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Transmission Projective Synchronization of Multiple Non-Identical Coupled Chaotic Systems Using Sliding Mode Control

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ABSTRACT This paper presents the control design method for the transmission projective synchronization (TPS) of the Multiple Non-identical Coupled Chaotic systems using sliding mode. A total of four different cases of chaotic systems are studied which are: 1) systems with known parameters without fault; 2) systems with known parameters with a fault; 3) systems with unknown parameters without fault; and 4) systems with unknown parameters and fault occurrence. In first and third cases, the controllers are designed using sliding mode, and adaptive integral sliding mode (AISM) is used to design controllers for the second and fourth cases. To employ AISM, the error dynamics is broken into a structure comprising a nominal and some unknown part, which are adaptively computed. The error dynamics are stabilized by AISM control which consists of a nominal and a compensator control. To avoid the chattering phenomenon, smooth continuous compensator control is used instead of the traditional discontinuous control. The stability of adaptive law and compensator is derived using a Lyapunov function which becomes strictly negative. Finally, the simulations of a numerical example verify the TPS behavior.

INDEX TERMS Multiple coupled chaotic system, transmission projective synchronization, adaptive integral sliding mode control, and Lyapunov function.

I. INTRODUCTION

There is a growing interest of scientists from various fields in chaos synchronization since the seminal work of Pecora and Carroll [1]. Due to broad range of applications, chaos synchronizing of multiple coupled chaotic systems (MCCS) has attracted interest in nonlinear research. Presently, two kinds of synchronization modes are being used. One, the conventional mode, in which multiple systems connect with one drive system, and two, the ring transmission synchronization—both have successful but complex applications in mathematics, physics, engineering sciences, etc. Some new schemes for MCCS have also been extensively investigated [2]–[6]. They have advantages over previous synchronization schemes in application domains such as communication, information science, etc. Therefore, an effective synchronization of MCCS is now an interesting area of research.

An improvement over conventional schemes in MCCS has been proposed in a scheme named transmission synchronization [7]. In this scheme, each system acts both as a drive system and a response system, while synchronization among MCCS is completed through a stepwise process. An occurrence of a synchronizing fault between two such systems in real world problems is unavoidable. With previous models, a fault can disrupt the synchronization process. This shortcoming hinders the efficacy of the system, hence needs to be addressed. To this end, we introduce a synchronization model that can avoid performance degradation of remaining systems. The aim of this work is to design an effective controller that increases the synchronization reliability in MCCS in the presence of faults to realize a transmission projective synchronization (TPS).

Recently, the synchronization of MCCS has attained a considerable attention from research community due to its

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wide range of potential applications such as multilateral communications, secret signaling etc. An adaptive synchronization of MCCS is studied using integral sliding surfaces, which can successfully stabilize the error systems against different disturbances and parameter variations [8]. However, the adaptive laws may suffer from parameter drift problem. Hybrid synchronization is investigated for coupled chaotic systems with a ring connection [9], [10]. The control strategy is based on adaptive integral sliding mode to achieve anti-synchronization and complete synchronization. The role of restorative coupling on synchronization of coupled identical systems in electrical networks is identified [11]. A class of impulsive synchronization problems can be addressed using (non)delayed couplings [12]. The controller is designed to synchronize a complex network to an isolate chaotic system. The synchronization problems of identical coupled chaotic systems is also analyzed and based on stability conditions, the appropriate control laws were designed to realize the TPS [13]. This paper extends the work [12], [13] in two aspects by proposing a technique. First, a proper TPS problem is discussed for Multiple Non-identical Coupled Chaotic (MNCC) systems, and then its error system is transformed into a nonlinear system with a special anti-symmetric structure. Second, sufficient asymptotic stability conditions are derived so that TPS behavior of multiple chaotic systems is realized. The results confirm the effectiveness of our proposed technique.

Consider a multiple coupled chaotic dynamic system for chaos synchronization. Such a system has been studied for a known parameter [13]. We extend this work under the assumption that all systems in the network have all parameters unknown. To reach TPS in this system, an adaptive integral sliding mode is used.

Sliding mode control (SMC) is a nonlinear control method [14]–[19]. It aims to drive the system states to a certain surface, known as the sliding manifold. Once the surface is reached, the system is forced to stay there. The closed loop dynamics of the system in SMC depends only on the design parameters of the switching sliding manifold. The main disadvantage of SMC is discontinuity across the sliding manifolds, which results in chattering and may cause harmful effects in real life systems. On the advantages side, it offers real time response, simplicity, and robustness against parameter variations and external disturbances. A variant of SMC, integral SMC (ISMC), guarantees the robustness in the whole state space because of elimination of the reaching phase [20]. The ISMC combines the nominal and discontinuous controls to stabilize the nominal system and reject the uncertainty, respectively.

Remaining paper is arranged as follows. System description and some preliminaries are introduced in section 2. The proposed control strategies for the general case of hybrid synchronization are discussed in Section 3. Application examples are presented in Section 4. In Section 5, simulation results are discussed. Finally, concluding remarks are made in Section 6.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

The MCCS are generally expressed as

$$\begin{cases} \dot{x}_1 = f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1) \\ \dot{x}_2 = f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) \\ \quad\vdots \\ \dot{x}_N = f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N), \end{cases} \tag{1}$$

where $x_1, x_2, \ldots, x_N$ are the state vectors and $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$, $f_i:\mathbb{R}^n \to \mathbb{R}^n, i = 1, 2, \ldots, N$ are the continuous nonlinear functions and $\theta_i \in \mathbb{R}^{p_i}$ are real vectors of parameters. $F_i(x_i) \in \mathbb{R}^{n\times p_i}$ are real matrices, $D_i = diag\{d_{i1}, d_{i2}, \ldots, d_{iN}\}$, $i = 1, 2, \ldots, N$ are n dimensional diagonal matrices and $d_{ij} \geq 0$ represent the coupled parameters of the diagonal matrices. If $f_i(\cdot) \neq f_j(\cdot), i, j == 1, 2, \ldots, N$ or $F_i(\cdot) \neq F_j(\cdot.), i, j = 1, 2, \ldots, N$, then (1) is an array of non-identical chaotic dynamic systems.

Applying the above coupling mode, the following TPS control problem is formulated:

$$\begin{cases} \dot{x}_1 = f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1) + u_1 \\ \dot{x}_2 = f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) + u_2 \\ \quad\vdots \\ \dot{x}_N = f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N) + u_N, \end{cases} \tag{2}$$

The TPS can be defined as follows.

Definition: If there is no fault in N coupled chaotic non-identical systems, given in (2), we say they are in TPS if the controllers $u_i(t), i = 1, 2, \ldots, N$ exist with trajectories $x_1(t), x_2(t), \ldots, x_N(t)$ in (2) and initial conditions $(x_1(0), x_2(0), \ldots, x_N(0))$ satisfy

$$\lim_{t\to\infty} \|e_i\| = \lim_{t\to\infty} \|x_{i+1}(t) - q_i x_i(t)\| = 0, \tag{3}$$

where $i = 1, 2, ..., N, e_i = (e_{i1}, e_{i2}, ..., e_{in})^T$ and the scaling parameters $q_i, i = 1, 2, ..., N$ are chosen such that $\prod_{i=1}^{N} q_i \neq 1$. If there is a fault then the scaling parameters $q_i, i = 1, 2, ..., N$ are chosen such that $\prod_{i=1}^{N} q_i = 1$.

III. TRANSMISSION PROJECTIVE SYNCHRONIZATION OF MNCC SYSTEMS

The TPS control problem is now selection of a proper controller, u_i to converge error vector $e_i = (e_{i1}, e_{i2}, \ldots, e_{in})^T$, $i = 1, 2, \ldots, N$ to zero asymptotically. For the TPS without fault, the errors are defined as

$$\begin{cases} e_1 = x_2 - q_1 x_1 \\ e_2 = x_3 - q_2 x_2 \\ \quad\vdots \\ e_{N-1} = x_N - qX_{N-1} \\ e_N = x_1 - q_n x_n, \end{cases}$$

where $\prod_{i=1}^{n} q_i \neq 1$. Therefore the error dynamics are (4), as shown at the top of the next page and can be written as (5), shown at the top of the next page.

$$
\begin{aligned}
\dot{e}_1 &= \dot{x}_2 - q_1\dot{x}_1 = f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) + u_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1)\} - q_1u_1 \\
\dot{e}_2 &= \dot{x}_3 - q_2\dot{x}_2 = f_3(x_3) + F_3(x_3)\theta_3 + D_3(x_2 - x_3) + u_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2)\} - q_2u_2 \\
&\vdots \\
\dot{e}_{N-1} &= \dot{x}_N - q_{N-1}\dot{x}_{N-1} \\
&= f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N) + u_N - q_{N-1}\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\theta_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\} \\
&\quad - q_{N-1}u_{N-1} \\
&= f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N) + u_N - q_{N-1}\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\theta_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\} \\
&\quad - q_{N-1}u_{N-1} \\
\dot{e}_N &= \dot{x}_1 - q_N\dot{x}_{N-1} = f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1) + u_1 - q_N\{f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N)\} - q_Nu_N
\end{aligned}
\tag{4}
$$

$$
\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{N-1} \\ \dot{e}_N \end{bmatrix} =
\begin{bmatrix}
f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) - q_1\{f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1)\} \\
f_3(x_3) + F_3(x_3)\theta_3 + D_3(x_2 - x_3) - q_2\{f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2)\} \\
\vdots \\
f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N) - q_{N-1}\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\theta_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\} \\
f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1) - q_N\{f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N)\}
\end{bmatrix}
+ \begin{bmatrix}
-q_1 & 1 & 0 & \dots & 0 \\
0 & -q_2 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & -q_{N-1} & \dots & 1 \\
1 & 0 & 0 & \dots & -q_N
\end{bmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix}
\tag{5}
$$

$$
\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{N-1} \\ \dot{e}_N \end{bmatrix} =
\begin{bmatrix}
-q_1 & 1 & 0 & \dots & 0 \\
0 & -q_2 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & -q_{N-1} & \dots & 1 \\
1 & 0 & \dots & 0 & -q_N
\end{bmatrix}^{-1}
\times \left\{ - \begin{bmatrix}
f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2) - q_1\{f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1)\} \\
f_3(x_3) + F_3(x_3)\theta_3 + D_3(x_2 - x_3) - q_2\{f_2(x_2) + F_2(x_2)\theta_2 + D_2(x_1 - x_2)\} \\
\vdots \\
f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N) \\
-q_{N-1}\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\theta_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\} \\
f_1(x_1) + F_1(x_1)\theta_1 + D_1(x_N - x_1) - q_N\{f_N(x_N) + F_N(x_N)\theta_N + D_N(x_{N-1} - x_N)\}
\end{bmatrix}
+ \begin{bmatrix} ee_1 \\ ee_2 \\ \vdots \\ ee_{N-1} \\ ee_N \end{bmatrix} \right\}
\tag{6}
$$

