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Stability Analysis of Switched Positive Nonlinear Systems by Mode-Dependent Average Dwell Time Method

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ABSTRACT The stability problem of switched positive nonlinear systems is presented in this paper. Both continuous-time and discrete-time systems are considered. Compared with the average dwell time (ADT) switching, the switching law designed here is more general in which each mode possesses its own ADT. Unlike most of the existing results, based on the approach in which the Lyapunov function method is not involved, several stability criteria are derived. In the end, a numerical example is presented to illustrate the main results. The simulation results demonstrate that the proposed MDADT switching signal can achieve better performance, which shows that the state trajectories converge faster than the ADT switching.

INDEX TERMS Stability, switched nonlinear positive system, mode-dependent average dwell time.

I. INTRODUCTION

In nature and society, there exist a class of systems with non-negative state variables. This kind of systems are called positive systems [1], which have wide applications in compartment systems, chemical systems and so on. In recent years, stability problem of positive systems has been studied extensively, see e.g., ([2]–[4]). Switched positive systems consisting of some positive subsystems and a switching signal have been paid more attention in different kind of areas. From the point of engineering application, it is essential to consider the qualitative theory of the system before designing the controller, such as oscillation ([5]–[7]) and stability ([8], [9]). Especially, stability is the typical feature which should to be considered in advance. Based on time driven or state driven, the method for studying stability of switched systems can be divided into time-dependent switching [10] and state-dependent switching [11].

In recent years, stability of switched positive systems has been paid more attention ([12]–[21]). As the time-dependent switching, the ADT method is usually utilized to investigate the stability of switched positive systems. To list a few, Zhao *et al.* [12] utilized ADT method to study the stability problem of switched positive linear systems, and the results were further improved in [16] with average dwell time switching. On the other hand, the MDADT method was firstly proposed in [22] to investigate stability and stabilization problem of switched linear systems, which illustrated that the proposed method was more effective than ADT method. Many stability criteria were established based on the MDADT method ([23]–[26]). For example, Zhang *et al.* [23] employed MDADT method to investigate stability and stabilization of switched linear systems and improved the main results in [12]. At present, most of existing results are based on switched positive linear systems. For the switching positive nonlinear systems, Dong [28] studied the stability problem under ADT switching, which was generalized to switched positive nonlinear systems with exogenous input in [29]. Zhang *et al.* [30] and Liu *et al.* [31], the authors considered the stability problem of a class of switched positive nonlinear systems with delays. Up to now, stability of switched positive nonlinear systems under MDADT switching receives less attention. This motivates us for the present study.

Stability of switched positive nonlinear systems is addressed in this paper, and the main contributions of this paper are attributed to two aspects: (i) Unlike ADT switching, the switching law designed here is more general in which each mode possesses its own ADT. (ii) Based on the approach in which Lyapunov function method is not involved, several stability criteria are derived under MDADT switching.

At the end, we provide an example to illustrate the results obtained in this paper.

Throughout the paper, the following notation is utilized. Let N_0, R, and R^n represent the set of natural numbers, real numbers, and n-dimensional Euclidean space, respectively. Vectors are defined by bold letters. For $\boldsymbol{x}_p \in R^n$ and $i = 1, 2, \ldots, n$, x_{pi} is the ith coordinate of $\boldsymbol{x}_p$ with $x_p = [x_{pi}, \ldots, x_{pn}]^T$. Let $R^n_+ = \{\boldsymbol{x} \in R^n : x_i \geq 0, 1 \leq i \leq n\}$. For $\boldsymbol{x}, \boldsymbol{y} \in R^n$, we write: $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_i \geq y_i$ for all $1 \leq i \leq n$; $\boldsymbol{x} > \boldsymbol{y}$ if $x_i \geq y_i$, and $\boldsymbol{x} \neq \boldsymbol{y}$; $\boldsymbol{x} \gg \boldsymbol{y}$ if $x_i > y_i$ for all $1 \leq i \leq n$. Given a positive vector $\boldsymbol{v} > \boldsymbol{0}$, we use the weighted l^∞-norm for a vector $\boldsymbol{x} \in R^n$: $\|\boldsymbol{x}\|_v = \max_{1 \leq i \leq n} \frac{x_i}{v_i}$. A matrix $\boldsymbol{A} \in R^{n\times n}$ is nonnegative if $a_{ij} \geq 0$ for $1 \leq i, j \leq n$, and the matrix $\boldsymbol{A}$ is *Metzler* if $a_{ij} \geq 0$ for $i \neq j$.

II. PROBLEM DESCRIPTION AND PRELIMINARIES

Consider the switched nonlinear system

$$\delta \boldsymbol{x}(t) = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x}(t)), \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$ is the state, δ represents the derivative operator under the case of continuous-time ($\delta x(t) = (d/dt)x(t)$) and the shift forward operator under the case of discrete-time ($\delta x(t) = x(t+1)$). A switching signal $\sigma(t) : [0, +\infty) \to \mathscr{P} = \{1, 2, \ldots, M\}$ is defined on a switching sequence $0 = t_0 < t_1 < \cdots < t_j < t_{j+1} < \cdots$, which is everywhere continuous from the right, when $t \in [t_j, t_{j+1})$, the $\sigma(t_j)$th subsystem is active, M is the number of subsystems. The following definitions, proposition, and assumption are needed to present the main results.

