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Synthesis of a Dynamical Output H_∞ Controller for the Fuzzy Input-Output Model

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ABSTRACT In this paper, the issue of dynamical output H_∞ controller designing is addressed for the fuzzy input–output (FIO) model. The FIO model is significantly distinctive from the conventional Mamdani and T-S or T-S-K fuzzy models and can be conveniently used to describe more complicated dynamical systems that cannot be easily handled by the conventional fuzzy models. By using the robust control theory available for both the linear and fuzzy systems, sufficient conditions in terms of the linear matrix inequalities (LMIs) are derived to synthesize a dynamical output feedback H_∞ controller for the FIO plant. These LMI conditions can be numerically and efficiently solved by the existing convex optimization software, e.g., the MATALB LMI toolbox. Moreover, a motor–spring-mass system abstracted from the real applications is provided to validate the applicability and efficiency of our method.

INDEX TERMS Fuzzy input-output (FIO) model, dynamical output feedback control, H_∞ control, linear matrix inequalities (LMIs).

I. INTRODUCTION

Since Mamdani firstly introduced the concept of fuzzy sets into the control community in 1974, fuzzy control has attracted great interests from both control theorists and control engineers, and substantial research progresses have been made [1]–[4]. Sugeno and Taniguchi [5] classifies all the existing fuzzy systems into three main types, i.e., Type I, Type II, and Type III. Type I, proposed by Mamdani [6], is characterized by a set of fuzzy rules, which are constructed by the linguistic terms both in the antecedent and consequent parts of fuzzy rules. Fuzzy sets can be used to mathematically quantify the linguistic terms, and fuzzy inference techniques are employed to draw the conclusions from these linguistic rules. Because the Mamdani fuzzy systems are built on the basis of linguistic terms, the human knowledge/experience can be easily embedded into the fuzzy rule bases. The mathematical properties of this type of fuzzy systems have been extensively investigated [1]. If the fuzzy variables in the consequent part of fuzzy rules are replaced by singletons, we have the Type II fuzzy systems. Therefore, Type II can be considered as a special case of Type I fuzzy systems. When the consequent part of fuzzy rules becomes an analytical function instead of linguistic terms, a new fuzzy model is constructed, namely T-S or T-S-K fuzzy systems, i.e., Type III fuzzy systems. The T-S fuzzy system proposed in 1985 [7] is a well-known landmark in the history of fuzzy control theory, which can be regarded as a fuzzy blending of local linear systems. In fact, it is a significant extension of the classical linear model. On the other hand, the classical linear model can be regarded as a special kind of T-S fuzzy model with all the local linear models chosen to be the same. Within the framework of the T-S fuzzy model, numerous fuzzy control issues, such as stability analysis [8]–[12], systematic controller design [12]–[16], robustness analysis [17]–[20], have been extensively investigated. In effect, the T-S fuzzy system based research still remains one of the hot topics in the field of nonlinear control [21]–[28]. In [29], the T-S fuzzy model is generalized to a more complicated case, where the local linear systems in the consequent part of fuzzy rules are replaced by T-S fuzzy systems. Although the approximation capability is greatly enhanced, the formulation of this model looks more complicated than the conventional T-S model and hence the controller design method also becomes sophisticated. Nowadays, much effort has been attracted to the type II fuzzy sets based fuzzy systems, which are argued to be more effectively in handling uncertainty. The readers are referred to [3] and the reference therein for the details.

Motivated by some real applications encountered in control engineering, a fuzzy input-output (FIO) model was preliminarily presented in our earlier work [30], in which the local linear system in the consequent part of fuzzy rules was replaced by an input-output relationship with the following form:

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau \tag{1-1}$$

where $u(\cdot)$ and $y(\cdot)$ are the functions of time, and they represent the input and output signals of this system, respectively. It can be seen that the output $y(\cdot)$ is determined by the convolution of $u(\cdot)$ and $h(\cdot)$. Actually (1-1) represents a linear time invariant system.

As we know, all the state variables for each local linear model are exactly the same for the conventional T-S fuzzy System. More specifically, the group of state variables for all the local linear models in a T-S fuzzy model is exactly the same set of variables both from the physical or mathematical point of view. Therefore, it is inconvenient if not impossible to deal with the complicated process for which the state variables might be distinctive under different operating conditions.

Motivated by this observation, the FIO model is constructed. Within the framework of FIO models, it is pretty easy to circumvent this inconvenience because the state variables of each local linear system could be distinct from others. This will include the T-S fuzzy model as a special case. The more in-depth comparison between the FIO system and the conventional T-S fuzzy system will be provided in Section III.

In this research, starting from two practical systems, the merit of the FIO model is emphasized and the motivation of the FIO model is further strengthened. Then the H_∞ controller designing problem is to be addressed, which is an extremely important theoretical issue in the field of control engineering. More specifically, a dynamical output H_∞ controller is synthesized by using the convex optimization technique to stabilize as well as to guarantee the H_∞ performance of the closed-loop FIO system. As far as we know, this issue is still open for the FIO system, even though it is well solved for the conventional T-S fuzzy system and the linear control system.

The rest of this paper is organized as follows. In Section II, two practical systems are provided to solidify the motivation of the FIO model. For the readability and completeness of this article, the three kinds of formulations for the FIO system are provided in Section III, which functions as a preliminary for the following H_∞ controller synthesis. Moreover, the H_∞ dynamical output feedback controller designing problem is discussed in Section IV. Finally, an application example is provided to validate our approach in Section V. Section VI concludes this article with some remarks.

II. MOTIVATION OF THE FIO MODEL

In this section, we explain the motivation of the FIO model. As could be seen, although T-S fuzzy systems have gained great success in dealing with nonlinear control problems, it encounters some difficulty when coping with an industrial process and a control equipment which will be detailed here in this section.

TABLE 1. The dynamics of superheated steam temperature process.

Operating Points	Load	Transfer functions
1	100% (D=527.8kg/s)	$\frac{1.276}{(1+18.4s)^6}$
2	75% (D=347.9kg/s)	$\frac{1.202}{(1+27.1s)^7}$
3	50% (D=242.2kg/s)	$\frac{1.119}{(1+42.1s)^7}$
4	37% (D=179.32kg/s)	$\frac{1.048}{(1+56.6s)^8}$

A. SUPERHEATED STEAM TEMPERATURE PROCESS

It has always been a challenge to regulate or control the temperature of superheated steam in power plants. Nowadays, large-scale coal-fired thermal power plants are required to operate in a cycling mode. This means the load of power plants increases during the daytime while decreases at night. Therefore, the control system should adapt to the variations of load as quickly as possible [31], [32]. Consequently, we obtain a set of linear approximations of the original complex dynamics of superheated steam temperature process around different operating points based on experiments as listed in TABLE 1 [33], [34].

An apparent observation from TABLE 1 is that the orders of the linear systems are different from each other. This makes it unreasonable to construct a conventional T-S fuzzy system by directly blending all the linear systems together. It may be argued it is due to the absence of observability that the orders the linear systems around different operating point are distinct, and we can still use an exactly same set of state variables to construct a conventional T-S fuzzy system. In fact, theoretically it is. However, the question is in that way some local linear systems have to be unobservable, which is extremely unusual for the current conventional T-S control theory. More importantly, it is impractical if not impossible to construct state-space equations based on the same set of state variables around different operating points simply from the measured input-output data.

B. A MOTOR-SPRING-MASS SYSTEM

Consider the motor-spring-mass system described in FIGURE 1, where a torsional spring with the spring constant K_2 is mounted on the top of the shaft of the motor and a metal string is fixed on the shaft with another end fixed on a mass-spring system. At the beginning, the metal string is loose. With the shaft rotating, the metal string gets tightened. For this system, we aim at controlling the angular position

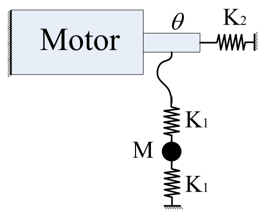

FIGURE 1. Configuration of the motor-spring-mass system.

of the shaft. More specifically, we want to make the shaft of the motor follow some positional instructions as accurately and quickly as possible. The system can be mathematically modeled as follows.