Case 1 (Known System Parameters): Choosing (6), as shown at the top of this page, where $ee = \left[e_2\ e_3\ \dots\ e_N\ v \right]^T$ and $v = \left[v_1\ v_2\ \dots\ v_n \right]^T$ is a new input vector, we have

$$
\begin{cases}
\dot{e}_1 = e_2 \\
\dot{e}_2 = e_3 \\
\quad\vdots \\
\dot{e}_{N-1} = e_N \\
\dot{e}_N = v.
\end{cases}
\tag{7}
$$

Define Hurwitz sliding surface as

$$
S = \left(1 + \frac{d}{dt}\right)^{N-1} e_1 = Ce,
$$

$$
S = e_1 + C_1e_2 +, \dots, + C_{N-2}e_{N-1} + e_N,
$$

where $C = diag\{I_n, C_1, \dots, C_{N-2}, I_n\}$, I_n is $n \times n$ identity matrix, $C_i = diag\{1\ c_{i1}\ \dots\ c_{in-2}1\}, i = 1, 2, \dots, N-2$, $e = [e_1\ e_2\ \dots\ e_N]^T$ and $S = [s_1\ s_2\ \dots\ s_N]^T$. Now $\dot{s}_i = e_{i2} + c_1e_{i3} + c_2e_{i4} + \dots + c_{n-2}e_{in} + v_i$. Choosing $v = -e_2 - C_1e_3 - C_2e_4 - \dots - C_{N-2}e_N - kS, k > 0$, we have $\dot{S} = -kS \Rightarrow e_1, e_2, \dots e_N \to 0$.

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{N-1} \\ \dot{e}_N \end{bmatrix} = \begin{bmatrix} f_2(x_2) + F_2(x_2)\hat{\theta}_2 + F_2(x_2)\tilde{\theta}_2 D_2(x_1 - x_2) - q_1\left\{f_1(x_1) + F_1(x_1)\hat{\theta}_1 + F_1(x_1)\tilde{\theta}_1 + D_1(x_N - x_1)\right\} \\ f_3(x_3) + F_3(x_3)\hat{\theta}_3 + F_3(x_3)\tilde{\theta}_{3+} D_3(x_2 - x_3) - q_2\left\{f_2(x_2) + F_2(x_2)\hat{\theta}_2 + F_2(x_2)\tilde{\theta}_2 + D_2(x_1 - x_2)\right\} \\ \vdots \\ f_N(x_N) + F_N(x_N)\hat{\theta}_N + F_N(x_N)\tilde{\theta}_N + D_N(x_{N-1} - x_N) \\ -q_{N-1}\left\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\hat{\theta}_{N-1} + F_{N-1}(x_{N-1})\tilde{\theta}_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\right\} \\ f_1(x_1) + F_1(x_1)\hat{\theta}_1 + F_1(x_1)\tilde{\theta}_1 + D_1(x_N - x_1) - q_N\left\{f_N(x_N) + F_N(x_N)\hat{\theta}_N + F_N(x_N)\tilde{\theta}_N + D_N(x_{N-1} - x_N)\right\} \end{bmatrix}$$
$$+ \begin{bmatrix} -q_1 & 1 & 0 & \dots & 0 \\ 0 & -q_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & -q_{N-1} & \dots & 1 \\ 1 & 0 & \dots & 0 & -q_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} -q_1 & 1 & 0 & \dots & 0 \\ 0 & -q_2 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & -q_{N-1} & 1 \\ 1 & 0 & \dots & 0 & -q_N \end{bmatrix}^{-1}$$
$$\times \left\{ - \begin{bmatrix} f_2(x_2) + F_2(x_2)\hat{\theta}_2 + D_2(x_1 - x_2) - q_1\left\{f_1(x_1) + F_1(x_1)\hat{\theta}_1 + D_1(x_N - x_1)\right\} \\ f_3(x_3) + F_3(x_3)\hat{\theta}_3 + D_3(x_2 - x_3) - q_2\left\{f_2(x_2) + F_2(x_2)\hat{\theta}_2 + D_2(x_1 - x_2)\right\} \\ \vdots \\ f_N(x_N) + F_N(x_N)\hat{\theta}_N + D_N(x_{N-1} - x_N) - q_{N-1} \\ \left\{f_{N-1}(x_{N-1}) + F_{N-1}(x_{N-1})\hat{\theta}_{N-1} + D_{N-1}(x_{N-2} - x_{N-1})\right\} \\ f_1(x_1) + F_1(x_1)\hat{\theta}_1 + D_1(x_N - x_1) - q_N\left\{f_N(x_N) + F_N(x_N)\hat{\theta}_N + D_N(x_{N-1} - x_N)\right\} \end{bmatrix} + \begin{pmatrix} e_2 \\ e_3 \\ \vdots \\ e_N \\ v \end{pmatrix} \right\} \tag{9}$$

Case 2 (Unknown System Parameters): For $i = 1, 2, \dots, N$, let $\hat{\theta}_i$ be the estimate vector of θ_i and $\tilde{\theta}_i, = \theta_i - \hat{\theta}_i$ be the estimation errors in θ_i, then error system (5) can be written as (8), shown at the top of this page, where v is new input vector. Choosing (9), as shown at the top of this page, we have

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{N-1} \\ \dot{e}_N \end{bmatrix} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_N \\ v \end{bmatrix} + \begin{bmatrix} F_2(x_2)\tilde{\theta}_2 - q_1 F_1(x_1)\tilde{\theta}_1 \\ F_3(x_3)\tilde{\theta}_3 - q_2 F_2(x_2)\tilde{\theta}_2 \\ \vdots \\ F_N(x_N)\tilde{\theta}_N - q_{N-1}F_{N-1}(x_{N-1})\tilde{\theta}_{N-1} \\ F_1(x_1)\tilde{\theta}_1 - q_N F_N(x_N)\tilde{\theta}_N \end{bmatrix} \tag{10}$$

or

$$\begin{cases} \dot{e}_1 = e_2 + F_2(x_2)\tilde{\theta}_2 - q_1 F_1(x_1)\tilde{\theta}_1 \\ \dot{e}_2 = e_3 + F_3(x_3)\tilde{\theta}_3 - q_2 F_2(x_2)\tilde{\theta}_2 \\ \dot{e}_3 = e_4 + F_4(x_4)\tilde{\theta}_4 - q_3 F_3(x_3)\tilde{\theta}_3 \\ \quad\vdots \\ \dot{e}_{N-2} = e_{N-1} + F_{N-1}(x_{N-1})\tilde{\theta}_{N-1} - q_{N-2} \\ \qquad \times F_{N-2}(x_{N-2})\tilde{\theta}_{N-2} \\ \dot{e}_{N-1} = e_N + F_N(x_N)\tilde{\theta}_N - q_{N-1}F_{N-1}(x_{N-1})\tilde{\theta}_{N-1} \\ \dot{e}_N = v + F_1(x_1)\tilde{\theta}_1 - q_N F_N(x_N)\tilde{\theta}_N. \end{cases} \tag{11}$$

To employ the ISMC, choose nominal system for (11) as

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = e_4 \\ \quad\vdots \\ \dot{e}_{N-1} = e_N \\ \dot{e}_N = v_0. \end{cases} \tag{12}$$

Define Hurwitz sliding surface vector for nominal systems (12) as:

$$S_0 = \left(1+\frac{d}{dt}\right)^{N-1} e_1 \quad \text{or}$$
$$S_0 = e_1 + C_1 e_2 + \cdots + C_{N-2} e_{N-1} + e_N = Ce,$$

then

$$\dot{S}_0 = C\dot{e} = e_2 + C_1 e_3 + C_2 e_4 + \cdots + C_{N-2} e_N + v_0.$$

By choosing

$$v_0 = -e_2 + C_1 e_3 + C_2 e_4 + \cdots + C_{N-2} e_N - k s_0, \quad k > 0,$$

we have $\dot{S}_0 = -kS_0$, therefore $S_0 \rightarrow 0$, and (12) is stable asymptotically. Sliding surface vector for the (11) is chosen as $S = S_0 + Z = Ce + Z$, where Z is an integral term computed later. Reaching phase is avoided by choosing $Z(0)$ such that $S(0) = 0$. Choose $v = v_0 + v_s$, where v_0 is nominal input and v_s is compensator term computed later, to get

