Received December 27, 2018, accepted January 20, 2019, date of publication January 30, 2019, date of current version February 14, 2019.

Digital Object Identifier 10.1109/ACCESS.2019.2896101

Minimum Neighborhood of Alternating Group Graphs

YANZE HUANG[1,2], LIMEI LIN[2,3], DAJIN WANG[4], AND LI XU[2,3]
[1]School of Mathematics and Physics, Fujian Provincial Key Laboratory of Big Data Mining and Applications, Fujian University of Technology, Fuzhou 350118, China
[2]College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350117, China
[3]Key Laboratory of Network Security and Cryptology, Fujian Normal University, Fuzhou 350117, China
[4]Department of Computer Science, Montclair State University, Upper Montclair, NJ 07043, USA

Corresponding author: Limei Lin (putianlinlimei@163.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 61702100, Grant 61771140, Grant 61702103, Grant 61572010, and Grant 61773415, in part by the China Postdoctoral Science Foundation under Grant 2018T110636 and Grant 2017M612107, in part by the Natural Science Foundation of Fujian Province under Grant 2017J01738, in part by the Fujian University of Technology under Grant GY-Z17008 and Grant GY-Z18145, and in part by the Education Department of Fujian Province under Grant JAT170397.

ABSTRACT The minimum neighborhood and combinatorial property are two important indicators of fault tolerance of a multiprocessor system. Given a graph G, $\theta_G(q)$ is the minimum number of vertices adjacent to a set of q vertices of G $(1 \leq q \leq |V(G)|)$. It is meant to determine $\theta_G(q)$, the minimum neighborhood problem (MNP). In this paper, we obtain $\theta_{AG_n}(q)$ for an independent set with size q in an n-dimensional alternating group graph AG_n, a well-known interconnection network for multiprocessor systems. We first propose some combinatorial properties of AG_n. Then, we study the MNP for an independent set of two vertices and obtain that $\theta_{AG_n}(2) = 4n - 10$. Next, we prove that $\theta_{AG_n}(3) = 6n - 16$. Finally, we propose that $\theta_{AG_n}(4) = 8n - 24$.

INDEX TERMS Minimum neighborhood, combinatorial property, fault tolerance, independent set, alternating group graphs.

I. INTRODUCTION

Minimum neighborhood problem of interconnection networks has become increasingly important due to the rapid development of multiprocessor systems. Given a graph G, the $\theta_G(q)$ is the minimum number of vertices adjacent to a set of q vertices of G $(1 \leq q \leq |V(G)|)$. It is meaningful to determine $\theta_G(q)$, the Minimum Neighborhood Problem (MNP). To ensure the stable running of the systems, we should consider the tolerant degree in term of MNP. Also, the MNP plays a crucial role in determining the connectivity, the diagnosability and the general relationship between them in interconnection networks. Nevertheless, it is rather complicated to determine the MNP for a general set of q vertices, since it depends intensely on the topological structure of the interconnection network, and it needs detailed analysis on the common neighbors and the private neighbors.

Minimum neighborhood problem is not only of interest in its own right, but also useful in system-level fault diagnosibility–traditional diagnosability [11], conditional diagnosability [13], extra conditional diagnosability [38], good-neighbor diagnosability [17], t/k-diagnosability [25], and in the analysis of fault tolerance of interconnection networks–traditional connectivity [8], extra connectivity [7], restricted connectivity [29], component connectivity [4], structure connectivity [23].

In 1996, Somani and Peleg [25] gave the lower bounds on the size of neighbors of a subset D with $|D| = q$ in n-hypercube Q_n with $1 \leq q \leq n+1$ and n-star graph S_n with $1 \leq q \leq n$, which are $qn - q(q+1)/2 + 1$ and $qn - 3q + 2$, respectively, and used them to obtain the t/k-diagnosability under the PMC model. Caruso *et al.* [1] used the minimum neighborhood of regular systems to obtain the diagnosability. Lai *et al.* [14] also applied the minimum neighborhood of matching composition network to the diagnosability. Fan and Lin [6] computed the minimum neighborhood of BC graphs which is applied to t/k-diagnosability. Zheng *et al.* [40] and Zheng and Zhou *et al.* [41] proposed the size of neighbors of a subset D with $2 \leq |D| \leq 4$ and used it to obtain the traditional diagnosability. Yang *et al.* [34]–[36] applied this results to

The associate editor coordinating the review of this manuscript and approving it for publication was Sun-Yuan Hsieh.

obtain the maximal component. In 2009, Yang and Meng [31] established the size of a subgraph A of Q_n which is $(q+1)n-2q-q(q-1)/2$ when $|V(A)|=q+1$ and used it to obtain the extra connectivity. In 2015, Lin *et al.* [18] proposed the size of neighbors of a subset D in n-shuffle-cube, which can be applied to conditional diagnosability. In 2016, Zhao *et al.* [39] proposed that the size of neighbors of an independent set D of vertices with $|D|=q$ in Q_n is $\geq -q^2/2+(2n-5/2)q-n^2+2n+1$ for $n+1\leq q\leq 2n-4$ and applied it to obtain the component connectivity. In 2016, Xu *et al.* [32] established the lower bound of size of neighbors of a subset D with $3\leq |D|\leq 5$ in (n,k)-arrangement graph, which can be applied to t/k-diagnosability and extra conditional diagnosability. Lin *et al.* [21] proposed the size of neighbors of a 2-path or a 3-cycle in general regular graphs, which can be applied to build the relationship between conditional diagnosability and extra connectivity under the comparison model. In 2018, under the PMC model, Lin *et al.* [16] proposed the size of neighbors of a 4-cycle in general regular graphs, which can be applied to build the relationship between conditional diagnosability and extra connectivity. Moreover, Lin *et al.* [22] also applied the size of neighbors of a subset D to propose the t/k-diagnosability for regular graphs, including n-alternating group graph, n-split-star network, l^n-hypermesh and (n,l)-star graph. Also, Lin *et al.* [19] proposed the size of neighbors of a subgraph B of order $q+1$ in the (n,k)-arrangement graph $A_{n,k}$, which can be applied to the g-good-neighbor conditional diagnosability. Lin *et al.* [15] used the size of neighbors of a subset in general regular networks to establish the relationship between g-good-neighbor diagnosability and g-restricted connectivity. To summarize, the minimum neighborhood problem in a network is very meaningful.

In 1993, Jwo *et al.* [12] first proposed the *n-alternating group graph* AG_n as a topology of interconnection network for multiprocessor systems. The n-alternating group graph AG_n possesses sufficient amount of good properties including cycle-embedding [2], [27], [33], and small diameter [12]. Moreover, the alternating group graph is not only pancyclic and hamiltonian-connected [3], but also panconnected [10]. It also have a fault-free longest path [26] and vertex pancyclicity [28]. Furthermore, Lin *et al.* [24] established the extra fault tolerance and conditional diagnosability of AG_n in 2015. There is another type of interconnection network based on alternating group called the n-alternating group network AN_n, which is different from the n-alternating group graph AG_n [12] investigated in this paper. Both of the AG_n and AN_n are Cayley graphs, but with different generating sets. Consequently, they have distinct adjacency manners. Roughly speaking, the edges of AG_n are generated by $(12i)$ and $(1i2)$ for $i=3,\ldots,n$, while the edges of AN_n are generated by (123), (132) and $(12)(3i)$ for $i=4,\ldots,n$.

Motivated by the research on minimum neighborhood in a generalized cube [30] and the size of neighbors of an independent set D of vertices with $|D|=q$ in Q_n [39], in this paper, we obtain $\theta_{AG_n}(q)$ for an independent set with size q in AG_n. We first propose some combinatorial properties of AG_n. Then we study the minimum neighborhood problem for an independent set of two vertices and obtain that $\theta_{AG_n}(2)=4n-10$. Next, we prove that $\theta_{AG_n}(3)=6n-16$. Finally, we propose that $\theta_{AG_n}(4)=8n-24$. Meanwhile, we present sufficient amount of figures to better illustrate the process of the proofs.

Organization: The remainder of this paper is organized as follows. Section II introduces some preliminaries used throughout this paper. Section III gives some basic combinatorial properties of AG_n. Section IV focuses on the minimum neighborhood of AG_n. Section V concludes this paper.

II. PRELIMINARIES

In this section, we will introduce some basic definitions for networks.

A. TERMINOLOGIES

The term $G=(V(G),E(G))$ represents a graph where $v\in V(G)$ is a vertex and $e\in E(G)$ is an edge. The numbers of vertices and edges of G are defined as $|V(G)|$ and $|E(G)|$.

A subgraph S of G denoted by $H\subseteq G$, is a graph where $V(S)\subseteq V(G)$ and $E(S)\subseteq E(G)$. Let $G_1,G_2,\ldots,G_t$ be t subgraphs of G, and we set $\bigcup_{i=1}^{t}G_i=G[\bigcup_{i=1}^{t}V(G_i)]$ and $\bigcap_{i=1}^{t}G_i=G[\bigcap_{i=1}^{t}V(G_i)]$. Let $D\subseteq V(G)$, the notation $G-D$ denotes a subgraph obtained by deleting all vertices of D from G and deleting those edges with at least one end-vertex in D, simultaneously. We use $E[M,N]$ to represent the set of all edges between M and N.

For any $v\in V(G)$, the neighborhood $N_G(v)$ of v in G is the set of all vertices which are adjacent to v. Let $N_G[v]=N_G(v)\cup\{v\}$. We define $N_G(D)=\{v\in V(G)-D\mid \exists u\in D, uv\in E(G)\}=(\bigcup_{u\in D}N(u))-D$. Let $N_G[D]=N_G(D)\cup D$. A *path* is a finite or infinite sequence of edges which connect a sequence of vertices that are all distinct from each another. A *cycle* is a path of edges and vertices wherein a vertex is reachable from itself. A k-*path* (or k-*cycle*) is a path (or cycle) of length k.

