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Width-Based Interval-Valued Distances and Fuzzy Entropies

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ABSTRACT In this paper, we study distance measures between interval-valued fuzzy sets and entropies of interval-valued fuzzy sets. These are well-known and widely used notions in the fuzzy sets theory. The novelty of our approach is twofold: on one hand, it considers the width of intervals in order to connect the uncertainty of the output with the uncertainty of the inputs. On the other hand, it makes use of total orders between intervals, instead of partial ones, so that the usefulness of the notions related with some kind of monotonicity is fully recovered in the interval-valued setting. The construction of distance measures and entropies is done by aggregating interval-valued restricted dissimilarity functions and interval-valued normal E_N-functions. For this reason, we first study these functions, both in line with the two above stated considerations. Finally, we present an illustrative example in image thresholding using an expression of the proposed interval-valued entropy to show the validity of our approach.

INDEX TERMS Admissible order, interval-valued distance, interval-valued entropy, interval-valued restricted dissimilarity function.

I. INTRODUCTION

Distance measures and entropies are significant notions in fuzzy sets theory due to their high applicability [18], [26], [29], [31], [35], [43], [45]. At the same time, interval-valued fuzzy sets [12] in many cases improve the results of fuzzy sets in different applications [1]–[4], [6], [7], [11], [14], [16], [17], [22], [34], [42], since they allow us to take into account the uncertainty linked to the construction of a precise membership function. For this reason, there exists a wide interest in the literature for extending the notions of distance measure and entropy to deal with interval-valued fuzzy sets and entropies [5], [23], [37], [38].

However, it is worth mentioning that in many recent developments in the field of interval-valued fuzzy sets we have found the following two problems, which, in our opinion, are an obstacle for the further development of the theory and the applications of interval-valued fuzzy sets:

1) In most of the cases, only a partial order between intervals is considered.
2) The widths of the intervals are not taken into account.

With these ideas in mind, the objective of this paper is to construct distance measures between interval-valued fuzzy sets and entropies of interval-valued fuzzy sets in such a way that:

1) A total order for intervals (not only partial) is used, since otherwise the usefulness of fundamental notions in the standard fuzzy set theory (aggregation functions, implications, inclusions, etc.) is not fully recovered in the interval-valued setting. The reason is that there may exist incomparable intervals, which means that all kinds of monotonicity are considerably weakened when working with intervals.
2) The widths of intervals are considered. We assume that the width of the membership interval of an element in a given set reflects the lack of knowledge of the precise membership degree of the element to the fuzzy set. So, if the real-valued membership degree is in fact an element inside the membership interval, then two elements

with the same interval membership degrees need not have the same real-valued membership degrees. Otherwise, the output of an interval function may display less uncertainty than its inputs.

To achieve this objective, we first introduce new definitions of interval-valued restricted dissimilarity functions and interval-valued normal E_N-functions, both in line with the above stated consideration. Then, interval-valued restricted dissimilarity functions and interval-valued normal E_N-functions are aggregated to obtain new distance measures and entropies of interval-valued fuzzy sets.

To show the validity of our developments, we present an application in image thresholding [8], [9], [25] using one of the expressions of the proposed interval-valued entropy. The results reveal that using the total order and taking into account the width of the intervals provide better results than other methods that can be found in the literature.

The paper is organized as follows. We start with some preliminaries. In Section III, we introduce new definitions of interval-valued restricted dissimilarity functions and interval-valued normal E_N-functions preserving the widths of intervals. In Section IV, the definition of distance measures and entropies of interval-valued fuzzy sets are introduced and different construction methods are studied. In Section V, we present an illustrative example of application of the proposed entropy in image thresholding. We finish with some conclusions and references.

II. PRELIMINARIES

In this section, we introduce several well known notions and results which are necessary for our subsequent developments. We consider closed subintervals of the unit interval [0, 1]. In this sense, we denote:

$$L([0,1]) = \{[\underline{X}, \overline{X}] \mid 0 \leq \underline{X} \leq \overline{X} \leq 1\}.$$

Capital letters will denote elements in $L([0,1])$ as well as the bounds of intervals. The width of the interval $X \in L([0,1])$ will be denoted by $w(X)$. Clearly $w(X) = \overline{X} - \underline{X}$. An interval function $f : (L([0,1]))^n \to L([0,1])$ is called w-preserving if $w(X_1) = \ldots = w(X_n)$ implies $w(f(X_1, \ldots, X_n)) = w(X_1)$.

A fuzzy set in an universe U is a mapping $A : U \to [0,1]$. An interval-valued fuzzy set is a mapping $A : U \to L([0,1])$. The class of all fuzzy sets in U is denoted by $FS(U)$ and the class of all interval-valued fuzzy sets in U, by $IVFS(U)$.

Another key notion in this work is that of order relation. We recall here its definition, adapted for the case of $L([0,1])$.

Definition 1: An order relation on $L([0,1])$ is a binary relation $\leq_L$ on $L([0,1])$ such that, for all $X, Y, Z \in L([0,1])$,

(i) *$X \leq_L X$, (reflexivity),*
(ii) *$X \leq_L Y$ and $Y \leq_L X$ imply $X = Y$, (antisymmetry),*
(iii) *$X \leq_L Y$ and $Y \leq_L Z$ imply $X \leq_L Z$, (transitivity).*

An order relation on $L([0,1])$ is called total or linear if any two elements of $L([0,1])$ are comparable, i.e., if for every $X, Y \in L([0,1])$, $X \leq_L Y$ or $Y \leq_L X$. An order relation on $L([0,1])$ is partial if it is not total.

We will denote by $\precsim_L$ the partial order relation on $L([0,1])$ induced by the usual partial order on $\mathbb{R}^2$, that is:

$$[\underline{X}, \overline{X}] \precsim_L [\underline{Y}, \overline{Y}] \quad \text{iff} \quad \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}. \tag{1}$$

This is the order relation most widely used in the literature [15].

We denote by $\leq_L$ any order in $L([0,1])$ (which can be partial or total) with $0_L = [0,0]$ as its minimal element and $1_L = [1,1]$ as its maximal element. To denote a total order in $L([0,1])$ with these minimal and maximal elements, we use the notation $\leq_{TL}$.