Definition 1 [27]: Assume that the vector field $\boldsymbol{f} : R^n \to R^n$ is continuous on R^n and continuously differentiable on $R^n \backslash \{\boldsymbol{0}\}$. If the Jacobian matrix $\frac{\partial f}{\partial \boldsymbol{a}}$ is Metzler for all $\boldsymbol{a} \in R^n_+ \backslash \{\boldsymbol{0}\}$, then it is said to be cooperative.

Proposition 2 [28]: Suppose that $\boldsymbol{f}$ is a cooperative vector filed. For any $\boldsymbol{x}, \boldsymbol{y} \in R^n \backslash \{\boldsymbol{0}\}$, if they are satisfied with $\boldsymbol{x} \geq \boldsymbol{y}$ and $x_i = y_i$, then we have $f_i(\boldsymbol{x}) \geq f_i(\boldsymbol{y})$.

Definition 3: For $\forall\, \boldsymbol{x} \in R^n$ and $\lambda > 0$, if $\boldsymbol{f}(\lambda \boldsymbol{x}) = \lambda^\alpha \boldsymbol{f}(\boldsymbol{x})$, then we call that the vector field $\boldsymbol{f}$ is homogeneous of degree $\alpha > 0$.

$\boldsymbol{f}$ is homogeneous of degree one under the case $\alpha = 1$.

Definition 4: If for any $\boldsymbol{x}, \boldsymbol{y} \in R^n$ satisfying $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\boldsymbol{f}(\boldsymbol{x}) \geq \boldsymbol{f}(\boldsymbol{y})$, then we call the vector field $\boldsymbol{f} : R^n \to R^n$ is order-preserving on R^n_+.

Assumption 5: $\boldsymbol{f}_p$ is cooperative and homogeneous of degree one for $\forall\, p \in \mathscr{P}$.

Assumption 5 guarantees that $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$, which implies the system (1) possesses zero solution. On the other hand, it is shown in [28] that system (1) which satisfies Assumption 5 is a switched positive nonlinear systems (SPNS).

Definition 6 [22]: The zero solution $\boldsymbol{x} = \boldsymbol{0}$ of system (1) is globally uniformly exponentially stable (GUES) under the switching signal $\sigma(t)$ and the initial conditions $x(t_0)$ (or $x(k_0)$), if there exist constants $\alpha > 0$ and $r > 0$ (respectively, $0 < \varsigma < 1$) such that the solution of the system satisfies $\|x(t)\| \leq \alpha e^{-r(t-t_0)} \|x(t_0)\|, t \geq t_0$ (respectively, $\|x(k)\| \leq \alpha \varsigma^{k-k_0} \|x(k_0)\|, k \geq k_0$).

Definition 7 [22]: Let $N_{\sigma p}(T, t)$ and $T_p(T, t)$ be the switching numbers and the total active time of the p^{th} subsystem on the interval $[t, T]$, $p \in \mathscr{P}$, respectively. We call that the switching signal $\sigma(t)$ possesses a mode-dependent average dwell time τ_{ap} under the condition that

$$N_{\sigma p}(T, t) \leq N_{0p} + \frac{T_p(T, t)}{\tau_{ap}}, \quad T \geq t \geq 0,$$

where N_{0p} and τ_{ap} are positive constants.

III. MAIN RESULTS

In this part, we consider the stability problem of both continuous-time and discrete-time SPNSs under MDADT switching. The continuous-time case is first considered.

Theorem 1 (Continuous-Time Case): Consider continuous-time switched systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x}(t)) \tag{2}$$

and let Assumption 5 hold. If there exists a vector $\boldsymbol{v}_p \gg 0$ satisfying $\boldsymbol{f}_p(\boldsymbol{v}_p) \ll 0$ for $\forall\, p \in \mathscr{P}$, then the SPNS (2) is GUES under a class of MDADT switching signal satisfying

$$\tau_{ap} > \tau^*_{ap} = \frac{\ln \mu_p}{\eta_p}, \tag{3}$$

where

$$\mu_p = \max_{p,q \in \mathscr{P}} \frac{\overline{v}_q}{\underline{v}_p} \quad \text{with } \overline{v}_q = \max_{1 \leq i \leq n} v_{qi}, \quad \underline{v}_p = \min_{1 \leq i \leq n} v_{pi},$$

and $\eta_p \in (0, \min_{1 \leq i \leq n} \eta_{pi})$ with η_{pi} satisfying

$$\frac{f_{pi}(\boldsymbol{v}_p)}{v_{pi}} + \eta_{pi} = 0. \tag{4}$$

Proof: Based on $\boldsymbol{v}_p \gg 0$ and $\boldsymbol{f}_p(\boldsymbol{v}_p) \ll 0$ for $\forall\, p \in \mathscr{P}$, (4) possesses a unique positive solution and

$$\frac{f_{pi}(\boldsymbol{v}_p)}{v_{pi}} + \eta_p < 0, \quad 1 \leq i \leq n.$$

Consider the interval $[t_k, t_{k+1})$, $k \geq 0$, without of generality, we may assume $\sigma(t) = \sigma(t_k) \equiv p,\ t \in [t_k, t_{k+1})$. SPNS (2) can be rewritten as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}_{\sigma(t_k)}(\boldsymbol{x}(t)), \quad t \in [t_k, t_{k+1}).$$

