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A Noise-Acceptable ZNN for Computing Complex-Valued Time-Dependent Matrix Pseudoinverse

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ABSTRACT The issue of complex-valued time-dependent pseudoinverse often exists in science and engineering fields. In the existing studies, many models were presented for solving complex-valued time-dependent pseudoinverse in the noiseless environments. However, the appearance of noise is unavoidable in practice. In this paper, a novel noise-acceptable zeroing neural network (NAZNN) model is first proposed for computing complex-valued time-dependent matrix pseudoinverse with different noise situations. For comparison, the traditional zeroing neural network and the gradient neural network are adopted to complete the same task. Theoretical analyses prove that the proposed NAZNN model obtains the global exponential convergence performance. Moreover, the proposed NAZNN is also proven to obtain strong resistance to various sorts of noise. Finally, the results of numerical experiments further substantiate the theoretical analysis and indicate the effectiveness and superiority of the proposed NAZNN model for computing complex-valued time-dependent matrix pseudoinverse in various kinds of noise.

INDEX TERMS Zeroing neural network, time-dependent, complex-valued, noise-acceptable, matrix pseudoinverse.

I. INTRODUCTION

Pseudoinverse is extended from the inverse of matrices. It is also known as the Moore-Penrose inverse, generalized from a rectangular or a singular full rank matrix. As an essential foundation of solution, pseudoinverse appears frequently in various mathematical and engineering fields. Its application also can be found in optimization [1], robotics [2], [3], image restoration [4], [5]. In view of its importance, many researchers put forward a lot of algorithms [6]–[11]. For instance, Perković and Stanimirović [6] showed that an iterated algorithm can be employed to solve the Moore-Penrose generalized inverse. Wei *et al.* [10] put forward Newton iteration to solve the Moore-Penrose generalized inverse of Toeplitz matrix. Hoyle [8] utilized pseudoinverse to improve the accuracy of classifiers. Based on the famous Greville formula, Zhou *et al.* [11] raised an order-recursive formula for computing the pseudoinverse of matrix. However, these existing algorithms are iterative and suitable for static matrix in real-value domain. These iterative algorithms are of serial-distributed processing performance, When applied to solving complex-valued time-dependent issue, which will fail to complete the calculation within every sampling period.

Because of the high-speed parallel-distributed processing performance and the convenient implementation of hardware, neural networks have been used widely in science and engineering areas [12]–[20]. For instance, Zhang *et al.* [12] utilized neural network to obtain near-optimal control of HJB equations; Arima and Hirose classified complex texture by complex-valued neural networks in millimeter-wave active imaging [18]. Thereinto, a lot of recurrent neural networks (RNNs) have been investigated for solution of scientific issues (including matrix pseudoinverse) online [21]–[24]. Zhang designed and investigated an RNN model for time-dependent matrix inversion [21]. The gradient neural network (GNN) is a classic RNN. GNN adopts the

Frobenius norm of error matrix as the performance criterion. It makes the error matrix norm converge to 0 along with the negative gradient direction. The GNN works well for solving the time-independent problems, while generates large errors for solving the time-dependent problems. Zhang *et al*. proposed a novel type of RNN, named zeroing neural network (ZNN), for tracking time-dependent solution [25]–[28]. By utilizing the time derivative of time-dependent parameters, the traditional ZNN (TZNN) tackles the errors of the GNN in solving time-dependent problems, which is seminal in neural network field.

In implementation of RNN, we generally assume that it is used in the noiseless situation. However, noises are inevitably in the practical environment. The noises may be fixable or random. Because of the noises, the accuracy of resolving time-dependent issue is impacted greatly. In some cases, they cause failure in online computing process. In view of this, Xiang *et al* [29] first proposed a noise-resistant neural dynamics for computing time-dependent lyapunov equation in real-domain. However, few literatures investigate complex-valued time-dependent matrix pseudoinverse with different noise. Therefore, it is urgently needed to improve the TZNN for computing complex-valued time-dependent matrix pseudoinverse. The improved neural network should have the ability to resist inherent noises.

The main contributions of this work are given as follows.

- Based on a new design formula, the NAZNN model is proposed for complex-valued time-dependent matrix pseudoinverse with various kinds of noise. Moreover, there is no literature has investigate the neural networks for complex-valued time-dependent matrix pseudoinverse with noise-acceptable performance.
- Theoretical derivations prove that the proposed NAZNN model obtain the global exponential convergence performance. In addition, the proposed NAZNN model has improved performance of anti-noise when it is used to solve complex-valued time-dependent matrix pseudoinverse in different noise situation.
- Numerical experiments demonstrate the efficacy and advantages of the proposed NAZNN model for complex-valued time-dependent matrix pseudoinverse with different noises.

The remaining parts of the paper are composed of the following sections. The relevant formulation and preliminaries are put forward in Section II. In Section III, a noise-acceptable ZNN (NAZNN) model is first proposed. For comparison, the GNN and the TZNN are also adopted and generalized to solve the same time-dependent problem in noise environment in this section. In Section IV, theoretical analyses are presented to demonstrate the convergence property of the NAZNN model in the environment of different types of noise in complex domain. In Section V, numerical examples to further illustrate the efficiency of the proposed NAZNN model for computing the complex-valued time-dependent matrix pseudoinversion. Section VI concludes this work.

II. FORMULATIONS AND PRELIMINARIES

In this section, the formulations and some important preliminaries of the complex-valued time-dependent matrix pseudoinverse are presented.

Definition 1 [30], [31]: Given a time-dependent complex-valued matrix $V(t) \in \mathbb{C}^{m\times n}$, if $X(t) \in \mathbb{C}^{n\times m}$ satisfies all the Penrose equations as following:

$$V(t)X(t)V(t) = V(t), \quad X(t)V(t)X(t) = X(t),$$
$$(V(t)X(t))^{\mathrm{H}} = V(t)X(t), \quad (X(t)V(t))^{\mathrm{H}} = X(t)V(t),$$

where H denotes the Hermitian transpose of a complex-valued matrix, $X(t)$ is termed the complex-valued pseudoinverse of $V(t)$, denoted with $V^{+}(t)$. It is worth noting that the pseudoinverse $V^{+}(t)$ exists and is unique.

As we know, if the matrix $V(t) \in \mathbb{C}^{m\times n}$ is a full rank matrix at any moment, i.e., $rank(V(t)) = \min\{m, n\}$, we can obtain the pseudoinverse of $V(t)$ from the following lemma.

