\exists^n_{(a_1,a_2)}\beta$ is provably equivalent to some $\mathcal{L}_K$-formula in nCCRML. To prove this, we require some auxiliary results and notions.

Proposition 16: Let $\alpha \in \mathcal{L}_p$ and $\psi \in \mathcal{L}^n_{CC}$. Then

$$\vdash \exists^n_{(a_1,a_2)}(\alpha \wedge \psi) \leftrightarrow (\alpha \wedge \exists^n_{(a_1,a_2)}\psi).$$

Proof: Analogous to the proof of [12, Proposition 4.13]. □

Now recall the notion of ***disjunctive formula in cover logic*** (***df***, for short) [17]. The ***df*** formulas are defined by the BNF grammar as follows:

$$\alpha ::= (\alpha \vee \alpha) \mid \alpha_0 \mid (\alpha_0 \wedge \bigwedge_{b\in B} \nabla_b\{\alpha, \cdots, \alpha\})$$

where $\emptyset \neq B \subseteq A$ and $\alpha_0 \in \mathcal{L}_p$.

*Proposition 17 ([17]): For each $\psi \in \mathcal{L}_K$, there is a **df** formula α such that $\vdash_{\boldsymbol{K}} \psi \leftrightarrow \alpha$.*

So far, we can prove that any formula being of the form $\exists^n_{(a_1,a_2)}\alpha$ $(\alpha \in \mathcal{L}_K)$ can be provably reduced to a $\mathcal{L}_K$-formula.

Proposition 18: Let $\alpha \in \mathcal{L}_K$. Then

$$\vdash \exists^n_{(a_1,a_2)}\alpha \leftrightarrow \xi \quad \text{for some } \xi \in \mathcal{L}_K.$$

Proof: Since the axiom system $\boldsymbol{K}$ is involved in nCCRML presented in this paper, by Proposition 17 and Proposition 13, we may w.l.o.g. suppose that α is a ***df*** formula. Below, we proceed inductively by the structure of α.

For $\alpha \in \mathcal{L}_p$, it follows from Proposition 15.

For $\alpha \equiv \alpha_1 \vee \alpha_2$, by Proposition 14 (2), it holds that

$$\vdash \exists^n_{(a_1,a_2)}\alpha \longleftrightarrow \exists^n_{(a_1,a_2)}\alpha_1 \vee \exists^n_{(a_1,a_2)}\alpha_2.$$

Thus, by the induction hypothesis and Proposition 13, we have

$$\vdash \exists^n_{(a_1,a_2)}\alpha \longleftrightarrow \xi_1 \vee \xi_2 \ \text{ for some } \xi_1,\ \xi_2 \in \mathcal{L}_K.$$

For $\alpha \equiv \alpha_0 \wedge \bigwedge_{b\in B} \nabla_b \Phi_b$ with $\alpha_0 \in \mathcal{L}_p$ and $\Phi_b \subseteq \boldsymbol{df}$ for every $b \in B$, by Proposition 16, we obtain

$$\vdash \exists^n_{(a_1,a_2)}\alpha \longleftrightarrow \alpha_0 \wedge \exists^n_{(a_1,a_2)} \bigwedge_{b\in B} \nabla_b \Phi_b.$$

Further, by **nCCRKconj** and Proposition 13, it holds that

$$\vdash \exists^n_{(a_1,a_2)}\alpha \longleftrightarrow \alpha_0 \wedge \bigwedge_{b\in B} \exists^n_{(a_1,a_2)} \nabla_b \Phi_b.$$

Then, to complete the proof, we will prove that, for every $b \in B$,

$$\vdash \exists^n_{(a_1,a_2)}\nabla_b\Phi_b \longleftrightarrow \xi_b \text{ for some } \xi_b \in \mathcal{L}_K. \, (*) \qquad (1)$$

Now we prove this by considering two cases based on n.

(I) $n = 0$

Since $\Phi_b \subseteq \boldsymbol{df} \subseteq \mathcal{L}_K$, we have that,

$$\vdash \exists^0_{(a_1,a_2)}\nabla_b\Phi_b \longleftrightarrow \bot$$

whenever $\vdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for some $\beta \in \Phi_b$, or, by **0CCRK**,

$$\vdash \exists^0_{(a_1,a_2)}\nabla_b\Phi_b \longleftrightarrow (\neg q \vee q)$$

whenever $\nvdash_{\boldsymbol{K}} \beta \leftrightarrow \bot$ for all $\beta \in \Phi_b$, where $q \in P$.

(II) $n > 0$

Due to $\Phi_b \subseteq \boldsymbol{df} \subseteq \mathcal{L}_K$, applying the axiom schemes **nCCRKco1**, **nCCRKco2**, **nCCRKcontra** and **nCCRKbis**, we obtain $\vdash \exists^n_{(a_1,a_2)}\nabla_b\Phi_b \longleftrightarrow \theta$ for some θ in which the quantifiers $\exists^{n-1}_{(a_1,a_2)}$ only bind the formulas in Φ_b. Finally, by the induction hypothesis and Proposition 13, the claim $(*)$ follows, as desired. □

It is time to verify that every $\mathcal{L}^n_{CC}$-formula can be provably reduced to a $\mathcal{L}_K$-formula. This is crucial to establish the completeness of nCCRML.