In the following, we will show that for $t \in [t_k, t_{k+1})$

$$\|\boldsymbol{x}(t)\|_{\boldsymbol{v}_{\sigma(t_k)}} \leq e^{-\eta_{\sigma(t_k)}(t-t_k)} \|\boldsymbol{x}(t_k)\|_{\boldsymbol{v}_{\sigma(t_k)}}.$$

Define

$$y_i(t) = \frac{x_i(t)}{v_{\sigma(t_k)i}} - e^{-\eta_{\sigma(t_k)}(t-t_k)} \|\boldsymbol{x}(t_k)\|_{\boldsymbol{v}_{\sigma(t_k)}},$$

where $t \in [t_k, t_{k+1})$. $y_i(t_k) \leq 0$ for all i due to the definition of $\|\boldsymbol{x}(t_k)\|_{\boldsymbol{v}_{\sigma(t_k)}}$. Next we will prove that

$$y_i(t) \leq 0, \quad t \in [t_k, t_{k+1}),\ 1 \leq i \leq n. \tag{5}$$

By contradiction, suppose (5) is not true for $t \in [t_k, t_{k+1})$, there exist an index $m \in \{1, 2, \ldots, n\}$ and a time constant

$t^* \in [t_k, t_{k+1})$ such that $y_i(t) \le 0,\ t \in [t_k, t^*)$ and $y_m(t^*) = 0$. Then

$$\dot{y}_m(t^*) \ge 0. \tag{6}$$

By the definition of $y_i(t)$, we have

$$x_m(t^*) = \|x(t_k)\|_{v_{\sigma(t_k)}} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} v_{\sigma(t_k)m}$$

and

$$\boldsymbol{x}(t^*) \le \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \boldsymbol{v}_{\sigma(t_k)}.$$

By Assumption 5 and Proposition 2,

$$\begin{aligned} f_{\sigma(t_k)m}(\boldsymbol{x}(t^*)) &\le f_{\sigma(t_k)m}\Big(\|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} \\ &\qquad \times e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \boldsymbol{v}_{\sigma(t_k)} \Big) \\ &= \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \\ &\qquad \times f_{\sigma(t_k)m}(\boldsymbol{v}_{\sigma(t_k)}). \end{aligned}$$

Therefore,

$$\begin{aligned} \dot{y}_m(t^*) &= \frac{\dot{x}_m(t^*)}{v_{\sigma(t_k)}m} + \eta_{\sigma(t_k)} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} \\ &= \frac{1}{v_{\sigma(t_k)}m} f_{\sigma(t_k)m}(\boldsymbol{x}(t^*)) \\ &\quad + \eta_{\sigma(t_k)} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} \\ &\le \frac{f_{\sigma(t_k)m}(\boldsymbol{v}_{\sigma(t_k)})}{v_{\sigma(t_k)}m} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \\ &\quad + \eta_{\sigma(t_k)} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} \\ &= \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}} e^{-\eta_{\sigma(t_k)}(t^*-t_k)} \\ &\quad \times \left(\frac{f_{\sigma(t_k)m}(\boldsymbol{v}_{\sigma(t_k)})}{v_{\sigma(t_k)}m} + \eta_{\sigma(t_k)} \right) \\ &< 0, \end{aligned}$$

which contradicts (6). Therefore, $y_i(t) \le 0, \quad t \in [t_k, t_{k+1})$, i.e.,

$$\|\boldsymbol{x}(t)\|_{v_{\sigma(t_k)}} \le e^{-\eta_{\sigma(t_k)}(t-t_k)} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_k)}},$$

where $t \in [t_k, t_{k+1})$. Moreover,

$$\begin{aligned} \|\boldsymbol{x}(t)\|_{v_{\sigma(t_k)}} &= \max_{1\le i\le n} \frac{x_i(t)}{v_{\sigma(t_k)}i} = \max_{1\le i\le n} \frac{v_{\sigma(t_k^-)}i}{v_{\sigma(t_k)}i} \frac{x_i(t)}{v_{\sigma(t_k^-)}i} \\ &\le \max_{1\le i\le n} \frac{\overline{v}_{\sigma(t_k^-)}}{\underline{v}_{\sigma(t_k)}} \|\boldsymbol{x}(t)\|_{v_{\sigma(t_k^-)}} \\ &= \mu_{\sigma(t_k)} \|\boldsymbol{x}(t)\|_{v_{\sigma(t_k^-)}}. \end{aligned}$$