$$\begin{aligned}
\dot{S} &= \dot{S}_0 + \dot{Z} = \dot{e}_1 + C_1\dot{e}_2 + \cdots C_{N-2}\dot{e}_{N-1} + \dot{e}_N,\\
\dot{S} &= e_2 + F_2(x_2)\tilde{\theta}_2 - q_1 F_1(x_1)\tilde{\theta}_1 + C_1 e_3 + C_1 F_3(x_3)\tilde{\theta}_3\\
&\quad - q_2 C_1 F_2(x_2)\tilde{\theta}_2 + C_2 e_4 + C_2 F_4(x_4)\tilde{\theta}_4 - q_3 C_2 F_3(x_3)\tilde{\theta}_3\\
&\quad + \cdots + C_{N-3} e_{N-1} + C_{N-3} F_{N-1}(x_{N-1})\tilde{\theta}_{N-1}\\
&\quad - q_{N-2} C_{N-3} F_{N-2}(x_{N-2})\tilde{\theta}_{N-2} + C_{N-2} e_N\\
&\quad + C_{N-2} F_N(x_N)\tilde{\theta}_N - q_{N-1} C_{N-2} F_{N-1}(x_{N-1})\tilde{\theta}_{N-1}\\
&\quad + v_0 + v_s + F_1(x_1)\tilde{\theta}_1 - q_N F_N(x_N)\tilde{\theta}_N + \dot{Z},\\
\dot{S} &= e_2 + C_1 e_3 + C_2 e_4 + \cdots + C_{N-3} e_{N-1} + C_{N-2} e_N + v_0\\
&\quad + v_s + \dot{Z} + \{F_1(x_1) - q_1 F_1(x_1)\}\tilde{\theta}_1\\
&\quad + \{F_2(x_2) - q_2 C_1 F_2(x_2)\}\tilde{\theta}_2\\
&\quad + \{C_1 F_3(x_3) - q_3 C_2 F_3(x_3)\}\tilde{\theta}_3 + \cdots + \{C_{N-3} F_{N-1}\\
&\quad \times (x_{N-1}) - q_{N-1} C_{N-2} F_{N-1}(x_{N-1})\}\tilde{\theta}_{N-1}\\
&\quad + \{C_{N-2} F_N(x_N) - q_N F_N(x_N)\}\tilde{\theta}_N.
\end{aligned} \tag{13}$$

By choosing a Lyapunov function (14), adaptive laws are designed for $\tilde{\theta}_i, \hat{\theta}_i, i = 1, 2, \ldots, N$ and v_s computed such that $\dot{V} < 0$,

$$V = \frac{1}{2}\left\{S^T S + \tilde{\theta}_1^T\tilde{\theta}_1 + \tilde{\theta}_2^T\tilde{\theta}_2 + \tilde{\theta}_3^T\tilde{\theta}_3 + \cdots + \tilde{\theta}_N^T\tilde{\theta}_N\right\} \tag{14}$$

Theorem: Consider a Lyapunov function

$$V = \frac{1}{2}\{S^T S + \tilde{\theta}_1^T\tilde{\theta}_1 + \tilde{\theta}_2^T\tilde{\theta}_2 + \tilde{\theta}_3^T\tilde{\theta}_3 + \cdots + \tilde{\theta}_N^T\tilde{\theta}_N\},$$

then $\dot{V} < 0$ if the adaptive laws for $\tilde{\theta}_i, \hat{\theta}_i, i = 1, 2, \ldots, N$ and the value of v_s are chosen as

$$\begin{aligned}
\dot{Z} &= -e_2 - C_1 e_3 - C_2 e_4 - \ldots - C_{N-2} e_N - v_0,\\
v_s &= -kS,\\
\dot{\tilde{\theta}}_1 &= -(1-q_1)F_1^T(x_1)S - k_1\tilde{\theta}_1, \quad \dot{\hat{\theta}}_1 = -\dot{\tilde{\theta}}_1,\\
\dot{\tilde{\theta}}_2 &= -\{F_2(x_2) - q_2 C_1 F_2(x_2)\}^T S - k_2\tilde{\theta}_2, \quad \dot{\hat{\theta}}_2 = -\dot{\tilde{\theta}}_2,\\
\dot{\tilde{\theta}}_3 &= -\{C_1 F_3(x_3) - q_3 C_2 F_3(x_3)\}^T S - k_3\tilde{\theta}_3, \quad \dot{\hat{\theta}}_3 = -\dot{\tilde{\theta}}_3,\\
&\vdots\\
\dot{\tilde{\theta}}_{N-1} &= -\left\{C_{N-3} F_{N-1}(x_{N-1}) - q_{N-1} C_{N-2} F_{N-1}(x_{N-1})\right\}^T\\
&\quad \times S - k_{N-1}\tilde{\theta}_{N-1},\\
\dot{\hat{\theta}}_{N-1} &= -\dot{\tilde{\theta}}_{N-1},\\
\dot{\tilde{\theta}}_N &= -\{C_{N-2} F_N(x_N) - q_N C_{N-1} F_N(x_N)\}^T S - k_N\tilde{\theta}_N,\\
\dot{\hat{\theta}}_N &= -\dot{\tilde{\theta}}_N.
\end{aligned} \tag{15}$$

Proof: Since

$$\begin{aligned}
\dot{V} &= \{S^T\dot{S} + \tilde{\theta}_1^T\dot{\tilde{\theta}}_1 + \tilde{\theta}_2^T\dot{\tilde{\theta}}_2 + \tilde{\theta}_3^T\dot{\tilde{\theta}}_3 + \cdots + \tilde{\theta}_N^T\dot{\tilde{\theta}}_N\},\\
\dot{V} &= S^T\{e_2 + C_1 e_3 + C_2 e_4 + \cdots + C_{N-2} e_N + v_0 + v_s + Z\\
&\quad + \{F_1(x_1) - q_1 F_1(x_1)\}\tilde{\theta}_1 + \{F_2(x_2) - q_2 C_1 F_2(x_2)\}\tilde{\theta}_2\\
&\quad + \{C_1 F_3(x_3) - q_3 C_2 F_3(x_3)\}\tilde{\theta}_3 + \cdots + \{C_{N-3} F_{N-1}\\
&\quad \times (x_{N-1}) - q_{N-1} C_{N-2} F_{N-1}(x_{N-1})\}\tilde{\theta}_{N-1}\\
&\quad + \{C_{N-2} F_N(x_N) - q_N C_{N-1} F_N(x_N)\}\tilde{\theta}_N\} + \tilde{\theta}_1^T\dot{\tilde{\theta}}_1\\
&\quad + \tilde{\theta}_2^T\dot{\tilde{\theta}}_2 + \tilde{\theta}_3^T\dot{\tilde{\theta}}_3 + \cdots + \tilde{\theta}_{N-1}^T\dot{\tilde{\theta}}_{N-1} + \tilde{\theta}_N^T\dot{\tilde{\theta}}_N,\\
\dot{V} &= S^T\{e_2 + C_1 e_3 + C_2 e_4 + \cdots + C_{N-3} e_{N-1} + C_{N-2} e_N\\
&\quad + v_0 + v_s + \dot{Z} + \tilde{\theta}_1^T\left\{\dot{\tilde{\theta}}_1 + (1-q_1)F_1^T(x_1)S\right\}\\
&\quad + \tilde{\theta}_2^T\left\{\dot{\tilde{\theta}}_2 + \{F_2(x_2) - q_2 C_1 F_2(x_2)\}\right\}^T S\}\\
&\quad + \tilde{\theta}_3^T\left\{\dot{\tilde{\theta}}_3 + \{F_3(x_3) - q_3 C_2 F_3(x_3)\}\right\}^T S\\
&\quad + \cdots + \tilde{\theta}_{N-1}^T\left\{\dot{\tilde{\theta}}_{N-1} + \{C_{N-3} F_{N-1}(x_{N-1}) - q_{N-1}\right.\\
&\quad \left.\times C_{N-2} F_{N-1}(x_{N-1})\}\right\}^T S\}\\
&\quad + \tilde{\theta}_N^T\left\{\dot{\tilde{\theta}}_N + \{C_{N-2} F_N(x_N) - q_N F_N(x_N)\}\right\}^T S\}.
\end{aligned}$$

By choosing

$$\begin{aligned}
\dot{Z} &= -e_2 - C_1 e_3 - C_2 e_4 - \cdots - C_{N-2} e_N - v_0,\\
v_s &= -kS,\\
\dot{\tilde{\theta}}_1 &= -(1-q_1)F_1^T(x_1)S - k_1\tilde{\theta}_1,\\
\dot{\hat{\theta}}_1 &= -\dot{\tilde{\theta}}_1,\\
\dot{\tilde{\theta}}_2 &= -\{F_2(x_2) - q_2 C_1 F_2(x_2)\}^T S - k_2\tilde{\theta}_2,\\
\dot{\hat{\theta}}_2 &= -\dot{\tilde{\theta}}_2,\\
\dot{\tilde{\theta}}_3 &= -\{C_1 F_3(x_3) - q_3 C_2 F_3(x_3)\}^T S - k_3\tilde{\theta}_3,\\
\dot{\hat{\theta}}_3 &= -\dot{\tilde{\theta}}_3,\\
&\vdots\\
\dot{\tilde{\theta}}_{N-1} &= -\left\{C_{N-3} F_{N-1}(x_{N-1}) - q_{N-1} C_{N-2} F_{N-1}(x_{N-1})\right\}^T\\
&\quad \times S - k_{N-1}\tilde{\theta}_{N-1},\\
\dot{\hat{\theta}}_{N-1} &= -\dot{\tilde{\theta}}_{N-1},\\
\dot{\tilde{\theta}}_N &= -\left\{C_{N-2} F_N(x_N) - q_N C_{N-1} F_N(x_N)\right\}^T S - k_N\tilde{\theta}_N,\\
\dot{\hat{\theta}}_N &= -\dot{\tilde{\theta}}_N.
\end{aligned}$$

We have

$$\dot{V} = -kS^T S - \sum_{i=1}^{N} k_i\tilde{\theta}_i^T\tilde{\theta}_i.$$

From this we conclude that $S, \tilde{\theta}_i \to 0, i = 1, \ldots, N$.

Since $S \to 0$, therefore $e_i \to 0, i = 1, \ldots.., N$.