Lemma 1 [30], [31]: For a time-dependent matrix $V(t) \in \mathbb{C}^{m\times n}$, if $rank(V(t)) = \min\{m, n\}$, then the unique pseudoinverse $V^{+}(t)$ can be given as:

$$V^{+}(t) = \begin{cases} (V^{\mathrm{H}}(t)V(t))^{-1}V^{\mathrm{H}}(t), & \textit{if} \;\; m > n, \\ V^{-1}(t), & \textit{if} \;\; m = n, \\ V^{\mathrm{H}}(t)(V(t)V^{\mathrm{H}}(t))^{-1}, & \textit{if} \;\; m < n. \end{cases} \tag{1}$$

In this paper, the full rank complex-valued time-dependent matrix $V(t) \in \mathbb{C}^{m\times n}$ (with $m > n$) is considered. For $m \leq n$, the process of computing pseudoinverse is homoplastical to that of $m > n$, and is omitted. Then, we consider the problem of a complex-valued time-dependent matrix pseudoinverse as

$$V^{\mathrm{H}}(t)V(t)X(t) = V^{\mathrm{H}}(t) \in \mathbb{C}^{n\times m}, \tag{2}$$

where $V(t)$ denotes a complex-valued time-dependent singular matrix, $X(t)$ is an unknown matrix. The purpose of our work is to design a ZNN model for solving the unknown matrix $X(t)$ in the noisy situations.

III. DESIGN OF NAZNN MODEL

This section provides the design process of NAZNN model. In addition, the TZNN and the GNN are also introduced. To controlling the process of solving the complex-valued time-dependent matrix pseudoinverse (2), the corresponding error-monitoring function is defined as

$$\Omega(t) = V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t) \in \mathbb{C}^{n\times m}. \tag{3}$$

For forcing $\Omega(t)$ to converge to zero, the evolution of $\Omega(t)$ is presented as [29]:

$$\dot{\Omega}(t) = -\alpha\Omega(t) - \beta\int_0^t \Omega(s)\,\mathrm{d}s, \tag{4}$$

where $\alpha > 0$ and $\beta > 0$ are scaling parameters used to measure the convergence rate of NAZNN model. By expanding the designed formula (4), the novel complex-valued time-dependent NAZNN model can be obtained as follows:

$$\begin{aligned} V^{\mathrm{H}}(t)V(t)\dot{X}(t) &= \dot{V}^{\mathrm{H}}(t) - \big(\dot{V}^{\mathrm{H}}(t)V(t) + V^{\mathrm{H}}(t)\dot{V}(t)\big)X(t) \\ &\quad - \alpha\big(V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\big) \\ &\quad - \beta\int_0^t (V^{\mathrm{H}}(s)V(s)X(s) - V^{\mathrm{H}}(s))\,\mathrm{d}s. \end{aligned} \tag{5}$$

For comparative purpose, the TZNN for computing complex-valued time-dependent matrix pseudoinverse is presented directly as [27]:

$$\begin{aligned} V^{\mathrm{H}}(t)V(t)\dot{X}(t) &= \dot{V}^{\mathrm{H}}(t) - \big(\dot{V}^{\mathrm{H}}(t)V(t) + V^{\mathrm{H}}(t)\dot{V}(t)\big)X(t) \\ &\quad - \alpha\big(V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\big). \end{aligned} \tag{6}$$

The GNN for computing complex-valued time-dependent matrix pseudoinverse is also presented as [32]:

$$\dot{X}(t) = -\alpha V^{\mathrm{H}}(t)V(t)\big(V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\big). \tag{7}$$

In order to further investigate the robustness of the proposed NAZNN model (5) in noisy environment, the proposed NAZNN model (5) with unknown noises is presented as follows:

$$\begin{aligned} V^{\mathrm{H}}(t)V(t)\dot{X}(t) &= \dot{V}^{\mathrm{H}}(t) - \big(\dot{V}^{\mathrm{H}}(t)V(t) + V^{\mathrm{H}}(t)\dot{V}(t)\big)X(t) \\ &\quad - \alpha\big(V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\big) \\ &\quad - \beta\int_0^t (V^{\mathrm{H}}(s)V(s)X(s) - V^{\mathrm{H}}(s))\,\mathrm{d}s + \Psi(t), \end{aligned} \tag{8}$$

where $\Psi(t) \in \mathbb{C}^{n\times m}$ denotes the unpredictable noise. $\Psi(t)$ is represented in matrix-form, the noises fall into three categories: the constant noise, the linear noise, the random noise [29].

It is worth pointing out that the preprocessing of resisting noises will increase additional time, possibly affecting the online computation. The proposed NAZNN model (5) can resist different noises and compute the complex-valued time-dependent matrix pseudoinverse simultaneously. The relevant theoretical derivations and results are presented in the ensuing section.

IV. THEORETICAL DERIVATIONS AND RESULTS

The global and exponential convergence property of complex-valued time-dependent TZNN have been proven in [27]. In this section, the pseudoinverse $X(t)$ generated by the proposed NAZNN model (5) converges to the theoretical pseudoinverse $V^{+}(t)$ globally and exponentially. In addition, the proposed NAZNN model (5) is proven to be robust with uncertainty noises.

A. CONVERGENCE OF NAZNN MODEL

Firstly, two theorems are given to investigate the convergence of NAZNN model (5) in noiseless environment.

Theorem 1: Given a complex-valued time-dependent matrix $V(t) \in \mathbb{C}^{m\times n}$ (with $m > n$) of full rank, the state matrix $X(t)$ generated by NAZNN model (5), from any initial state $X(0)$, globally converges to the theoretical pseudoinverse $V^{+}(t)$, i.e., the solution of (2).

Proof: The formula

$$\dot{\Omega}(t) = -\alpha\Omega(t) - \beta\int_0^t \Omega(s)\,\mathrm{d}s$$

can be rewritten as the following set of $n \times m$ neuron's dynamic equations

$$\dot{\omega}_{ab}(t) = -\alpha\omega_{ab}(t) - \beta\int_0^t \omega_{ab}(s)\,\mathrm{d}s, \quad \forall a \in \{1, 2, \ldots, n\} \text{ and } \forall b \in \{1, 2, \ldots, m\}, \tag{9}$$

where $\omega_{ab}(t) \in \mathbb{C}$ denotes the abth item of $\Omega(t)$. For the equation (9), we define a Lyapunov function as

$$e_{ab}(t) = \omega_{ab}(t)\omega^{*}_{ab}(t) + \beta\int_0^t \omega_{ab}(s)\,\mathrm{d}s\int_0^t \omega^{*}_{ab}(s)\,\mathrm{d}s, \tag{10}$$