Denote $t_0 = 0$ and $t_1, t_2 \ldots, t_k$ the switching times on the interval $[0, T]$, where $k = N_\sigma(T, 0) = \sum_{p=1}^{M} N_{\sigma p}(T, 0)$. Therefore,

$$\begin{aligned} &\|\boldsymbol{x}(T)\|_{v_{\sigma(t_k)}} \\ &\le e^{-\eta_{\sigma(t_k)}(T-t_k)} \mu_{\sigma(t_k)} \|\boldsymbol{x}(t_k)\|_{v_{\sigma(t_{k-1})}} \\ &\le \prod_{i=1}^{N_\sigma} \mu_{\sigma(t_i)} e^{-\eta_{\sigma(t_k)}(T-t_k)} \\ &\quad \times e^{-\sum_{i=0}^{k-1} \eta_{\sigma(t_i)}(t_{i+1}-t_i)} \|\boldsymbol{x}(0)\|_{v_{\sigma(0)}} \\ &\le \prod_{P=1}^{M} \mu_p^{N_{\sigma p}} \exp\Big\{ -\eta_p \sum_{s\in\psi(p)} (t_{s+1} - t_s) \\ &\quad - \eta_{\sigma(t_k)}(T - t_k) \Big\} \|\boldsymbol{x}(0)\|_{v_{\sigma(0)}} \\ &\le \exp\left\{ \sum_{p=1}^{M} N_{0p} \ln \mu_p \right\} \\ &\quad \times \exp\left\{ \sum_{p=1}^{M} \frac{T_p}{\tau_{ap}} \ln \mu_p - \sum_{p=1}^{M} \eta_p T_p \right\} \|\boldsymbol{x}(0)\|_{v_{\sigma(0)}} \\ &= \exp\left\{ \sum_{p=1}^{M} N_{0p} \ln \mu_p \right\} \\ &\quad \times \exp\left\{ \sum_{p=1}^{M} (\frac{\ln \mu_p}{\tau_{ap}} - \eta_p) T_p \right\} \|\boldsymbol{x}(0)\|_{v_{\sigma(0)}}, \end{aligned}$$

where $\psi(p)$ represents the set s with $\sigma(t_s) = p,\ t_s \in \{t_0, t_1, \ldots, t_{N_\sigma - 1}\}$, i.e.,

$$\begin{aligned} \|\boldsymbol{x}(T)\|_{v_{\sigma(t_k)}} &\le \exp\left\{ \sum_{p=1}^{M} N_{0p} \ln \mu_p \right\} \\ &\quad \times \exp\left\{ \max_{p\in S} (\frac{\ln \mu_p}{\tau_{ap}} - \eta_p) T \right\} \|\boldsymbol{x}(0)\|_{v_{\sigma(0)}}. \end{aligned}$$

This combined with (3) yields $\|\boldsymbol{x}(T)\|_{v_{\sigma(t_k)}}$ converges to zero as $T \to \infty$. Therefore, SPNS (1) is GUES under any MDADT switching signal satisfying $\tau_{ap} > \tau_{ap}^* = \frac{\ln \mu_p}{\eta_p}$.

Remark 1: In [28, Theorem 3.4], the switched rule satisfies $\tau_a > \tau_a^* = \frac{\ln \mu}{\eta}$, where $\mu = \max_{1\le i\le n} \frac{\overline{v}_i}{\underline{v}_i}$, $\overline{v}_i = \max_{p\in\mathscr{P}} v_{pi}$, $\underline{v}_i = \min_{p\in\mathscr{P}} v_{pi}$, $\eta \in (0, \min_{1\le i\le n, p\in\mathscr{P}} \eta_{pi})$ with η_{pi} satisfying $f_{pi}(\boldsymbol{v}_p)/v_{pi} + \eta_{pi} = 0$. We have $\tau_a^* = \frac{\ln \mu}{\eta} \ge \frac{\ln \mu_p}{\eta_p} = \tau_{ap}^*$, which allows that the ADT associated with some of subsystems is less than the ADT of all subsystems. Therefore, it is a typical advantage of the MDADT method.

In the following, we consider the discrete-time case.

Theorem 2 (Discrete-Time Case): Consider the discrete-time switched nonlinear systems

$$\boldsymbol{x}(k+1) = \boldsymbol{f}_{\sigma(k)}(\boldsymbol{x}(k)), \quad k \in N_0 \tag{7}$$

and let Assumption 5 is satisfied. If there exists a vector $\boldsymbol{v}_p \gg 0$ such that $\boldsymbol{f}_p(\boldsymbol{v}_p) \ll \boldsymbol{v}_p,\ p \in \mathscr{P}$, then SPNS (7) is GUES under the MDADT switching signal satisfying

$$\tau_{ap} > \tau_{ap}^{*} = -\frac{\ln \mu_p}{\ln \gamma_p}, \tag{8}$$

where

$$\mu_p = \max_{p,q\in\mathscr{P}} \frac{\overline{v}_q}{\underline{v}_p} \quad \text{with} \quad \overline{v}_q = \max_{1\le i\le n} v_{qi}, \quad \underline{v}_p = \min_{1\le i\le n} v_{pi},$$