Example 2: (i) *A total order on $L([0,1])$ is, for example, Xu and Yager's order (see [44]). $[\underline{X}, \overline{X}] \leq_{XY} [\underline{Y}, \overline{Y}]$ if*

$$\begin{cases} \underline{X} + \overline{X} < \underline{Y} + \overline{Y} \text{ or} \\ \underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}. \end{cases} \tag{2}$$

This definition of Xu and Yager's order was originally provided for Atanassov intuitionistic fuzzy pairs [44].

(ii) *Another example of total order is provided by the lexicographical orders with respect to the first variable, $\leq_{lex1}$ and with respect to the second variable, $\leq_{lex2}$, which are defined, respectively, by:*

$$[\underline{X}, \overline{X}] \leq_{lex1} [\underline{Y}, \overline{Y}] \quad \text{if} \begin{cases} \underline{X} < \underline{Y} \text{ or} \\ \underline{X} = \underline{Y} \text{ and } \overline{X} \leq \overline{Y}. \end{cases}$$

$$[\underline{X}, \overline{X}] \leq_{lex2} [\underline{Y}, \overline{Y}] \quad \text{if} \begin{cases} \overline{X} < \overline{Y} \text{ or} \\ \overline{X} = \overline{Y} \text{ and } \underline{X} \leq \underline{Y}. \end{cases}$$

Regarding total orders in $L([0,1])$, we are going to consider the so-called admissible orders, whose definition we recall now.

Definition 3 [13]: An admissible order on $L([0,1])$ is a total order $\leq_{TL}$ that refines the partial order $\precsim_L$; that is, for every $X, Y \in L([0,1])$, if $X \precsim_L Y$ then $X \leq_{TL} Y$.

An interesting feature of admissible orders is that they can be built using aggregation functions, as stated in the following result. Recall that an aggregation function is a non-decreasing function $M : [0,1]^n \to [0,1]$ with $M(0, \ldots, 0) = 0$ and $M(1, \ldots, 1) = 1$, see [28]. An aggregation function is called idempotent if $M(x, \ldots, x) = x$ for all $x \in [0,1]$; and it is called symmetric if $M(x_1, \ldots, x_n) = M(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $x_1, \ldots, x_n \in [0,1]$ and all permutations σ on $\{1, \ldots, n\}$.

Proposition 4 [13]: Let $M_1, M_2 : [0,1]^2 \to [0,1]$ be two aggregation functions such that for all $X, Y \in L([0,1])$, the equalities $M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y})$ and $M_2(\underline{X}, \overline{X}) = M_2(\underline{Y}, \overline{Y})$ can only hold simultaneously if $X = Y$. The order $\leq_{M_1,M_2}$ on $L([0,1])$ given by

$$X \leq_{M_1,M_2} Y \quad \text{if} \begin{cases} M_1(\underline{X}, \overline{X}) < M_1(\underline{Y}, \overline{Y}) \text{ or} \\ M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y}) \text{ and} \\ M_2(\underline{X}, \overline{X}) \leq M_2(\underline{Y}, \overline{Y}) \end{cases}$$

is an admissible order on $L([0,1])$.

Example 5: (i) *Xu and Yager's order is an example of admissible order with* $M_1(x, y) = \frac{x+y}{2}$ *and* $M_2(x, y) = y$.

(ii) *The lexicographical orders* $\leq_{lex1}$ ($\leq_{lex2}$) *are also examples of admissible orders with* $M_1(x, y) = x$ ($M_1(x, y) = y$) *and* $M_2(x, y) = y$ ($M_2(x, y) = x$).

(iii) *More generally, if, for* $\alpha \in [0, 1]$ *we define the aggregation function*

$$K_\alpha(x, y) = (1 - \alpha)x + \alpha y$$

then, for $\alpha, \beta \in [0, 1]$ *with* $\alpha \neq \beta$, *we can obtain an admissible order* $\leq_{\alpha,\beta}$ *just taking* $M_1(x, y) = K_\alpha(x, y)$ *and* $M_2(x, y) = K_\beta(x, y)$. *See [13] for more details.*

Definition 6: *Let* $\leq_L$ *be an order relation in* $L([0, 1])$. *A function* $N: L([0, 1]) \to L([0, 1])$ *is an interval-valued negation function (IV negation) if it is a decreasing function with respect to the order* $\leq_L$ *such that* $N(0_L) = 1_L$ *and* $N(1_L) = 0_L$. *A negation* N *is called a strong negation if* $N(N(X)) = X$ *for every* $X \in L([0, 1])$. *An interval* $\varepsilon \in L([0, 1])$ *is called an equilibrium point of the IV negation if* $N(\varepsilon) = \varepsilon$.

A. INTERVAL-VALUED AGGREGATION FUNCTIONS WITH RESPECT TO A PARTIAL ORDER

The definition of aggregation function has been extended to the interval-valued setting with respect to the order $\precsim_L$ in a straightforward way [24].

Definition 7: *Let* $n \geq 2$. *An (n-dimensional) interval-valued (IV) aggregation function in* $L([0, 1])$ *with respect to* $\precsim_L$ *is a mapping* $M_{IV} : (L([0, 1]))^n \to L([0, 1])$ *satisfying:*

(i) $M_{IV}(0_L, \cdots, 0_L) = 0_L$.

(ii) $M_{IV}(1_L, \cdots, 1_L) = 1_L$.

(iii) M_{IV} *is a non-decreasing function in each variable with respect to* $\precsim_L$.

Remark 8: *Note that this definition does not fully recover the usefulness of the usual definition of aggregation functions in the real setting since there may exist intervals which are not comparable by means of the order* $\precsim_L$, *so the full meaning of monotonicity is lost.*

Example 9 [32]: *If* $A: [0, 1]^2 \to [0, 1]$ *is an aggregation function, then the function* $M_A: L([0, 1])^2 \to L([0, 1])$ *given by*

$$M_A([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = [A(\underline{X}, \underline{Y}), A(\overline{X}, \overline{Y})],$$

is an IV aggregation function in $L([0, 1])$ *with respect to the order* $\precsim_L$.