IV. NUMERICAL EXAMPLE

As an application of the derived behavior, numerical examples of three non-identical coupled chaotic systems show the effectiveness of method. These are Chen system, Lorenz system and Lu system and described as follows:

$$\begin{cases} \dot{x}_{11} = a_1 x_{11} + a_2 x_{12} + d_{11}(x_{31} - x_{11}) + u_{11} \\ \dot{x}_{12} = a_3 x_{11} + a_4 x_{12} + x_{11}x_{13} + d_{12}(x_{32} - x_{12}) + u_{12} \\ \dot{x}_{13} = a_5 x_{13} + x_{11}x_{12} + d_{13}(x_{33} - x_{13}) + u_{13}, \end{cases} \tag{16}$$

where $a_1 = -35, a_2 = 35, a_3 = -7, a_4 = 28, a_5 = -3$.

$$\begin{cases} \dot{x}_{21} = b_1 x_{21} + b_2 x_{22} + d_{21}(x_{11} - x_{21}) + u_{21} \\ \dot{x}_{22} = b_3 x_{22} - x_{21}x_{23} + d_{22}(x_{12} - x_{22}) + u_{22} \\ \dot{x}_{23} = b_4 x_{23} + x_{21}x_{22} + d_{23}(x_{13} - x_{23}) + u_{23}, \end{cases} \tag{17}$$

where $b_1 = -36, b_2 = 36, b_3 = 20, b_4 = -3$.

$$\begin{cases} \dot{x}_{31} = c_1 x_{31} + c_2 x_{32} + d_{31}(x_{21} - x_{31}) + u_{31} \\ \dot{x}_{32} = c_3 x_{31} - x_{32} - x_{31}x_{33} + d_{32}(x_{22} - x_{32}) + u_{32} \\ \dot{x}_{33} = c_4 x_{33} + x_{31}x_{32} + d_{33}(x_{23} - x_{33}) + u_{33}, \end{cases} \tag{18}$$

where $c_1 = -10, c_2 = 10, c_3 = 28, c_4 = -\frac{8}{3}$.

Define

$$x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \quad x_3 = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix},$$

$$u_1 = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix}, \quad u_2 = \begin{bmatrix} u_{21} \\ u_{22} \\ u_{23} \end{bmatrix}, \quad u_3 = \begin{bmatrix} u_{31} \\ u_{32} \\ u_{33} \end{bmatrix},$$

$$\theta_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \theta_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

The equations (16) – (18) are written as a vector

$$\begin{cases} \dot{x}_1 = f_1(x_1) + F_1(x_1)\theta_1 + u_1 \\ \dot{x}_2 = f_2(x_2) + F_2(x_2)\theta_2 + u_2 \\ \dot{x}_3 = f_3(x_3) + F_3(x_3)\theta_3 + u_3, \end{cases} \tag{19}$$

where

$$f_1(x_1) = \begin{bmatrix} d_{11}(x_{31} - x_{11}) \\ -x_{11}x_{13} + d_{12}(x_{32} - x_{12}) \\ x_{11}x_{12} + d_{13}(x_{33} - x_{13}) \end{bmatrix},$$

$$F_1(x_1) = \begin{bmatrix} x_{11} & x_{12} & 0 & 0 & 0 \\ 0 & 0 & x_{11} & x_{12} & 0 \\ 0 & 0 & 0 & 0 & x_{13} \end{bmatrix},$$

$$f_2(x_2) = \begin{bmatrix} d_{21}(x_{11} - x_{21}) \\ -x_{21}x_{23} + d_{22}(x_{12} - x_{22}) \\ x_{21}x_{22} + d_{23}(x_{13} - x_{23}) \end{bmatrix},$$

$$F_2(x_2) = \begin{bmatrix} x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{22} & 0 \\ 0 & 0 & 0 & x_{23} \end{bmatrix},$$

$$f_3(x_3) = \begin{bmatrix} d_{31}(x_{21} - x_{31}) \\ -x_{32} - x_{31}x_{33} + d_{32}(x_{22} - x_{32}) \\ x_{31}x_{32} + d_{33}(x_{23} - x_{33}) \end{bmatrix},$$

$$F_3(x_3) = \begin{bmatrix} x_{31} & x_{32} & 0 & 0 \\ 0 & 0 & x_{31} & 0 \\ 0 & 0 & 0 & x_{33} \end{bmatrix}.$$

Case I (Known System Parameters With No Fault): For TPS, the errors

$$e_1 = \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix}, \quad e_2 = \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} \text{ and } e_3 = \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix}$$

are $e_1 = x_2 - q_1 x_1$, $e_2 = x_3 - q_2 x_2$ and $e_3 = x_1 - q_3 x_3$, where $q_1 q_2 q_3 \neq 1$.

Therefore

$$\begin{aligned} \dot{e}_1 &= \dot{x}_2 - q_1 \dot{x}_1 \\ \dot{e}_2 &= \dot{x}_3 - q_2 \dot{x}_2 \\ \dot{e}_1 &= \dot{x}_1 - q_3 \dot{x}_3, \end{aligned}$$

which can also be written as (20), shown at the top of the next page. By choosing (21), as shown at the top of the next page, with $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$ as the new input vector, we write

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = v. \end{cases} \tag{22}$$

Define the Hurwitz sliding surface as

$$S = e_1 + 2e_2 + e_3,$$

where S $= \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} e_{11} + 2e_{21} + e_{31} \\ e_{12} + 2e_{22} + e_{32} \\ e_{13} + 2e_{23} + e_{33} \end{bmatrix}$.

Now $\dot{S} = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = e_2 + 2e_3 + v$.

By choosing

$$v = -e_2 - 2e_3 - ks, \quad k > 0,$$

we have $\dot{S} = -kS$, therefore $S \to 0$ which gives $e_1, e_2, e_3 \to 0$.

Case II (Known System Parameters With a Fault): Assume that there exists a fault between the systems (16) and (18), then the TPS errors

$$e_1 = \begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix}, \quad e_2 = \begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} \text{ and } e_3 = \begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix}$$

are $e_1 = x_2 - q_1 x_1$, $e_2 = x_3 - q_2 x_2$ and $e_3 = x_1 - q_3 x_3$, where $q_1 q_2 q_3 = 1$. We set $u_1 = 0$ in e_3 so that

$$\begin{cases} \dot{e}_1 = \dot{x}_2 - q_1 \dot{x}_1 \\ \dot{e}_2 = \dot{x}_3 - q_2 \dot{x}_2 \\ \dot{e}_3 = \dot{x}_1 - q_3 \dot{x}_3. \end{cases} \tag{23}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} f_2(x_2) + F_2(x_2)\theta_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1\} \\ f_3(x_3) + F_3(x_3)\theta_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2\} \\ f_1(x_1) + F_1(x_1)\theta_1 - q_3\{f_3(x_3) + F_3(x_3)\theta_3\} \end{bmatrix} + \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix}^{-1} \left\{ - \begin{bmatrix} f_2(x_2) + F_2(x_2)\theta_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1\} \\ f_3(x_3) + F_3(x_3)\theta_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2\} \\ f_1(x_1) + F_1(x_1)\theta_1 - q_3\{f_3(x_3) + F_3(x_3)\theta_3\} \end{bmatrix} + \begin{bmatrix} e_2 \\ e_3 \\ v \end{bmatrix} \right\} \tag{21}$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} f_2(x_2) + F_2(x_2)\theta_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1\} \\ f_3(x_3) + F_3(x_3)\theta_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2\} \\ f_1(x_1) + F_1(x_1)\theta_1 - q_3\{f_3(x_3) + F_3(x_3)\theta_3\} \end{bmatrix} + \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix}^{-1} \left\{ - \begin{bmatrix} f_2(x_2) + F_2(x_2)\theta_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1\} \\ f_3(x_3) + F_3(x_3)\theta_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2\} \\ f_1(x_1) + F_1(x_1)\theta_1 - q_3\{f_3(x_3) + F_3(x_3)\theta_3\} \end{bmatrix} + \begin{bmatrix} e_2 \\ e_3 \\ v \end{bmatrix} \right\} \tag{25}$$

It can be represented as (24), shown at the top of this page. By choosing (25), as shown at the top of this page, with $v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T$ as the new input vector, we write

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = v. \end{cases} \tag{26}$$

Define the Hurwitz sliding surface as

$$S = e_1 + 2e_2 + e_3,$$

where $S = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} e_{11} + 2e_{21} + e_{31} \\ e_{12} + 2e_{22} + e_{32} \\ e_{13} + 2e_{23} + e_{33} \end{bmatrix}$, then $\dot{S} = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = e_2 + 2e_3 + v$.

By choosing $v = -e_2 - 2e_3 - ks$, $k > 0$, we have $\dot{S} = -kS$, therefore $S \to 0$, which gives $e_1, e_2, e_3 \to 0$.

Case III (Unknown System Parameters With No Fault): For $i = 1, 2, 3$, let $\hat{\theta}_i$ be the estimates and $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ be the estimation error of θ_i Vector form of (16) – (18) can now be written as

$$\begin{cases} \dot{x}_1 = f_1(x_1) + F_1(x_1)\hat{\theta}_1 + F_1(x_1)\tilde{\theta}_1 + u_1 \\ \dot{x}_2 = f_2(x_2) + F_2(x_2)\hat{\theta}_2 + F_2(x_2)\tilde{\theta}_2 + u_2 \\ \dot{x}_3 = f_3(x_3) + F_3(x_3)\hat{\theta}_3 + F_3(x_3)\tilde{\theta}_3 + u_3, \end{cases} \tag{27}$$

where

$$\hat{\theta}_1 = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \\ \hat{a}_4 \\ \hat{a}_5 \end{bmatrix}, \quad \tilde{\theta}_1 = \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \\ \tilde{a}_5 \end{bmatrix}, \hat{\theta}_2 = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \\ \hat{b}_4 \end{bmatrix}, \tilde{\theta}_2 = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \end{bmatrix},$$

$$\hat{\theta}_3 = \hat{c} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{bmatrix} \text{ and } \tilde{\theta}_3 = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \tilde{c}_4 \end{bmatrix}.$$

For TPS, errors are $e_1 = x_2 - q_1x_1$, $e_2 = x_3 - q_2x_2$ and $e_3 = x_1 - q_3x_3$, where $q_1q_2q_3 \neq 1$.