and $\gamma_p = \max_{1\le i\le n} \gamma_{pi}$ with $\gamma_{pi} \in (0,1)$ satisfying

$$\frac{f_{pi}(\boldsymbol{v}_p)}{v_{pi}} = \gamma_{pi}. \tag{9}$$

Proof: Since there exists a vector $\boldsymbol{v}_p \gg 0$ such that $\boldsymbol{f}_p(\boldsymbol{v}_p) \ll \boldsymbol{v}_p$, $p \in \mathscr{P}$. From (9), it is not difficult to get that $\gamma_{pi} \in (0,1)$, and

$$\frac{f_{pi}(\boldsymbol{v}_p)}{v_{pi}} = \gamma_{pi} \le \gamma_p.$$

In the following, $0 = k_0 < k_1 < k_2 < \ldots < k_l < \ldots$ represent the switching points and $N_\sigma(K,k)$ denotes the number of switching times on the interval $[k,K]$. Without of generality, we may assume $\sigma(k) = \sigma(k_l) \equiv p$, $k \in [k_l, k_{l+1})$. SPNS (7) can be rearranged as

$$\boldsymbol{x}(k+1) = \boldsymbol{f}_{\sigma(k_l)}(\boldsymbol{x}(k)), \quad k \in [k_l, k_{l+1}).$$

Next, we deduce that

$$\|\boldsymbol{x}(k)\|_{v_{\sigma(k_l)}} \le \gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}}, \quad k \in [k_l, k_{l+1}). \tag{10}$$

It is trivial that (10) is satisfied with $k = k_l$, we will prove that (10) holds by induction. Suppose it is true for $k \ge k_l$. According to (10), we have

$$\boldsymbol{x}(k) \le \gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}} \boldsymbol{v}_{\sigma(k_l)}, \quad k \in [k_l, k_{l+1}).$$

Since $\boldsymbol{f}_{\sigma(k_l)}$ satisfies Assumption 5, by Proposition 2, we get

$$\begin{aligned} f_{\sigma(k_l)}(\boldsymbol{x}(k)) &\le f_{\sigma(k_l)}\left(\gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}} \boldsymbol{v}_{\sigma(k_l)}\right) \\ &= \gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}} f_{\sigma(k_l)}(\boldsymbol{v}_{\sigma(k_l)}). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} \|\boldsymbol{x}(k+1)\|_{\boldsymbol{v}_{\sigma(k_l)}} &= \|\boldsymbol{f}_{\sigma(k_l)}(\boldsymbol{x}(k))\|_{\boldsymbol{v}_{\sigma(k_l)}} \\ &\le \gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}} \|f_{\sigma(k_l)}(\boldsymbol{v}_{\sigma(k_l)})\|_{\boldsymbol{v}_{\sigma(k_l)}} \\ &\le \gamma_{\sigma(k_l)}^{k+1-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}}. \end{aligned}$$

By induction, we have

$$\|\boldsymbol{x}(k)\|_{v_{\sigma(k_l)}} \le \gamma_{\sigma(k_l)}^{k-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}}, \quad k \in [k_l, k_{l+1}).$$

In addition,

$$\begin{aligned} \|\boldsymbol{x}(k)\|_{\boldsymbol{v}_{\sigma(k_l)}} &= \max_{1\le i\le n} \frac{x_i(k)}{\boldsymbol{v}_{\sigma(k_l)}i} = \max_{1\le i\le n} \frac{v_{\sigma(t_k^-)}i}{\boldsymbol{v}_{\sigma(t_k)}i} \frac{x_i(t)}{v_{\sigma(t_k^-)}i} \\ &\le \max_{1\le i\le n} \frac{\overline{v}_{\sigma(t_k^-)}}{\underline{v}_{\sigma(t_k)}} \frac{x_i(t)}{v_{\sigma(t_k^-)}i} \\ &= \mu_{\sigma(t_k)} \|\boldsymbol{x}(t)\|_{\boldsymbol{v}_{\sigma(t_k^-)}}. \end{aligned}$$

Since $k_1, k_2 \ldots, k_l$ are the switching points on $[0,K]$, we have $l = N_\sigma(K,0) = \sum_{p=1}^{M} N_{\sigma p}(K,0)$. Therefore,