where, the symbol $*$ denotes complex-valued conjugate. As we known, $\omega_{ab}(t)$ can be denoted as $x(t) + y(t)i$, $x(t)$, $y(t) \in \mathbb{R}$, for $X(t) + Y(t)i = \int_0^t \omega_{ab}(s)\,\mathrm{d}s$, then $\int_0^t \omega^{*}_{ab}(s)\,\mathrm{d}s = X(t) - Y(t)i$, where $X(t)$, $Y(t) \in \mathbb{R}$, we have $e_{ab}(t) = x^2(t)+y^2(t)+\beta(X^2(t)+Y^2(t))$. Obviously, the Lyapunov function $e_{ab}(t)$ is positive, i.e., $e_{ab}(t) > 0$ for any $\omega_{ab}(t) \neq 0$ or $\int_0^t \omega_{ab}(s)\,\mathrm{d}s \neq 0$, and $e_{ab}(t) = 0$ just for $\omega_{ab}(t) = \int_0^t \omega_{ab}(s)\,\mathrm{d}s = 0$. Based on Eqs.(9) and (10), we obtain

$$\begin{aligned} \dot{e}_{ab}(t) &= \dot{\omega}_{ab}(t)\omega^{*}_{ab}(t) + \omega_{ab}(t)\dot{\omega}^{*}_{ab}(t) \\ &\quad + \beta\omega_{ab}(t)\int_0^t \omega^{*}_{ab}(s)\,\mathrm{d}s + \beta\omega^{*}_{ab}(t)\int_0^t \omega_{ab}(s)\,\mathrm{d}s \\ &= -2\alpha\omega_{ab}(t)\omega^{*}_{ab}(t) \le 0. \end{aligned}$$

Based on the Lyapunov stability theory, we can conclude that the abth item of $\Omega(t)$ globally converges to zero, for $\forall a \in \{1, 2, \ldots, n\}$ and $\forall b \in \{1, 2, \ldots, m\}$. Therefore, the error-monitoring function $\Omega(t)$ globally converges to 0, it means that, the $X(t)$ generated by NAZNN model (5) globally converges the theoretical pseudoinverse. The global convergence property of NAZNN model (5) is proven. ■

For the convergence speed of the NAZNN model (5), we have the following theorem.

Theorem 2: Given a full-rank time-dependent matrix $V(t) \in \mathbb{C}^{m\times n}$ (with $m > n$), the state matrix $X(t) \in \mathbb{C}^{n\times m}$ generated by NAZNN model (5) exponentially converges to the theoretical pseudoinverse $V^{+}(t)$ of (2).

Proof: Let $R(t) = \int_0^t \Omega_{ab}(s)\,ds$, its first-order time derivative and second-order time derivative are

$$\dot{R}(t) = \Omega(t) \text{ and } \ddot{R}(t) = \dot{\Omega}(t),$$

$\omega_{ab}(t)$, $r_{ab}(t)$, $\dot{r}_{ab}(t)$, and $\ddot{r}_{ab}(t)$ are the abth item of $\Omega(t)$, $R(t)$, $\dot{R}(t)$, and $\ddot{R}(t)$ respectively. Therefore, the formula

$$\dot{\Omega}(t) = -\alpha\Omega(t) - \beta\int_0^t \Omega(s)\,\mathrm{d}s$$

can be rewritten as

$$\ddot{R}(t) = -\alpha\dot{R}(t) - \beta R(t). \tag{11}$$

the abth item of (11) can be obtained as

$$\ddot{r}_{ab}(t) = -\alpha\dot{r}_{ab}(t) - \beta r_{ab}(t).$$

Let $\lambda_1 = 0.5(-\alpha+(\alpha^2-4\beta)^{0.5})$, and $\lambda_2 = 0.5(-\alpha+(\alpha^2-4\beta)^{0.5})$. In view of the initial values $r_{ab}(0) = 0$, and $\dot{r}_{ab}(0) = \omega_{ab}(0)$. According to the solution of the above differential equation, The solution to the abth item of Eq.(11) is investigated in below three situations.

1) If $\alpha^2-4\beta > 0$, then $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, we have

$$r_{ab}(t) = \frac{\omega_{ab}(0)(e^{\lambda_1 t} - e^{\lambda_2 t})}{(\alpha^2-4\beta)^{0.5}},$$

and

$$\omega_{ab}(t) = \dot{r}_{ab}(t) = \frac{\omega_{ab}(0)(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t})}{(\alpha^2-4\beta)^{0.5}}.$$

Therefore, the error-monitoring matrix is derived as

$$\Omega(t) = \frac{\Omega(0)(\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t})}{(\alpha^2-4\beta)^{0.5}}.$$

2) If $\alpha^2-4\beta = 0$, then $\lambda_1 = \lambda_2 = -0.5\alpha$, we have

$$r_{ab}(t) = \omega_{ab}(0)te^{-0.5\alpha t},$$

and

$$\omega_{ab}(t) = \dot{r}_{ab}(t) = \omega_{ab}(0)(1-0.5\alpha t)e^{-0.5\alpha t}.$$

Therefore, the error-monitoring matrix is derived as

$$\Omega(t) = \Omega(0)(1-0.5\alpha t)e^{-0.5\alpha t}.$$

3) If $\alpha^2-4\beta < 0$, then $\lambda_1 = \xi + \eta i$, $\lambda_2 = \xi - \eta i$, we have

$$r_{ab}(t) = \frac{\omega_{ab}(0)\sin(\eta t)e^{\xi t}}{\eta},$$

and

$$\omega_{ab}(t) = \omega_{ab}(0)e^{\xi t}(\frac{\xi\sin(\eta t)}{\eta} + \cos(\eta t)).$$

Therefore, the error-monitoring matrix is derived as

$$\Omega(t) = \Omega(0)e^{\xi t}(\frac{\xi\sin(\eta t)}{\eta} + \cos(\eta t)).$$

According to the previous analysis and [33, Proof of Theorem 1], $\Omega(t)$ exponentially converges to 0. It means, beginning from any initial state $X(0)$, the state matrix $X(t)$ exponentially converges to theoretical pseudoinverse. The proof completes. ■

B. CONVERGENCE OF NAZNN MODEL WITH NOISES

As the previous discussion, the matrix-form $\Psi(t) \in \mathbb{C}^{n\times m}$ in the noise-disturbed NAZNN model (8) is divided into three categories, i.e., the constant noise, the linear noise, the random noise. In this section, the convergence of the noise-disturbed NAZNN model (8) with different noises is investigated in theory.

Constant Noise : Both the real part and the imaginary part of $\Psi(t) \in \mathbb{C}^{n\times m}$ are constants, the matrix-form $\Psi(t)$ is noted with $\Psi \in \mathbb{C}^{n\times m}$, i.e., $\psi_{ab} = c_1 + c_2 i$, where $c_1, c_2 \in \mathbb{R}$, ψ_{ab} noted the abth item of Ψ. the following theorem is presented.