Therefore

$$\begin{cases} \dot{e}_1 = \dot{x}_2 - q_1\dot{x}_1 \\ \dot{e}_2 = \dot{x}_3 - q_2\dot{x}_2 \\ \dot{e}_1 = \dot{x}_1 - q_3\dot{x}_3, \end{cases} \tag{28}$$

which can be written as (29), shown at the bottom of the next page. By choosing (30), as shown at the bottom of the next page, with v as the new input vector, we write

$$\begin{cases} \dot{e}_1 = e_2 + F_2(x_2)\tilde{\theta}_2 - q_1F_1(x_1)\tilde{\theta}_1 \\ \dot{e}_2 = e_3 + F_3(x_3)\tilde{\theta}_3 - q_2F_2(x_2)\tilde{\theta}_2 \\ \dot{e}_3 = v + F_1(x_1)\tilde{\theta}_1 - q_3F_3(x_3)\tilde{\theta}_3, \end{cases} \tag{31}$$

To employ the integral sliding mode, choose the nominal system for (31) as

$$\begin{cases} \dot{e}_1 = e_2 \\ \dot{e}_2 = e_3 \\ \dot{e}_3 = v_0, \end{cases} \tag{32}$$

where v_0 is the nominal input vector.

Define the Hurwitz sliding surface for nominal system (32) as

$$S_0 = e_1 + 2e_2 + e_3,$$

where $S_0 = \begin{bmatrix} s_{01} \\ s_{02} \\ s_{03} \end{bmatrix} = \begin{bmatrix} e_{11} + 2e_{21} + e_{31} \\ e_{12} + 2e_{22} + e_{32} \\ e_{13} + 2e_{23} + e_{33} \end{bmatrix}$, then $\dot{S_0} = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = e_2 + 2e_3 + v$.

By choosing

$$v = -e_2 - 2e_3 - kS_0, \quad k > 0,$$

we have $\dot{S_0} = -kS_0$, therefore $S_0 \to 0$, which shows that (32) is asymptotically stable. The sliding manifold for (31) is chosen as

$$S = S_0 + Z = e_1 + 2e_2 + e_3 + Z,$$

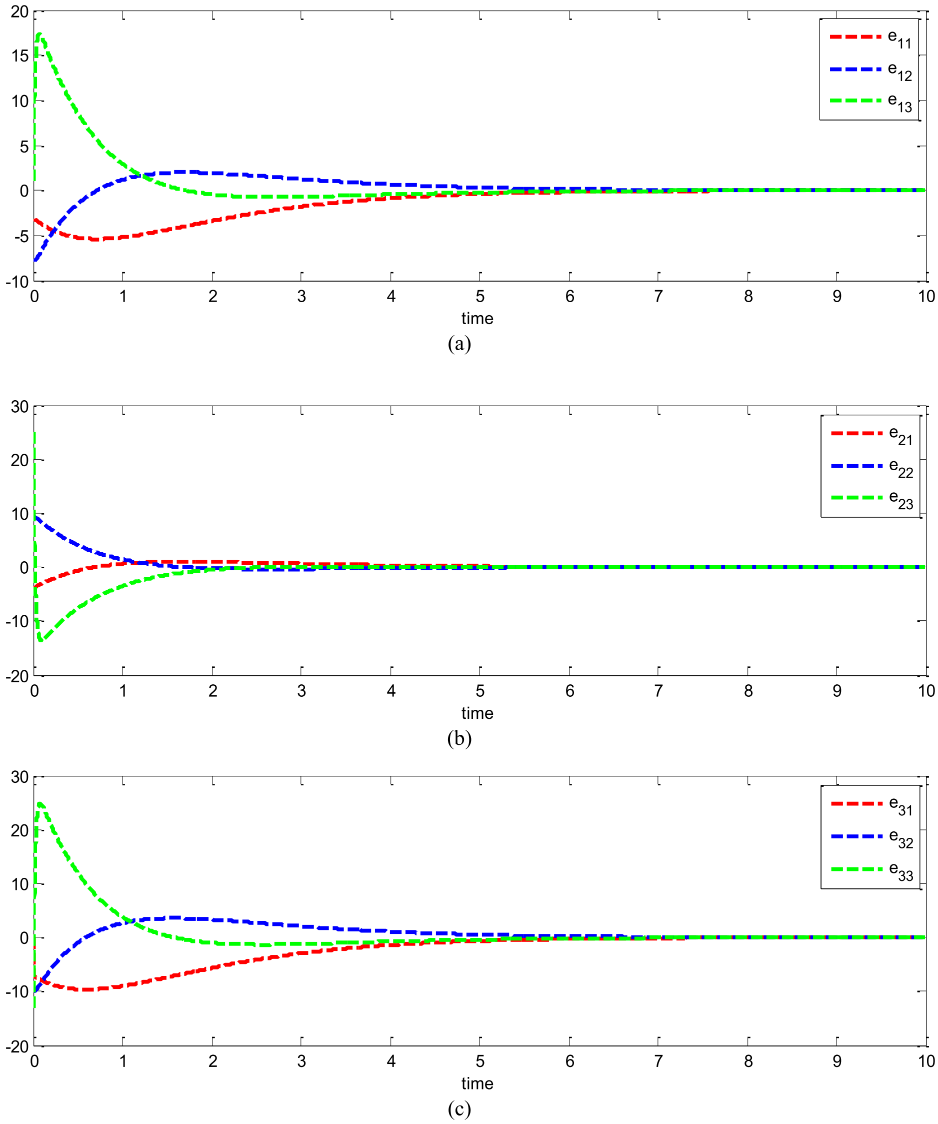


FIGURE 1. Time history of errors when there is no fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3) and (c) shows error between the systems (1) and (3).

where $Z = \begin{bmatrix} z_1 & z_3 & z_3 \end{bmatrix}^T$ is an integral term. To eliminate the reaching phase, choose $Z(0)$ such that $S(0) = 0$. Set $v = v_0 + v_s$, where $v_0 = \begin{bmatrix} v_{01} & v_{02} & v_{03} \end{bmatrix}^T$ is an input and $v_s = \begin{bmatrix} v_{s1} & v_{s2} & v_{s3} \end{bmatrix}^T$ is a compensator which is computed later, so that

$$\dot{S} = \dot{S}_0 + \dot{Z} = \dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 + \dot{Z},$$

$$\dot{S} = e_2 + F_2(x_2)\tilde{\theta}_2 - q_1F_1(x_1)\tilde{\theta}_1 + 2e_3 + 2F_3(x_3)\tilde{\theta}_3 - q_22F_2(x_2)\tilde{\theta}_2 + v_0 + v_s + F_1(x_1)\tilde{\theta}_1 - q_3F_3(x_3)\tilde{\theta}_3 + \dot{Z},$$

$$\dot{S} = e_2 + 2e_3 + v_0 + v_s + \dot{Z} + \{F_1(x_1)\tilde{\theta}_1 - q_1F_1(x_1)\}\tilde{\theta}_1 + \left\{F_2(x_2)\tilde{\theta}_2 - q_22F_2(x_2)\right\}\tilde{\theta}_2 + \{2F_3(x_3)\tilde{\theta}_3 - q_3F_3(x_3)\}\tilde{\theta}_3. \quad (33)$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} f_2(x_2) + F_2(x_2)\theta_2 - q_1\{f_1(x_1) + F_1(x_1)\theta_1\} \\ f_3(x_3) + F_3(x_3)\theta_3 - q_2\{f_2(x_2) + F_2(x_2)\theta_2\} \\ f_1(x_1) + F_1(x_1)\theta_1 - q_3\{f_3(x_3) + F_3(x_3)\theta_3\} \end{bmatrix} + \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} F_2(x_2)\tilde{\theta}_2 - q_1F_1(x_1)\tilde{\theta}_1 \\ F_3(x_3)\tilde{\theta}_3 - q_2F_2(x_2)\tilde{\theta}_2 \\ F_1(x_1)\tilde{\theta}_1 - q_3F_3(x_3)\tilde{\theta}_3 \end{bmatrix} \quad (29)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix}^{-1} \left\{ - \begin{bmatrix} f_2(x_2) + F_2(x_2)\hat{\theta}_2 - q_1\{f_1(x_1) + F_1(x_1)\hat{\theta}_1\} \\ f_3(x_3) + F_3(x_3)\hat{\theta}_3 - q_2\{f_2(x_2) + F_2(x_2)\hat{\theta}_2\} \\ f_1(x_1) + F_1(x_1)\hat{\theta}_1 - q_3\{f_3(x_3) + F_3(x_3)\hat{\theta}_3\} \end{bmatrix} + \begin{bmatrix} e_2 \\ e_3 \\ v \end{bmatrix} \right\} \quad (30)$$