Moreover, if $A, B: [0, 1]^2 \to [0, 1]$ *are two aggregation functions such that* $A(x, y) \leq B(x, y)$ *for each* $x, y \in [0, 1]$, *then*

$$M_{A,B}([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = [A(\underline{X}, \underline{Y}), B(\overline{X}, \overline{Y})],$$

is an IV aggregation function in $L([0, 1])$ *with respect to the order* $\precsim_L$.

Example 10: *The following functions are IV aggregation functions in* $L([0, 1])$ *with respect to the order* $\precsim_L$.

- $M_{IV}([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = [(\underline{X} \cdot \underline{Y})^2, (\overline{X} \cdot \overline{Y})^2]$,
- $M_{IV}([\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}]) = [(\underline{X} \cdot \underline{Y})^{1/2}, (\overline{X} + \overline{Y})/2]$.

B. RESTRICTED DISSIMILARITY AND E_N FUNCTIONS IN THE FUZZY SETTING

We recall here the usual notions of dissimilarity and E_N functions when we are dealing with fuzzy sets.

Definition 11 [8]: *A function* $d : [0, 1]^2 \to [0, 1]$ *is called a restricted dissimilarity function if it satisfies:*

1) $d(x, y) = d(y, x)$ *for all* $x, y \in [0, 1]$;
2) $d(x, x) = 0$ *for all* $x \in [0, 1]$;
3) $d(x, y) = 1$ *if and only if* $\{x, y\} = \{0, 1\}$;
4) *If* $x \leq y \leq z$, *then* $d(x, z) \geq d(x, y)$ *and* $d(x, z) \geq d(y, z)$ *for all* $x, y, z \in [0, 1]$.

Remark 12: *For any* $p \in]0, \infty[$, *the function* $d^p(x, y) = |x - y|^p$ *is a restricted dissimilarity function. Note that* d^p, *for all* $p \in]0, \infty[$, *also satisfies a stronger condition than the second one in Definition 11:*

2′. $d(x, y) = 0$ *if and only if* $x = y$.

However, for our purpose the weaker condition used in our definition is sufficient.

Definition 13 [8]: *Let* $n : [0, 1] \to [0, 1]$ *be a strong negation with the equilibrium point* e *(i.e., an involutive decreasing function such that* $n(e) = e$*). A function* $E_N : [0, 1] \to [0, 1]$ *is called a normal* E_N*-function w.r.t.* n *if it satisfies the following conditions:*

1) $E_N(e) = 1$;
2) $E_N(x) = 0$ *if and only if* $x = 0$ *or* $x = 1$;
3) *If* $y \leq x \leq e$ *or* $y \geq x \geq e$, *then* $E_N(x) \geq E_N(y)$.

Remark 14: *For any* $p \in]0, \infty[$, *the function* $E_N^p(x) = 1 - |2x - 1|^p$ *is a normal* E_N*-function w.r.t. any strong negation* n *with the equilibrium point* $e = 1/2$.

III. FUNCTIONS PRESERVING THE WIDTH OF INTERVALS

In this section we propose new definitions of restricted dissimilarity functions and normal E_N-functions in the interval-valued setting which take into account the width of the inputs.

A. INTERVAL-VALUED RESTRICTED DISSIMILARITY FUNCTIONS

Definition 15: *A function* $d_{IV} : (L([0, 1]))^2 \to L([0, 1])$ *is called an interval-valued restricted dissimilarity function w.r.t. an order* $\leq_L$ *if it satisfies the following conditions:*

1) $d_{IV}(X, Y) = d_{IV}(Y, X)$ *for all* $X, Y \in L([0, 1])$;
2) $d_{IV}(X, X) = [0, w(X)]$ *for all* $X \in L([0, 1])$;
3) $d_{IV}(X, Y) = 1_L$ *if and only if* $\{X, Y\} = \{0_L, 1_L\}$;
4) *If* $X \leq_L Y \leq_L Z$ *and* $w(X) = w(Y) = w(Z)$, *then* $d_{IV}(X, Y) \leq_L d_{IV}(X, Z)$ *and* $d_{IV}(Y, Z) \leq_L d_{IV}(X, Z)$ *for all* $X, Y, Z \in L([0, 1])$.

Remark 16: *The main point in which this definition differs from the one of restricted dissimilarity functions in the fuzzy setting is in axiom 2. Note that we can consider that the width of the membership interval of an element in a given set is a measure of the lack of knowledge of the precise (real-valued) membership degree of that element. That is, we can assume* *that it exists a precise membership value in* $[0, 1]$*, but, due to uncertainty or lack of knowledge, we are not able to provide that exact value, but only to say that it is inside the provided membership interval. This means that the smaller the width of the membership interval is, the smaller the uncertainty about the actual membership value is, too. But, with this interpretation of interval-valued membership functions, if two elements have the same interval memberships, this does not mean that their corresponding real-valued memberships are the same. To consider an extremal, if two elements have as interval-valued membership value the interval* $[0, 1]$*, this means that we do not know at all which is their actual real-valued membership values, which could be* 0 *for the first one and* 1 *for the second one, for instance. Hence it is natural to expect that this uncertainty is not lost when comparing them. Observe that for this extremal case, and due to axiom 2, the result of the IV restricted dissimilarity function is* $[0, 1]$*, which can be understood as reflecting that we do not know at all how similar the two elements actually are.*

Example 17: Let $X_0 \in L([0, 1])$ *where* $X_0 \geq_{TL} [0, 1]$ *and* $X_0 \neq 1_L$*. Then the function* $d_{IV} : L([0, 1])^2 \to L([0, 1])$ *defined by:*

$$d_{IV}(X, Y) = \begin{cases} 1_L, & \text{if } \{X, Y\} = \{0_L, 1_L\}, \\ [0, w(X)], & \text{if } X = Y, \\ X_0, & \text{otherwise,} \end{cases}$$

is a trivial example of IV restricted dissimilarity function w.r.t. any admissible order $\leq_{TL}$*.*

The following result shows that our definition is monotone with respect to the width of the intervals.