Theorem 3: The state matrix $X(t)$ generated by the noise-disturbed NAZNN model (8), with the constant noise $\Psi(t) = \Psi \in \mathbb{C}^{n\times m}$, converges globally to the theoretical pseudoinverse $V^+(t)$ of (2).

Proof: The abth item of the noise-disturbed NAZNN model (8) can be noted below by adopting Laplace transform [34].

$$\tau\omega_{ab}(\tau) = \omega_{ab}(0) - \alpha\omega_{ab}(\tau) - \frac{\beta}{\tau}\omega_{ab}(\tau) + \psi_{ab}(\tau),$$

that is

$$\omega_{ab}(\tau) = \frac{\tau(\omega_{ab}(0) + \psi_{ab}(\tau))}{\tau^2+\alpha\tau+\beta}. \quad (12)$$

Obviously, $\tau/(\tau^2+\alpha\tau+\beta)$ is the transfer function, and its poles are $\tau_1 = 0.5(-\alpha+(\alpha^2-4\beta)^{0.5})$ and $\tau_2 = 0.5(-\alpha-(\alpha^2-4\beta)^{0.5})$. For $\alpha > 0$ and $\beta > 0$, τ_1 and τ_2 locate on the left half-plane, which means the neural system is stable. The final value theorem can be applied in this stable system. Note that $\psi_{ab}(\tau) = \psi_{ab}/\tau$ as $\psi_{ab}(t) = \psi_{ab}$ adds to a step single for complex-valued constant matrix Ψ. The final value theorem [34] is applied to (12), we have

$$\begin{aligned}\lim_{t\to\infty}\omega_{ab}(t) &= \lim_{\tau\to 0}\tau\omega_{ab}(\tau)\\ &= \lim_{\tau\to 0}\frac{\tau^2(\omega_{ab}(0)+\psi_{ab}/\tau)}{\tau^2+\alpha\tau+\beta}\\ &= 0.\end{aligned}$$

Then, we can conclude that $\|\Omega(t)\|_{\mathrm{F}} \to 0$ as $t \to \infty$. The proof is completed. ■

Linear Noise: In actual applications, the neural system may be exposed to linear time-dependent noise, both the real part and the imaginary part of $\Psi(t)$ are time linear functions, its matrix-form is noted with $\Psi(t) = t\Psi$, Ψ has been mentioned in constant situation. For linear time-dependent noise, the theorem below demonstrates the convergence of the proposed noise-disturbed NAZNN model (8).

Theorem 4: The state matrix $X(t)$ generated by the noise-disturbed NAZNN model (8) with a linear time-dependent noise $\Psi(t) = t\Psi$, converges to the theoretical pseudoinverse $V^+(t)$ of (2). Furthermore, the upper bound of the steady-state computing error $\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}}$ is $\|\Psi\|_{\mathrm{F}}/\beta$. i.e., as $\beta \to +\infty$, the computing error $\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}} \to 0$.

Proof: The abth item of the noise-disturbed NAZNN model (8) is transformed as following by employing Laplace transform [34].

$$\tau\omega_{ab}(\tau) = \omega_{ab}(0) - \alpha\omega_{ab}(\tau) - \frac{\beta}{\tau}\omega_{ab}(\tau) + \frac{\psi_{ab}}{\tau^2},$$

i.e.,

$$\omega_{ab}(\tau) = \frac{\tau^2(\omega_{ab}(0) + \psi_{ab}/\tau^2)}{\tau^2+\alpha\tau+\beta}. \quad (13)$$

where ψ_{ab}/τ^2 is the Laplace transform of $t\Psi$. The final value theorem is used in Eq.(13), we have

$$\begin{aligned}\lim_{t\to\infty}\omega_{ab}(t) &= \lim_{\tau\to 0}\tau\omega_{ab}(\tau)\\ &= \lim_{\tau\to 0}\frac{\tau^2(\omega_{ab}(0)+\psi_{ab}/\tau^2)}{\tau^2+\alpha\tau+\beta}\\ &= \frac{\psi_{ab}}{\beta}.\end{aligned}$$

Then, we can conclude that $\|\Omega\|_{\mathrm{F}} \to \|\Psi\|_{\mathrm{F}}/\beta$ as $t\to+\infty$. The proof is thus completed. ■

Random Noise: In practical applications, the noises may be nonlinear. The nonlinear time-dependent noise can be deemed as a bounded random noise in real-time computation process, we have below theorem for investigating the property of the proposed noise-disturbed NAZNN model (8).

Theorem 5: The computing error $\|\Omega(t)\|_{\mathrm{F}}$ of noise-disturbed NAZNN model (8) is bounded for the bounded random noise $\Psi(t) = \zeta(t) \in \mathbb{C}^{n\times m}$. In addition the steady-state computing estimate error $\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}}$ is bounded by $2(mn)^{\frac{1}{2}}\sup_{0\le s\le t}|\zeta_{ab}(s)|/(\alpha^2-4\beta)^{0.5}$ if $\alpha^2 > 4\beta$, or $4\beta(mn)^{\frac{1}{2}}\sup_{0\le s\le t}|\zeta_{ab}(s)|/\alpha(4\beta-\alpha^2)^{0.5}$ if $\alpha^2 < 4\beta$. $\zeta_{ab}(t)$ is the abth item of $\zeta(t)$, that means, the upper bound of $\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}}$ is almost inversely proportional to α, and the computing error $\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}}$ close to 0 for β is large enough with suitable α.

Proof: The noise-disturbed NAZNN model (8) with random noise can be rewritten as

$$\dot{\Omega}(t) = -\alpha\Omega(t) - \beta\int_0^t \Omega(s)\,\mathrm{d}s + \zeta(t),$$

its abth item can be expressed as

$$\dot{\omega}_{ab}(t) = -\alpha\omega_{ab}(t) - \beta\int_0^t \omega_{ab}(s)\,\mathrm{d}s + \zeta_{ab}(t). \quad (14)$$

Based on the values of α and β, three situation is analyzed.