When the metal string is completely loose, according to Newton's laws of mechanics, the following equations hold:

$$T_e = J\ddot{\theta} + \mu_0\dot{\theta} + K_2\theta \tag{2-1}$$

$$T_e = K_e U \tag{2-2}$$

where T_e denotes the electromagnetic torque supplied by the motor, J is the moment of inertia of the motor shaft, μ_0 denotes the factor of friction and K_e is the coefficient between the output electromagnetic torque of the motor and the voltage applied to the motor. With the state variables chosen as $x_1 = \theta$, $x_2 = \dot{\theta}$, we get the following second order state-space equations of the system:

$$\dot{x}_1 = x_2 \tag{2-3}$$

$$\dot{x}_2 = -\frac{K_2}{J}x_1 - \frac{\mu_0}{J}x_2 + \frac{K_e}{J}u \tag{2-4}$$

$$y = x_1 \tag{2-5}$$

When the metal string gets tightened, it follows from the Newton's laws that the following equations hold:

$$K_e U = J\ddot{\theta} + \mu_0\dot{\theta} + K_2\theta + K_1 R(\theta R - Z) \tag{2-6}$$

$$K_1(\theta R - Z) - K_1 Z = M\ddot{Z} + \mu_1\dot{Z} \tag{2-7}$$

where R is the radius of the motor shaft; Z represents the position of the mass M. With the state variables chosen as $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = Z$, $x_4 = \dot{Z}$, we get the following fourth order state-space equations:

$$\dot{x}_1 = x_2 \tag{2-8}$$

$$\dot{x}_2 = -\frac{K_2 + R^2K_1}{J}x_1 - \frac{\mu_0}{J}x_2 + \frac{RK_1}{J}x_3 + \frac{K_e}{J}u \tag{2-9}$$

$$\dot{x}_3 = x_4 \tag{2-10}$$

$$\dot{x}_4 = \frac{RK_1}{M}x_1 - \frac{2K_1}{M}x_3 - \frac{\mu_1}{M}x_4 \tag{2-11}$$

$$y = x_1 \tag{2-12}$$

The order of the state space equations (2-3)~(2-4) are apparently distinct with that of (2-8)~(2-11) when the motor-spring-mass system works in different conditions according to whether the string is loose or tensed. Thus dynamics of the above movement equipment cannot be easily described by the general T-S fuzzy model. One might describe the system by a switching system and design a corresponding switching controller. While considering the continuous tension building process along the string, we prefer a fuzzy model for this practical equipment since it is hard to tell to what position the tension along the string is build and when it disappears suddenly.

III. DESCRIPTION OF THE FIO MODEL

The FIO model can be mathematically formulated in three kinds different formations, including the integral, the transfer function based, and the state space equation based formations. These three formations are essentially equivalent to each other. Further more, a thorough comparison is made between the FIO and the conventional T-S fuzzy models [30].

A. THE INTEGRAL FORMATION OF THE FIO MODEL

The ith rule of a FIO model is of the following linguistic form:

Plant Rule i

IF $v_1(t)$ is M_i^1 and $\cdots$ and $v_g(t)$ is M_i^g,
THEN

$$\boldsymbol{y}_i(t) = \int_0^t h_i(t-\tau)\boldsymbol{u}(\tau)\,d\tau \quad i = 1, 2, \ldots, \eta \tag{3-1}$$

where h_i: R→R, $i = 1, \ldots, \eta$, are integral functions corresponding to the local linear systems; $v_k(t)$, $k = 1, \ldots, g$ are antecedent variables, which are all or part of the measurable state variables or output of the FIO model; $\boldsymbol{y}_i(t)$, $i = 1, \ldots, \eta$, represent the outputs of the local single-input-single-output (SISO) linear systems; η denotes the whole number of fuzzy rules involved; M_i^k, $i = 1, \ldots, \eta$, are fuzzy terms, which can be quantified by certain kinds of membership functions.

By using some specific fuzzy inference methods, the output of the above fuzzy input-output model can be formulized as follows:

$$\begin{aligned}\boldsymbol{y}(t) &= \frac{\sum_{i=1}^{\eta}\omega_i(\boldsymbol{v}(t))\,y_i(t)}{\sum_{i=1}^{\eta}\omega_i(\boldsymbol{v}(t))} \\ &= \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\,y_i(t) \\ &= \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\int_0^t h_i(t-\tau)\boldsymbol{u}(\tau)\,d\tau \end{aligned} \tag{3-2}$$

where

$$\mu_i(\boldsymbol{v}(t)) = \frac{\omega_i(\boldsymbol{v}(t))}{\sum_{j=1}^{\eta}\omega_j(\boldsymbol{v}(t))} \tag{3-3}$$

$$\omega_i(\boldsymbol{v}(t)) = \prod_{k=1}^{g} M_i^k(v_k(t)) \tag{3-4}$$

$$\boldsymbol{v}(t) = [v_1(t),\ v_2(t),\ \cdots,\ v_g(t)] \tag{3-5}$$

for all t. The term $M_i^k(v_k(t))$ is the grade of membership of $v_k(t)$ in M_i^k.

Since

$$\begin{cases} \sum_{i=1}^{\eta} \omega_i(\boldsymbol{v}(t)) > 0 \\ \omega_i(\boldsymbol{v}(t)) \geq 0, \quad i = 1, 2, \ldots, \eta \end{cases} \tag{3-6}$$

we have

$$\begin{cases} \sum_{i=1}^{\eta} \mu_i(\boldsymbol{v}(t)) = 1 \\ \mu_i(\boldsymbol{v}(t)) \geq 0, \quad i = 1, 2, \ldots, \eta \end{cases} \tag{3-7}$$

for all t.

B. THE TRANSFER FUNCTION-BASED FIO MODEL

Assuming that functions $\boldsymbol{u}(\cdot)$, $\boldsymbol{y}(\cdot)$ and $\boldsymbol{h}(\cdot)$ are all Laplace transformable and transforming the local linear mapping in (3-1) into the Laplace form, the FIO model can be described as follows:

Plant Rule *i*

IF $v_1(t)$ is M_i^1 and $\cdots$ and $v_g(t)$ is M_i^g,

THEN

$$\boldsymbol{Y}_i(s) = \boldsymbol{H}_i(s)\boldsymbol{U}(s) \quad i = 1, 2, \ldots, \eta \tag{3-8}$$

where $\boldsymbol{Y}_i(s)$, $\boldsymbol{H}_i(s)$, and $\boldsymbol{U}(s)$ are Laplace transform of $\boldsymbol{y}_i(\cdot)$,$\boldsymbol{h}_i(\cdot)$ and $\boldsymbol{u}(\cdot)$, respectively.

Note that a so-called Takagi-Sugeno fuzzy transfer model has been proposed in [35], where the inference result of the antecedent part of a linguistic fuzzy rule is directly combined with the coefficients of the consequent transfer function. Unfortunately, such a direct combination is difficult if not impossible in the complex domain [36]. Different from this Takagi-Sugeno fuzzy transfer model, the transfer function-based form of our FIO model implies that its local input and output can be described by the transfer function $\boldsymbol{H}_i(s)$.

C. THE STATE SPACE EQUATION BASED FIO MODEL

It follows from the classical control theory that the FIO model can be transformed from transfer-function form (3-8) into the following state-space form:

Plant Rule *i*

IF $v_1(t)$ is M_i^1 and $\ldots$ and $v_g(t)$ is M_i^g,

THEN

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{A}_i\boldsymbol{x}_i(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \\ \boldsymbol{y}_i(t) = \boldsymbol{C}_i\boldsymbol{x}_i(t) + \boldsymbol{D}_i\boldsymbol{u}(t), \end{cases} \quad i = 1, 2, \ldots, \eta \tag{3-9}$$

where $\boldsymbol{x}_i(t) = \left[x_i^1, x_i^2, \cdots, x_i^{n_i}\right]^{\mathrm{T}}$ is the state vector of the local linear system of the ith rule; n_i represents the system order of the ith local linear system; $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$ and $\boldsymbol{D}_i$, are the corresponding matrices with compatible dimensions. Here, it is assumed that the pairs $(\boldsymbol{A}_i, \boldsymbol{B}_i)$ and $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ are controllable and observable, respectively.