$$\begin{aligned} &\|\boldsymbol{x}(K)\|_{\boldsymbol{v}_{\sigma(k_l)}} \\ &\quad \le \gamma_{\sigma(k_l)}^{K-k_l} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_l)}} \\ &\quad \le \gamma_{\sigma(k_l)}^{K-k_l} \mu_{\sigma(k_l)} \|\boldsymbol{x}(k_l)\|_{\boldsymbol{v}_{\sigma(k_{l-1})}} \\ &\quad \le \prod_{i=1}^{N_\sigma} \mu_{\sigma(k_i)} \gamma_{\sigma(k_l)}^{K-k_l} \prod_{i=0}^{l-1} \gamma_{\sigma(k_i)}^{k_{i+1}-k_i} \|\boldsymbol{x}(k_0)\|_{\boldsymbol{v}_{\sigma(0)}} \\ &\quad \le \prod_{P=1}^{M} \mu_p^{N_{\sigma p}} \exp\{\ln \gamma_p \sum_{s\in\psi(p)} (k_{s+1} - k_s) \\ &\qquad + \ln \gamma_{\sigma(k_l)}(K - k_l)\} \|\boldsymbol{x}(0)\|_{\boldsymbol{v}_{\sigma(0)}} \\ &\quad \le \exp\left\{\sum_{p=1}^{M} N_{0p} \ln \mu_p\right\} \\ &\qquad \times \exp\left\{\sum_{p=1}^{M} \frac{T_p}{\tau_{ap}} \ln \mu_p + \sum_{p=1}^{M} \ln \gamma_p T_p\right\} \|\boldsymbol{x}(0)\|_{\boldsymbol{v}_{\sigma(0)}} \\ &\quad = \exp\left\{\sum_{p=1}^{M} N_{0p} \ln \mu_p\right\} \\ &\qquad \times \exp\left\{\sum_{p=1}^{M} (\frac{\ln \mu_p}{\tau_{ap}} + \ln \gamma_p) T_p\right\} \|\boldsymbol{x}(0)\|_{\boldsymbol{v}_{\sigma(0)}}, \end{aligned}$$

where $\psi(p)$ denotes the set s satisfying $\sigma(k_s) = p$, $k_s \in \{k_0, k_1, \ldots, k_{N_\sigma - 1}\}$. Then

$$\begin{aligned} \|\boldsymbol{x}(K)\|_{\boldsymbol{v}_{\sigma(k_l)}} &\le \exp\left\{\sum_{p=1}^{M} N_{0p} \ln \mu_p\right\} \\ &\quad \times \exp\left\{\max_{p\in S}(\frac{\ln \mu_p}{\tau_{ap}} + \ln \gamma_p) T\right\} \|\boldsymbol{x}(0)\|_{\boldsymbol{v}_{\sigma(0)}}. \end{aligned}$$

This together with (8) implies that $\|\boldsymbol{x}(K)\|_{\boldsymbol{v}_{\sigma(k_l)}}$ is exponentially convergent. Therefore, SPNS (7) is GUES under a certain class of MDADT switching signals satisfying $\tau_{ap} > \tau_{ap}^{*} = -\frac{\ln \mu_p}{\ln \gamma_p}$, which completes the proof. □

Remark 2: In Theorem 1 and Theorem 2, we always assume that each subsystem of the switched systems is Lyapunov asymptotic stable.

IV. NUMERICAL EXAMPLE

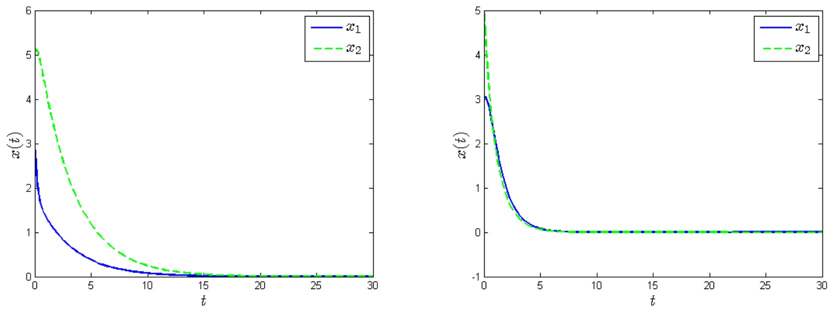
In this part, an illustrative example is presented.

Example 1: Consider SPNS (1) with the following parameters

$$f_1(x_1, x_2) = \begin{bmatrix} -3x_1 + 4x_2 - 3\sqrt{x_1^2 + x_2^2} \\ 2x_1 - 2x_2 + \sqrt{x_1^2 + x_2^2} \end{bmatrix}$$

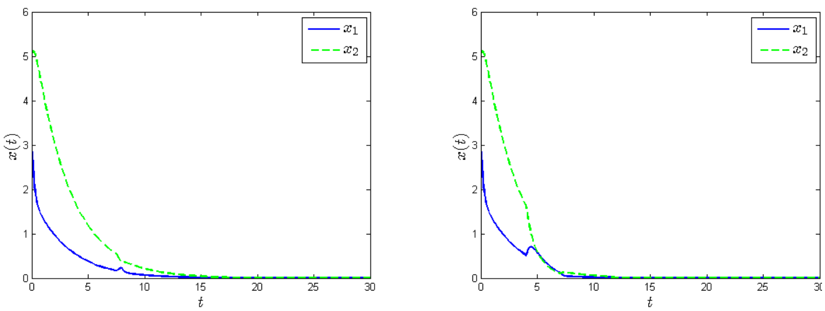
and

$$f_2(x_1, x_2) = \begin{bmatrix} -2.5x_1 + 0.5x_2 + \sqrt{x_1^2 + x_2^2} \\ x_1 - 0.5x_2 - \sqrt{x_1^2 + x_2^2} \end{bmatrix}.$$