B. ALTERNATING GROUP GRAPHS

Let $\langle n\rangle=\{1,\ldots,n-1,n\}$ and let $\zeta=\zeta_1\zeta_2\ldots\zeta_n$ be a permutation of elements in $\langle n\rangle$ where $\zeta_\alpha\in\langle n\rangle$ for $1\leq\alpha\leq n$ and $\zeta_\alpha\neq\zeta_\beta$ for $1\leq\alpha\neq\beta\leq n$. A pair of elements ζ_α and ζ_β is said to be an inversion of ζ if $\zeta_\alpha<\zeta_\beta$ whenever $1\leq\beta<\alpha\leq n$. An even permutation is a permutation with an even number of inversions. Let A_n denote the set of all even permutations over $\langle n\rangle$.

Definition 1 [12]: The n-dimensional alternating group graph AG_n consists of vertex-set $V(AG_n)=A_n$, and edge-set $E(AG_n)=\{\zeta\eta\mid\eta$ is obtained from ζ by rotating the symbols in positions 1, 2, and α from left to right or from right to left for some $\alpha\in\{3,4,\ldots,n-1,n\}\}$.

It can be seen that AG_n is regular of degree $2n-4$, $|V(AG_n)|=\frac{n!}{2}$, and $|E(AG_n)|=\frac{(n-2)n!}{2}$. Fig. 1 describes an example of AG_4.

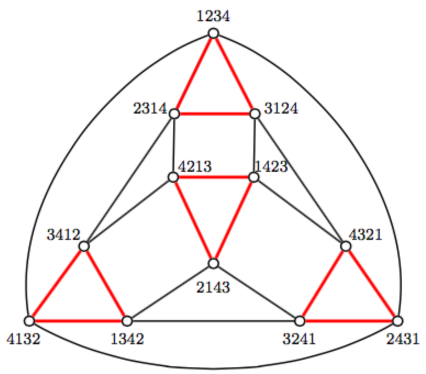


FIGURE 1. An illustration of a 4-alternating group graph.

Denote by A_n^α ($n \geq 3$ and $1 \leq \alpha \leq n$) the subset of A_n consisting of all even permutations with α in the n-th position, and denote by AG_n^α the subgraph of AG_n induced by A_n^α. It implies that AG_n^α is isomorphic to AG_{n-1} for any $\alpha \in \langle n\rangle$. The n-alternating group graph is composed of n disjoint copies of $(n-1)$-alternating group graphs such that AG_n^α connects AG_n^β $(1 \leq \alpha \neq \beta \leq n)$ by $(n-2)!$ disjointed edges due to the hierarchical structure. These $(n-2)!$ disjointed edges are called *outside edges* with the form $\zeta\eta$ where $\zeta = \gamma\beta\cdots\alpha, \eta = \alpha\gamma\cdots\beta$ or $\zeta = \beta\gamma\cdots\alpha, \eta = \gamma\alpha\cdots\beta$ for $\gamma \in \langle n\rangle - \{\alpha, \beta\}$. Let $E_n^{\alpha,\beta}(AG_n)$ be the set of edges in AG_n connecting AG_n^α and AG_n^β for $1 \leq \alpha \neq \beta \leq n$. In particular, for each inside edge $\zeta\eta$ with $\zeta = \gamma\beta\cdots\delta\cdots\alpha$ and $\eta = \beta\delta\cdots\gamma\cdots\alpha$ in AG_n^α, there are two adjacent vertices $\zeta' = \alpha\gamma\cdots\delta\cdots\beta$ and $\eta' = \delta\alpha\cdots\gamma\cdots\beta$ in AG_n^β such that $\{\zeta, \zeta', \eta', \eta, \zeta\}$ is a 4-cycle in AG_n. Note that ζ' and η' are uniquely determined by the 4-cycle structure of $\zeta\eta$. Moreover, every vertex $\zeta \in V(AG_n^\alpha)$ lies on exactly $2n-6$ inside edges and two outside edges. Furthermore, the two end-vertices, connecting one node via these two outside edges are in distinct induced subgraphs.

III. COMBINATORIAL PROPERTIES OF ALTERNATING GROUP GRAPHS

In this section, we give some basic combinatorial properties of AG_n.

Lemma 1 [12]: An n-alternating group graph AG_n has the following basic properties.

(1) AG_n is $(2n-4)$-regular and $\kappa(AG_n) = \delta(AG_n) = 2n-4$ for $n \geq 3$.

(2) Every vertex of AG_n^α has two outside neighbors, whose are in distinct AG_n^β and AG_n^γ for $1 \leq \alpha \neq \beta \leq n$, $1 \leq \alpha \neq \gamma \leq n$, $1 \leq \beta \neq \gamma \leq n$ and $n \geq 4$.

(3) There exist $(n-2)!$ disjoint edges between any two AG_n^α and AG_n^β, i.e., $|E_n^{\alpha,\beta}(AG_n)| = (n-2)!$, for $1 \leq \alpha \neq \beta \leq n$ and $n \geq 4$.

Lemma 2: Given any two vertices ζ, η of AG_n.

(1) [9] If $\zeta\eta \notin E(AG_n)$, then $|N(\zeta) \cap N(\eta)| \leq 2$.

(2) [24] If $\zeta\eta \in E(AG_n)$, then $|N(\zeta) \cap N(\eta)| = 1$.

Lemma 3: Let D be a subset of $V(AG_n)$ $(n \geq 5)$.

(1) [5], [9] If $|D| \leq 4n-11$, then $AG_n - D$ possesses one of the following results.

- one component (connected);
- two components, the small one being a node;
- two components, the small one being an edge. Furthermore, $|D| = 4n-11$ and D consists of all neighbors of the edge.

(2) [5] If $|D| \leq 6n-20$, then $AG_n - D$ possesses one of the following results.

- one component (connected);
- two components, the small one being a node or an edge;
- three components, the small two being both a node, respectively.

(3) [9] If $|D| \leq 6n-19$, then $AG_n - D$ possesses one of the following results.

- one component (connected);
- two components, the small one being a 2-path, or a node, or an edge;
- three components, the small two being both a node, respectively.

(4) [24] If $|D| \leq 8n-29$, then $AG_n - D$ possesses one of the following results.

- one component (connected);
- two components, the small one being a 3-cycle, or a 2-path, or a node, or an edge;
- three components, the small two being both a node, respectively, or a node and an edge;
- four components, the small three being all a node, respectively.

Lemma 4 [37]: (1) Any 3-cycle in AG_n has the form $u_1u_2u_3u_1$, where $u_1 = u_2(12i)$, $u_2 = u_3(12i)$, $u_3 = u_1(12i)$ for some i. Each edge is in a unique triangle.

(2) Any 4-cycle in AG_n has the form $u_1u_2u_3u_4u_1$ where $u_2 = u_1(12i)$, $u_3 = u_2(12j)$, $u_4 = u_3(12i)$, $u_1 = u_4(12j)$ for some i, j with $i \neq j$.

It should be noted that a 3-cycle may also have the form $u_2 = u1(1i2)$, $u_3 = u_2(1i2)$ and $u_1 = u_3(1i2)$. Moreover, a 4-cycle may also have the form $u_2 = u_1(1i2)$, $u_3 = u_2(1j2)$, $u_4 = u_3(1i2)$, $u_1 = u_4(1j2)$ for $i \neq j$. That is to say, if $u = v(1i2)$, then $v = u(12i)$.

Theorem 1: Let u, v be two independent vertices in a 4-cycle of an $(n-1)$-subgraph of AG_n. Then the four outside neighbors of u, v are in four different subgraphs.

Proof: Assume that $uxvyu$ is a 4-cycle in subgraph $AG_n^{k_n}$. By Lemma 4 (2) and the symmetry of AG_n, $x = u(12i)$, $v = x(12j)$, $y = v(12i)$ and $u = y(12j)$ where $i \neq j$ (see Fig. 2). Assume that $u = k_1k_2\cdots k_i\cdots k_j\cdots k_n$. We have $v = k_ik_j\cdots k_1\cdots k_2\cdots k_n$. By Lemma 1 (2), u has two outside neighbors in subgraph $AG_n^{k_1}$ and subgraph $AG_n^{k_2}$. Moreover, v has two outside neighbors in subgraphs $AG_n^{k_i}$ and $AG_n^{k_j}$, respectively. By the fact that any two values in set $\{k_1, k_2, k_i, k_j\}$ are distinct, the four outside neighbors of u, v are in four different subgraphs. ■

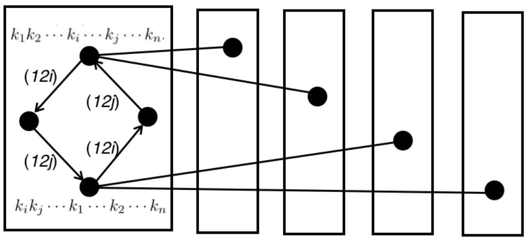


FIGURE 2. An illustration of outside neighbors of two independent vertices in a 4-cycle of an $(n-1)$-subgraph of AG_n.

Theorem 2: Let u, v be two independent vertices in a 2-path of an $(n-1)$-subgraph of AG_n such that $uv \notin E(AG_n)$. If $|N(u) \cap N(v)| = 1$, then the four outside neighbors of u, v are in three different subgraphs.

Proof: Assume that uxv is a 2-path in subgraph $AG_n^{k_n}$ such that $uv \notin E(AG_n)$. By the symmetry of AG_n, let $x = u(12i)$. Hence, $u = x(1i2)$.