1) If $\alpha^2-4\beta > 0$, the solution to (14) can be gotten as

$$\begin{aligned}\omega_{ab}(t) = \frac{1}{\lambda_1-\lambda_2}\big(&\omega_{ab}(0)(\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t})\\ &+\int_0^t(\lambda_1 e^{\lambda_1(t-s)}-\lambda_2 e^{\lambda_2(t-s)})\zeta_{ab}(s)\,\mathrm{d}s\big),\end{aligned}$$

where $\lambda_{1,2} = \frac{1}{2}(-\alpha\pm(\alpha^2-4\beta)^{\frac{1}{2}})$. According to the triangle inequality, we obtain

$$\begin{aligned}|\omega_{ab}(t)| \le \frac{1}{\lambda_1-\lambda_2}\big(&|\omega_{ab}(0)(\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t})|\\ &+\int_0^t(|\lambda_1 e^{\lambda_1(t-s)}||\zeta_{ab}(s)|\,\mathrm{d}s\\ &+\int_0^t|\lambda_2 e^{\lambda_2(t-s)})||\zeta_{ab}(s)|\,\mathrm{d}s\big),\end{aligned}$$

further, we have

$$\begin{aligned}|\omega_{ab}(t)| &\le \frac{1}{\lambda_1-\lambda_2}\big(|\omega_{ab}(0)(\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t})|\\ &\quad+2\max_{0\le s\le t}|\zeta_{ab}(s)|\big)\\ &= (\alpha^2-4\beta)^{-\frac{1}{2}}\big(|\omega_{ab}(0)(\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t})|\\ &\quad+2\max_{0\le s\le t}|\zeta_{ab}(s)|\big).\end{aligned}$$

Finally, we obtain

$$\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}} \le 2(\frac{mn}{\alpha^2-4\beta})^{\frac{1}{2}}\sup_{0\le s\le t}|\zeta_{ab}(s)|.$$

2) If $\alpha^2-4\beta = 0$, the solution to (14) can be gotten as

$$\begin{aligned}\omega_{ab}(t) &= \omega_{ab}(0)e^{-\frac{1}{2}\alpha t}(1-\frac{1}{2}\alpha t)\\ &\quad-\frac{\alpha}{2}\int_0^t(t-s)e^{-\frac{\alpha}{2}(t-s)}\zeta_{ab}(s)\,\mathrm{d}s\\ &\quad+\int_0^t e^{-\frac{\alpha}{2}(t-s)}\zeta_{ab}(s)\,\mathrm{d}s.\end{aligned}$$

According to [33, Proof of Theorem 1], existing $\mu > 0$, $\nu > 0$, so that

$$|\frac{\alpha}{2}|te^{-\frac{1}{2}\alpha t} \le \mu e^{-\nu t}. \quad (15)$$

Based on the inequality (15) and the triangle inequality, we obtain

$$\begin{aligned}|\omega_{ab}(t)| &\le |\omega_{ab}(0)e^{-\frac{1}{2}\alpha t}(1-\frac{1}{2}\alpha t)|\\ &\quad+\int_0^t|\mu e^{-\nu(t-s)}||\zeta_{ab}(s)|\,\mathrm{d}s\\ &\quad+\int_0^t|e^{-\frac{\alpha}{2}(t-s)}||\zeta_{ab}(s)|\,\mathrm{d}s,\end{aligned}$$

further, we have

$$|\omega_{ab}(t)| \le |\omega_{ab}(0)e^{-\frac{1}{2}\alpha t}(1-\tfrac{1}{2}\alpha t)| + (\tfrac{\mu}{\nu}+\tfrac{2}{\alpha})\max_{0\le s\le t}|\zeta_{ab}(s)|.$$

Finally, we obtain

$$\lim_{t\to\infty}\|\Omega(t)\|_{\mathrm{F}} \le (\frac{\mu}{\nu}+\frac{2}{\alpha})(mn)^{\frac{1}{2}}\sup_{0\le s\le t}|\zeta_{ab}(s)|.$$

3) If $\alpha^2-4\beta < 0$, the solution to (14) can be gotten as

$$\begin{aligned}\omega_{ab}(t) &= \omega_{ab}(0)e^{\xi t}(\xi\sin(\eta t)/\eta+\cos(\eta t))\\ &\quad+\int_0^t\big(\xi\sin(\eta(t-s))e^{\xi(t-s)}/\eta\\ &\quad+\cos(\eta(t-s))e^{\xi(t-s)}\big)\zeta_{ab}(s)\,\mathrm{d}s,\end{aligned}$$

where ξ, η is defined as $-\alpha/2$, $-1/2(4\beta-\alpha^2)^{\frac{1}{2}}$ respectively. According to the triangle inequality, we have

$$\begin{aligned}|\omega_{ab}(t)| &\le |\omega_{ab}(0)e^{\xi t}(\xi\sin(\eta t)/\eta+\cos(\eta t))|\\ &\quad-\frac{\sqrt{\xi^2+\eta^2}}{\xi\eta}\max_{0\le s\le t}|\zeta_{ab}(s)|\\ &= |\omega_{ab}(0)e^{\xi t}(\xi\sin(\eta t)/\eta+\cos(\eta t))|\\ &\quad+\frac{4\beta}{\alpha\sqrt{4\beta-\alpha^2}}\max_{0\le s\le t}|\zeta_{ab}(s)|.\end{aligned}$$

Finally, we obtain

$$\lim_{t\to\infty} \|\Omega(t)\|_{\mathrm{F}} \leq \frac{4\beta}{\alpha}\left(\frac{mn}{4\beta-\alpha^2}\right)^{\frac{1}{2}} \sup_{0\leq s\leq t} |\zeta_{ab}(s)|.$$

■

V. SIMULATION AND VERIFICATION

In the Section IV, the convergence and robustness property of the proposed NAZNN model (5) for computing complex-valued time-dependent matrix pseudoinverse are analysed. This section, tests are conducted on two complex-valued time-dependent matrices to validate the superiority of the presented NAZNN model (5) for solving complex-valued time-dependent matrix pseudoinverse in various noises.

Example 1. The complex-valued time-dependent matrix $V(t)$ (with full rank) is considered as follow:

$$V(t) = \begin{bmatrix} i\sin(2t) & i\cos(2t) \\ -i\cos(2t) & i\sin(2t) \\ i\sin(2t) & i\cos(2t) \end{bmatrix} \in \mathbb{C}^{3\times 2}. \quad (16)$$

According to (1), the theoretical pseudoinverse of (16) can be gotten as

$$V^{+}(t) = \begin{bmatrix} -i\sin(2t)/2 & -i\sin(2t) & -i\cos(2t)/2 \\ -i\sin(2t)/2 & -i\cos(2t) & -i\cos(2t)/2 \end{bmatrix} \in \mathbb{C}^{2\times 3}. \quad (17)$$

Four noise scenarios are considered as following to operate the simulations for complex-valued time-dependent matrix pseudoinversion (16).