(a) (b)

FIGURE 1. **The state trajectories of subsystem 1 and subsystem 2 asymptotically converge to zero, i.e., each subsystem of SPNS (1) is Lyapunov asymptotic stable.**



(a) (b)

FIGURE 2. **The state trajectories of SPNS (1) converge to zero at 15s under ADT switching, while the convergence time is 10s under MDADT switching, the proposed MDADT switching signal can achieve better performance.**

TABLE 1. **Compare ADT switching with MDADT switching for Example 1.**

Switching schemes	ADT switching	MDADT switching
Stability criteria	Theorem 3.4 in [28]	Theorem 1 in this paper
Switching signals	$\tau_a^* = 3.9344$ ($\mu = 3$)	$\tau_{a1}^* = 3.9344$, $\tau_{a2}^* = 1.6588$ ($\mu_1 = \mu_2 = 3$)

It is easy to verify that f_1 and f_2 satisfy assumption 5. Select $v_1 = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 3 & 1 \end{bmatrix}^T$, we can check that

$$f_1(1, 3) = \begin{bmatrix} -0.4868 & -0.8377 \end{bmatrix}^T \ll \mathbf{0}$$

and

$$f_2(3, 1) = \begin{bmatrix} -3.8377 & -0.6623 \end{bmatrix}^T \ll \mathbf{0}.$$

Table 1 provides the comparison between ADT switching and MDADT switching for SPNS (1). The state trajectories of subsystem 1 and the state trajectories of subsystem 2 after Figure 1 (a) and Figure 1 (b) respectively. For the switching law in Theorem 3.4 in [28], we choose to activate subsystems 1 and 2 with a periodic switching rule which is 7.5s and 0.5s, respectively. It is easy to obtain that the ADT of SPNS (1) is $\tau_a = 4$. Based on Theorem 3.4 in [28], SPNS (1) is GUES. The state trajectories under ADT and the state trajectories of under MDADT after Figure 2 (a) and Figure 2 (b) respectively. Fig. 2 (a) shows that SPNS (1) under the ADT switching signal with $\tau_a = 4$.

For the switching rule used in Theorem 1, we activate subsystems 1 and 2 with a periodic switching which is 4s and 3s, respectively. It is easy to obtain that the MDADT of SPNS (1) is $\tau_{a1} = 4$ and $\tau_{a2} = 3$. However, the ADT of SPNS (1) is $\tau_a = 3.5 < 3.9344$, Theorem 3.4 in [28] fails to apply. By Theorem 1, SPNS (1) is GUES. Fig. 2 (b) shows that SPNS (1) under the MDADT switching signal with $\tau_{a1} = 4$ and $\tau_{a2} = 3$. From Fig. 2, it is easy to obtain that the state trajectories of SPNS (1) under the MDADT switching signal converges faster than the ADT switching.

V. CONCLUSION

In this paper, stability analysis of switched positive nonlinear systems with mode-dependent average dwell time approach is investigated in both continuous-time and discrete-time cases. Based on non-Lyapunov function method developed in positive system, several stability criteria are derived. A numerical example is presented to illustrate the main results.

REFERENCES

[1] D. G. Luenberger, *Positive Linear Systems Introduction to Dynamic Systems: Theory, Models, and Applications*. New York, NY, USA: Wiley, 1979.

[2] J. Shen and S. Chen, "Stability and L_∞-gain analysis for a class of nonlinear positive systems with mixed delays," *Int. J. Robust Nonlinear Control*, vol. 27, no. 1, pp. 39–49, Jan. 2017.

[3] H. R. Feyzmahdavian, T. Charalambous, and M. Johansson, "Exponential stability of homogeneous positive systems of degree one with time-varying delays," *IEEE Trans. Autom. Control*, vol. 59, no. 6, pp. 1594–1599, Jun. 2014.

[4] J.-G. Dong, "On the decay rates of homogeneous positive systems of any degree with time-varying delays," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2983–2988, Nov. 2015.

[5] F. Meng and Y. Huang, "Interval oscillation criteria for a forced second-order nonlinear differential equations with damping," *Appl. Math. Comput.*, vol. 218, no. 5, pp. 1857–1861, Nov. 2011.

[6] H. Liu, F. Meng, and P. Liu, "Oscillation and asymptotic analysis on a new generalized Emden–Fowler equation," *Appl. Math. Comput.*, vol. 219, no. 5, pp. 2739–2748, Nov. 2012.

[7] J. Shao, F. Meng, and Z. Zheng, "Oscillation theorems for linear matrix Hamiltonian systems," *Appl. Math. Comput.*, vol. 253, pp. 402–409, Feb. 2015.

[8] Y. Tian, Y. Cai, Y. Sun, and H. Gao, "Finite-time stability for impulsive switched delay systems with nonlinear disturbances," *J. Franklin Inst.*, vol. 353, no. 14, pp. 3578–3594, Sep. 2016.