Proposition 18: Let $X, Y \in L([0, 1])$*. If* $w(X) < w(Y)$*, then, for any admissible order* $\leq_{TL}$*, it follows that*

$$d_{IV}(X, X) \leq_{TL} d_{IV}(Y, Y).$$

Proof: It follows straightforwardly from Definition 15. □

Now we give a construction method for IV restricted dissimilarity functions which preserves the width of the input intervals. We start with a lemma which shows how intervals of the same length behave with respect to admissible orders.

Lemma 19: Let $X, Y \in L([0, 1])$ *be intervals such that* $w(X) = w(Y)$*. Then*

$$X \precsim_L Y \quad \Leftrightarrow \quad X \leq_{TL} Y$$

for any admissible order $\leq_{TL}$*.*

Proof: The proof follows from the observations: 1. intervals with the same width are always comparable by the partial order $\precsim_L$; 2. an admissible order refines the partial order $\precsim_L$. □

Proposition 20: Let $\alpha \in]0, 1[$*, let* $M : [0, 1]^2 \to [0, 1]$ *be an idempotent symmetric aggregation function and let* $d : [0, 1]^2 \to [0, 1]$ *be a restricted dissimilarity function. Then, the function* $d_{IV} : L([0, 1])^2 \to L([0, 1])$ *given by*

$$\begin{aligned} d_{IV}(X, Y) &= \big[\min\big(d\,(K_\alpha(X), K_\alpha(Y))\,, 1 - M(w(X), w(Y))\big), \\ &\quad \min\big(1, d\,(K_\alpha(X), K_\alpha(Y)) + M(w(X), w(Y))\big)\big] \end{aligned} \quad (3)$$

is an IV restricted dissimilarity function w.r.t. any admissible order $\leq_{TL}$*. Moreover,* d_{IV} *is w-preserving.*

Proof: For simplicity we write $\mathcal{D}$ instead of $d\,(K_\alpha(X), K_\alpha(Y))$, and $\mathcal{M}$ instead of $M(w(X), w(Y))$. Then (3) can be simplified:

$$\begin{aligned} d_{IV}(X, Y) &= \big[\min\big(1 - \mathcal{M}, \mathcal{D}\big), \min\big(1, \mathcal{D} + \mathcal{M}\big)\big] \\ &= \begin{cases} [\mathcal{D}, \mathcal{D} + \mathcal{M}], & \text{if } \mathcal{D} + \mathcal{M} \leq 1, \\ [1 - \mathcal{M}, 1], & \text{otherwise,} \end{cases} \end{aligned} \quad (4)$$

By (4) it is clear that d_{IV} is well-defined.

Symmetry of d_{IV} directly follows from the symmetry of d and M.

The second condition in Definition 15 follows from the observations: $d\,(K_\alpha(X), K_\alpha(X)) = 0$ and $M(w(X), w(X)) = w(X)$.

Observe that $d_{IV}(X, Y) = 1_L$ if and only if $\mathcal{D} = 1$ and $\mathcal{M} = 0$. The former holds if and only if $\{K_\alpha(X), K_\alpha(Y)\} = \{0, 1\}$, which may happen if and only if $\{X, Y\} = \{0_L, 1_L\}$. So it follows that $w(X) = w(Y) = 0$ and we get the third condition in Definition 15.

Monotonicity w.r.t. any admissible order is obvious due to the monotonicity of d, Lemma 19 and the observation: if $X \leq_{TL} Y \leq_{TL} Z$ and $w(X) = w(Y) = w(Z)$, then $K_\alpha(X) \leq K_\alpha(Y) \leq K_\alpha(Z)$.

Finally, the fact that d_{IV} is w-preserving directly follows from Equation (4) and idempotency of M. □

To construct an IV restricted dissimilarity function, any restricted dissimilarity function d and any idempotent symmetric aggregation function M can be applied in Equation (3). However, using some additional assumptions on M and d, the construction given by Proposition (20) can be simplified.

Corollary 21: Let $\alpha \in]0, 1[$*, let* $M : [0, 1]^2 \to [0, 1]$ *be an idempotent symmetric aggregation function such that* $M(x, y) \leq \min\big((1 - \alpha)x + \alpha y, \alpha x + (1 - \alpha)y\big)$ *for all* $x, y \in [0, 1]$ *and let* $d : [0, 1]^2 \to [0, 1]$ *be a restricted dissimilarity function such that* $d(x, y) \leq |x - y|$ *for all* $x \in [0, 1]$*. Then, the function* $d_{IV} : L([0, 1])^2 \to L([0, 1])$ *given by*

$$\begin{aligned} d_{IV}(X, Y) = [&d\,(K_\alpha(X), K_\alpha(Y))\,, d\,(K_\alpha(X), K_\alpha(Y)) \\ &+ M(w(X), w(Y))] \end{aligned} \quad (5)$$

is an IV restricted dissimilarity function w.r.t. any admissible order $\leq_{TL}$*. Moreover,* d_{IV} *is w-preserving.*

Proof: We only need to prove that $M(w(X), w(Y)) \leq 1 - d\,(K_\alpha(X), K_\alpha(Y))$ for all $X, Y \in L([0, 1])$, since in that case Equation (5) is a special case of Equation (3). Assume that $K_\alpha(X) \geq K_\alpha(Y)$. Then, due to the assumptions on M and d, we have

$$\begin{aligned} 1 - d\,(K_\alpha(X), K_\alpha(Y)) &\geq 1 - |K_\alpha(X) - K_\alpha(Y)| \\ &= 1 - (1 - \alpha)\underline{X} - \alpha\overline{X} + (1 - \alpha)\underline{Y} + \alpha\overline{Y} \\ &\geq (1 - \alpha)w(X) + \alpha w(Y) \\ &\geq \min\big((1 - \alpha)w(X) + \alpha w(Y),\ \alpha w(X) \end{aligned}$$

$$+(1-\alpha)w(Y)\big) \geq M\left(w(X), w(Y)\right).$$

Now, let $K_\alpha(X) \geq K_\alpha(Y)$. Then

$$\begin{aligned}
1 - d\left(K_\alpha(X), K_\alpha(Y)\right) \\
&\geq 1 + (1-\alpha)\underline{X} + \alpha\overline{X} - (1-\alpha)\underline{Y} - \alpha\overline{Y} \\
&\geq \alpha w(X) + (1-\alpha)w(Y) \geq \min\big((1-\alpha)w(X) \\
&\quad + \alpha w(Y),\ \alpha w(X) + (1-\alpha)w(Y)\big) \geq M\left(w(X), w(Y)\right).
\end{aligned}$$