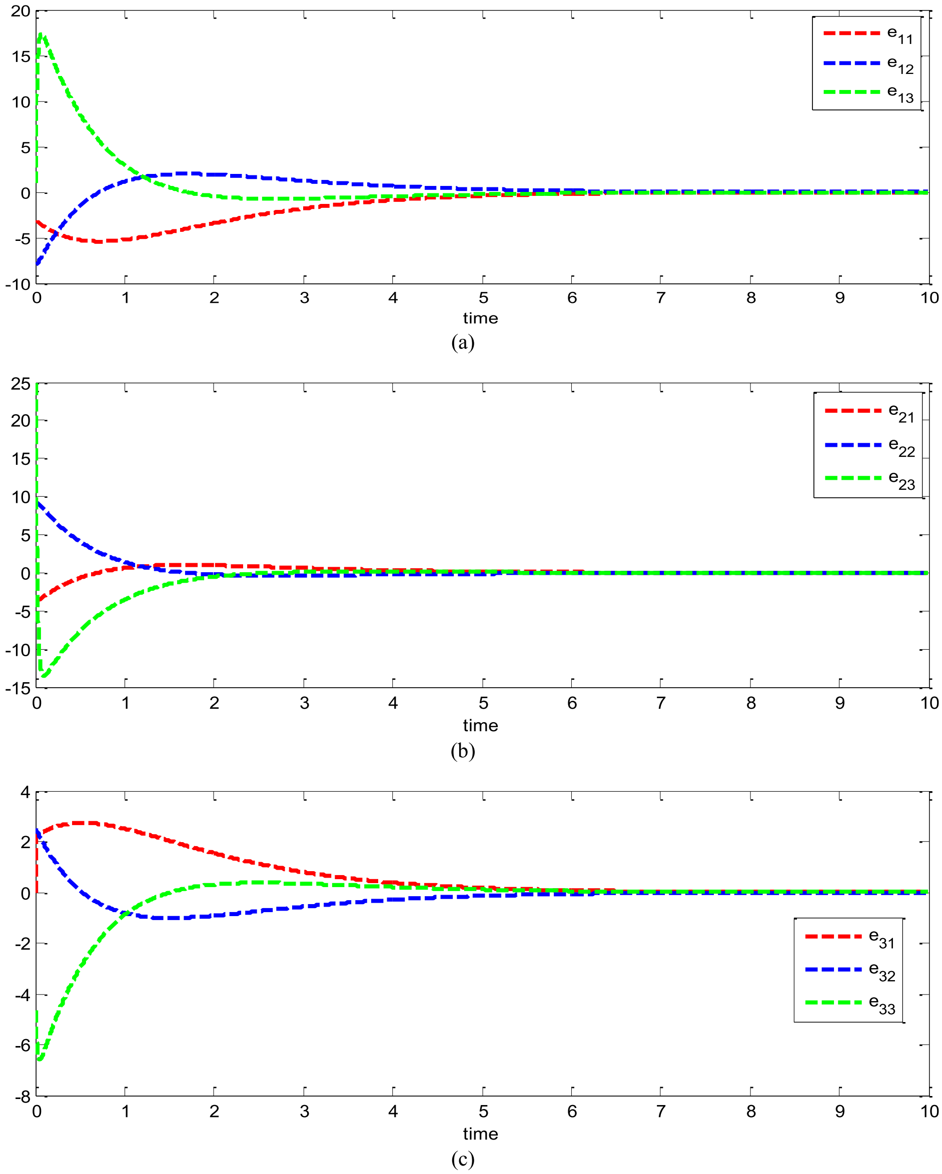


FIGURE 2. Time history of errors when there is a fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3) and (c) shows error between the systems (1) and (3).

By choosing a Lyapunov function

$$V=\frac{1}{2}\{S_T S+\tilde{\theta}_1^T\tilde{\theta}_1+\tilde{\theta}_2^T\tilde{\theta}_2+\tilde{\theta}_3^T\tilde{\theta}_3,$$

the following adaptive laws for $\tilde{\theta}_i, \hat{\theta}_{i,} i=1,2,3$ and v_s gives $\dot{V}<0$:

$$\begin{cases}\dot{Z}=-e_2-2e_3-v_0, & v_s=-kS\\ \dot{\tilde{\theta}}_1=-(F_1(x_1)-q_1F_1(x_1))^T S-k_1\tilde{\theta}_1, & \dot{\hat{\theta}}_1=-\dot{\tilde{\theta}}_1\\ \dot{\tilde{\theta}}_2=-(F_2(x_2)-q_2 2F_2(x_2))^T S-k_2\tilde{\theta}_2, & \dot{\hat{\theta}}_2=-\dot{\tilde{\theta}}_2\\ \dot{\tilde{\theta}}_3=-(2F_3(x_3)-q_3F_3(x_3))^T S-k_3\tilde{\theta}_3, & \dot{\hat{\theta}}_3=-\dot{\tilde{\theta}}_3.\end{cases} \tag{34}$$

Since

$$\dot{V}=S^T\dot{S}+\tilde{\theta}_1^T\dot{\tilde{\theta}}_1+\tilde{\theta}_2^T\dot{\tilde{\theta}}_2+\tilde{\theta}_3^T\dot{\tilde{\theta}}_3,$$

$$\begin{aligned}\dot{V}&=S^T[e_2+2e_3+v_0+v_s+\dot{Z}+\{F_1(x_1)-q_1F_1(x_1)\}\tilde{\theta}_1\\&\quad+\{F_2(x_2)-q_2 2F_2(x_2)\}\tilde{\theta}_2\\&\quad+\{2F_3(x_3)-q_3F_3(x_3)\}\tilde{\theta}_3]+\tilde{\theta}_1^T\dot{\tilde{\theta}}_1+\tilde{\theta}_2^T\dot{\tilde{\theta}}_2+\tilde{\theta}_3^T\dot{\tilde{\theta}}_3,\\ \dot{V}&=S^T[e_2+2e_3+v_0+v_s+\dot{Z}]\\&\quad+\tilde{\theta}_1^T\{\dot{\tilde{\theta}}_1+(F_1(x_1)-q_1F_1(x_1))^T S\}\\&\quad+\tilde{\theta}_2^T\{\dot{\tilde{\theta}}_2+(F_2(x_2)-q_2 2F_2(x_2))^T S\}\\&\quad+\tilde{\theta}_3^T\{\dot{\tilde{\theta}}_3+(2F_3(x_3)-q_3F_3(x_3))^T S\}.\end{aligned}$$

Using (34)

$$\begin{aligned}\dot{\tilde{\theta}}_1&=-(F_1(x_1)-q_1F_1(x_1))^T S-K_1\tilde{\theta}_1, \quad \dot{\hat{\theta}}_1=\dot{\tilde{\theta}}_1\\ \Rightarrow\dot{\tilde{\theta}}_1&=(q_1-1)F_1^T(x_1)S-K_1\tilde{\theta}_1, \quad \dot{\hat{\theta}}_1=\dot{\tilde{\theta}}_1,\end{aligned}$$

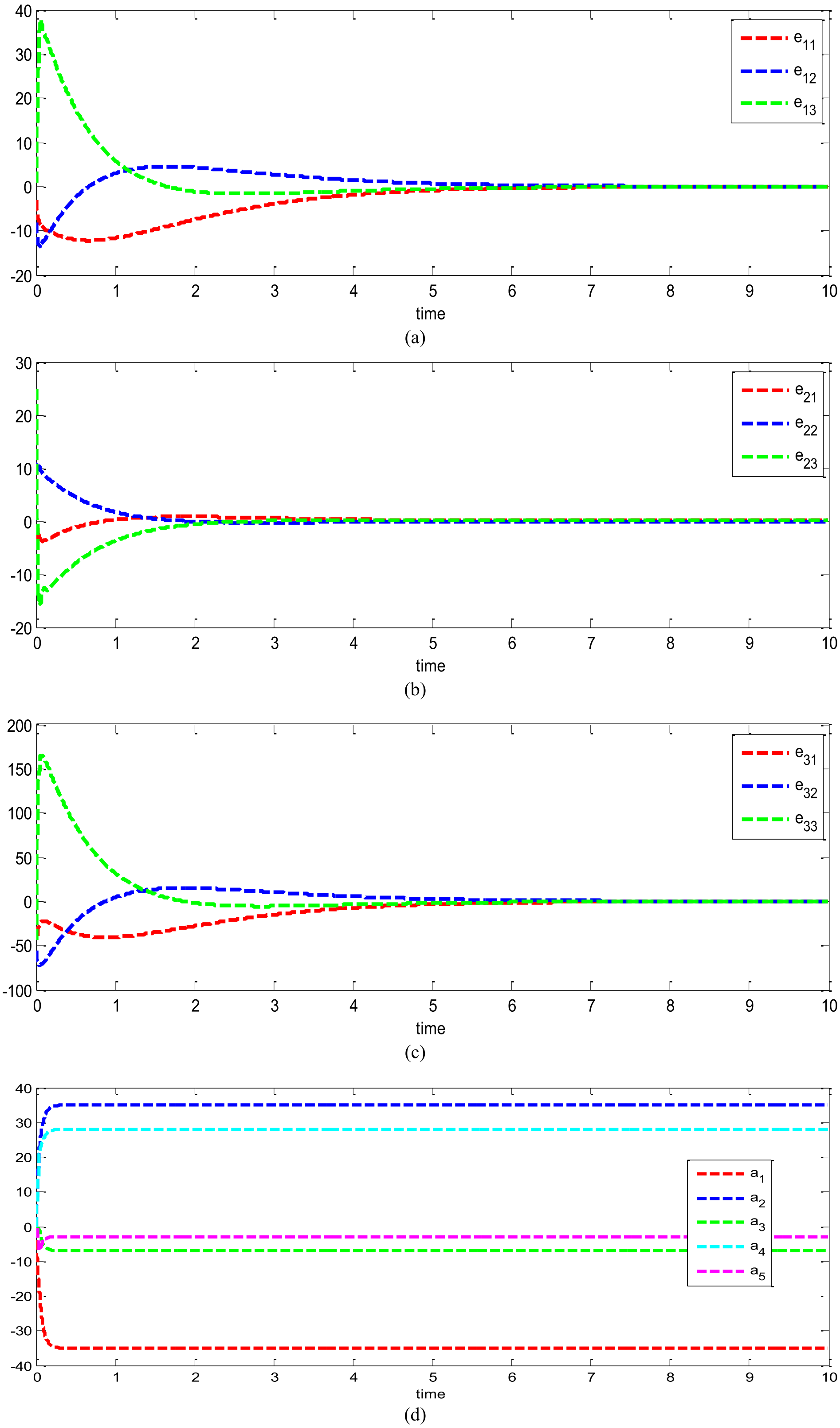
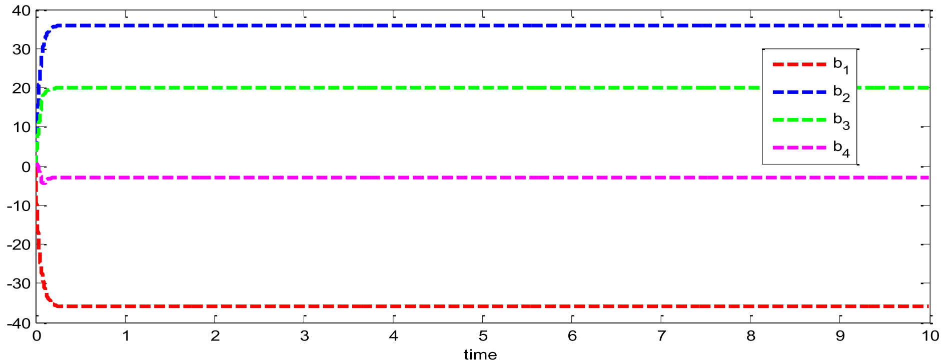
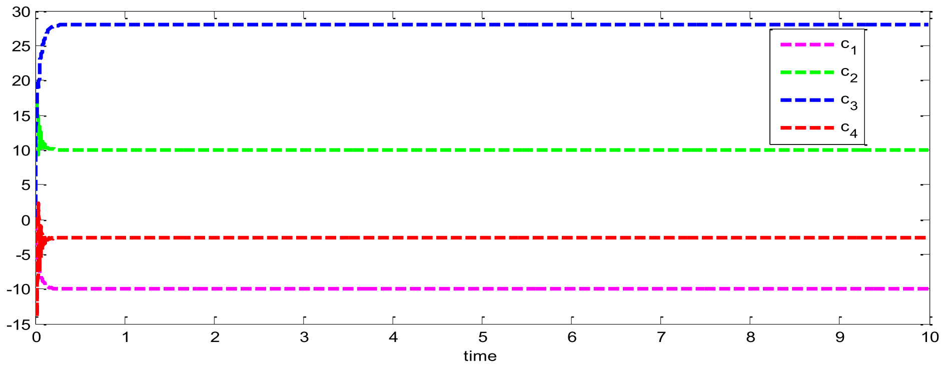