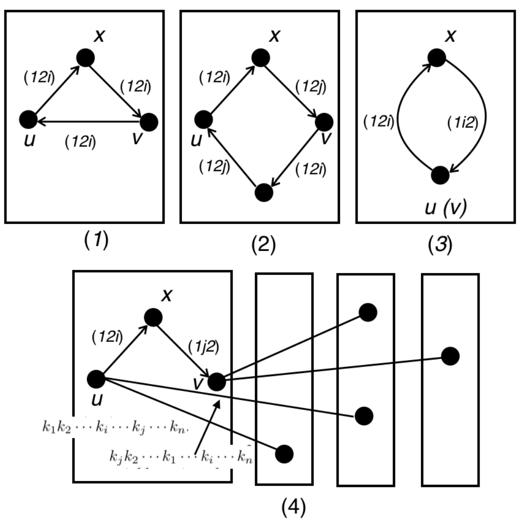


FIGURE 3. Illustrations of outside neighbors of two independent vertices in a 2-path of an $(n-1)$-subgraph of AG_n.

If $v = x(12i)$ (see Fig. 3 (1)), then uxv is a 3-cycle by Lemma 4 (1). Hence $uv \in E(AG_n)$, which is a contradiction.

If $v = x(12j)$ $(j \neq i)$ (see Fig. 3 (2)), then there exists a vertex $y = v(12i)$ such that $uxvyu$ be a 4-cycle by Lemma 4 (2). Hence $|N(u) \cap N(v)| = 2$, which is a contradiction.

If $v = x(1i2)$ (see Fig. 3 (3)), then $u = v$, which is a contradiction.

Therefore, $v = x(1j2)$ $(j \neq i)$ (see Fig. 3 (4)). Assume that $u = k_1k_2\cdots k_i\cdots k_j\cdots k_n$. We have $v = k_jk_2\cdots k_1\cdots k_i\cdots k_n$. By Lemma 1 (2), u has two outside neighbors in subgraphs $AG_n^{k_1}$ and $AG_n^{k_2}$, respectively. Moreover, v has two outside neighbors in subgraphs $AG_n^{k_j}$ and $AG_n^{k_2}$, respectively. By the fact that any two values in set $\{k_1, k_2, k_j\}$ are distinct, the four outside neighbors of u, v are in three different subgraphs. ■

IV. MINIMUM NEIGHBORHOOD OF ALTERNATING GROUP GRAPHS

In this section, we will compute the minimum neighborhood of an independent set in $V(AG_n)$.

Theorem 3: Let D be an independent set in $V(AG_n)$ $(n \geq 4)$. We have the following results:

(1) If $|D| = 2$, then $|N(D)| \geq 4n - 10$, and the minimum neighborhood for D in AG_n is $\theta_{AG_n}(2) = 4n - 10$.

(2) If $|D| = 3$, then $|N(D)| \geq 6n - 16$, and the minimum neighborhood for D in AG_n is $\theta_{AG_n}(3) = 6n - 16$.

Proof: Let D be an independent set of $V(AG_n)$.

(1) Let $D = \{u, v\}$. By Lemma 2 (1), $|N(u) \cap N(v)| \leq 2$. By Lemma 1 (1), $|N(u)| = |N(v)| = 2n-4$. Hence, $|N(D)| \geq 2(2n-4) - 2 = 4n - 10$ and $\theta_{AG_n}(2) = 4n - 10$.

(2) Let $D = \{u, v, w\}$. We divide it into the following three cases depending on the distribution of D.

Case 1 (D Is in a Subgraph (see Fig. 4)): By the symmetry of AG_n, assume that $D \in V(AG_n^1)$. We will prove the result by mathematical induction. First, the result holds for $n = 4$. Assume that the result holds for $n - 1$. By inductive assumption, $|N_{AG_n^1}(D)| \geq 6(n-1) - 16 = 6n - 22$. By Lemma 1 (2), any vertex of D has two outside neighbors outside subgraph AG_n^1 and these two neighbors are in different subgraphs. Hence, $|N_{AG_n^{<n>-\{1\}}}(D)| \geq 2 \times 3 = 6$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^1}(D)| + |N_{AG_n^{<n>-\{1\}}}(D)| \\ &\geq 6n - 22 + 6 \\ &= 6n - 16. \end{aligned}$$

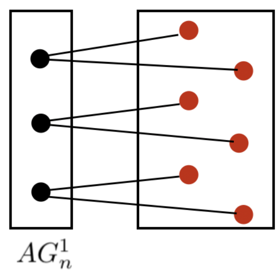


FIGURE 4. An illustration of outside neighbors of D distributed in an $(n-1)$-subgraph with $|D| = 3$.

Case 2 (D Is Distributed in Two Distinct Subgraphs): By the symmetry of AG_n, assume that $\{u, v\} \subseteq V(AG_n^n)$ and $w \in V(AG_n^1)$.

(2.1) $|N(u) \cap N(v)| = 2$ **(see Fig. 5 (1)-(3)).**

$|N_{AG_n^n}(u)| = |N_{AG_n^n}(v)| = 2(n-1)-4 = 2n-6$ by Lemma 1 (1). Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-6) - 2 = 4n - 14$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Therefore, there exists at most one outside neighbor of u, v in subgraph AG_n^1. Suppose that there exists one outside neighbor x of u, v in subgraph AG_n^1 such that $x \in N_{AG_n^1}(w)$. Assume that $u = 12\cdots i\cdots j\cdots k\cdots n$ and $v = ij\cdots 1\cdots 2\cdots k\cdots n$ by Theorem 1. Hence, $x = 2n\cdots i\cdots j\cdots k\cdots 1$. The other outside neighbor of u is

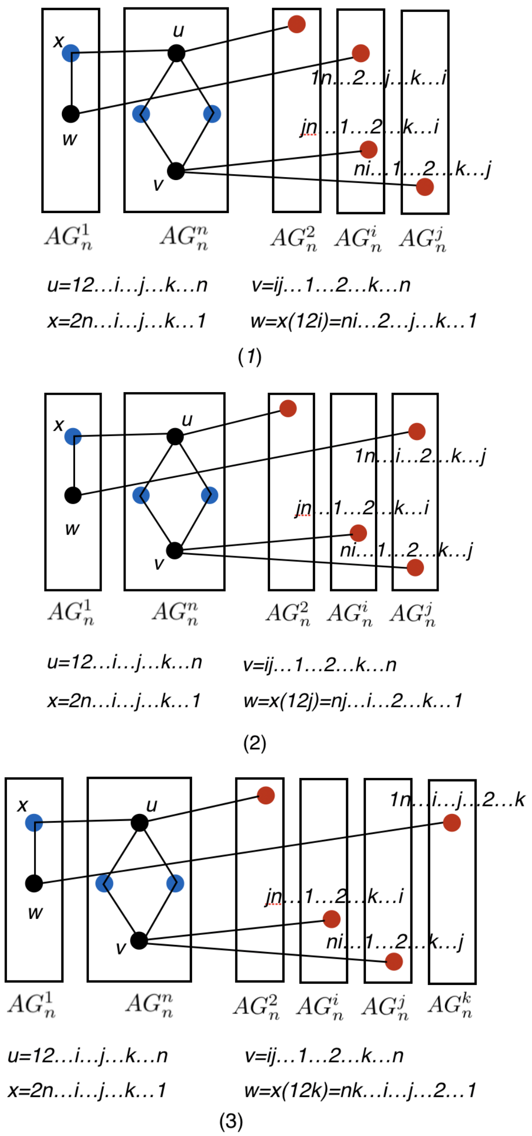


FIGURE 5. Illustrations of outside neighbors of $D = \{u, v, w\}$ where $\{u, v\} \subseteq V(AG_n^n)$ and $w \in V(AG_n^1)$ such that $|N(u) \cap N(v)| = 2$.

$n1 \cdots i \cdots j \cdots k \cdots 2$ and two outside neighbors of v are $ni \cdots 1 \cdots 2 \cdots k \cdots j$ and $jn \cdots 1 \cdots 2 \cdots k \cdots i$.

AG_n^k

By the symmetry of AG_n, assume that $w = x(12i)$ (see Fig. 5 (1)), we have $w = ni \cdots 2 \cdots j \cdots k \cdots 1$. Hence, two outside neighbors of w are $i1 \cdots 2 \cdots j \cdots k \cdots n$ in AG_n^n and $1n \cdots 2 \cdots j \cdots k \cdots i$ in AG_n^i. Since $1n \cdots 2 \cdots j \cdots k \cdots i \neq jn \cdots 1 \cdots 2 \cdots k \cdots i$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \geq 4$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \\ &\geq 4n - 14 + 2n - 6 + 4 \\ &= 6n - 16.\end{aligned}$$

By the symmetry of AG_n, assume that $w = x(12j)$ (see Fig. 5 (2)), we have $w = nj \cdots i \cdots 2 \cdots k \cdots 1$. Hence, two outside neighbors of w are $j1 \cdots i \cdots 2 \cdots k \cdots n$ and $1n \cdots i \cdots 2 \cdots k \cdots j$. Since $2n \cdots i \cdots 2 \cdots k \cdots j \neq ni \cdots 1 \cdots 2 \cdots k \cdots j$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \geq 4$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \\ &\geq 4n - 14 + 2n - 6 + 4 \\ &= 6n - 16.\end{aligned}$$

We can find that the result still holds when $w = x(1j2)$.

By the symmetry of AG_n, assume that $w = x(12k)$ ($k \neq i$ and $k \neq j$) (see Fig. 5 (3)), we have $w = nk \cdots i \cdots j \cdots 2 \cdots 1$. Thus, two outside neighbors of w are in subgraphs AG_n^n and AG_n^k, respectively. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \geq 4$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \\ &\geq 4n - 14 + 2n - 6 + 4 \\ &= 6n - 16.\end{aligned}$$

We can find that the result still holds when $w = x(1k2)$.