□

Corollary 22: Consider the interval-valued restricted dissimilarity function d_{IV} constructed in Corollary 21. Then, for all $X, Y \in L([0,1])$ it holds that

$$\begin{aligned}
&\min(w(X), w(Y)) \leq w(d_{IV}(X,Y)) \\
&\quad \leq \min\big((1-\alpha)w(X) + \alpha w(Y), \alpha w(X) + (1-\alpha)w(Y)\big).
\end{aligned}$$

Proof: The first inequality follows from the fact that an idempotent aggregation function is always greater than or equal to the minimum, and the second inequality follows from the property of M assumed in Corollary 21. □

Example 23: Consider the construction of IV restricted dissimilarity function given by Corollary 21. It is easy to see that the restricted dissimilarity d^p defined in Remark 12 satisfies the assumptions in Proposition III.6: $d^p(x,y) \leq d^1 = |x-y|$ for all $x \in [0,1]$ if and only if $p \in [1,\infty[$.

(i) For $\alpha = \frac{1}{2}$ and $M(x,y) = \frac{x+y}{2}$ we get a class of IV restricted dissimilarity functions w.r.t. any admissible order:

$$\begin{aligned}
d^p_{IV}(X,Y) = \Bigg[& d^p\left(\frac{\underline{X}+\overline{X}}{2}, \frac{\underline{Y}+\overline{Y}}{2}\right), \\
& d^p\left(\frac{\underline{X}+\overline{X}}{2}, \frac{\underline{Y}+\overline{Y}}{2}\right) + \frac{w(X)+w(Y)}{2}\Bigg]
\end{aligned}$$

for $p \in [1,\infty[$.

(ii) For $M(x,y) = \min(x,y)$, a class of IV restricted dissimilarity functions w.r.t. any admissible order arises:

$$\begin{aligned}
d^{p,\alpha}_{IV}(X,Y) = \big[& d^p\left(K_\alpha(X), K_\alpha(Y)\right), d^p\left(K_\alpha(X), K_\alpha(Y)\right) \\
& + \min(w(X), w(Y))\big].
\end{aligned}$$

for $\alpha \in]0,1[$ and $p \in [1,\infty[$.

(iii) Finally, we get a more general class of IV restricted dissimilarity functions w.r.t. any admissible order, if we take $\alpha \in]0,1[$ and

$$M(x,y) = \min\big((1-\beta)x + \beta y, \beta x + (1-\beta)y\big)$$

for $\beta \in [\max(\alpha, 1-\alpha), 1]$ (or equivalently for $\beta \in [0, \min(\alpha, 1-\alpha)]$) and $p \in [1,\infty[$:

$$\begin{aligned}
d^{p,\alpha,\beta}_{IV}(X,Y) = \big[& d^p\left(K_\alpha(X), K_\alpha(Y)\right), d^p\left(K_\alpha(X), K_\alpha(Y)\right) \\
& + \min\big((1-\beta)w(X) + \beta w(Y), \beta w(X) \\
& + (1-\beta)w(Y)\big)\big].
\end{aligned}$$

It is easy to see that for $\beta = 1$ (or equivalently for $\beta = 0$) we get the class described in item (ii).

B. INTERVAL-VALUED NORMAL E_N-FUNCTIONS

Definition 24: Let $N : L([0,1]) \to L([0,1])$ be an interval-valued strong negation w.r.t. a total order $\leq_{TL}$ with the equilibrium point ε (i.e., a decreasing involutive function such that $N(\varepsilon) = \varepsilon$). A function $EN_{IV} : L([0,1]) \to L([0,1])$ is called an interval-valued normal E_N-function w.r.t. N if it satisfies the following conditions:

1) $EN_{IV}(\varepsilon) = [1 - w(\varepsilon), 1]$;
2) $EN_{IV}(X) = 0_L$ *if and only if* $X = 0_L$ *or* $X = 1_L$;
3) *If* $Y \leq_{TL} X \leq_{TL} \varepsilon$ *or* $Y \geq_{TL} X \geq_{TL} \varepsilon$, *where* $w(X) = w(Y)$, *then* $EN_{IV}(X) \geq_{TL} EN_{IV}(Y)$.

Example 25: Let $X_0 \in L([0,1])$ where $X_0 \leq_{TL} [0,1]$ and $X_0 \neq 0_L$. Then the function $EN_{IV} : L([0,1]) \to L([0,1])$ defined by:

$$EN_{IV}(X) = \begin{cases} 0_L, & \text{if } X = 0_L \text{ or } X = 1_L, \\ [1 - w(X), 1], & \text{if } X = \varepsilon, \\ X_0, & \text{otherwise,} \end{cases}$$

is a trivial example of IV normal E_N-function w.r.t. any IV strong negation and any admissible order.

The task is now to find a construction method for IV E_N-functions which preserve the width of input intervals.

Proposition 26: Let $n : [0,1] \to [0,1]$ be a strong negation with equilibrium point e. Let $\alpha, \beta \in]0,1[$, $\beta \neq \alpha$ and $N : L([0,1]) \to L([0,1])$ be a strong IV negation w.r.t. $\leq_{\alpha,\beta}$ with equilibrium point ε and such that $K_\alpha(\varepsilon) = e$. Let $E_N : [0,1]^2 \to [0,1]$ be a normal E_N-function w.r.t. n. Then, the function $EN_{IV} : L([0,1]) \to L([0,1])$ given by

$$\begin{aligned}
EN_{IV}(X) = \big[& \max\left(0, E_N\left(K_\alpha(X)\right) - w(X)\right), \\
& \max\left(E_N\left(K_\alpha(X)\right), w(X)\right)\big] \quad (6)
\end{aligned}$$

is an IV normal E_N-function w.r.t. N. Moreover, EN_{IV} is w-preserving.