FIGURE 3. Time history of errors and identification of parameters when there is no fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3), (c) shows error between the systems (1) and (3) and (d)-(f) show estimated parameters.



(e)



(f)

FIGURE 3. ***(Continued)*** **Time history of errors and identification of parameters when there is no fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3), (c) shows error between the systems (1) and (3) and (d)-(f) show estimated parameters.**

gives

$$\begin{bmatrix} \dot{\tilde{a}}_1 \\ \dot{\tilde{a}}_2 \\ \dot{\tilde{a}}_3 \\ \dot{\tilde{a}}_4 \\ \dot{\tilde{a}}_5 \end{bmatrix} = \begin{bmatrix} s_1(q_1-1)x_{11} \\ s_1(q_1-1)x_{12} \\ s_2(q_1-1)x_{11} \\ s_2(q_1-1)x_{12} \\ s_3(q_1-1)x_{13} \end{bmatrix} - \begin{bmatrix} k_1\tilde{a}_1 \\ k_1\tilde{a}_2 \\ k_1\tilde{a}_3 \\ k_1\tilde{a}_4 \\ k_1\tilde{a}_5 \end{bmatrix}, \quad \begin{bmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \\ \dot{\hat{\theta}}_3 \\ \dot{\hat{\theta}}_4 \\ \dot{\hat{\theta}}_5 \end{bmatrix} = - \begin{bmatrix} \dot{\tilde{a}}_1 \\ \dot{\tilde{a}}_2 \\ \dot{\tilde{a}}_3 \\ \dot{\tilde{a}}_4 \\ \dot{\tilde{a}}_5 \end{bmatrix}.$$

Further

$$\Rightarrow \dot{\tilde{\theta}}_2 = \left(2q_2 - 1\right) F_2^T\left(x_2\right) S - K_2\tilde{\theta}_2, \dot{\hat{\theta}}_2 = \dot{\tilde{\theta}}_2,$$

gives

$$\begin{bmatrix} \dot{\tilde{b}}_1 \\ \dot{\tilde{b}}_2 \\ \dot{\tilde{b}}_3 \\ \dot{\tilde{b}}_4 \end{bmatrix} = \begin{bmatrix} \left(2q_2-1\right) x_{21}s_1 \\ \left(2q_2-1\right) x_{22}s_1 \\ \left(2q_2-1\right) x_{22}s_2 \\ \left(2q_2-1\right) x_{23}s_3 \end{bmatrix} - \begin{bmatrix} k_2\tilde{a}_1 \\ k_2\tilde{a}_2 \\ k_2\tilde{a}_3 \\ k_2\tilde{a}_4 \end{bmatrix},$$

$$\begin{bmatrix} \dot{\hat{b}}_1 \\ \dot{\hat{b}}_2 \\ \dot{\hat{b}}_3 \\ \dot{\hat{b}}_4 \end{bmatrix} = - \begin{bmatrix} \dot{\tilde{b}}_1 \\ \dot{\tilde{b}}_2 \\ \dot{\tilde{b}}_3 \\ \dot{\tilde{b}}_4 \end{bmatrix}.$$

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} f_2\left(x_2\right) + F_2\left(x_2\right)\hat{\theta}_2 - q_1\{f_1\left(x_1\right) + F_1\left(x_1\right)\hat{\theta}_1\} \\ f_3\left(x_3\right) + F_3\left(x_3\right)\hat{\theta}_3 - q_2\{f_2\left(x_2\right) + F_2\left(x_2\right)\hat{\theta}_2\} \\ f_1\left(x_1\right) + F_1\left(x_1\right)\hat{\theta}_1 - q_3\{f_3\left(x_3\right) + F_3\left(x_3\right)\hat{\theta}_3\} \end{bmatrix} + \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 0 & 0 & -q_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} F_2(x_2)\tilde{\theta}_2 - q_1F_1(x_1)\tilde{\theta}_1 \\ F_3(x_3)\tilde{\theta}_3 - q_2F_2(x_2)\tilde{\theta}_2 \\ F_1(x_1)\tilde{\theta}_1 - q_3F_3(x_3)\tilde{\theta}_3 \end{bmatrix} \tag{36}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ 1 & 0 & -q_3 \end{bmatrix}^{-1} \left\{ - \begin{bmatrix} f_2\left(x_2\right) + F_2\left(x_2\right)\hat{\theta}_2 - q_1\{f_1\left(x_1\right) + F_1\left(x_1\right)\hat{\theta}_1\} \\ f_3\left(x_3\right) + F_3\left(x_3\right)\hat{\theta}_3 - q_2\{f_2\left(x_2\right) + F_2\left(x_2\right)\hat{\theta}_2\} \\ f_1\left(x_1\right) + F_1\left(x_1\right)\hat{\theta}_1 - q_3\{f_3\left(x_3\right) + F_3\left(x_3\right)\hat{\theta}_3\} \end{bmatrix} + \begin{bmatrix} e_2 \\ e_3 \\ v \end{bmatrix} \right\} \tag{37}$$

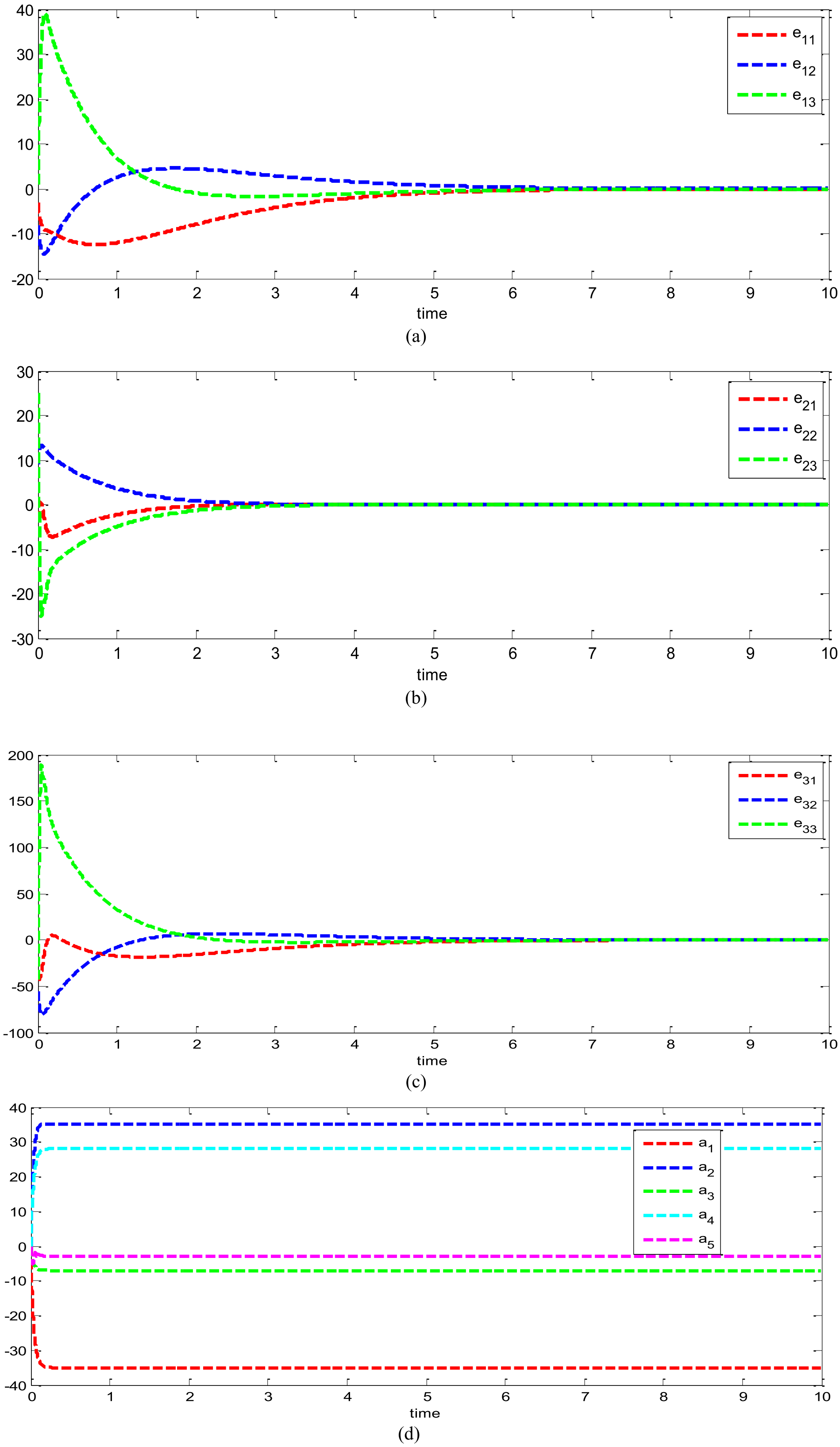
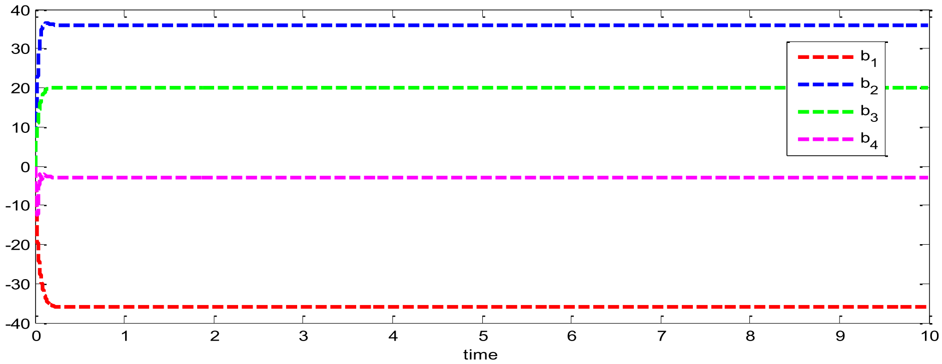
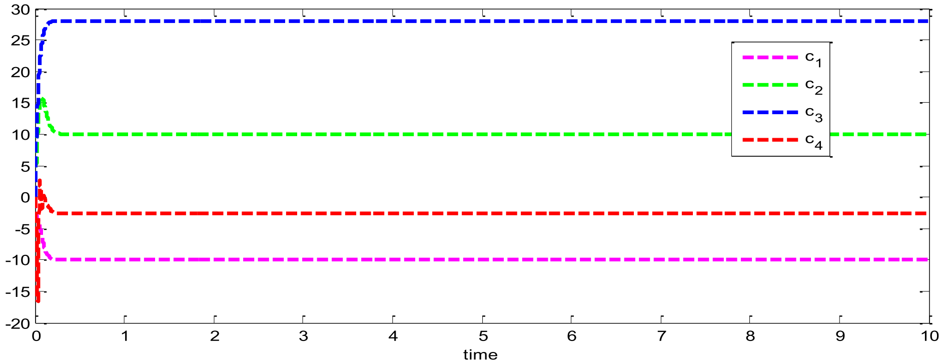