Otherwise, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| = 4$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| \\ &\geq 4n - 14 + 2n - 6 + 4 \\ &= 6n - 16.\end{aligned}$$

(2.2) $|N(u) \cap N(v)| = 1$ **(see Fig. 6 (1)).**

We have $|N_{AG_n^n}(u)| = |N_{AG_n^n}(v)| = 2(n-1) - 4 = 2n - 6$ by Lemma 1 (1). Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-7) + 1 = 4n - 13$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraph AG_n^1. If there exist two outside neighbors x, y of u, v in subgraph AG_n^1 such that $x, y \in N_{AG_n^1}(w)$, then assume that $u = 12 \cdots i \cdots j \cdots n$ and $v = 1j \cdots 2 \cdots i \cdots n$ by Theorem 2. Hence, $x = 2n \cdots i \cdots j \cdots 1$ and $y = jn \cdots 2 \cdots i \cdots 1$. The other outside neighbor of u is $n1 \cdots i \cdots j \cdots 2$ and the other outside neighbor of v is $n1 \cdots 2 \cdots i \cdots j$. We have $w = ni \cdots 2 \cdots j \cdots 1$. Therefore, two outside neighbors of w are $i1 \cdots 2 \cdots j \cdots n$ and $1n \cdots 2 \cdots j \cdots i$. Thus, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \geq 3$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Hence,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(w)| \\ &\geq 4n - 13 + 2n - 6 + 3 \\ &= 6n - 16.\end{aligned}$$

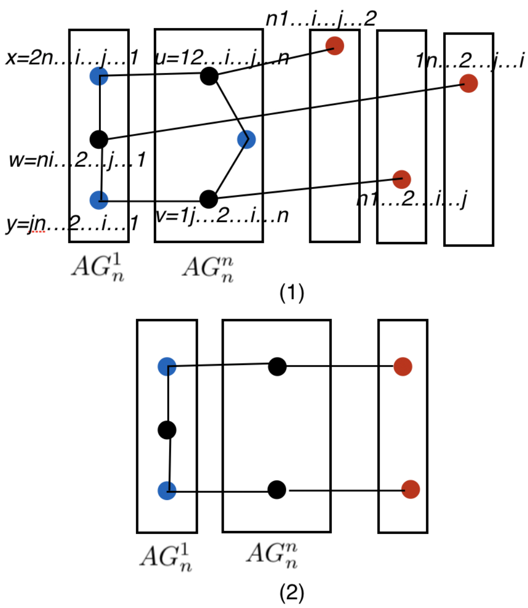


FIGURE 6. **Illustrations of outside neighbors of $D = \{u, v, w\}$ where $\{u, v\} \subseteq V(AG_n^n)$ and $w \in V(AG_n^1)$ such that $|N(u) \cap N(v)| = 1$ or $|N(u) \cap N(v)| = 0$.**

Otherwise, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| \geq 3$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| \\ &\geq 4n - 13 + 2n - 6 + 3 \\ &= 6n - 16. \end{aligned}$$

(2.3) $|N(u) \cap N(v)| = 0$ (see Fig. 6 (2)).

$|N_{AG_n^n}(u)| = |N_{AG_n^n}(v)| = 2(n-1) - 4 = 2n - 6$ by Lemma 1 (1). Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-6) = 4n - 12$. By Lemma 1 (2), the four outside neighbors of u, v are in at least two different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| \geq 2$. By Lemma 1 (1), $|N_{AG_n^1}(w)| \geq 2(n-1) - 4 = 2n - 6$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(w)| \\ &\geq 4n - 12 + 2n - 6 + 2 \\ &= 6n - 16. \end{aligned}$$

Case 3 (D Is Distributed in Three Distinct Subgraphs): By the symmetry of AG_n, assume that $u \in V(AG_n^n)$, $v \in V(AG_n^1)$ and $w \in V(AG_n^2)$.

By Lemma 1 (1), $|N_{AG_n^n}(u)| = |N_{AG_n^1}(v)| = |N_{AG_n^2}(w)| = 2(n-1) - 4 = 2n - 6$. By Lemma 1 (2), any one of u, v, w has two outside neighbors and these six outside neighbors are distinct. Otherwise, there exists at least one edge among u, v, w by Lemma 4 (1). Let these six outside neighbors of u, v, w be $u', u'', v', v'', w', w''$.

If $|\{u', u'', v', v'', w', w''\} \cap (N_{AG_n^n}(u) \cup N_{AG_n^1}(v) \cup N_{AG_n^2}(w))| \leq 4$, then $|\{u', u'', v', v'', w', w''\} - (N_{AG_n^n}(u) \cup N_{AG_n^1}(v) \cup N_{AG_n^2}(w))| \geq 2$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(u)| + |N_{AG_n^1}(v)| + |N_{AG_n^2}(w)| \\ &\quad + |\{u', u'', v', v'', w', w''\} \\ &\quad - (N_{AG_n^n}(u) \cup N_{AG_n^1}(v) \cup N_{AG_n^2}(w))| \\ &\geq 3(2n-6) + 2 \\ &= 6n - 16. \end{aligned}$$

If $|\{u', u'', v', v'', w', w''\} \cap (N_{AG_n^n}(u) \cup N_{AG_n^1}(v) \cup N_{AG_n^2}(w))| \geq 5$, then there exist at least two 4-cycles between u, v, w (see Fig. 7). By the symmetry of AG_n, assume that $v' \in N_{AG_n^n}(u)$, $v'' \in N_{AG_n^2}(w)$, $u' \in N_{AG_n^1}(v)$ and $w' \in N_{AG_n^1}(v)$ and assume that $v' = v(12n)$ and $v'' = v(1n2)$. By Lemma 4 (2), $u' = v(1i2)$ and $w' = v(12j)$ (i may be equal to j). Let $v = n2\cdots i\cdots j\cdots 1$. If $i \neq j$ (see Fig. 7 (1)), then $v' = 21\cdots i\cdots j\cdots n$ in AG_n^n and $v'' = 1n\cdots i\cdots j\cdots 2$ in AG_n^2. By Lemma 4 (1), $u = v'(12i) = 1i\cdots 2\cdots j\cdots n$ and $w = v''(1j2) = j1\cdots i\cdots n\cdots 2$. Thus, $u'' = u(1n2) = n1\cdots 2\cdots j\cdots i$ in AG_n^i and $w'' = w(12n) = 12\cdots i\cdots n\cdots j$ in AG_n^j. Hence, $|\{u', u'', v', v'', w', w''\} \cap (N_{AG_n^n}(u) \cup N_{AG_n^1}(v) \cup N_{AG_n^2}(w))| = 4$, which is a contradiction. If $i = j$ (see Fig. 7 (2)), then $v' = 21\cdots i\cdots n$ in AG_n^n and

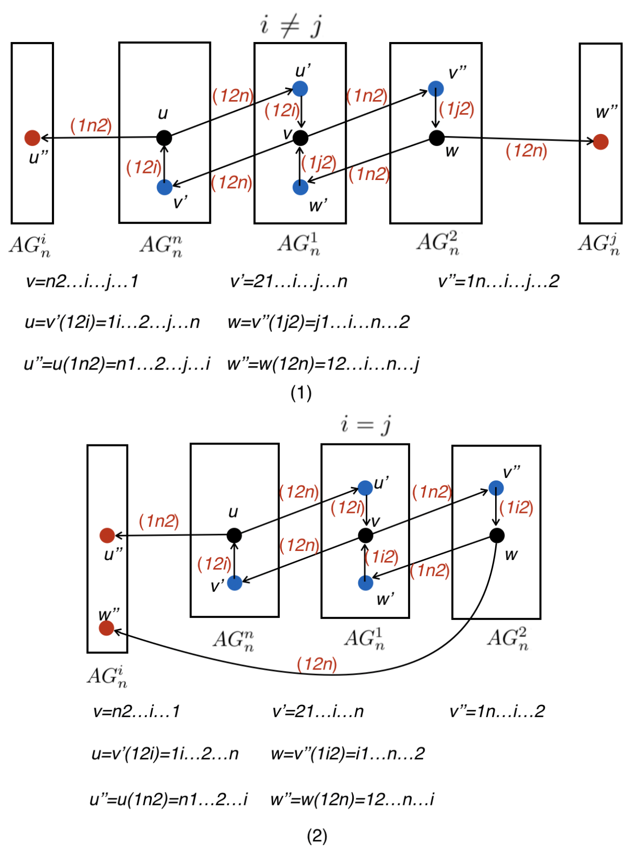


FIGURE 7. **Illustrations of outside neighbors of D distributed in three distinct $(n-1)$-subgraphs with $|D| = 3$.**

$v'' = 1n\cdots i\cdots 2$ in AG_n^2. By Lemma 4 (1), $u = v'(12i) = 1i\cdots 2\cdots n$ and $w = v''(1i2) = i1\cdots n\cdots 2$. Therefore, $u'' = u(1n2) = n1\cdots 2\cdots i$ in AG_n^i and $w'' = w(12n) = 12\cdots n\cdots i$ in AG_n^i. Since $n1\cdots 2\cdots i \neq 12\cdots n\cdots i$, $|\{u', u'', v', v'', w', w''\}\cap(N_{AG_n^n}(u)\cup N_{AG_n^1}(v)\cup N_{AG_n^2}(w))| = 4$, which is a contradiction.

From the above, $|N(D)| \geq 6n - 16$ when $|D| = 3$ for $n \geq 4$. ■

We give two examples in AG_4 to explain Theorem 3 as follows (see Table 1 and Table 2).

TABLE 1. Minimum neighborhood of an independent set of two vertices in AG_4.

Two independent vertices	Common neighbors	Neighborhood	No. of neighbors
1234, 2143	$\emptyset$	2314, 3124, 4132, 2431, 4213, 1423, 1342, 3241,	8
1234, 1423	3124	2314, 3124, 4132, 2431, 4213, 2143, 4321	7
1234, 3412	2314, 4132	2314, 3124, 4132, 2431, 4213, 1342	6

TABLE 2. Minimum neighborhood of an independent set of three vertices in AG_4.