Proof: Equation (6) can be simplified:

$$EN_{IV}(X) = \begin{cases} \left[E_N\left(K_\alpha(X)\right) - w(X), E_N\left(K_\alpha(X)\right)\right], \\ \quad \text{if } E_N\left(K_\alpha(X)\right) \geq w(X), \\ [0, w(X)], \quad \text{otherwise.} \end{cases} \quad (7)$$

Clearly, EN_{IV} is well-defined and w-preserving, so it remains to prove the three conditions in Definition 24:

1. Since $E_N(K_\alpha(\varepsilon)) = E_N(e) = 1 \geq w(\varepsilon)$, we have

$$EN_{IV}(\varepsilon) = \left[E_N\left(K_\alpha(\varepsilon)\right) - w(\varepsilon), E_N\left(K_\alpha(\varepsilon)\right)\right] = [1 - w(\varepsilon), 1].$$

2. $EN_{IV}(X) = 0_L$ if and only if $E_N(K_\alpha(X)) = 0$ and $w(X) = 0$ if and only if $X \in \{0_L, 1_L\}$.

3. Let $w(X) = w(Y)$ and $Y \leq_{\alpha,\beta} X \leq_{\alpha,\beta} \varepsilon$. Then $K_\alpha(Y) \leq K_\alpha(X) \leq K_\alpha(\varepsilon) = e$, hence $E_N(K_\alpha(Y)) \leq E_N(K_\alpha(X))$ and consequently $EN_{IV}(Y) \leq_{\alpha,\beta} EN_{IV}(X)$. The same conclusion can be drawn for $Y \geq_{\alpha,\beta} X \geq_{\alpha,\beta} \varepsilon$. □

Corollary 27: Let $\leq_{XY}$ be the Xu and Yager order and $N : L([0,1]) \to L([0,1])$ be an IV strong negation w.r.t. $\leq_{XY}$ with the equilibrium point ε such that $\underline{\varepsilon} + \overline{\varepsilon} = 1$. Let $E_N : [0,1]^2 \to [0,1]$ be a normal E_N-function w.r.t. a strong negation n with the equilibrium point $e = 1/2$ such that

$E_N(x) \geq 1 - |2x - 1|$ *for all* $x \in [0, 1]$. *Then, the function* $EN_{IV} : L([0, 1]) \to L([0, 1])$ *given by*

$$EN_{IV}(X) = \left[E_N\left(\frac{\underline{X}+\overline{X}}{2}\right) - w(X), E_N\left(\frac{\underline{X}+\overline{X}}{2}\right)\right] \quad (8)$$

is an IV normal E_N*-function w.r.t.* N. *Moreover,* EN_{IV} *is w-preserving.*

Proof: Since (8) is a special case of (6), we only need to show that $E_N\left(\frac{\underline{X}+\overline{X}}{2}\right) - w(X) \geq 0$. Let $\frac{\underline{X}+\overline{X}}{2} \geq 0.5$. Then

$$E_N\left(\frac{\underline{X}+\overline{X}}{2}\right) - w(X) \geq 1 - |\underline{X} + \overline{X} - 1| - w(X)$$
$$= 1 - \underline{X} - \overline{X} + 1 - \overline{X} + \underline{X}$$
$$= 2 - 2\overline{X} \geq 0.$$

Similarly, $E_N\left(\frac{\underline{X}+\overline{X}}{2}\right) - w(X) \geq 2\underline{X} \geq 0$ for $\frac{\underline{X}+\overline{X}}{2} < 0.5$, hence the the proof is completed. □

Remark 28: (i) It is worth pointing out that the assumption $\underline{\varepsilon}+\overline{\varepsilon} = 1$ *imposed on IV negation N and its equilibrium point* ε *in Corollary 27 is not particularly restrictive. Almost all strong negations defined in paper [1] (in which a deep study of IV strong negations can be found) satisfy the condition.*

(ii) Similarly, although the condition, $K_\alpha(\varepsilon) = e$, *imposed on IV negation N in Proposition 26, looks too restrictive, in [1] a wide class of IV negations satisfying this condition was defined, see the following Proposition.*

From now on, $d_\alpha(c)$ denotes the maximal possible length of an interval $X \in L([0, 1])$ such that $K_\alpha(X) = c$ where $c \in [0, 1]$ and $\alpha \in]0, 1[$. Then (see [1, Proposition 7]):

$$d_\alpha(K_\alpha(X)) = \wedge\left(\frac{K_\alpha(X)}{\alpha}, \frac{1 - K_\alpha(X)}{1 - \alpha}\right).$$

Proposition 29: Let $\alpha \in]0, 1[$. *For any* $X \in L([0, 1])$ *let*

$$\lambda_\alpha(X) = \frac{\overline{X} - \underline{X}}{d_\alpha(K_\alpha(X))} = \begin{cases} (\overline{X} - \underline{X})\dfrac{\alpha}{K_\alpha(X)} & \text{if } K_\alpha(X) \leq \alpha, \\ (\overline{X} - \underline{X})\dfrac{1 - \alpha}{1 - K_\alpha(X)} & \text{if } K_\alpha(X) \geq \alpha. \end{cases} \quad (9)$$

If $n: [0, 1] \to [0, 1]$ *is a strong negation, then the mapping* $N_{\alpha,n}: L([0, 1]) \to L([0, 1])$ *given by*

$$\begin{cases} N_{\alpha,n}(0_L) = 1_L, \\ N_{\alpha,n}(1_L) = 0_L, \\ N_{\alpha,n}(X) = Y, \quad \text{if } X \in L([0, 1])\setminus\{0_L, 1_L\}, \end{cases} \quad (10)$$

where

$$\begin{cases} K_\alpha(Y) = n(K_\alpha(X)), \\ \lambda_\alpha(Y) = n_s(\lambda_\alpha(X)) = 1 - \lambda_\alpha(X), \end{cases}$$

is a strong IV negation on $L([0, 1])$ *with respect to the order* $\leq_{\alpha,\beta}$ *for any* $\beta \neq \alpha$.