FIGURE 4. Time history of errors and identification of parameters when there is a fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3), (c) shows error between the systems (1) and (3) and (d)-(f) show estimated parameters.



(e)



(f)

FIGURE 4. ***(Continued)*** **Time history of errors and identification of parameters when there is a fault, where (a) shows error between the systems (1) and (2), (b) shows error between the systems (2) and (3), (c) shows error between the systems (1) and (3) and (d)-(f) show estimated parameters.**

$$\dot{\tilde{\theta}}_3 = -\left(2F_3\left(x_3\right) - q_3F_3\left(x_3\right)\right)^T S - K_3\tilde{\theta}_3, \quad \dot{\hat{\theta}}_3 = \dot{\tilde{\theta}}_3$$
$$\Rightarrow \dot{\tilde{\theta}}_3 = \left(q_3 - 2\right) F_3^T\left(x_3\right) S - K_3\tilde{\theta}_3, \quad \dot{\hat{\theta}}_3 = \dot{\tilde{\theta}}_3,$$

gives

$$\begin{bmatrix} \dot{\tilde{c}}_1 \\ \dot{\tilde{c}}_2 \\ \dot{\tilde{c}}_3 \\ \dot{\tilde{c}}_4 \end{bmatrix} = \begin{bmatrix} (q_3-2)\,x_{31}s_1 \\ (q_3-2)\,x_{32}s_1 \\ (q_3-2)\,x_{31}s_2 \\ (q_3-2)\,x_{33}s_3 \end{bmatrix} - \begin{bmatrix} k_3\tilde{c}_1 \\ k_3\tilde{c}_2 \\ k_3\tilde{c}_3 \\ k_3\tilde{c}_4 \end{bmatrix},$$
$$\begin{bmatrix} \dot{\hat{c}}_1 \\ \dot{\hat{c}}_2 \\ \dot{\hat{c}}_3 \\ \dot{\hat{c}}_4 \end{bmatrix} = -\begin{bmatrix} \dot{\tilde{c}}_1 \\ \dot{\tilde{c}}_2 \\ \dot{\tilde{c}}_3 \\ \dot{\tilde{c}}_4 \end{bmatrix}.$$

Case IV (Unknown System Parameters With a Fault): Assume, there is a fault occurring between systems (16) and (18), then TPS errors are

$e_1 = x_2 - q_1x_1$, $e_2 = x_3 - q_2x_2$ and $e_3 = x_1 - q_3x_3$, where $q_1q_2q_3 = 1$ and $u_1 = 0$ in e_3. Therefore

$$\begin{cases} \dot{e}_1 = \dot{x}_2 - q_1\dot{x}_1 \\ \dot{e}_2 = \dot{x}_3 - q_2\dot{x}_2 \\ \dot{e}_3 = \dot{x}_1 - q_3\dot{x}_3, \end{cases} \tag{35}$$

which can also be represented as (36), shown at the bottom of the 11 page. Setting (37), as shown at the bottom of the 11 page, with v as the new input vector, we have (38). Remaining procedure is similar to Case III.

$$\begin{cases} \dot{e}_1 = e_2 + F_2\left(x_2\right)\tilde{\theta}_2 - q_1F_1\left(x_1\right)\tilde{\theta}_1 \\ \dot{e}_2 = e_3 + F_3\left(x_3\right)\tilde{\theta}_3 - q_2F_2\left(x_2\right)\tilde{\theta}_2 \\ \dot{e}_3 = v + F_1\left(x_1\right)\tilde{\theta}_1 - q_3F_3\left(x_3\right)\tilde{\theta}_3. \end{cases} \tag{38}$$

V. NUMERICAL SIMULATIONS

Initial conditions for simulation are chosen as

$$\left(x_{11}(0), x_{12}(0), x_{13}(0)\right) = (10, 20, 30),$$
$$\left(x_{21}(0), x_{22}(0), x_{23}(0)\right) = (-5.8, 8, 30) \quad \text{and}$$
$$\left(x_{31}(0), x_{32}(0), x_{33}(0)\right) = (11, 25, 26).$$

We choose $d_{11} = d_{21} = d_{13} = d_{23} = d_{31} = d_{33} = 0$, $d_{12} = 10$, $d_{22} = 11$ and $d_{32} = 1$.

For Case I, when systems parameters are known and there is no fault, we choose $q_1 = 2, q_2 = 3, q_3 = 2$. The synchronization errors between the systems (1) and (2), systems (2) and (3), and systems (1) and (3) are shown in Figures 1(a), 1(b) and 1(c), respectively.

For Case II, when system parameters are known and there is a fault, we choose $q_1 = 2, q_2 = 3, q_3 = \frac{1}{6}$. The synchronization errors between the systems (1) and (2), systems (2) and (3), and systems (1) and (3) are shown in Figures 2(a), 2(b) and 2(c), respectively.

From Figures 1 and 2, it is quite clear that the error state trajectories converge to zero asymptotically for controllers (21) and (25).

For Case III, when system parameters are unknown and there is no fault, we choose $q_1 = 2, q_2 = 3, q_3 = 4$ and true values of the parameters are chosen as

$$a_1 = -35, \quad a_2 = 35,\ a_3 = -7,\ a_4 = 28,\ a_5 = -3,$$
$$b_1 = -35, \quad b_2 = 36,\ b_3 = 20,\ b_4 = -3, \quad \text{and}$$
$$c_1 = -10, \quad c_2 = 10,\ c_3 = 28,\ c_4 = -\frac{8}{3}.$$

The synchronization errors are depicted in Figure 3.

For Case IV, when system parameters are unknown with a fault, we choose $q_1 = 2, q_2 = 2, q_3 = 0.25$. The synchronization errors between the systems (1) and (2), systems (2) and (3) and systems (1) and (3) are shown in Figures 4(a), 4(b) and 4(c), respectively.

It is clear from Figures 3 and 4, that the error state trajectory reaches to zero asymptotically and the estimates reach their true values, i.e.,

$$\hat{a}_1 = a_1, \quad \hat{a}_2 = a_2,\ \hat{a}_3 = a_3,\ \hat{a}_4 = a_4,\ \hat{a}_5 = a_5,$$
$$\hat{b}_1 = b_1, \quad \hat{b}_2 = b_2,\ \hat{b}_3 = b_3,\ \hat{b}_4 = b_4,$$
$$\hat{c}_1 = c_1, \quad \hat{c}_2 = c_2,\ \hat{c}_3 = c_3,\ \hat{c}_4 = c_4.$$

Hence, the TPS is realized.

VI. CONCLUSION

In this work, the behavior of transmission projective synchronization is investigated for multiple coupled chaotic systems. Based on sliding mode approach, the controllers are developed. Four cases of chaotic systems are considered. Case I: Known system parameters with no fault, Case II: Known system parameters with a fault, Case III: Unknown system parameters with no fault, and Case IV: Unknown system parameters with a fault. In Cases I and III, controllers are designed using sliding mode control, and in Cases II and IV, controllers are proposed using an adaptive integral sliding mode. The control laws for anti-synchronization and complete synchronization are also designed based on stability theory such that the uncertain parameters can effectively be identified. Numerical simulations results verify the theoretical analysis.

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