The above fuzzy rules can be formulated by the following nonlinear state equations

$$\begin{cases} \dot{\boldsymbol{x}}^1(t) = \boldsymbol{A}^1\boldsymbol{x}^1(t) + \boldsymbol{B}^1\boldsymbol{u}(t) \\ \dot{\boldsymbol{x}}^2(t) = \boldsymbol{A}^2\boldsymbol{x}^2(t) + \boldsymbol{B}^2\boldsymbol{u}(t) \\ \quad\vdots \\ \dot{\boldsymbol{x}}^i(t) = \boldsymbol{A}^i\boldsymbol{x}^i(t) + \boldsymbol{B}^i\boldsymbol{u}(t) \\ \quad\vdots \\ \dot{\boldsymbol{x}}^\eta(t) = \boldsymbol{A}^\eta\boldsymbol{x}^\eta(t) + \boldsymbol{B}^\eta\boldsymbol{u}(t) \end{cases} \tag{3-10}$$

$$\boldsymbol{y}^i(t) = \boldsymbol{C}^i\boldsymbol{x}^i(t) + \boldsymbol{D}^i\boldsymbol{u}(t) \tag{3-11}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{\eta} \mu_i(\boldsymbol{v}(t))\boldsymbol{y}^i(t) \tag{3-12}$$

where $\boldsymbol{x}^i \in \mathbf{R}^{n_i}$, $\boldsymbol{u} \in \mathbf{R}^m$, $\boldsymbol{y}^i \in \mathbf{R}^p$, $\boldsymbol{y} \in \mathbf{R}^p$, $\boldsymbol{A}^i \in \mathbf{R}^{n_i\times n_i}$, $\boldsymbol{B}^i \in \mathbf{R}^{n_i\times m}$, $\boldsymbol{C}^i \in \mathbf{R}^{p\times n_i}$, $\boldsymbol{D}^i \in \mathbf{R}^{p\times m}$ and n_{i} represents the system order of the ith local linear system $i = 1, \ldots, \eta$.

D. COMPARISON BETWEEN THE FIO AND T-S MODEL

In this section, we make a detailed comparison between these two models. Generally speaking, the T-S model is of the following form:

Plant Rule *i*

IF $v_1(t)$ is M_i^1 and $\ldots$ and $v_g(t)$ is M_i^g,

THEN

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{u}(t), \end{cases} \quad i = 1, 2, \ldots, \eta \tag{3-13}$$

By using some specific fuzzy inference and defuzzification methods, the above linguistic fuzzy model can be expressed by the following analytical equation:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{\eta} \mu_i(\boldsymbol{v}(t))\left[\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t)\right] \tag{3-14}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{\eta} \mu_i(\boldsymbol{v}(t))\left[\boldsymbol{C}_i\boldsymbol{x}(t) + \boldsymbol{D}_i\boldsymbol{u}(t)\right] \tag{3-15}$$

According to (3-14) and (3-15), it can be seen that both the local state and the output are blended in a fuzzy way, and the local state variables for the different rules are always the same, i.e., $\boldsymbol{x}(t)$.

For a FIO model, however, the state vectors $\boldsymbol{x}_i(t) = \left[x_i^1, x_i^2, \cdots, x_i^{n_i}\right]^{\mathrm{T}}$ of the local linear dynamical system differ from each other, i.e., the vector $\boldsymbol{x}_i(t)$, may comprise different elements from the vector $\boldsymbol{x}_j(t)$, $i \neq j$. The number of the local state variables may be also different, which means the order of the local linear systems can be different from each other.

Based on the above feature, it is possible for us to model the plant accurately under some work conditions, while we can roughly model the plant under other conditions. For example, we can model a plant by a second order differential equation under some condition, while under other conditions, we can use a fourth order differential equation, which is more accurate than the former. More importantly, all kinds of identification methods developed for linear systems can be directly used to identify the local linear input-output relationship. For example, the frequency response methods can be used to identify the parameters of the local transfer functions. As we know, one of the foremost reasons that the classical control theory is widely used in practice is that transfer functions can be easily obtained by experiments. Therefore, we could conveniently obtain a nonlinear model which is much accurate than the linear one.

IV. H_∞ OUTPUT FEEDBACK CONTROLLER DESIGN FOR THE FIO MODEL

A. THE FIO GENERALIZED MODEL

According to the standard H_∞ control problem discussed in [37] and [38], the generalized plant with η rules is constructed as follows:

$$\begin{cases} \dot{\boldsymbol{x}}^1(t) = \boldsymbol{A}^1\boldsymbol{x}^1(t) + \boldsymbol{B}_1^1\boldsymbol{w}(t) + \boldsymbol{B}_2^1\boldsymbol{u}(t) \\ \dot{\boldsymbol{x}}^2(t) = \boldsymbol{A}^2\boldsymbol{x}^2(t) + \boldsymbol{B}_1^2\boldsymbol{w}(t) + \boldsymbol{B}_2^2\boldsymbol{u}(t) \\ \qquad \vdots \\ \dot{\boldsymbol{x}}^i(t) = \boldsymbol{A}^i\boldsymbol{x}^i(t) + \boldsymbol{B}_1^i\boldsymbol{w}(t) + \boldsymbol{B}_2^i\boldsymbol{u}(t) \\ \qquad \vdots \\ \dot{\boldsymbol{x}}^\eta(t) = \boldsymbol{A}^\eta\boldsymbol{x}^\eta(t) + \boldsymbol{B}_1^\eta\boldsymbol{w}(t) + \boldsymbol{B}_2^\eta\boldsymbol{u}(t) \end{cases} \tag{4-1}$$

$$\boldsymbol{z}^i(t) = \boldsymbol{C}_1^i\boldsymbol{x}^i(t) + \boldsymbol{D}_{11}^i\boldsymbol{w}(t) + \boldsymbol{D}_{12}^i\boldsymbol{u}(t), \quad i = 1, 2, \ldots, \eta \tag{4-2}$$

$$\boldsymbol{y}^i(t) = \boldsymbol{C}_2^i\boldsymbol{x}^i(t) + \boldsymbol{D}_{21}^i\boldsymbol{w}(t) + \boldsymbol{D}_{22}^i\boldsymbol{u}(t), \quad i = 1, 2, \ldots, \eta \tag{4-3}$$

$$\boldsymbol{z}(t) = \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\boldsymbol{z}^i(t) \tag{4-4}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\boldsymbol{y}^i(t) \tag{4-5}$$

where $\boldsymbol{x}_i$ $(i = 1, 2, \ldots, \eta)$ are the state variables of the system; $\boldsymbol{w} \in \mathbf{R}^{m_1}$ are the exogenous inputs; $\boldsymbol{u} \in \mathbf{R}^{m_2}$ are the control inputs; $z_i, z \in \mathbf{R}^{p_1}$ $(i = 1, 2, \ldots, \eta)$ are the regulated outputs; $\boldsymbol{y}_i, \boldsymbol{y} \in \mathbf{R}^{p_2}$ $(i = 1, 2, \ldots, \eta)$ are measured outputs; $\boldsymbol{v} \in \mathbf{R}^g$ are the fuzzy premise variables; $\boldsymbol{A}^i \in \mathbf{R}^{n_i\times n_i}$, $\boldsymbol{B}_1^i \in \mathbf{R}^{n_i\times m_1}$, $\boldsymbol{B}_2^i \in \mathbf{R}^{n_i\times m_2}$, $\boldsymbol{C}_1^i \in \mathbf{R}^{p_1\times n_i}$, $\boldsymbol{C}_2^i \in \mathbf{R}^{p_2\times n_i}$, $\boldsymbol{D}_{11}^i \in \mathbf{R}^{p_1\times m_1}$, $\boldsymbol{D}_{12}^i \in \mathbf{R}^{p_1\times m_2}$, $\boldsymbol{D}_{21}^i \in \mathbf{R}^{p_2\times m_1}$, $\boldsymbol{D}_{22}^i \in \mathbf{R}^{p_2\times m_2}$.