Moreover, $N_{\alpha,n}$ *has a unique equilibrium point* ε, *given by* $K_\alpha(\varepsilon) = e$ *and* $\lambda_\alpha(\varepsilon) = \frac{1}{2}$, *where e is the equilibrium point of the strong negation n.*

Proof: See [1, Theorem 4 and Proposition 8]. □

Example 30: (i) Let us consider the construction of IV normal E_N*-functions given by Corollary 27. Consider the normal* E_N*-function* $E_N^p(x) = 1 - |2x - 1|^p$ *where* $p \in [1, \infty[$. *Let* $\leq_{XY}$ *be the Xu and Yager order. Then (see [1, Theorem 2])* $N(X) = [c' - r', c' + r']$ *where* $c = \frac{\underline{X}+\overline{X}}{2}$, $r = \frac{\overline{X}-\underline{X}}{2}$, $a = \min(c, 1 - c)$, $c' = 1 - c$ *and* $r' = a - r$, *is a strong IV negation w.r.t.* $\leq_{XY}$ *with the unique equilibrium point* $[1/4, 3/4]$. *Then*

$$EN_{IV}^p(X) = \left[E_N^p\left(\frac{\underline{X}+\overline{X}}{2}\right) - w(X), E_N^p\left(\frac{\underline{X}+\overline{X}}{2}\right)\right]$$

is a class of IV normal E_N*-functions w.r.t.* N.

(ii) Now let us consider the construction of IV normal E_N*-functions given by Proposition 26. Consider the normal* E_N*-function* $E_N^p(x) = 1 - |2x - 1|^p$ *where* $p \in]0, \infty[$. *Let* $N_{\alpha,n}$ *be the strong IV negation given in Proposition 29. Then*

$$EN_{IV}^{p,\alpha}(X) = \left[\max\left(0, E_N^p\left(K_\alpha(X)\right) - w(X)\right), \max\left(E_N^p\left(K_\alpha(X)\right), w(X)\right)\right],$$

for $\alpha \in]0, 1[$, *is a class of IV normal* E_N*-functions w.r.t.* $N_{\alpha,n}$.

IV. AGGREGATION OF IV RESTRICTED DISSIMILARITY FUNCTIONS AND IV NORMAL E_N-FUNCTIONS

In this section, we propose a definition of dissimilarity measure and entropy for interval-valued fuzzy sets. We also discuss a construction method based on the aggregation of IV restricted dissimilarity functions and IV normal E_N-functions, respectively.

A. w-PRESERVING IV AGGREGATION FUNCTIONS

First we summarize the main results in [27] about w-preserving IV aggregation functions which are going to be useful in this section.

Definition 31: Let $n \geq 2$. *An (n-dimensional) interval-valued (IV) aggregation function in* $L([0, 1])$ *with respect to* $\leq_L$ *is a mapping* $M_{IV} : (L([0, 1]))^n \to L([0, 1])$ *which verifies:*

(i) $M_{IV}(0_L, \cdots, 0_L) = 0_L$.
(ii) $M_{IV}(1_L, \cdots, 1_L) = 1_L$.
(iii) M_{IV} *is a non-decreasing function with respect to* $\leq_L$.

We say that $M_{IV} : (L([0, 1]))^n \to L([0, 1])$ is a decomposable n-dimensional IV aggregation function associated with M_L and M_U, if there exist n-dimensional aggregation functions $M_L, M_U : [0, 1]^n \to [0, 1]$ such that $M_L \leq M_U$ and

$$M_{IV}(X_1, \ldots, X_n) = \left[M_L\left(\underline{X_1}, \ldots, \underline{X_n}\right), M_U\left(\overline{X_1}, \ldots, \overline{X_n}\right)\right] \quad (11)$$

for all $X_1, \ldots, X_n \in L([0, 1])$.

We propose now a construction method of IV aggregation functions w.r.t. $\leq_{\alpha,\beta}$.

Theorem 32: Let $\alpha, \beta \in [0, 1]$, $\beta \neq \alpha$. *Let* $M_1, M_2 : [0, 1]^n \to [0, 1]$ *be aggregation functions where* M_1 *is strictly* *increasing. Then* $M_{IV} : (L([0,1]))^n \to L([0,1])$ *defined by:*

$$M_{IV}(X_1, \ldots, X_n) = Y,$$

where

$$\begin{cases} K_\alpha(Y) = M_1\left(K_\alpha(X_1), \ldots, K_\alpha(X_n)\right), \\ \lambda_\alpha(Y) = M_2\left(\lambda_\alpha(X_1), \ldots, \lambda_\alpha(X_2)\right), \end{cases}$$

for all $X_1, \ldots, X_n \in L([0,1])$*, is an IV aggregation function with respect to* $\leq_{\alpha,\beta}$*.*

Now we give a construction method of IV aggregation functions w.r.t. $\leq_{\alpha,\beta}$ which preserve the width of the input intervals. To do so, we take into account the following two properties.

(P1) $M(cx_1, \ldots, cx_n) \geq cM(x_1, \ldots, x_n)$ for all $c \in [0,1], x_1, \ldots, x_n \in [0,1]$.

(P2) $M(x_1, \ldots, x_n) \leq 1 - M(1 - x_1, \ldots, 1 - x_n)$ for all $x_1, \ldots, x_n \in [0,1]$.

Theorem 33: Let $\alpha, \beta \in [0,1]$*,* $\beta \neq \alpha$*. Let* $M_1, M_2 : [0,1]^n \to [0,1]$ *be aggregation functions such that* M_1 *is strictly increasing,* $M_1(x_1, \ldots, x_n) \geq M_2(x_1, \ldots, x_n)$ *for all* $x_1, \ldots, x_n \in [0,1]$*,* M_1 *or* M_2 *satisfy property (P1) and* M_1 *or* M_2 *satisfy property (P2). Then* $M_{IV} : (L([0,1]))^n \to L([0,1])$ *defined by:*

$$M_{IV}(X_1, \ldots, X_n) = Y,$$

where

$$\begin{cases} K_\alpha(Y) = M_1\left(K_\alpha(X_1), \ldots, K_\alpha(X_n)\right), \\ w(Y) = M_2\left(w(X_1), \ldots, w(X_n)\right), \end{cases}$$

for all $X_1, \ldots, X_n \in L([0,1])$*, is an IV aggregation function with respect to* $\leq_{\alpha,\beta}$*.*