Three independent vertices	Common neighbors	Neighborhood	No. of neighbors
1234, 2143, 3412	2314, 4213, 4132, 1342	2314, 3124, 4132, 2431, 4213, 1423, 1342, 3241,	8
1234, 1423, 1342	3124, 2143, 4132	2314, 3124, 4132, 2431, 4213, 2143, 4321, 3412, 3241	9
1234, 1423, 3241	3124, 2143, 4321, 2431	2314, 3124, 4132, 2431, 4213, 2143, 4321, 1342	8

Theorem 4: Let D be an independent set in $V(AG_n)$ $(n \geq 4)$ with $|D| = 4$, then $|N(D)| \geq 8n - 24$ and the minimum neighborhood for D in AG_n is $\theta_{AG_n}(4) = 8n - 24$.

Proof: Let $D = \{u, v, x, y\}$. We divide it into the following four cases depending on the distribution of D.

Case 1 (D Is in a Subgraph (see Fig. 8)): By the symmetry of AG_n, assume that $D \in V(AG_n^1)$. We will prove the result by mathematical induction. First, the result holds for $n = 4$. Assume that the result holds for $n - 1$. By inductive assumption, $|N_{AG_n^1}(D)| \geq 8(n-1) - 24 = 8n - 32$. By Lemma 1 (2), any one of D has two outside neighbors outside subgraph AG_n^1 and these two neighbors are in different subgraphs. Hence, $|N_{AG_n^{<n>-\{1\}}}(D)| \geq 2 \times 4 = 8$. Therefore, $|N(D)| = |N_{AG_n^1}(D)| + |N_{AG_n^{<n>-\{1\}}}(D)| \geq 8n - 32 + 8 = 8n - 24$.

Case 2 (D Is Distributed in Two Distinct Subgraphs):

Case 2.1 (By the Symmetry of AG_n, Assume That $\{u, v, x\} \subseteq V(AG_n^n)$ and $y \in V(AG_n^1)$): By Lemma 1 (1),

$$\begin{aligned}|N_{AG_n^n}(u)| &= |N_{AG_n^n}(v)|\\ &= |N_{AG_n^n}(x)|\\ &= 2(n-1) - 4\\ &= 2n - 6.\end{aligned}$$

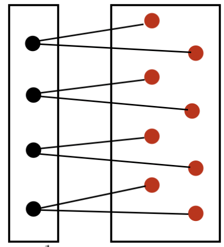


FIGURE 8. An illustration of outside neighbors of D distributed in an $(n-1)$-subgraph with $|D| = 4$.

and $|N_{AG_n^1}(y)| \geq 2(n-1) - 4 = 2n - 6$.

(1) Any two of $\{u, v, x\}$ have no common neighbors (see Fig. 9 (1)).

Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 3(2n-6) = 6n - 18$. By Lemma 1 (2), the six outside neighbors of u, v, x are in at least two different subgraphs. Thus, there exist at most three outside neighbors of u, v, x in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 3$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)|\\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)|\\ &\geq 6n - 18 + 2n - 6 + 3\\ &= 8n - 21.\end{aligned}$$

(2) There exists one common neighbor between two vertices of u, v, x (see Fig. 9 (2)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 1$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 2(2n-7) + (2n-6) + 1 = 6n - 19$. By Lemma 1 (2), the six outside neighbors of u, v, x are in at least two different subgraphs. Thus, there exist at most three outside neighbors of u, v, x in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 3$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)|\\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)|\\ &\geq 6n - 19 + 2n - 6 + 3\\ &= 8n - 22.\end{aligned}$$

(3) There exist two common neighbors between two vertices of u, v, x (see Fig. 9 (3)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 2$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 2(2n-8) + (2n-6) + 2 = 6n - 20$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. By Lemma 1 (2), x has at most one outside neighbor in AG_n^1. Moreover, the other three outside neighbors of u, v are different from the other outside neighbor of x. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 4$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \\ &\geq 6n - 20 + 2n - 6 + 4 \\ &= 8n - 22.\end{aligned}$$

(4) There both exists one common neighbor in two pairs of vertices of u, v, x (see Fig. 9 (4)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 1$ and $|N(v) \cap N(x)| = 1$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 2(2n - 7) + (2n - 8) + 2 = 6n - 20$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Moreover, the four outside neighbors of v, x are in three different subgraphs. Thus, there exist at least two outside neighbors of u, v outside subgraph AG_n^1 and there also exist at least two outside neighbors of v, x outside subgraph AG_n^1. By Lemma 1 (2), the three outside neighbors of u, v, x are different. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 3$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \\ &\geq 6n - 20 + 2n - 6 + 3 \\ &= 8n - 23.\end{aligned}$$

(5) There all exists one common neighbor in three pairs of vertices of u, v, x (see Fig. 9 (5)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 1$, $|N(v) \cap N(x)| = 1$ and $|N(u) \cap N(x)| = 1$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 3(2n - 8) + 3 = 6n - 21$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Moreover, the four outside neighbors of v, x are in three different subgraphs. Furthermore, the four outside neighbors of u, x are in three different subgraphs. Thus, there exist at least three outside neighbors of u, v, x outside subgraph AG_n^1 and these neighbors are different by Lemma 1 (2). Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 3$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \\ &\geq 6n - 21 + 2n - 6 + 3 \\ &= 8n - 24.\end{aligned}$$

(6) There exist two common neighbors in one pair and one common neighbor in another pair of vertices of u, v, x (see Fig. 9 (6)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 2$ and $|N(v) \cap N(x)| = 1$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq (2n - 8) + (2n - 9) + (2n - 7) + 3 = 6n - 21$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. By Lemma 1 (2), x has at most one outside neighbor in AG_n^1. Moreover, by Lemma 1 (2), the other three outside neighbors of u, v are different from the other outside

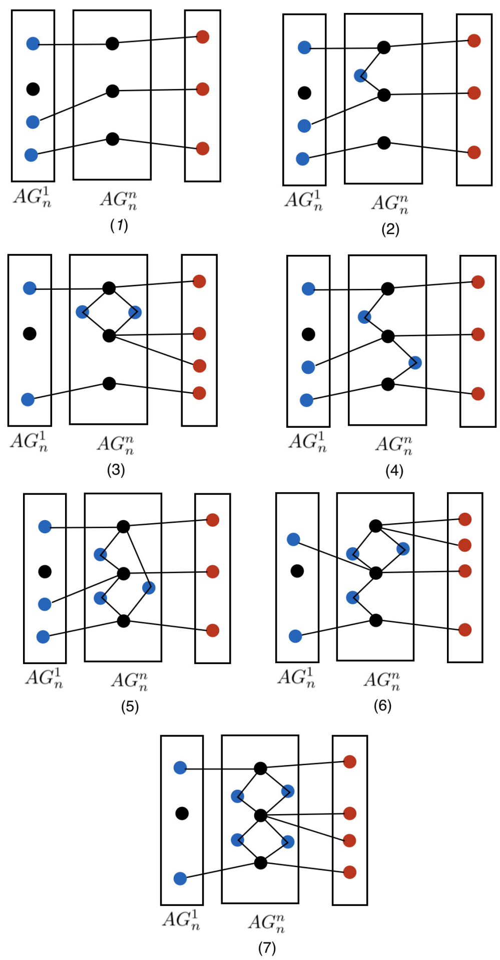


FIGURE 9. Illustrations of outside neighbors of D distributed in two distinct subgraphs with $|D| = 4$.

neighbor of x. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 4$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \\ &\geq 6n - 21 + 2n - 6 + 4 \\ &= 8n - 23.\end{aligned}$$

(7) There both exist two common neighbors in two pairs of vertices of u, v, x (see Fig. 9 (7)).

By the symmetry of AG_n, assume that $|N(u) \cap N(v)| = 2$ and $|N(v) \cap N(x)| = 2$. Hence, $|N_{AG_n^n}(\{u, v, x\})| \geq 2(2n - 8) + (2n - 10) + 4 = 6n - 22$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. By Lemma 1 (2), x exists at most one outside neighbor in AG_n^1. Moreover, by Lemma 1 (2), the other three outside neighbors of u, v are different from the other outside neighbor of x. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)| \geq 4$. Therefore,

$$\begin{aligned}|N(D)| &= |N_{AG_n^n}(\{u, v, x\})| + |N_{AG_n^1}(w)| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u, v, x\}) - N_{AG_n^1}(w)|\end{aligned}$$

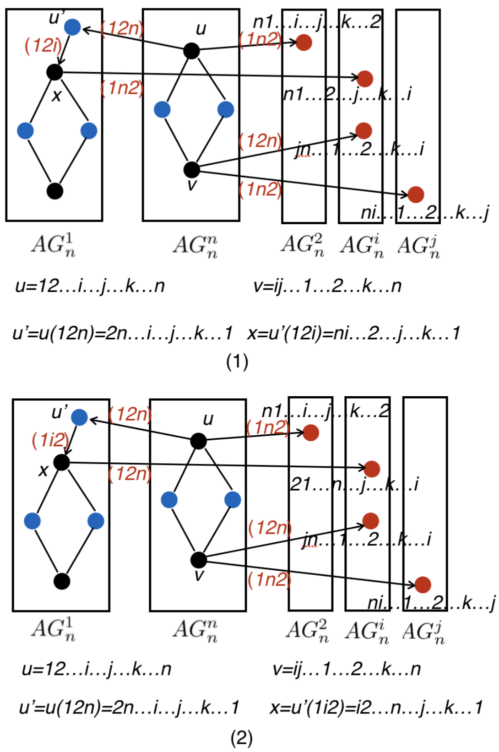


FIGURE 10. Illustrations of outside neighbors of $x = u'(12i)$ or $x = u'(1i2)$.