Equations (4-1)~(4-5) can be cast into a more compact form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_1\boldsymbol{w}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) \tag{4-6}$$

$$\boldsymbol{z}(t) = \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\left(\tilde{\boldsymbol{C}}_1^i\boldsymbol{x}(t) + \boldsymbol{D}_{11}^i\boldsymbol{w}(t) + \boldsymbol{D}_{12}^i\boldsymbol{u}(t)\right) \tag{4-7}$$

$$\boldsymbol{y}(t) = \sum_{i=1}^{\eta}\mu_i(\boldsymbol{v}(t))\left(\tilde{\boldsymbol{C}}_2^i\boldsymbol{x}(t) + \boldsymbol{D}_{21}^i\boldsymbol{w}(t) + \boldsymbol{D}_{22}^i\boldsymbol{u}(t)\right) \tag{4-8}$$

where

$$\boldsymbol{x}(t) = \left[\left(\boldsymbol{x}^1(t)\right)^{\mathrm{T}}, \left(\boldsymbol{x}^2(t)\right)^{\mathrm{T}}, \cdots, \left(\boldsymbol{x}^\eta(t)\right)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\boldsymbol{x} \in \mathbf{R}^{\sum_{k=1}^{\eta} n_k},$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}^1 & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}^2 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{A}^i & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \boldsymbol{A}^\eta \end{bmatrix},$$

$$\boldsymbol{A} \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times(\sum_{k=1}^{\eta} n_k)},$$

$$\boldsymbol{B}_1 = \left[\left(\boldsymbol{B}_1^1\right)^{\mathrm{T}}, \left(\boldsymbol{B}_1^2\right)^{\mathrm{T}}, \cdots, \left(\boldsymbol{B}_1^\eta\right)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\boldsymbol{B}_1 \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times m_1},$$

$$\boldsymbol{B}_2 = \left[\left(\boldsymbol{B}_2^1\right)^{\mathrm{T}}, \left(\boldsymbol{B}_2^2\right)^{\mathrm{T}}, \cdots, \left(\boldsymbol{B}_2^\eta\right)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\boldsymbol{B}_2 \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times m_2},$$

$$\tilde{\boldsymbol{C}}_1^i = \left[\delta_{i,1}\boldsymbol{C}_1^1,\ \delta_{i,2}\boldsymbol{C}_1^2, \cdots,\ \delta_{i,i}\boldsymbol{C}_1^i,\ \cdots,\ \delta_{i,\eta}\boldsymbol{C}_1^\eta\right],$$

$$\tilde{\boldsymbol{C}}_1^i \in \mathbf{R}^{p_1\times(\sum_{k=1}^{\eta} n_k)},$$

$$\tilde{\boldsymbol{C}}_2^i = \left[\delta_{i,1}\boldsymbol{C}_2^1,\ \delta_{i,2}\boldsymbol{C}_2^2, \cdots,\ \delta_{i,i}\boldsymbol{C}_2^i,\ \cdots,\ \delta_{i,\eta}\boldsymbol{C}_2^\eta\right],$$

$$\tilde{\boldsymbol{C}}_2^i \in \mathbf{R}^{p_2\times(\sum_{k=1}^{\eta} n_k)},$$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

is Kronecker delta function.

For convenience, the two assumptions are assumed throughout this article:

(i) $(\boldsymbol{A}^i, \boldsymbol{B}_2^i, \tilde{\boldsymbol{C}}_2^i)$ $(i = 1, \ldots, \eta)$ is stabilizable and detectable,

(ii) $\boldsymbol{D}_{22}^i = \mathbf{0}$.

B. MAIN RESULTS

Given the generalized plant (4-6)~(4-8), our goal is to synthesize a controller in the form of (4-9)(4-10) that could render the L_2 gain from the disturbance w to the error signal z less than γ as well as guarantee the stability of the closed-loop system.

$$\dot{\boldsymbol{x}}_{\mathrm{c}} = \sum_{j=1}^{\eta}\mu_j\left(\boldsymbol{A}_{\mathrm{K}}^j\boldsymbol{x}_{\mathrm{c}} + \boldsymbol{B}_{\mathrm{K}}^j\boldsymbol{y}\right) \tag{4-9}$$

$$\boldsymbol{u} = \sum_{j=1}^{\eta}\mu_j\left(\boldsymbol{C}_{\mathrm{K}}^j\boldsymbol{x}_{\mathrm{c}} + \boldsymbol{D}_{\mathrm{K}}^j\boldsymbol{y}\right) \tag{4-10}$$

where $\boldsymbol{x}_{\mathrm{c}} = \left[\left(\boldsymbol{x}_{\mathrm{c}}^1\right)^{\mathrm{T}}, \left(\boldsymbol{x}_{\mathrm{c}}^2\right)^{\mathrm{T}}, \cdots, \left(\boldsymbol{x}_{\mathrm{c}}^\eta\right)^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{x}_{\mathrm{c}} \in \mathbf{R}^{\sum_{k=1}^{\eta} n_k}$, $\boldsymbol{x}_{\mathrm{c}}^j \in \mathbf{R}^{n_j}$ $(j = 1, \ldots, \eta)$ are the states of controllers, $\boldsymbol{u}^j \in \mathbf{R}^{m_2}$ $(j = 1, \ldots, \eta)$ the outputs of controllers, $\boldsymbol{A}_{\mathrm{K}}^j \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times(\sum_{k=1}^{\eta} n_k)}$, $\boldsymbol{B}_{\mathrm{K}}^j \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times p_2}$, $\boldsymbol{C}_{\mathrm{K}}^j \in \mathbf{R}^{m_2\times(\sum_{k=1}^{\eta} n_k)}$, $\boldsymbol{D}_{\mathrm{K}}^j \in \mathbf{R}^{m_2\times p_2}$ $(j = 1, \ldots, \eta)$ are the matrices, which will be designed.

Applying the controller (4-9)(4-10) into the generalized plant (4-6)~(4-8) gives the following closed-loop system (4-11)~(4-13), as shown at the top of the next page.

$$\dot{x} = \sum_{j=1}^{\eta}\sum_{i=1}^{\eta}\mu_j(v)\mu_i(v)\left\{\begin{bmatrix} A + B_2 D_K^j \tilde{C}_2^i & B_2 C_K^j \end{bmatrix}\begin{bmatrix} x \\ x_c \end{bmatrix} + \left(B_1 + B_2 D_K^j D_{21}^i\right) w\right\} \quad (4\text{-}11)$$

$$\dot{x}_c = \sum_{j=1}^{\eta}\sum_{i=1}^{\eta}\mu_j(v)\mu_i(v)\left\{\begin{bmatrix} B_K^j \tilde{C}_2^i & A_K^j \end{bmatrix}\begin{bmatrix} x \\ x_c \end{bmatrix} + B_K^j D_{21}^i w\right\} \quad (4\text{-}12)$$

$$z = \sum_{j=1}^{\eta}\sum_{i=1}^{\eta}\mu_j(v)\mu_i^2(v)\left\{\begin{bmatrix} \tilde{C}_1^i + D_{12}^i D_K^j \tilde{C}_2^i & D_{12}^i C_K^j \end{bmatrix}\begin{bmatrix} x \\ x_c \end{bmatrix} + \left(D_{11}^i + D_{12}^i D_K^j D_{21}^i\right) w\right\} \quad (4\text{-}13)$$

Moreover, we have

$$\dot{\xi} = \sum_{j=1}^{\eta}\sum_{i=1}^{\eta}\mu_j(v)\mu_i(v)\left(A_{cl}^{i,j}\xi + B_{cl}^{i,j} w\right) \quad (4\text{-}14)$$

$$z = \sum_{j=1}^{\eta}\sum_{i=1}^{\eta}\mu_j(v)\mu_i^2(v)\left(C_{cl}^{i,j}\xi + D_{cl}^{i,j} w\right) \quad (4\text{-}15)$$

where

$$\xi = \begin{bmatrix} x \\ x_c \end{bmatrix},\quad A_{cl}^{i,j} = \begin{bmatrix} A + B_2 D_K^j \tilde{C}_2^i & B_2 C_K^j \\ B_K^j \tilde{C}_2^i & A_K^j \end{bmatrix},$$

$$B_{cl}^{i,j} = \begin{bmatrix} B_1 + B_2 D_K^j D_{21}^i \\ B_K^j D_{21}^i \end{bmatrix},$$

$$C_{cl}^{i,j} = \begin{bmatrix} \tilde{C}_1^i + D_{12}^i D_K^j \tilde{C}_2^i & D_{12}^i C_K^j \end{bmatrix},$$

$$D_{cl}^{i,j} = D_{11}^i + D_{12}^i D_K^j D_{21}^i,$$

for $h_i \cap h_j \neq \emptyset$, $i, j = 1, \ldots, \eta$.