Zero Noise: The simulation results generated by the proposed NAZNN model (5) for computing complex-valued time-dependent pseudoinverse of (16) are shown in Fig. 1 and Fig. 2. As displayed in Fig. 1, beginning from random initial states $X(0) \in \mathbb{C}^{2\times 3}$, the state matrix $X(t) \in \mathbb{C}^{2\times 3}$ synthesized by the proposed NAZNN model (5) converges to the time-dependent theoretical pseudoinverse (17) accurately

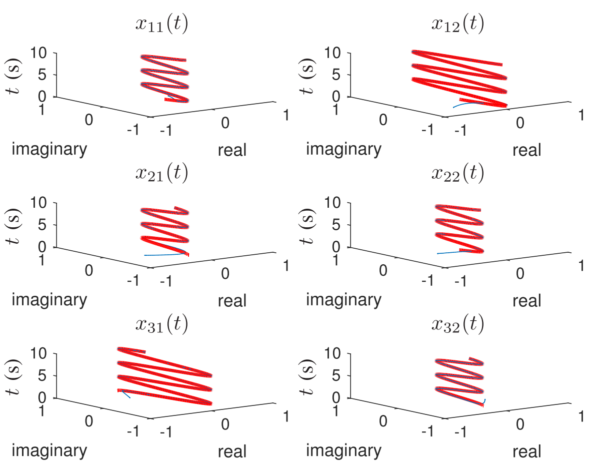

FIGURE 1. State trajectories of NAZNN model (5) with $\alpha = \beta = 10$, where red dash-dotted curves denote theoretical solution pseudoinverse of (16) and blue solid curves denote the neural-state solution (16).

and rapidly within a short time. The trajectories of residual error $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5) are shown in Fig. 2. In Fig. 2(a), the residual error synthesized by NAZNN model (5) with $\alpha = \beta = 10$ diminish to zero with 3 s. Furthermore, from Fig. 2(b), the residual error synthesized by NAZNN model (5) with $\alpha = \beta = 100$ diminish to zero with 0.6 s. That is, the convergence time can be shorten by increasing α and β, these results are consistent with the theoretical proof of Theorem 2. These simulation results illustrate the efficacy of NAZNN model (5) for computing complex-valued time-dependent pseudoinverse without noise existing, in the noise-free situation, the convergence property of TZNN for computing complex-valued time-dependent matrix pseudoinverse has been conducted in [27], the convergence property of GNN for computing time-dependent matrix inverse has been investigated in [32].

Constant Noise: For demonstration and comparison, TZNN (6) and GNN (7) are employed to compute complex-valued time-dependent matrix pseudoinverse (16). The residual errors synthesized by NAZNN model (5), TZNN (6) and GNN (7) with constant noise, the matrix-form noise is set to be $\Psi(t) = [6+8i]^{2\times 3}$, the corresponding simulation results are displayed in Fig. 3. As shown in Fig. 3(a), beginning from random initial states $X(0) \in \mathbb{C}^{2\times 3}$, the residual errors $\|\Omega(t)\|_{\mathrm{F}}$ generated by NAZNN model (5) with $\alpha = \beta = 10$ diminish to zero with 4.5 s, which are consistent with the theoretical proof of Theorem 3. On the contrary, the residual errors generated by TZNN (6) and GNN (7) do not converge to zero. From Fig. 3(b), with $\alpha = \beta = 100$, the residual error of NAZNN model (5) decrease to zero with 4 s. With $\alpha = 100$, the residual errors generated by TZNN (6) and GNN (7) can not diminish to zero yet, which is smaller than the situation with $\alpha = 10$. These simulation results demonstrate the robust performance of NAZNN model (5) for computing complex-valued time-dependent matrix pseudoinverse with constant noise situation.

Linear Noise: Linear time-dependent noise is investigated in this section, each element of the matrix-form linear time-dependent noise is $t + ti$. The relevant simulation results of NAZNN model (5), TZNN (6) and GNN (7) are shown in Fig. 4. From Fig. 4(a), the residual error of NAZNN (5) with $\alpha = \beta = 10$ diminishes toward near zero rapidly and keeps stable at 0.3 s, while the residual errors of TZNN (6) and GNN (7) with $\alpha = 10$ increase over time and each of them is over ten times larger than that of NAZNN model (5) at $t = 9$ s, which substantiates the superiority of the presented NAZNN model (5). Fig. 4(b) displays the residual errors of three model with $\alpha = \beta = 100$, the residual error of NAZNN model (5) also converges to near zero quickly and keeps stable at 0.08s. In addition, the residual error of TZNN (6) and GNN (7) $\alpha = 100$ also increase over time. Overall, these simulation results verify Theorem 4.

Random Noise: In this section, we investigate the property of NAZNN model (5) in a random noise situation.

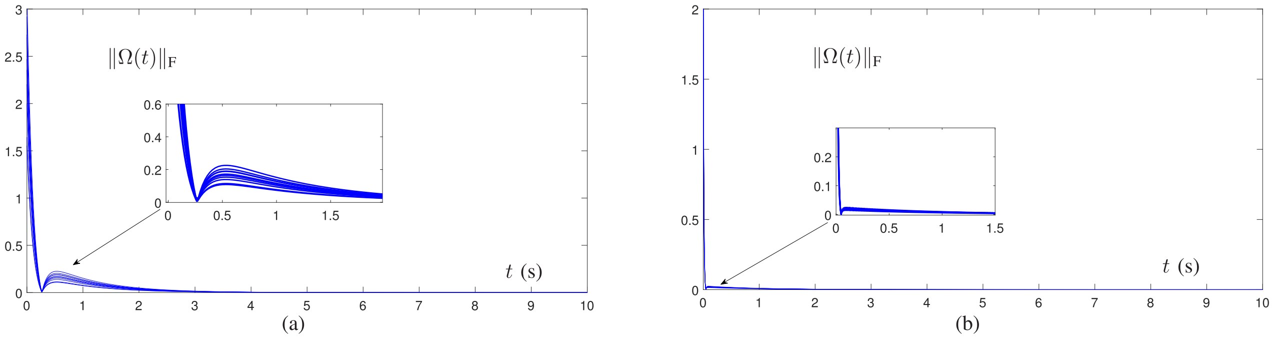

FIGURE 2. Residual errors $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5) for the pseudoinverse of (16). (a) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with $\alpha = \beta = 10$. (b) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with $\alpha = \beta = 100$.