[9] Y. Tian and Y. Sun, "Stability of switched positive homogeneous nonlinear systems with mode-dependent average dwell time," in *Proc. 18th Int. Conf. Control Autom. Syst.*, Daegwallyeong, South Korea, Oct. 2018, pp. 1030–1034.

[10] Y. Li, Y. Sun, F. Meng, and Y. Tian, "Exponential stabilization of switched time-varying systems with delays and disturbances," *Appl. Math. Comput.*, vol. 324, pp. 131–140, May 2018.

[11] H. Liu, Y. Shen, and X. Zhao, "Finite-time stabilization and boundedness of switched linear system under state-dependent switching," *J. Franklin Inst.*, vol. 350, no. 14, pp. 541–555, Apr. 2013.

[12] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, no. 6, pp. 1132–1137, 2012.

[13] X. Liu and C. Dang, "Stability analysis of positive switched linear systems with delays," *IEEE Trans. Autom. Control*, vol. 56, no. 7, pp. 1684–1690, Jul. 2011.

[14] Y. Sun, "Stability analysis of positive switched systems via joint linear copositive Lyapunov functions," *Nonlinear Anal. Hybrid Syst.*, vol. 19, pp. 146–152, Feb. 2016.

[15] Y. Sun, Y. Tian, and X.-J. Xie, "Stabilization of positive switched linear systems and its application in consensus of multiagent systems," *IEEE Trans. Autom. Control*, vol. 62, no. 12, pp. 6608–6613, Dec. 2017.

[16] Y. Yin, G. Zong, and X. Zhao, "Improved stability criteria for switched positive linear systems with average dwell time switching," *J. Franklin Inst.*, vol. 354, no. 8, pp. 3472–3484, May 2017.

[17] X. Ding and X. Liu, "On stabilizability of switched positive linear systems under state-dependent switching," *Appl. Math. Comput.*, vol. 307, pp. 92–101, Aug. 2017.

[18] R. Ma, X. Wang, and Y. Liu, "Robust stability of switched positive linear systems with interval uncertainties via multiple time-varying linear copositive Lyapunov functions," *Nonlinear Anal. Hybrid Syst.*, vol. 30, pp. 285–292, Nov. 2018.

[19] Y. Xu, J.-G. Dong, R. Lu, and L. Xie, "Stability of continuous-time positive switched linear systems: A weak common copositive Lyapunov functions approach," *Automatica*, vol. 97, pp. 278–285, Nov. 2018.

[20] Y. Liu, W. Tao, L. Lee, and J. Lu, "Finite-time boundedness and L_2-gain analysis for switched positive linear systems with multiple time delays," *Int. J. Robust Nonlinear Control*, vol. 27, no. 17, pp. 3508–3523, Nov. 2017.

[21] W. Tao, Y. Liu, and J. Lu, "Stability and L_2-gain analysis for switched singular linear systems with jumps," *Math. Methods Appl. Sci.*, vol. 40, no. 3, pp. 589–599, 2017.

[22] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability and stabilization of switched linear systems with mode-dependent average dwell time," *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1809–1815, Jul. 2012.

[23] J. Zhang, Z. Han, F. Zhu, and J. Huang, "Stability and stabilization of positive switched systems with mode-dependent average dwell time," *Nonlinear Anal. Hybrid Syst.*, vol. 9, pp. 42–55, Aug. 2013.

[24] C. Huang, J. Cao, and J. Cao, "Stability analysis of switched cellular neural networks: A mode-dependent average dwell time approach," *Neural Netw.*, vol. 82, pp. 84–99, Oct. 2016.

[25] S. Shi, Z. Fei, and J. Li, "Finite-time H_∞ control of switched systems with mode-dependent average dwell time," *J. Franklin Inst.*, vol. 353, no. 1, pp. 221–234, Jan. 2016.

[26] Y. Ren, G. Sun, and Z. Feng, "Observer-based stabilization for switched positive system with mode-dependent average dwell time," *ISA Trans.*, vol. 70, pp. 37–45, Sep. 2017.

[27] H. L. Smith, *Monotone Dynamical Systems, an Introduction to the Theory of Competitive and Cooperative Systems* (Mathematical Surveys and Monographs), vol. 41. Providence, RI, USA: AMS, 1995.

[28] J.-G. Dong, "Stability of switched positive nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 26, no. 14, pp. 3118–3129, Sep. 2016.

[29] N. Zhang, Y. Sun, and F. Meng, "State bounding for switched homogeneous positive nonlinear systems with exogenous input," *Nonlinear Anal., Hybrid Syst.*, vol. 29, pp. 363–372, Aug. 2018.

[30] J. Zhang, X. Zhao, and X. Cai, "Absolute exponential L_1-gain analysis and synthesis of switched nonlinear positive systems with time-varying delay," *Appl. Math. Comput.*, vol. 284, pp. 24–36, Jul. 2016.

[31] X. Liu, "Stability analysis of a class of nonlinear positive switched systems with delays," *Nonlinear Anal., Hybrid Syst.*, vol. 16, pp. 1–12, May 2015.

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