It follows from the bounded real lemma [39], [40], internal stability and the H_∞-norm constraint are jointly equivalent to the existence of $X_{cl} > 0$ of the dimension $\left(2\sum_{k=1}^{\eta} n_k\right) \times \left(2\sum_{k=1}^{\eta} n_k\right)$ such that

$$\begin{bmatrix} \left(A_{cl}^{i,j}\right)^T X_{cl} + X_{cl} A_{cl}^{i,j} & X_{cl} B_{cl}^{i,j} & \left(C_{cl}^{i,j}\right)^T \\ \left(B_{cl}^{i,j}\right)^T X_{cl} & -\gamma I & \left(D_{cl}^{i,j}\right)^T \\ C_{cl}^{i,j} & D_{cl}^{i,j} & -\gamma I \end{bmatrix} < 0 \quad (4\text{-}16)$$

for $h_i \cap h_j \neq \emptyset$, $i, j = 1, \ldots, \eta$.

Notice that in the above inequality the unknown Lyapunov matrix X_{cl} and the controller matrices A_K^j, B_K^j, C_K^j, and D_K^j are coupled together in a nonlinear way. Therefore, those unknowns cannot be directly solved by using the convex optimization method in its current form. However, by some manipulations it can be reduced to an LMI in terms of unknowns and can be solved efficiently by some numerical algorithms, which leads to our main result.

Theorem 1: The fuzzy dynamical output feedback H_∞ controller can be synthesized for the generalized plant (4-6)∼(4-8), by solving the following optimization problem:

$$\min\ \gamma \quad (4\text{-}17)$$

$$\text{s.t.} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (4\text{-}18)$$

and (4-19), as shown at the top of the next page, where

$$\hat{A}_{i,j} = Y\left(A + B_2 D_K^j \tilde{C}_2^i\right) X + N B_K^j \tilde{C}_2^i X + Y B_2 C_K^j M^T + N A_K^j M^T,\quad \hat{A}_{i,j} \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)(\sum_{k=1}^{\eta} n_k)} \quad (4\text{-}20)$$

$$\hat{B}_j = Y B_2 D_K^j + N B_K^j,\quad \hat{B}_j \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times p_2} \quad (4\text{-}21)$$

$$\hat{C}_{i,j} = D_K^j \tilde{C}_2^i X + C_K^j M^T,\quad \hat{C}_{i,j} \in \mathbf{R}^{m_2\times(\sum_{k=1}^{\eta} n_k)} \quad (4\text{-}22)$$

$$\hat{D}_j = D_K^j,\quad \hat{D}_j \in \mathbf{R}^{m_2\times p_2} \quad (4\text{-}23)$$

Proof: Inspired by the procedure in [41], we have the following proof.

Partition X_{cl} in (4-16) and X_{cl}^{-1} as $X_{cl} = \begin{bmatrix} Y & N \\ N^T & W \end{bmatrix}$ and $X_{cl}^{1} = \begin{bmatrix} X & M \\ M^T & Z \end{bmatrix}$, and define $F_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$ and $F_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$, where $X, Y \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times(\sum_{k=1}^{\eta} n_k)}$ are symmetric, $M, N \in \mathbf{R}^{(\sum_{k=1}^{\eta} n_k)\times(\sum_{k=1}^{\eta} n_k)}$ have full column rank. Then we have

$$F_1^T X_{cl} = F_2^T \quad (4\text{-}24)$$

$$I = YX + NM^T \quad (4\text{-}25)$$

$$I = N^T M + WZ \quad (4\text{-}26)$$

$$0 = YM + NZ \quad (4\text{-}27)$$

$$0 = N^T X + WM^T \quad (4\text{-}28)$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \quad (4\text{-}29)$$

and $\begin{bmatrix} F_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$ has full column rank.

So the inequality (4-16) is equivalent to

$$\begin{bmatrix} F_1^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \left(A_{cl}^{i,j}\right)^T X_{cl} + X_{cl} A_{cl}^{i,j} & X_{cl} B_{cl}^{i,j} & \left(C_{cl}^{i,j}\right)^T \\ \left(B_{cl}^{i,j}\right)^T X_{cl} & -\gamma I & \left(D_{cl}^{i,j}\right)^T \\ C_{cl}^{i,j} & D_{cl}^{i,j} & -\gamma I \end{bmatrix} \times \begin{bmatrix} F_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \quad (4\text{-}30)$$

$$\begin{bmatrix} XA^{\mathrm{T}}+AX+B_2\hat{C}_{i,j}+\left(B_2\hat{C}_{i,j}\right)^{\mathrm{T}} & \hat{A}_{i,j}^{\mathrm{T}}+\left(A+B_2\hat{D}_j\tilde{C}_2^i\right) & B_1+B_2\hat{D}_jD_{21}^i & \left(\tilde{C}_1^iX+D_{12}^i\hat{C}_{i,j}\right)^{\mathrm{T}} \\ \Theta_{21} & A^{\mathrm{T}}Y+YA+\hat{B}_j\tilde{C}_2^i+\left(\hat{B}_j\tilde{C}_2^i\right)^{\mathrm{T}} & YB_1+\hat{B}_jD_{21}^i & \left(\tilde{C}_1^i+D_{12}^i\hat{D}_j\tilde{C}_2^i\right)^{\mathrm{T}} \\ \Theta_{31} & \Theta_{32} & -\gamma\mathbf{I} & \left(D_{11}^i+D_{12}^i\hat{D}_jD_{21}^i\right)^{\mathrm{T}} \\ \Theta_{41} & \Theta_{42} & \Theta_{43} & -\gamma\mathbf{I} \end{bmatrix}<\mathbf{0} \tag{4-19}$$

Then we get

$$\begin{bmatrix} F_1^{\mathrm{T}}\left(A_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}}X_{\mathrm{cl}}F_1+F_1^{\mathrm{T}}X_{\mathrm{cl}}A_{\mathrm{cl}}^{i,j}F_1 & F_1^{\mathrm{T}}X_{\mathrm{cl}}B_{\mathrm{cl}}^{i,j} & F_1^{\mathrm{T}}\left(C_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}} \\ \left(B_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}}X_{\mathrm{cl}}F_1 & -\gamma\mathbf{I} & \left(D_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}} \\ C_{\mathrm{cl}}^{i,j}F_1 & D_{\mathrm{cl}}^{i,j} & -\gamma\mathbf{I} \end{bmatrix}<\mathbf{0} \tag{4-31}$$

Multiplying (4-24) by $A_{\mathrm{cl}}F_1$ on the right, we get (4-32), as shown at the top of the next page, further (4-33), as shown at the top of the next page is obtained.

Equation (4-24) is multiplied by B_{cl} on the right, then we get (4-34).

$$\begin{aligned} F_1^{\mathrm{T}}X_{\mathrm{cl}}B_{\mathrm{cl}}^{i,j}&=F_2^{\mathrm{T}}B_{\mathrm{cl}}^{i,j}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ Y & N\end{bmatrix}\begin{bmatrix}B_1+B_2D_{\mathrm{K}}^jD_{21}^i\\ B_{\mathrm{K}}^jD_{21}^i\end{bmatrix}\\ &=\begin{bmatrix}B_1+B_2D_{\mathrm{K}}^jD_{21}^i\\ YB_1+\left(YB_2D_{\mathrm{K}}^j+NB_{\mathrm{K}}^j\right)D_{21}^i\end{bmatrix} \end{aligned} \tag{4-34}$$

Moreover, we have

$$\begin{aligned} &\left(B_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}}X_{\mathrm{cl}}F_1\\ &=\left(F_1^{\mathrm{T}}X_{\mathrm{cl}}B_{\mathrm{cl}}^{i,j}\right)^T\\ &=\Big[\left(B_1+B_2D_{\mathrm{K}}^jD_{21}^i\right)^T\\ &\qquad (YB_1)^T+\left(D_{21}^i\right)^T\left(YB_2D_{\mathrm{K}}^j+NB_{\mathrm{K}}^j\right)^T\Big] \end{aligned} \tag{4-35}$$