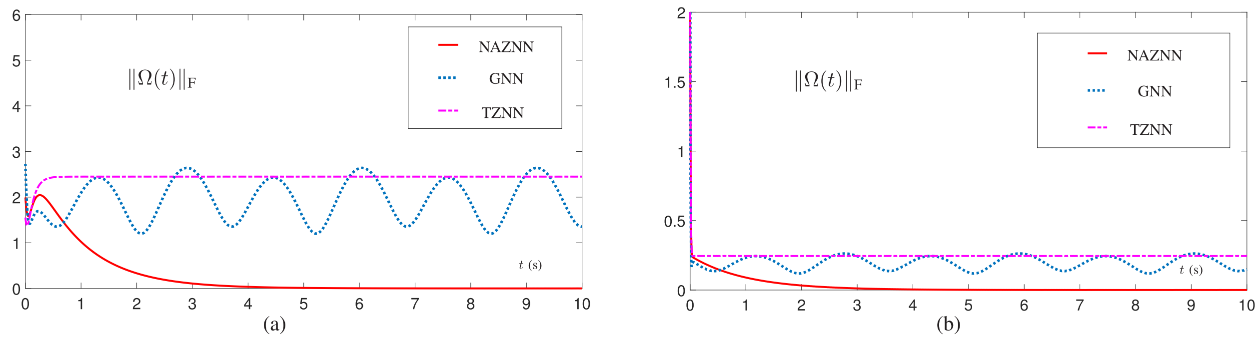

FIGURE 3. Residual errors $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5), TZNN (6) and GNN (7) for the pseudoinverse of (16) with the constant noise $\Psi(t) = [6 + 8i]^{2\times 3}$. (a) With $\alpha = 10$ for three models and $\beta = 10$ for NAZNN model (5). (b) With $\alpha = 100$ for three models and $\beta = 100$ for NAZNN model (5).

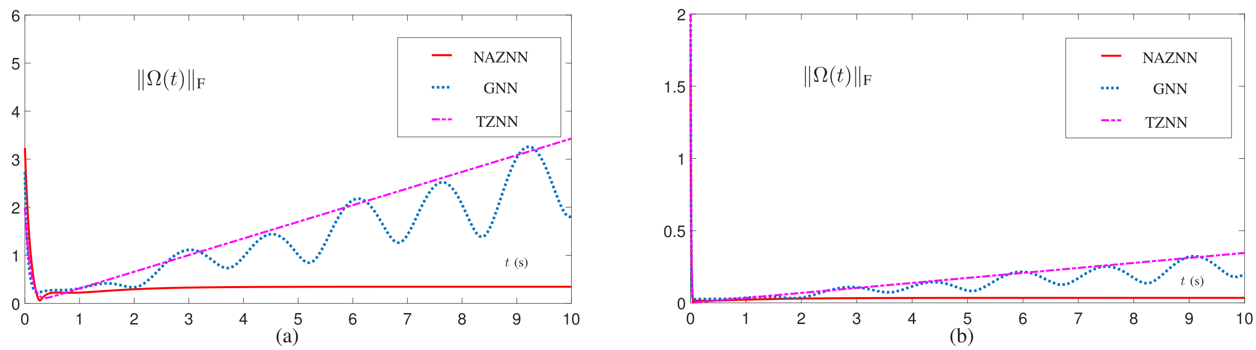

FIGURE 4. Residual errors $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5), TZNN (6) and GNN (7) for the pseudoinverse of (16) with the linear time-dependent noise $\Psi(t) = [t + ti]^{2\times 3}$. (a) With $\alpha = 10$ for three models and $\beta = 10$ for NAZNN model (5). (b) With $\alpha = 100$ for three models and $\beta = 100$ for NAZNN model (5).

The experimental results of NAZNN model (5), TZNN (6) and GNN (7) with matrix-form random noise $\Psi(t) = \zeta(t)$, each element $\zeta_{ab}(t) \in [18 + 18i, 22 + 22i]$ are displayed in Fig. 5, Fig. 5(a) illustrates the residual error of NAZNN model (5), TZNN (6) and GNN (7) with $\alpha = \beta = 10$, the residual error decreases to zero within 5.5 s and keeps very stable at 0.01, while the residual errors of TZNN (6) and GNN (7) have a big gap with zero. Besides, from Fig. 5(b) the residual error of NAZNN model (5) with $\alpha = \beta = 100$ also decreases to zero rapidly and remain stable in 5 s, in contrast, the residual errors of TZNN (6) and GNN (7) with $\alpha = 100$ do not diminish to zero as time goes on.

The above simulation results of Example 1, i.e, Figs.1-5, have demonstrate the superiority and the effectiveness of the proposed NAZNN model (5) for solving complex-valued time-dependent matrix pseudoinverse in the situation of different noise.

Example 2. For further investigation, a more complicated complex-valued time-dependent matrix is considered.

$$V(t) = \begin{bmatrix} v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\ v_{21}(t) & v_{22}(t) & \cdots & v_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1}(t) & v_{m2}(t) & \cdots & v_{mn}(t) \end{bmatrix} \in \mathbb{C}^{m\times n}. \quad (18)$$

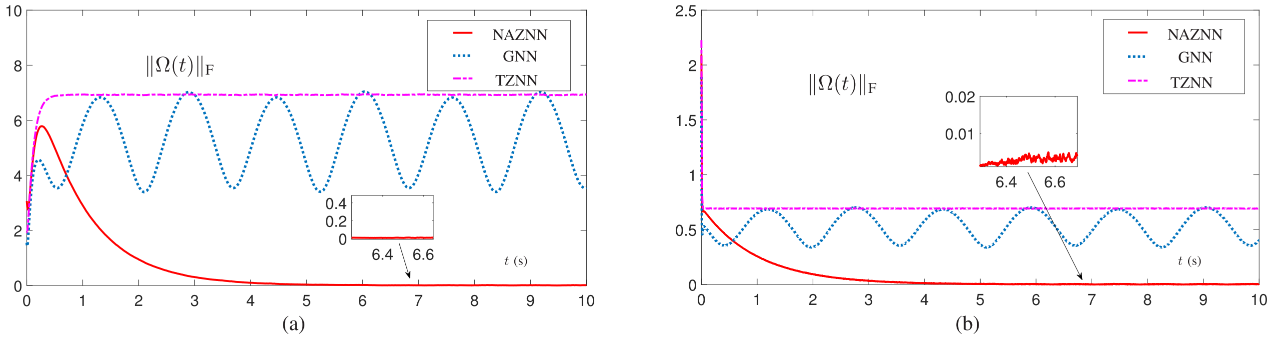

FIGURE 5. Residual errors $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5), TZNN (6) and GNN (7) for the pseudoinverse of (16) with the random time-dependent noise $\zeta(t)$, each element $\zeta_{ab}(t) \in [18 + 18i, 22 + 22i]$. (a) With $\alpha = 10$ for three models and $\beta = 10$ for NAZNN model (5). (b) With $\alpha = 100$ for three models and $\beta = 100$ for NAZNN model (5).