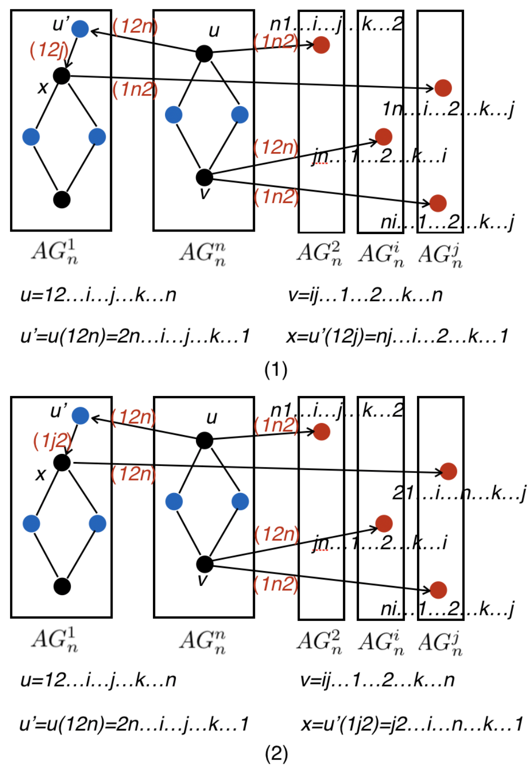


FIGURE 11. Illustrations of outside neighbors of $x = u'(12j)$ or $x = u'(1j2)$.

$$\geq 6n - 22 + 2n - 6 + 4$$
$$= 8n - 24.$$

(8) There exist in total at least five common neighbors in three pairs of vertices of u, v, x.

According to the process of proof of Case 3 in Theorem 3, the situation does not exist.

Case 2.2 (By the Symmetry of AG_n, Assume That $\{u, v\} \subseteq V(AG_n^n)$ and $x, y \in V(AG_n^1)$): By Lemma 1 (1), $|N_{AG_n^n}(u)| = |N_{AG_n^n}(v)| = 2(n-1) - 4 = 2n - 6$ and $|N_{AG_n^1}(x)| = |N_{AG_n^1}(y)| \geq 2(n-1) - 4 = 2n - 6$.

(1) $|N(u) \cap N(v)| = 2$ **and** $|N(x) \cap N(y)| = 2$.

Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-8) + 2 = 4n - 14$ and $|N_{AG_n^1}(\{x, y\})| \geq 2(2n-8) + 2 = 4n - 14$. By Lemma 1 (2), the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. Moreover, the four outside neighbors of x, y are in four different subgraphs and in which there exists at most one outside neighbor of u, v in subgraph AG_n^1. If one of four outside neighbors u, v is in $N_{AG_n^1}(\{x, y\})$, say u', then assume that $u = 12\cdots i\cdots j\cdots k\cdots n$ $(i \neq j)$ and $v = ij\cdots 1\cdots 2\cdots n$. By Lemma 1 (2), $u' = u(12n) = 2n\cdots i\cdots j\cdots k\cdots 1$ and two outside neighbors of v are $jn\cdots 1\cdots 2\cdots k\cdots i$ and $ni\cdots 1\cdots 2\cdots k\cdots j$.

Assume that $x = u'(12i)$ (see Fig. 10 (1)). We have $x = ni\cdots 2\cdots j\cdots k\cdots 1$. Hence, x has one outside neighbor $1n\cdots 2\cdots j\cdots k\cdots i$. By the fact that $jn\cdots 1\cdots 2\cdots k\cdots i \neq 1n\cdots 2\cdots j\cdots k\cdots i$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Assume that $x = u'(1i2)$ (see Fig. 10 (2)). We have $x = i2\cdots n\cdots j\cdots k\cdots 1$. Hence, x has one outside neighbor $21\cdots n\cdots j\cdots k\cdots i$. By the fact that $jn\cdots 1\cdots 2\cdots k\cdots i \neq 21\cdots n\cdots j\cdots k\cdots i$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Assume that $x = u'(12j)$ (see Fig. 11 (1)). We have $x = nj\cdots i\cdots 2\cdots k\cdots 1$. Hence, x has one outside neighbor $1n\cdots i\cdots 2\cdots k\cdots j$. By the fact that $ni\cdots 1\cdots 2\cdots k\cdots j \neq 1n\cdots i\cdots 2\cdots k\cdots j$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Assume that $x = u'(1j2)$ (see Fig. 11 (2)). We have $x = j2\cdots i\cdots n\cdots k\cdots 1$. Hence, x has one outside neighbor $21\cdots i\cdots n\cdots k\cdots j$. By the fact that $ni\cdots 1\cdots 2\cdots k\cdots j \neq 21\cdots i\cdots n\cdots k\cdots j$, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Assume that $x = u'(12k)$ $(k \neq i$ and $k \neq j)$ (see Fig. 12 (1)). We have $x = nk\cdots i\cdots j\cdots 2\cdots 1$. Hence, x has one outside neighbor AG_n^k. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Assume that $x = u'(1k2)$ $(k \neq i$ and $k \neq j)$ (see Fig. 12 (2)). We have $x = k2\cdots i\cdots j\cdots n\cdots 1$. Hence, x has one outside neighbor AG_n^k. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})| \geq 4$.

Therefore,

$$|N(D)| = |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(\{x, y\})| + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \cup N_{AG_n^{<n>-\{1\}}}(\{x, y\})|$$

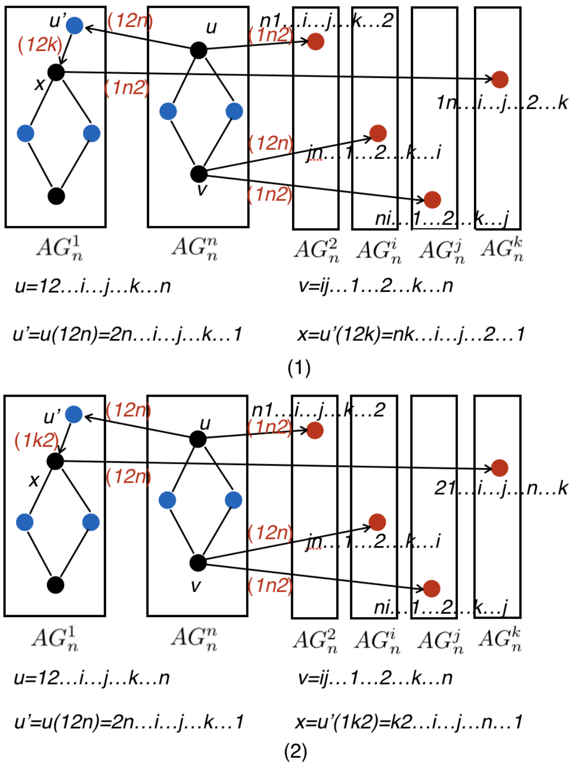


FIGURE 12. Illustrations of outside neighbors of $x = u'(12k)$ or $x = u'(1k2)$ ($k \neq i$ and $k \neq j$).

$$\begin{aligned} &\geq 2(4n-14)+4 \\ &= 8n-24. \end{aligned}$$

Otherwise, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 4$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq 2(4n-14)+4 \\ &= 8n-24. \end{aligned}$$

(2) $|N(u) \cap N(v)| = 2$ **and** $|N(x) \cap N(y)| = 1$ **(see Fig. 13 (1)).**

Hence, $|N_{AG_n^n}(\{u,v\})| \geq 2(2n-8)+2 = 4n-14$ and $|N_{AG_n^1}(\{x,y\})| \geq 2(2n-7)+1 = 4n-13$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 3$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq (4n-14)+(4n-13)+3 \\ &= 8n-24. \end{aligned}$$

(3) $|N(u) \cap N(v)| = 2$ **and** $|N(x) \cap N(y)| = 0$ **(see Fig. 13 (2)).**

Hence, $|N_{AG_n^n}(\{u,v\})| \geq 2(2n-8)+2 = 4n-14$ and $|N_{AG_n^1}(\{x,y\})| \geq 2(2n-6) = 4n-12$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exists at most one outside neighbor of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 3$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq (4n-14)+(4n-12)+3 \\ &= 8n-23. \end{aligned}$$

(4) $|N(u) \cap N(v)| = 1$ **and** $|N(x) \cap N(y)| = 1$ **(see Fig. 13 (3)).**

Hence, $|N_{AG_n^n}(\{u,v\})| \geq 2(2n-7)+1 = 4n-13$ and $|N_{AG_n^1}(\{x,y\})| \geq 2(2n-7)+1 = 4n-13$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 2$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq 2(4n-13)+2 \\ &= 8n-24. \end{aligned}$$

(5) $|N(u) \cap N(v)| = 1$ **and** $|N(x) \cap N(y)| = 0$ **(see Fig. 13 (4)).**

Hence, $|N_{AG_n^n}(\{u,v\})| \geq 2(2n-7)+1 = 4n-13$ and $|N_{AG_n^1}(\{x,y\})| \geq 2(2n-6) = 4n-12$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 2$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq (4n-13)+(4n-12)+2 \\ &= 8n-23. \end{aligned}$$

(6) $|N(u) \cap N(v)| = 0$ **and** $|N(x) \cap N(y)| = 0$**.**

Hence, $|N_{AG_n^n}(\{u,v\})| \geq 2(2n-6) = 4n-12$ and $|N_{AG_n^1}(\{x,y\})| \geq 2(2n-6) = 4n-12$. By Lemma 1 (2), the four outside neighbors of u, v are in at least two different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraph AG_n^1. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \geq 2$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u,v\})| + |N_{AG_n^1}(\{x,y\})| \\ &\quad + |N_{AG_n^{<n>-\{n\}}}(\{u,v\}) - N_{AG_n^1}(\{x,y\})| \\ &\geq 2(4n-12)+2 \\ &= 8n-22. \end{aligned}$$