By matrix operation, we get

$$\begin{aligned} C_{\mathrm{cl}}^{i,j}F_1&=\begin{bmatrix}\tilde{C}_1^i+D_{12}^iD_{\mathrm{K}}^j\tilde{C}_2^i & D_{12}^iC_{\mathrm{K}}^j\end{bmatrix}\begin{bmatrix}X & \mathbf{I}\\ M^{\mathrm{T}} & \mathbf{0}\end{bmatrix}\\ &=\Big[\tilde{C}_1^iX+D_{12}^i\left(D_{\mathrm{K}}^j\tilde{C}_2^iX+C_{\mathrm{K}}^jM^{\mathrm{T}}\right)\\ &\qquad \tilde{C}_1^i+D_{12}^iD_{\mathrm{K}}^j\tilde{C}_2^i\Big] \end{aligned} \tag{4-36}$$

$$\begin{aligned} F_1^{\mathrm{T}}&=\left(C_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}}=\left(C_{\mathrm{cl}}^{i,j}F_1\right)^{\mathrm{T}}\\ &=\begin{bmatrix}\left\{\tilde{C}_1^iX+D_{12}^i\left(D_{\mathrm{K}}^j\tilde{C}_2^iX+C_{\mathrm{K}}^jM^{\mathrm{T}}\right)\right\}^{\mathrm{T}}\\ \left(\tilde{C}_1^i+D_{12}^iD_{\mathrm{K}}^j\tilde{C}_2^i\right)^{\mathrm{T}}\end{bmatrix} \end{aligned} \tag{4-37}$$

$$\left(D_{\mathrm{cl}}^{i,j}\right)^{\mathrm{T}}=\left(D_{11}^i+D_{12}^iD_{\mathrm{K}}^jD_{21}^i\right)^{\mathrm{T}} \tag{4-38}$$

So the inequality (4-31) is transformed into

$$\begin{bmatrix}\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14}\\ \Theta_{21} & \Theta_{22} & \Theta_{23} & \Theta_{24}\\ \Theta_{31} & \Theta_{32} & \Theta_{33} & \Theta_{34}\\ \Theta_{41} & \Theta_{42} & \Theta_{43} & \Theta_{44}\end{bmatrix}<\mathbf{0} \tag{4-39}$$

for $h_i\cap h_j\neq\emptyset$, $i,j=1,\ldots,\eta$, where

$$\begin{aligned} \Theta_{11}&=XA^{\mathrm{T}}+AX+\left(D_{\mathrm{K}}^j\tilde{C}_2^iX+C_{\mathrm{K}}^jM^{\mathrm{T}}\right)^{\mathrm{T}}B_2^{\mathrm{T}}\\ &\quad+B_2\left(D_{\mathrm{K}}^j\tilde{C}_2^iX+C_{\mathrm{K}}^jM^{\mathrm{T}}\right)\\ \Theta_{12}&=\Big\{Y\left(A+B_2D_{\mathrm{K}}^j\tilde{C}_2^i\right)X+NB_{\mathrm{K}}^j\tilde{C}_2^iX\\ &\qquad+YB_2C_{\mathrm{K}}^jM^{\mathrm{T}}+NA_{\mathrm{K}}^jM^{\mathrm{T}}\Big\}^{\mathrm{T}}+A+B_2D_{\mathrm{K}}^j\tilde{C}_2^i\\ \Theta_{13}&=B_1+B_2D_{\mathrm{K}}^jD_{21}^i\\ \Theta_{14}&=\left\{\tilde{C}_1^iX+D_{12}^i\left(D_{\mathrm{K}}^j\tilde{C}_2^iX+C_{\mathrm{K}}^jM^{\mathrm{T}}\right)\right\}^{\mathrm{T}}\\ \Theta_{22}&=A^{\mathrm{T}}Y+YA+\left(\tilde{C}_2^i\right)^{\mathrm{T}}\left(YB_2D_{\mathrm{K}}^j+NB_{\mathrm{K}}^j\right)^{\mathrm{T}}\\ &\quad+\left(YB_2D_{\mathrm{K}}^j+NB_{\mathrm{K}}^j\right)\tilde{C}_2^i\\ \Theta_{23}&=YB_1+\left(YB_2D_{\mathrm{K}}^j+NB_{\mathrm{K}}^j\right)D_{21}^i\\ \Theta_{24}&=\left(\tilde{C}_1^i+D_{12}^iD_{\mathrm{K}}^j\tilde{C}_2^i\right)^{\mathrm{T}}\\ \Theta_{33}&=-\gamma\mathbf{I}\\ \Theta_{34}&=\left(D_{11}^i+D_{12}^iD_{\mathrm{K}}^jD_{21}^i\right)^{\mathrm{T}}\\ \Theta_{44}&=-\gamma\mathbf{I} \end{aligned}$$

and Θ_{21}, Θ_{31}, Θ_{32}, Θ_{41}, Θ_{42}, Θ_{43} can be inferred by symmetry.

In order to transform the matrix inequality (4-39) into an equivalent LMI, for $i,j=1,\ldots,\eta$, the changes of controller variables are defined as (4-20)~(4-23), then the matrix inequality (4-39) is transformed as (4-19).

Obviously the Inequality (4-19) is a LMI in terms of $\hat{A}_{i,j}$, $\hat{B}_j$, $\hat{C}_{i,j}$, $\hat{D}_j$, X and Y. If M and N have full row rank, and if $\hat{A}_{i,j}$, $\hat{B}_j$, $\hat{C}_{i,j}$, $\hat{D}_j$ and X, $Y$are given, the controller matrices A_{K}^j, B_{K}^j, C_{K}^j, D_{K}^j $(j=1,\ldots,\eta)$ can always be computed, which meet (4-39), i.e. (4-16) is satisfied. For full order design, one can always assume that M and N have full row rank. Hence the variables A_{K}^j, B_{K}^j, C_{K}^j, D_{K}^j can be replaced by $\hat{A}_{i,j}$, $\hat{B}_j$, $\hat{C}_{i,j}$, $\hat{D}_j$ without loss of generality [41], [42].

$$F_1^{\mathrm{T}} X_{\mathrm{cl}} A_{\mathrm{cl}} F_1 = F_2^{\mathrm{T}} A_{\mathrm{cl}} F_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ Y & N \end{bmatrix} \begin{bmatrix} A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i & B_2 C_{\mathrm{K}}^j \\ B_{\mathrm{K}}^j \tilde{C}_2^i & A_{\mathrm{K}}^j \end{bmatrix} \begin{bmatrix} X & \mathbf{I} \\ M^{\mathrm{T}} & \mathbf{0} \end{bmatrix}$$
$$= \begin{bmatrix} AX + B_2 \left(D_{\mathrm{K}}^j \tilde{C}_2^i X + C_{\mathrm{K}}^j M^{\mathrm{T}} \right) & A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \\ Y \left(A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \right) X + N B_{\mathrm{K}}^j \tilde{C}_2^i X + Y B_2 C_{\mathrm{K}}^j M^{\mathrm{T}} + N A_{\mathrm{K}}^j M^{\mathrm{T}} & YA + \left(Y B_2 D_{\mathrm{K}}^j + N B_{\mathrm{K}}^j \right) \tilde{C}_2^i \end{bmatrix} \tag{4-32}$$

$$F_1^{\mathrm{T}} A_{\mathrm{cl}}^{\mathrm{T}} X_{\mathrm{cl}} F_1 + F_1^{\mathrm{T}} X_{\mathrm{cl}} A_{\mathrm{cl}} F = \left(F_1^{\mathrm{T}} X_{\mathrm{cl}} A_{\mathrm{cl}} F_1 \right)^T + F_1^{\mathrm{T}} X_{\mathrm{cl}} A_{\mathrm{cl}} F$$
$$= \begin{bmatrix} \begin{array}{c} X^T A^T + AX + \left(D_{\mathrm{K}}^j \tilde{C}_2^i X + C_{\mathrm{K}}^j M^{\mathrm{T}} \right)^{\mathrm{T}} B_2^{\mathrm{T}} \\ + B_2 \left(D_{\mathrm{K}}^j \tilde{C}_2^i X + C_{\mathrm{K}}^j M^{\mathrm{T}} \right) \end{array} & \begin{array}{c} \left\{ Y \left(A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \right) X + N B_{\mathrm{K}}^j \tilde{C}_2^i X + Y B_2 C_{\mathrm{K}}^j M^{\mathrm{T}} + N A_{\mathrm{K}}^j M^{\mathrm{T}} \right\}^{\mathrm{T}} \\ + A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \end{array} \\ \begin{array}{c} \left\{ A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \right\}^{\mathrm{T}} + Y \left(A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \right) X \\ + N B_{\mathrm{K}}^j \tilde{C}_2^i X + Y B_2 C_{\mathrm{K}}^j M^{\mathrm{T}} + N A_{\mathrm{K}}^j M^{\mathrm{T}} \end{array} & A^{\mathrm{T}} Y + YA + \left(\tilde{C}_2^i \right)^{\mathrm{T}} \left(Y B_2 D_{\mathrm{K}}^j + N B_{\mathrm{K}}^j \right)^{\mathrm{T}} + \left(Y B_2 D_{\mathrm{K}}^j + N B_{\mathrm{K}}^j \right) \tilde{C}_2^i \end{bmatrix} \tag{4-33}$$

The variable γ in (4-19) can be directly minimized by LMI optimization to find the smallest achievable H_∞ norm [41], which ends the proof.