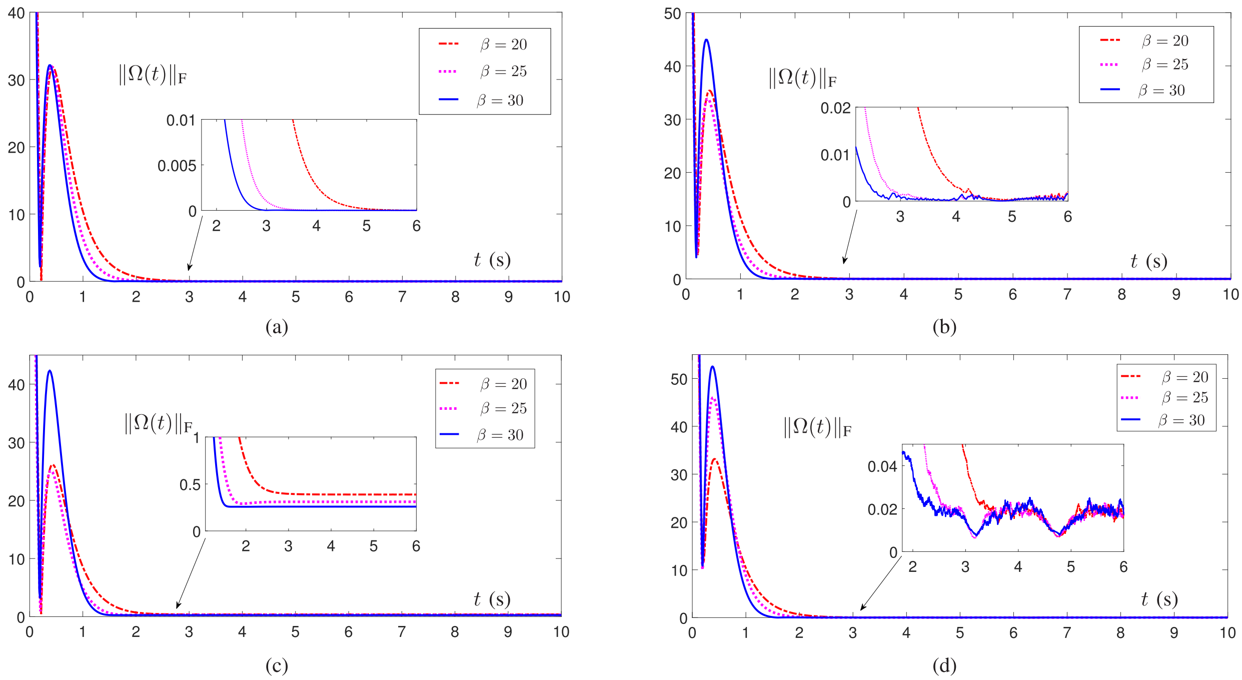

FIGURE 6. Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ with randomly initial state $X(0) \in \mathbb{C}^{m\times n}$, $\alpha = 10$ and different values of β for computing complex-valued time-dependent pseudoinverse of matrix (18). (a) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with noiseless. (b) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with matrix-form constant noise $\Psi(t) = [6 + 8i]^{5\times 6}$. (c) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with matrix-form time-dependent linear noise $\Psi(t) = [t + ti]^{5\times 6}$. (d) Residual errors $\|\Omega(t)\|_{\mathrm{F}}$ of NAZNN model (5) with matrix-form random noise $\Psi(t)$, each element $\zeta_{ab}(t) \in [18 + 18i, 22 + 22i]$.

with $v_{ab}(t)$ denotes the abth element of $V(t)$. Thereinto

$$v_{ab}(t) = \begin{cases} e^{2it}, & a = b, \\ a + e^{-2it}, & a < b, \\ b + e^{-2it}, & a > b. \end{cases}$$

Due to the large dimensions (with $m = 6$ and $n = 5$ in this paper) of matrix (18), the theoretical pseudoinversion is hard to be obtained. Thus, we just present the residual errors $\|\Omega(t)\|_{\mathrm{F}} = \|V^{\mathrm{H}}(t)V(t)X(t) - V^{\mathrm{H}}(t)\|_{\mathrm{F}}$ generated by NAZNN model (5) with different situations of $\alpha^2 - 4\beta$. That is, with α being 10, β being 20, 25, 30 denote the situation of $\alpha^2 - 4\beta > 0$, $\alpha^2 - 4\beta = 0$, $\alpha^2 - 4\beta < 0$, respectively. The simulation results of NAZNN model (5) are shown in Fig. 6. From Fig. 6(a), the residual errors of NAZNN model (5) setting different values of β decrease to zero rapidly in noiseless situation. From Fig. 6(b) to Fig. 6(d), the simulation results are inspiring, the residual errors of NAZNN model (5) setting different values of β also decrease to zero rapidly in different noises situation, it is worth pointing out that the residual errors of NAZNN model (5) remain very stable after converging to zero, which demonstrate excellent anti-noise performance of NAZNN model (5). In addition, as shown in Fig. 6, we can observe that the convergence rate of residual errors $\|\Omega(t)\|_{\mathrm{F}}$ is accelerated as the value of β increases with a suitable α. In summary, all these results the theorems presented in previous section and further substantiate the superior anti-noise performance of NAZNN model (5) for solving the complex-valued time-dependent matrix pseudoinverse once again.

VI. CONCLUSION

To deal with noises when resolving time-dependent issue in complex-valued domain, a novel NAZNN model (5) has been first proposed in this paper. Then, the NAZNN model (5) has been investigated for computing complex-valued time-dependent pseudoinverse. For comparison, TZNN (6) and GNN (7) have been adopted to solve the same task. Besides, the globally and exponential convergence performance has been researched in theoretical analyses. Moreover, in different noise situation, the proposed NAZNN model (5) has been proven to possess an superior property. That is, the complex-valued time-dependent pseudoinverse synthesized by the NAZNN model (5) converge to the theoretical pseudoinverse, and it owns strong resistance to various sorts of noise. Finally, simulation results have illustrated the efficacy and superiority of the NAZNN model (5) for solving complex-valued time-dependent matrix pseudoinverse with various kinds of noises. It is worth pointing out that the novel zeroing neural network model is firstly proposed for solving complex-valued time-dependent problem, moreover, it shows superior stability in the aspect of anti-noise, which is a breakthrough in the research of ZNN and time-dependent issue resolving.

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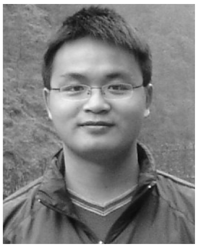

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