Case 3 (D Is Distributed in Three Distinct Subgraphs): By the symmetry of AG_n, assume that $u, v \in V(AG_n^n)$,

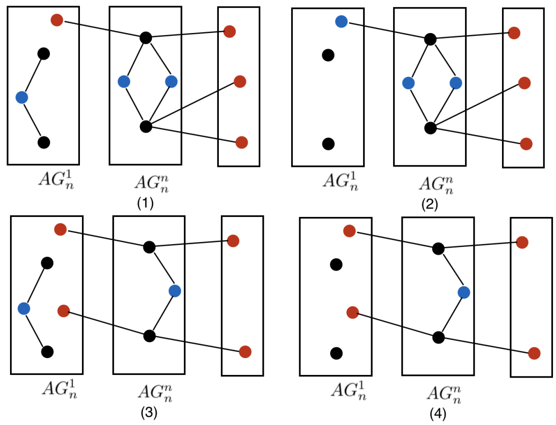


FIGURE 13. **Illustrations of Theorem 4 Case 2: (2)-(5).**

$x \in V(AG_n^1)$ and $y \in V(AG_n^2)$. By Lemma 1 (1), $|N_{AG_n^n}(u)| = |N_{AG_n^n}(v)| = |N_{AG_n^1}(x)| = |N_{AG_n^2}(y)| = 2(n-1)-4 = 2n-6$.

(1) $|N(u) \cap N(v)| = 2$ **(see Fig. 14 (1)).**

Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-8)+2 = 4n-14$. By Theorem 1, the four outside neighbors of u, v are in four different subgraphs. Thus, there exist at most two outside neighbors of u, v in subgraphs AG_n^1 and AG_n^2, respectively. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(x) - N_{AG_n^2}(y)| \geq 2$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(x)| \\ &\quad + |N_{AG_n^2}(y)| + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \\ &\quad - N_{AG_n^1}(x) - N_{AG_n^2}(y)| \\ &\geq (4n-14) + 2(2n-6) + 2 \\ &= 8n-24. \end{aligned}$$

(2) $|N(u) \cap N(v)| = 1$ **(see Fig. 14 (2)).**

Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-7)+1 = 4n-13$. By Theorem 2, the four outside neighbors of u, v are in three different subgraphs. Thus, there exists at least one outside neighbor of u, v outside in subgraphs AG_n^1 and AG_n^2, respectively. Hence, $|N_{AG_n^{<n>-\{n\}}}(\{u, v\}) - N_{AG_n^1}(x) - N_{AG_n^2}(y)| \geq 1$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(x)| \\ &\quad + |N_{AG_n^2}(y)| + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \\ &\quad - N_{AG_n^1}(x) - N_{AG_n^2}(y)| \\ &\geq (4n-13) + 2(2n-6) + 1 \\ &= 8n-24. \end{aligned}$$

(3) $|N(u) \cap N(v)| = 0$**.**

Hence, $|N_{AG_n^n}(\{u, v\})| \geq 2(2n-6) = 4n-12$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(\{u, v\})| + |N_{AG_n^1}(x)| \\ &\quad + |N_{AG_n^2}(y)| + |N_{AG_n^{<n>-\{n\}}}(\{u, v\}) \\ &\quad - N_{AG_n^1}(x) - N_{AG_n^2}(y)| \\ &\geq (4n-12) + 2(2n-6) + 0 \\ &= 8n-24. \end{aligned}$$

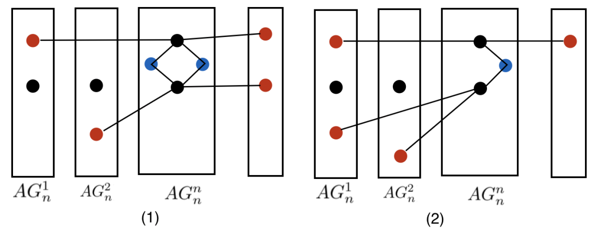


FIGURE 14. **Illustrations of Theorem 4 Case 3: (1)-(2).**

TABLE 3. **Minimum neighborhood of an independent set of four vertices in AG_4.**

Four independent vertices	Common neighbors	Neighborhood	No. of neighbors
1234, 2143, 3412, 4321	2314, 3124, 4132, 2431, 4213, 1423, 1342, 3241	2314, 3124, 4132, 2431, 4213, 1423, 1342, 3241	8
1234, 1423 3241, 3412	3124, 2143, 4321, 2431, 2314, 4213, 1342, 4132	3124, 2143, 4321, 2431, 2314, 4213, 1342, 4132	8

Case 4 (D Is Distributed in Four Distinct Subgraphs): By the symmetry of AG_n, assume that $u \in V(AG_n^n)$, $v \in V(AG_n^1)$, $x \in V(AG_n^2)$ and $y \in V(AG_n^3)$.

By Lemma 1 (1), $|N_{AG_n^n}(u)| = 2(n-1)-4 = 2n-6$, $|N_{AG_n^1}(v)| = 2(n-1)-4 = 2n-6$, $|N_{AG_n^2}(x)| = 2(n-1)-4 = 2n-6$ and $|N_{AG_n^3}(y)| = 2(n-1)-4 = 2n-6$. Therefore,

$$\begin{aligned} |N(D)| &= |N_{AG_n^n}(u)| + |N_{AG_n^1}(v)| \\ &\quad + |N_{AG_n^2}(x)| + |N_{AG_n^3}(y)| \\ &\geq 4(2n-6) \\ &= 8n-24. \end{aligned}$$

From the above, $|N(D)| \geq 8n-24$ when $|D| = 4$ for $n \geq 4$. ■

We give one example in AG_4 to explain Theorem 4 as follows (see Table 3).

V. CONCLUSION

In this paper, we obtain $\theta_{AG_n}(q)$ for an independent set with size q in AG_n. We first propose some combinatorial properties of AG_n. Then we study the minimum neighborhood problem for an independent set of two vertices and obtain that $\theta_{AG_n}(2) = 4n-10$. Next, we prove that $\theta_{AG_n}(3) = 6n-16$. Finally, we propose that $\theta_{AG_n}(4) = 8n-24$. Meanwhile, we present sufficient amount of figures to better illustrate the process of the proofs.

We will further applied the minimum neighborhood of AG_n to obtain the fault tolerance of the n-split-star network [18] in the future. Meanwhile, the minimum neighborhood can be applied to obtain all kinds of conditional fault tolerance and a variety of conditional diagnosability. Furthermore, the minimum neighborhood will be applied to solve the privacy protection and the optimization of various networks.

REFERENCES

[1] A. Caruso, S. Chessa, P. Maestrini, and P. Santi, "Diagnosability of regular systems," *J. Algorithms*, vol. 45, no. 2, pp. 126–143, 2002.

[2] J.-M. Chang and J.-S. Yang, "Fault-tolerant cycle-embedding in alternating group graphs," *Appl. Math. Comput.*, vol. 197, no. 2, pp. 760–767, 2008.

[3] J.-M. Chang, J.-S. Yang, Y.-L. Wang, and Y. Cheng, "Panconnectivity, fault-tolerant hamiltonicity and Hamiltonian-connectivity in alternating group graphs," *Networks*, vol. 44, no. 4, pp. 302–310, 2004.

[4] G. Chartrand, S. F. Kapoor, L. Lesniak, and D. R. Lick, "Generalized connectivity in graphs," *Bull. Bombay Math. Colloq.*, vol. 2, no. 1, pp. 1–6, 1984.

[5] E. Cheng, L. Lipták, and F. Sala, "Linearly many faults in 2-tree-generated networks," *Netw., Int. J.*, vol. 55, no. 2, pp. 90–98, 2010.

[6] J. Fan and X. Lin, "The t/k-diagnosability of the BC graphs," *IEEE Trans. Comput.*, vol. 54, no. 2, pp. 176–184, Feb. 2005.

[7] J. Fàbrega and M. A. Fiol, "On the extraconnectivity of graphs," *Discrete Math.*, vol. 155, nos. 1–3, pp. 49–57, 1996.

[8] M. Fiedler, "Algebraic connectivity of graphs," *Czechoslovak Math. J.*, vol. 23, no. 2, pp. 298–305, 1973.

[9] R.-X. Hao, Y.-Q. Feng, and J.-X. Zhou, "Conditional diagnosability of alternating group graphs," *IEEE Trans. Comput.*, vol. 62, no. 4, pp. 827–831, Apr. 2013.

[10] H.-C. Hsu, T.-K. Li, J. J. M. Tan, and L.-H. Hsu, "Fault hamiltonicity and fault Hamiltonian connectivity of the arrangement graphs," *IEEE Trans. Comput.*, vol. 53, no. 1, pp. 39–53, Jan. 2004.

[11] K. Huang, V. K. Agarwal, and K. Thulasiraman, "Diagnosis of clustered faults and wafer testing," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 17, no. 2, pp. 136–148, Feb. 1998.

[12] J.-S. Jwo, S. Lakshmivarahan, and S. K. Dhall, "A new class of interconnection networks based on the alternating group," *Networks*, vol. 23, no. 4, pp. 315–326, 1993.

[13] P.-L. Lai, J. J. M. Tan, C.-P. Chang, and L.-H. Hsu, "Conditional diagnosability measures for large multiprocessor systems," *IEEE Trans. Comput.*, vol. 54, no. 2, pp. 165–175, Feb. 2005.

[14] P.-L. Lai, J. J. M. Tan, C.-H. Tsai, and L.-H. Hsu, "The diagnosability of the matching composition network under the comparison diagnosis model," *IEEE Trans. Comput.*, vol. 53, no. 8, pp. 1064–1069, Aug. 2004.