After solving the synthesis LMIs (4-19), $\hat{A}_{i,j}$, $\hat{B}_j$, $\hat{C}_{i,j}$, $\hat{D}_j$, X and Y are obtained. The parameters of the controller can be constructed according to the following procedure.

1. We need to find two invertible matrices M, $N \in \mathbf{R}^{\left(\sum_{k=1}^{\eta} n_k\right)^2}$ via singular value decomposition (SVD) such that

$$MN^{\mathrm{T}} = \mathbf{I} - XY \tag{4-40}$$

2. The controller can be constructed by

$$D_{\mathrm{K}}^j = \hat{D}_j \tag{4-41}$$

$$C_{\mathrm{K}}^j = \left(\hat{C}_{i,j} - D_{\mathrm{K}}^j \tilde{C}_2^i X \right) \left(M^{\mathrm{T}} \right)^{-1} \tag{4-42}$$

$$B_{\mathrm{K}}^j = N^{-1} \left(\hat{B}_j - Y B_2 D_{\mathrm{K}}^j \right) \tag{4-43}$$

$$A_{\mathrm{K}}^j = N^{-1} \left[\hat{A}_{i,j} - Y \left(A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^i \right) X \right] \left(M^{\mathrm{T}} \right)^{-1} - B_{\mathrm{K}}^j \tilde{C}_2^i X \left(M^{\mathrm{T}} \right)^{-1} - N^{-1} Y B_2 C_{\mathrm{K}}^j \tag{4-44}$$

for $i, j = 1, \ldots, \eta$. It should be mentioned that A_{K}^j, C_{K}^j in (4-44) and (4-42) are not unique. Here, A_{K}^j, C_{K}^j are computed only when $i = j$, so

$$C_{\mathrm{K}}^j = \left(\hat{C}_{j,j} - D_{\mathrm{K}}^j \tilde{C}_2^j X \right) \left(M^{\mathrm{T}} \right)^{-1} \tag{4-45}$$

$$A_{\mathrm{K}}^j = N^{-1} \left[\hat{A}_{j,j} - Y \left(A + B_2 D_{\mathrm{K}}^j \tilde{C}_2^j \right) X \right] \left(M^{\mathrm{T}} \right)^{-1} - B_{\mathrm{K}}^j \tilde{C}_2^j X \left(M^{\mathrm{T}} \right)^{-1} - N^{-1} Y B_2 C_{\mathrm{K}}^j \tag{4-46}$$

V. SIMULATION EXAMPLES

In this section, a numerical simulation is presented to illustrate the effectiveness of the above results.

For the motor-spring-mass system described in Section II-B, in order to make the angle error (y-r) and the control voltage u as small as possible, a two degree-of-freedom feedback controller is designed.

$$u = u_{\mathrm{fb}} + \sum_{j=1}^{2} \mu_j(v)\, k_{\mathrm{ff},j} r \tag{5-1}$$

where the forward controllers $k_{\mathrm{ff},1} = \frac{K_2}{K_e}$ and $k_{\mathrm{ff},2} = \frac{K_2 + R^2 K_1}{K_e}$ are deduced, and the feedback controller u_{fb}, a fuzzy dynamical output feedback H_∞ controller, will be designed according to Theorem 1. It follows from Section II-B, we get the following generalized plant for the motor-spring-mass system.

Plant Rule 1

IF $y(t)$ is M_1,

THEN

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{x}_{1,2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K_2}{J} & -\frac{\mu_0}{J} \end{bmatrix} \begin{bmatrix} \xi_1 \\ x_{1,2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ \frac{K_e}{J} \end{bmatrix} u_{\mathrm{fb}} \tag{5-2}$$

$$\begin{bmatrix} \xi_1 \\ u_{\mathrm{fb}} + \frac{K_2}{K_e} r \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ x_{1,2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_2}{K_e} \end{bmatrix} r + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\mathrm{fb}} \tag{5-3}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 & x_{1,2} \end{bmatrix}^{\mathrm{T}} + [1]\, r + [0]\, u_{\mathrm{fb}} \tag{5-4}$$

Plant Rule 2

IF $y(t)$ is M_2,

THEN

$$\begin{bmatrix} \dot{\xi}_2 \\ \dot{x}_{2,2} \\ \dot{x}_{2,3} \\ \dot{x}_{2,4} \end{bmatrix}$$

$$
= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_2 + R^2 K_1}{J} & -\dfrac{\mu_0}{J} & \dfrac{RK_1}{J} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{RK_1}{M} & 0 & -\dfrac{2K_1}{M} & -\dfrac{\mu_1}{M} \end{bmatrix} \begin{bmatrix} \xi_2 \\ x_{2,2} \\ x_{2,3} \\ x_{2,4} \end{bmatrix}
+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{RK_1}{M} \end{bmatrix} r + \begin{bmatrix} 0 \\ \dfrac{K_e}{J} \\ 0 \\ 0 \end{bmatrix} u_{\mathrm{fb}} \tag{5-5}
$$

$$
\begin{bmatrix} \xi_2 \\ u_{\mathrm{fb}} + \dfrac{K_2 + R^2 K_1}{K_e} r \end{bmatrix}
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_2 \\ x_{2,2} \\ x_{2,3} \\ x_{2,4} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_2 + R^2 K_1}{K_e} \end{bmatrix} r
+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{\mathrm{fb}} \tag{5-6}
$$

$$
y_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_2 & x_{2,2} & x_{2,3} & x_{2,4} \end{bmatrix}^{\mathrm{T}}
+ [1]\, r + [0]\, u_{\mathrm{fb}} \tag{5-7}
$$

where the parameters of the system are chosen as $J = 0.02\mathrm{kg{\cdot}m^2}$, $K_2 = 0.1\mathrm{N{\cdot}m/rad}$, $K_{\mathrm{e}} = 1\mathrm{N{\cdot}m/V}$, $\mu_0 = 0.1\mathrm{N{\cdot}m{\cdot}s/rad}$, $R = 0.05\mathrm{m}$, $M = 1\mathrm{kg}$, $K_1 = 10\mathrm{N/m}$, $\mu_1 = 0.01\mathrm{N{\cdot}m{\cdot}s/rad}$, and the membership function M_1 and M_2 are

$$
M_1(y) = \begin{cases} 1 & y < \pi/2 \\ -\dfrac{y}{\pi} + 1.5 & \pi/2 \le y \le 3\pi/2 \\ 0 & y > 3\pi/2 \end{cases} \tag{5-8}
$$

$$
M_2(y) = \begin{cases} 0 & y < \pi/2 \\ \dfrac{y}{\pi} - 0.5 & \pi/2 \le y \le 3\pi/2 \\ 1 & y > 3\pi/2 \end{cases} \tag{5-9}
$$

The final outputs of the fuzzy input-output model of the generalized plant can be formulized as

$$
\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1 \boldsymbol{w} + \boldsymbol{B}_2 \boldsymbol{u}_{\mathrm{fb}} \tag{5-10}
$$

$$
\boldsymbol{z} = \sum_{i=1}^{2} \mu_i(\boldsymbol{v}) \left(\tilde{\boldsymbol{C}}_1^i \boldsymbol{x} + \boldsymbol{D}_{11}^i \boldsymbol{w} + \boldsymbol{D}_{12}^i \boldsymbol{u}_{\mathrm{fb}} \right) \tag{5-11}
$$

$$
\boldsymbol{y} = \sum_{i=1}^{2} \mu_i(\boldsymbol{v}) \left(\tilde{\boldsymbol{C}}_2^i \boldsymbol{x} + \boldsymbol{D}_{21}^i \boldsymbol{w} + \boldsymbol{D}_{22}^i \boldsymbol{u}_{\mathrm{fb}} \right) \tag{5-12}
$$

where

$$
\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix}, \quad \boldsymbol{x}_1 = \begin{bmatrix} \xi_1 \\ x_{1,2} \end{bmatrix},
$$

$$
\boldsymbol{x}_2 = \begin{bmatrix} \xi_2 & x_{2,2} & x_{2,3} & x_{2,4} \end{bmatrix}^{\mathrm{T}},
$$

$$
\boldsymbol{w} = [r], \quad \boldsymbol{u}_{\mathrm{fb}} = [u_{\mathrm{fb}}],
$$

$$
\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}^1 & 0 \\ 0 & \boldsymbol{A}^2 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{B}_1^1 \\ \boldsymbol{B}_1^2 \end{bmatrix}, \boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{B}_2^1 \\ \boldsymbol{B}_2^2 \end{bmatrix},
$$

TABLE 2. The parameters of controller.