Proposition 19: For each $\psi \in \mathcal{L}^n_{CC}$, we have that

$$\vdash \psi \leftrightarrow \alpha \qquad \textit{for some } \alpha \in \mathcal{L}_K.$$

Proof: Due to the axiom scheme **nCCRD** and Proposition 13, it suffices to deal with the formulas where all CC-n-refinement quantifiers are of the form $\exists^n_{(a,b)}$ with $n < \omega$ and $a, b \in A$. We prove by the induction on the number $num(\psi)$ of the occurrences of the CC-n-refinement quantifiers in ψ. For $num(\psi) = 0$, trivially.

For $num(\psi) > 0$, we can always find a subformula of ψ being of the form $\exists^n_{(a,b)}\delta$ with $\delta \in \mathcal{L}_K$. Further, by Proposition 18 and Proposition 13, it is easy to see that $\vdash \psi \leftrightarrow \psi'$ for some $\psi' \in \mathcal{L}^n_{CC}$ with $num(\psi') < num(\psi)$, which enables the induction proof to work well. □

Proposition 20: Let $\psi \in \mathcal{L}^n_{CC}$ and $\alpha \in \mathcal{L}_K$ such that $\vdash \psi \leftrightarrow \alpha$. If α is a theorem in $\boldsymbol{K}$, so is ψ in nCCRML.

Proof: Since nCCRML contains the axiom system $\boldsymbol{K}$, $\vdash \alpha$ follows immediately from $\vdash_{\boldsymbol{K}} \alpha$. Further, we get $\vdash \psi$ due to $\vdash \psi \leftrightarrow \alpha$. □

Theorem 2 (Completeness): For each $\psi \in \mathcal{L}^n_{CC}$, $\models \psi$ implies $\vdash \psi$.

Proof: By Proposition 19, it follows that $\vdash \psi \leftrightarrow \alpha$ for some $\alpha \in \mathcal{L}_K$. So, by Theorem 1 (the soundness of nCCRML), $\models \psi \leftrightarrow \alpha$, next, we get $\models \alpha$ due to $\models \psi$. Thus $\vdash_{\boldsymbol{K}} \alpha$ due to the completeness of $\boldsymbol{K}$. Finally, by Proposition 20, we get $\vdash \psi$, as desired. □

In order to establish the completeness of nCCRML, we have checked that all $\mathcal{L}^n_{CC}$-formulas can be provably reduced to $\mathcal{L}_K$-formulas. From our proof, we easily see that there is an algorithm to transform every $\mathcal{L}^n_{CC}$-formula φ into a $\mathcal{L}_K$-formula α such that $\vdash \varphi$ iff $\vdash_{\boldsymbol{K}} \alpha$. Then, due to the decidability of the system $\boldsymbol{K}$, we get

Theorem 3 (Decidability): nCCRML is decidable. □

V. CONCLUSIONS AND DISCUSSION

The notion of CC-n-refinement finitely approximates the notion of CC-refinement, based on which, this paper considers CC-n-refinement modal logic (nCCRML). In addition to the standard modal operators, the language $\mathcal{L}^n_{CC}$ of nCCRML contains CC-n-refinement operator $\exists^n_{(A_1,A_2)}$, where A_1 (A_2) is a set of all covariant (contravariant, resp.) actions. Intuitively, the formula $\exists^n_{(A_1,A_2)}\psi$ represents that we can CC-n-refine the current model so that ψ is satisfied in the obtained CC-n-refined model. CC-refinement requires two related models to be always able to provide matched transitions each other, nevertheless, for CC-n-refinement, two related models are required to match each other just within n steps and the farther future is not important. Focused on this approximation, CC-n-refinement operator may be used to formalize some interesting problems in the field of formal method.

For example, given a specification expressed by a LTS M which refers to the set A_1 (A_2) of passive (generative, resp.) actions, we consider the problem: whether this specification has an implementation to realize a given property ψ. In this situation, the transitions of passive actions in the given specification M should be simulated by any correct implementation and the transitions of generative actions in an implementation must be allowed by the specification M. In some applications, e.g. programming verification online, the above requirement in the far future is not as important as that in the near future. Thus, a CC-n-refined model of M may be considered to be an implementation of M if n is big enough. So the above problem may be formalized as the model checking problem:

$$M \models \exists^n_{(A_1,A_2)}\psi,$$

where realizing the given property ψ is described by the satisfiability of the formula ψ on a CC-refined model N. Hence, based on CC-n-refinement quantifiers, the problem whether there exists a desired approximative implementation such that some property of a given specification may be solved using model checking technique.

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