[15] L. Lin, S.-Y. Hsieh, R. Chen, L. Xu, and C.-W. Lee, "The relationship between g-restricted connectivity and g-good-neighbor fault diagnosability of general regular networks," *IEEE Trans. Rel.*, vol. 67, no. 1, pp. 285–296, Mar. 2018.

[16] L. Lin, S.-Y. Hsieh, L. Xu, S. Zhou, and R. Chen, "The relationship between extra connectivity and conditional diagnosability of regular graphs under the PMC model," *J. Comput. Syst. Sci.*, vol. 95, pp. 1–18, Aug. 2018.

[17] C.-K. Lin, T.-L. Kung, S.-L. Peng, J. J. M. Tan, and L.-H. Hsu, "The diagnosability of g-good-neighbor conditional-faulty hypercube under the PMC model," in *Proc. PDPTA*, 2010, pp. 494–499.

[18] L. Lin, L. Xu, and S. Zhou, "Conditional diagnosability and strong diagnosability of shuffle-cubes under the comparison model," *Int. J. Comput. Math.*, vol. 92, no. 2, pp. 230–249, 2015.

[19] L. Lin, L. Xu, D. Wang, and S. Zhou, "The g-good-neighbor conditional diagnosability of arrangement graphs," *IEEE Trans. Dependable Secure Comput.*, vol. 15, no. 3, pp. 542–548, May 2018.

[20] L. Lin, L. Xu, and S. Zhou, "Conditional diagnosability and strong diagnosability of split-star networks under the PMC model," *Theor. Comput. Sci.*, vol. 562, pp. 565–580, Jan. 2015.

[21] L. Lin, L. Xu, and S. Zhou, "Relating the extra connectivity and the conditional diagnosability of regular graphs under the comparison model," *Theor. Comput. Sci.*, vol. 618, pp. 21–29, Mar. 2016.

[22] L. Lin, L. Xu, S. Zhou, and S.-Y. Hsieh, "The t/k-diagnosability for regular networks," *IEEE Trans. Comput.*, vol. 65, no. 10, pp. 3157–3170, Oct. 2016.

[23] C.-K. Lin, L. Zhang, J. Fan, and D. Wang, "Structure connectivity and substructure connectivity of hypercubes," *Theor. Comput. Sci.*, vol. 634, pp. 97–107, Jun. 2016.

[24] L. Lin, S. Zhou, L. Xu, and D. Wang, "The extra connectivity and conditional diagnosability of alternating group networks," *IEEE Trans. Parallel Distrib. Syst.*, vol. 26, no. 8, pp. 2352–2362, Aug. 2015.

[25] A. K. Somani and O. Peleg, "On diagnosability of large fault sets in regular topology-based computer systems," *IEEE Trans. Comput.*, vol. 45, no. 8, pp. 892–903, Aug. 1996.

[26] A. Szepietowski, "Fault tolerance of vertex pancyclicity in alternating group graphs," *Appl. Math. Comput.*, vol. 217, no. 16, pp. 6785–6791, 2011.

[27] P.-Y. Tsai, "A note on an optimal result on fault-tolerant cycle-embedding in alternating group graphs," *Inf. Process. Lett.*, vol. 111, no. 8, pp. 375–378, 2011.

[28] P.-Y. Tsai, J.-S. Fu, and G.-H. Chen, "Fault-free longest paths in star networks with conditional link faults," *Theor. Comput. Sci.*, vol. 410, nos. 8–10, pp. 766–775, 2009.

[29] M. Wan and Z. Zhang, "A kind of conditional vertex connectivity of star graphs," *Appl. Math. Lett.*, vol. 22, no. 2, pp. 264–267, 2009.

[30] X. Yang, J. Cao, G. M. Megson, and J. Luo, "Minimum neighborhood in a generalized cube," *Inf. Process. Lett.*, vol. 97, no. 3, pp. 88–93, 2006.

[31] W. Yang and J. Meng, "Extraconnectivity of hypercubes," *Appl. Math. Lett.*, vol. 22, no. 6, pp. 887–891, 2009.

[32] L. Xu, L. Lin, S. Zhou, and S.-Y. Hsieh, "The extra connectivity, extra conditional diagnosability, and t/m-diagnosability of arrangement graphs," *IEEE Trans. Rel.*, vol. 65, no. 3, pp. 1248–1262, Sep. 2016.

[33] Z.-J. Xue and S.-Y. Liu, "An optimal result on fault-tolerant cycle-embedding in alternating group graphs," *Inf. Process. Lett.*, vol. 109, nos. 21–22, pp. 1197–1201, 2009.

[34] X. Yang, D. J. Evans, B. Chen, G. M. Megson, and H. Lai, "On the maximal connected component of hypercube with faulty vertices," *Int. J. Comput. Math.*, vol. 81, pp. 515–525, 2004.

[35] X. Yang, D. J. Evans, and G. M. Megson, "On the maximal connected component of hypercube with faulty vertices (II)," *Int. J. Comput. Math.*, vol. 81, no. 10, pp. 1175–1185, 2004.

[36] X. Yang, D. J. Evans, and G. M. Megson, "On the maximal connected component of a hypercube with faulty vertices III," *Int. J. Comput. Math.*, vol. 83, no. 1, pp. 27–37, 2006.

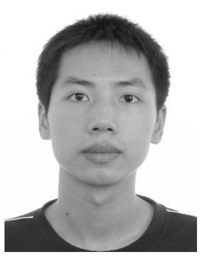
[37] Z. Zhang, W. Xiong, and W. Yang, "A kind of conditional fault tolerance of alternating group graphs," *Inf. Process. Lett.*, vol. 110, no. 22, pp. 998–1002, 2010.

[38] S. Zhang and W. Yang, "The g-extra conditional diagnosability and sequential t/k-diagnosability of hypercubes," *Int. J. Comput. Math.*, vol. 93, no. 3, pp. 482–497, Mar. 2016.

[39] S. Zhao, W. Yang, and S. Zhang, "Component connectivity of hypercubes," *Theor. Comput. Sci.*, vol. 640, pp. 115–118, Aug. 2016.

[40] J. Zheng, S. Latifi, E. Regentova, K. Luo, and X. Wu, "Diagnosability of star graphs under the comparison diagnosis model," *Inf. Process. Lett.*, vol. 93, no. 1, pp. 29–36, Jan. 2005.

[41] S. Zheng and S. Zhou, "Diagnosability of the incomplete star graphs," *Tsinghua Sci. Technol.*, vol. 177, no. 8, pp. 1771–1781, Apr. 2007.



YANZE HUANG received the B.S. and M.S. degrees from Xiamen University, in 2008 and 2011, respectively, and the Ph.D. degree from The Hong Kong University of Science and Technology, in 2015, all in mathematics. Since 2015, he has been a Faculty Member with the School of Mathematics and Physics, Fujian University of Technology, where he is also a member of the Fujian Provincial Key Laboratory of Big Data Mining and Applications, Fujian University of Technology. He has published several papers in *Information Sciences*, *Theoretical Computer Science*, and *The Computer Journal*. His research interests include graph theory, fault diagnosis, fault-tolerant computing, network security, network reliability, and representation theory.

LIMEI LIN received the M.S. and Ph.D. degrees from the College of Mathematics and Informatics, Fujian Normal University, in 2013 and 2016, respectively, and the Postdoctoral degree from Fujian Agriculture and Forestry University, in 2018. She was a Visiting Scholar with Montclair State University, in 2015. She is currently an Associate Professor with the College of Mathematics and Informatics, Fujian Normal University. Her research interests include interconnection networks, graph theory, mobile social networks, network security, fault diagnosis, fault-tolerant computing, combinatorial networks, and bioinformatics. She has published over 30 papers in these areas. She has also published over 10 papers in IEEE journals, including the IEEE TRANSACTIONS ON COMPUTERS, the IEEE TRANSACTIONS ON PARALLEL AND DISTRIBUTED SYSTEMS, the IEEE TRANSACTIONS ON DEPENDABLE AND SECURE COMPUTING, and the IEEE TRANSACTIONS ON RELIABILITY.

DAJIN WANG received the B.Eng. degree in computer engineering from the Shanghai University of Science and Technology, in 1982, and the Ph.D. degree in computer science from the Stevens Institute of Technology, in 1990. Since 1990, he has been with the Department of Computer Science, Montclair State University, Montclair, NJ, USA, where he is currently a Professor. He has held visiting positions in other universities, and has consulted in industry. His main research interests include interconnection networks, fault-tolerant computing, algorithmic robotics, parallel processing, and wireless ad hoc and sensor networks. He has published over 60 papers in these areas. He received several university awards for his scholarly accomplishments. He was an Associate Editor of the IEEE TRANSACTIONS ON PARALLEL AND DISTRIBUTED SYSTEMS, from 2010 to 2014.

LI XU received the B.S. and M.S. degrees from Fujian Normal University, Fuzhou, China, in 1992 and 2001, respectively, and the Ph.D. degree from the Nanjing University of Posts and Telecommunications, Nanjing, China, in 2004. He is currently a Professor and a Doctoral Supervisor with the College of Mathematics and Informatics, Fujian Normal University, where he is also the Director of the Information Construction and Management Office, Key Laboratory of Network Security and Cryptography. He has authored or co-authored over 150 papers in international journals and conferences, including the IEEE TRANSACTIONS ON COMPUTERS, the *ACM Transactions on Sensor Network*, the IEEE TRANSACTIONS ON RELIABILITY, the IEEE TRANSACTIONS ON PARALLEL AND DISTRIBUTED SYSTEMS, *Information Sciences*, and *Computer Network*. His current research interests include network and information security, wireless networks and communication, complex networks and systems, and intelligent information in communication networks. He is a member of ACM and a Senior Member of CCF and CIE, China. He has been invited to act as the PC Chair or a member at more than 30 international conferences.