	Values of the parameters					
$\boldsymbol{A}_{\mathrm{K}}^1$	-7897.10	45.29384	14.02529	11.85418	15.64852	-86844.8
	160.3101	-6.49200	0.021624	-0.44313	-0.53723	3107.724
	-40.4777	-3.90822	0.438589	-2.17105	-1.69596	-156.720
	169.0423	1.682324	0.842035	8.775678	7.910897	-355.288
	-267.014	2.337871	-1.94007	-11.6172	-8.06342	565.2251
	38130.02	-182.857	425.8791	-47.8573	-159.25	-80118.5
$\boldsymbol{A}_{\mathrm{K}}^2$	-6356.80	37.96556	27.59935	9.85560	9.833354	-86420.5
	105.1907	-6.23877	-0.46389	-0.37162	-0.32918	3092.558
	-37.7192	-4.55071	0.480818	-2.17505	-1.70986	-155.974
	175.2735	-0.46385	0.957098	8.766191	7.875646	-353.551
	-277.085	0.998228	-1.98942	-11.605	-8.03307	562.4459
	39551.04	-187.212	438.3463	-49.7009	-164.604	-79724.4
$\left(\boldsymbol{B}_{\mathrm{K}}^1\right)^{\mathrm{T}}$	-586.148	-9.38297	-2124.08	-7141.67	-4676.46	7654.134
$\left(\boldsymbol{B}_{\mathrm{K}}^2\right)^{\mathrm{T}}$	178.4326	-36.7501	-2122.68	-7138.54	-4681.44	8359.954
$\boldsymbol{C}_{\mathrm{K}}^1$	5.661497	-0.03246	-0.01006	-0.00851	-0.01123	62.26378
$\boldsymbol{C}_{\mathrm{K}}^2$	4.557176	-0.02721	-0.01979	-0.00708	-0.00706	61.95952
$\boldsymbol{D}_{\mathrm{K}}^1$	0.414542					
$\boldsymbol{D}_{\mathrm{K}}^2$	-0.133627					

$$
\tilde{\boldsymbol{C}}_1^1 = \begin{bmatrix} \boldsymbol{C}_1^1, & 0 \end{bmatrix}, \quad \tilde{\boldsymbol{C}}_1^2 = \begin{bmatrix} \boldsymbol{C}_1^2, & 0 \end{bmatrix}, \tilde{\boldsymbol{C}}_2^1 = \begin{bmatrix} 0, & \boldsymbol{C}_2^1 \end{bmatrix},
$$

$$
\tilde{\boldsymbol{C}}_2^2 = \begin{bmatrix} 0, & \boldsymbol{C}_2^2 \end{bmatrix},
$$

$$
\boldsymbol{A}^1 = \begin{bmatrix} 0 & 1 \\ -\dfrac{K_2}{J} & -\dfrac{\mu_0}{J} \end{bmatrix}, \quad \boldsymbol{B}_1^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{B}_2^1 = \begin{bmatrix} 0 \\ \frac{K_e}{J} \end{bmatrix},
$$

$$
\boldsymbol{C}_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{11}^1 = \begin{bmatrix} 0 \\ \dfrac{K_2}{K_e} \end{bmatrix}, \boldsymbol{D}_{12}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

$$
\boldsymbol{C}_2^1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{21}^1 = [1], \boldsymbol{D}_{22}^1 = [0],
$$

$$
\boldsymbol{A}^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_2 + R^2 K_1}{J} & -\dfrac{\mu_0}{J} & \dfrac{RK_1}{J} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{RK_1}{M} & 0 & -\dfrac{2K_1}{M} & -\dfrac{\mu_1}{M} \end{bmatrix},
$$

$$
\boldsymbol{B}_1^2 = \begin{bmatrix} 0 & 0 & 0 & \dfrac{RK_1}{M} \end{bmatrix}^{\mathrm{T}}, \quad \boldsymbol{B}_2^2 = \begin{bmatrix} 0 & \dfrac{K_e}{J} & 0 & 0 \end{bmatrix}^{\mathrm{T}},
$$

$$
\boldsymbol{C}_1^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{11}^2 = \begin{bmatrix} 0 \\ \dfrac{K_2 + R^2 K_1}{K_e} \end{bmatrix},
$$

$$
\boldsymbol{D}_{12}^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

$$
\boldsymbol{C}_2^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{21}^2 = [1], \boldsymbol{D}_{22}^2 = [0].
$$

By solving the LMIs in Theorem 1, we get the following dynamical output feedback H_∞ controller,

$$
\dot{\boldsymbol{x}}_{\mathrm{c}} = \sum_{j=1}^{2} \mu_j(\boldsymbol{v}) \left(\boldsymbol{A}_{\mathrm{K}}^j \boldsymbol{x}_{\mathrm{c}} + \boldsymbol{B}_{\mathrm{K}}^j \boldsymbol{y} \right) \tag{5-13}
$$

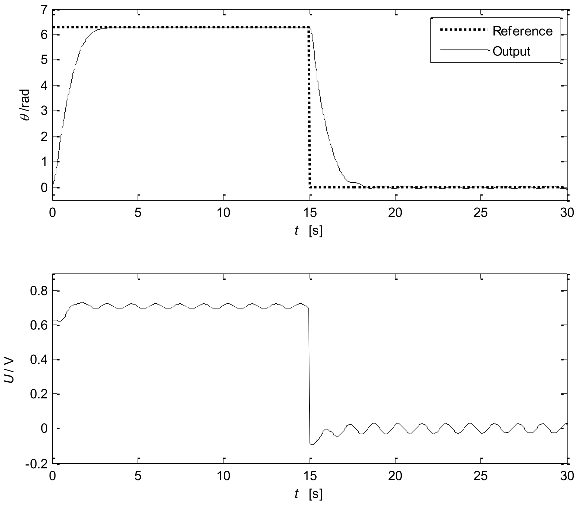

FIGURE 2. The output of the closed-loop fuzzy system and the control input.

$$u_{\mathrm{fb}} = \sum_{j=1}^{2} \mu_j (\boldsymbol{v}) \left(\boldsymbol{C}_{\mathrm{K}}^{j} \boldsymbol{x}_{\mathrm{c}} + \boldsymbol{D}_{\mathrm{K}}^{j} \boldsymbol{y} \right) \quad (5\text{-}14)$$

with the parameters listed in TABLE 2.

Then the fuzzy controllers (5-1) is employed to let the fuzzy input-output system in (5-2)∼(5-7) track the reference input in the form of positive and negative step. The output of the system and the control input are shown in FIGURE 2.

VI. CONCLUSION

In this article, the issue of dynamical output H_∞ controller designing is successfully addressed for the fuzzy input-output (FIO) model. The FIO model is different from the conventional Mamdani and T-S or T-S-K fuzzy models. So far there is no report on the H_∞ controller designing for this kind of nonlinear system. For this problem, a sufficient condition in terms of LMIs has been derived in this article to design a dynamical output feedback FIO controller. This condition can be efficiently solved by some commercial softwares, e. g., MATLAB. Moreover, a motor-spring-mass system abstracted from the real applications is provided to validate the applicability and efficiency of our method.

Our FIO model may encounter the problem of ''the curse of dimensionality'' because the dimensions of the matrices $\boldsymbol{A}$ in the plant and $\boldsymbol{A}_{\mathrm{k}}$ in the controller will be increasing significantly with the growth of the numbers of fuzzy rules. Consequently, numerical issues might be encountered when solving the controller in those scenarios.

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