Moreover, if M_2 *is idempotent, then* M_{IV} *is w-preserving.*

Lemma 34: Let $M_{IV} : (L([0,1]))^n \to L([0,1])$ *be defined as in Theorem 33.*

(i) *If*
- $M_1(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$*, and*
- $M_2(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$*,*

then $M_{IV}(X_1, \ldots, X_n) = 0_L$ *if and only if* $X_1 = \ldots = X_n = 0_L$*. Moreover, if* $\alpha \neq 0$*, then the restriction on* M_2 *can be skipped.*

(ii) *If*
- $M_1(x_1, \ldots, x_n) = 1$ *if and only if* $x_1 = \ldots = x_n = 1$ *and*
- $M_2(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$*,*

then $M_{IV}(X_1, \ldots, X_n) = 1_L$ *if and only if* $X_1 = \ldots = X_n = 1_L$*. Moreover, if* $\alpha \neq 1$*, then the restriction on* M_2 *can be skipped.*

(iii) M_{IV} *is idempotent if and only if* M_1 *and* M_2 *are idempotent.*

B. WIDTH-BASED IV DISTANCE MEASURES

In [26], Liu introduced a distance measure for fuzzy sets. We adapt the definition to the interval-valued setting. However, we change the second axiom in line with Remark 16, and we relax the fourth axiom in a similar way as the fourth axiom in Definition 15.

Definition 35: Let $\leq_L$ *be an order in* $L([0,1])$*. An interval-valued distance measure on* $IVFS(U)$ *w.r.t.* $\leq_L$ *is a mapping* $D : IVFS(U) \times IVFS(U) \to L([0,1])$ *such that, for every* $A, B, A', B' \in IVFS(U)$*,*

(D1) $D(A,B) = D(B,A)$*;*

(D2) $D(A,B) = 0_L$ *if and only if* $A = B$ *and* $A, B \in FS(U)$*;*

(D3) $D(A,B) = 1_L$ *if and only if* $\{A(u), B(u)\} = \{0_L, 1_L\}$ *for all* $u \in U$*;*

(D4) *If* $A \subseteq A' \subseteq B' \subseteq B$ *w.r.t.* $\leq_L$ *and* $w(A(u)) = w(A'(u)) = w(B'(u)) = w(B(u))$ *for all* $u \in U$*, then* $D(A,B) \geq_L D(A',B')$*.*

In the following proposition, we propose a construction method of IV distance measures by aggregating IV restricted dissimilarity functions.

Proposition 36: Let $U = \{u_1, \ldots, u_n\}$*. Let* $M_{IV} : (L([0,1]))^n \to L([0,1])$ *be an IV aggregation function w.r.t.* $\leq_L$ *such that* $M_{IV}(X_1, \ldots, X_n) = 1_L$ *if and only if* $X_1 = \ldots = X_n = 1_L$*, and* $M_{IV}(X_1, \ldots, X_n) = 0_L$ *if and only if* $X_1 = \ldots = X_n = 0_L$*. Let* $d_{IV} : L([0,1])^2 \to L([0,1])$ *be a function satisfying axioms 1, 3, 4 from Definition 15 and such that* $d_{IV}(X,Y) = 0_L$ *if and only if* $X = Y$ *and* $w(X) = 0$ *for all* $X, Y \in L([0,1])$*. Then the function* $D : IVFS(U) \times IVFS(U) \to L([0,1])$*, defined by:*

$$D(A,B) = M_{IV}\left(d_{IV}\left(A(u_1), B(u_1)\right), \ldots, d_{IV}\left(A(u_n), B(u_n)\right)\right)$$

for all $A, B \in IVFS(U)$*, is an IV distance measure on* $IVFS(U)$ *w.r.t.* $\leq_L$*.*

Proof: The proof is straightforward. ☐

In the following corollary, we show under which conditions the function d_{IV} given by Equation (3) can be used in the previous proposition to obtain an IV distance measure.

Corollary 37: Let $U = \{u_1, \ldots, u_n\}$ *and* $\alpha, \beta \in]0,1[$ *where* $\beta \neq \alpha$*. Let* $M_{IV} : (L([0,1]))^n \to L([0,1])$ *be an IV aggregation function w.r.t.* $\leq_{\alpha,\beta}$*, defined in terms of two aggregation functions* M_1, M_2*, as in Proposition 33. Let* $d_{IV} : L([0,1])^2 \to L([0,1])$ *be an IV restricted dissimilarity function constructed by means of an idempotent symmetric aggregation function* M *and a restricted dissimilarity function* d*, as in Proposition 20. Let* $D : IVFS(U) \times IVFS(U) \to L([0,1])$ *be a function defined by:*

$$D(A,B) = M_{IV}\left(d_{IV}\left(A(u_1), B(u_1)\right), \ldots, d_{IV}\left(A(u_n), B(u_n)\right)\right)$$

for all $A, B \in IVFS(U)$*. Then*

(i) *D satisfies axiom (D1).*

(ii) *D satisfies axiom (D2), if*
- $M_1(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$*;*
- $d(x,y) = 0$ *if and only if* $x = y$*;*
- $M(x,y) = 0$ *if and only if* $x = 0$ *and* $y = 0$*.*

(iii) *D satisfies axiom (D3), if*

- $M_1(x_1, \ldots, x_n) = 1$ *if and only if* $x_1 = \ldots = x_n = 1$.

(iv) *D satisfies axiom (D4) w.r.t.* $\leq_{\alpha,\beta}$.

(v) *Let M_2 be idempotent. Then, for all $A, B \in IVFS(U)$, $w(A(u_1)) = w(B(u_1)) = \ldots = w(A(u_n)) = w(B(u_n))$ implies $w(D(A, B)) = w(A(u_1))$.*

Proof: Items (i), (iv) and (v) are straightforward.

(ii) By Lemma 34 (i), $M_{IV}\big(d_{IV}(A(u_1), B(u_1)), \ldots, d_{IV}(A(u_n), B(u_n))\big) = 0_L$ if and only if $d_{IV}(A(u_i), B(u_i)) = 0_L$ for all $i = 1, \ldots, n$, which holds if and only if $K_\alpha(A(u_i)) = K_\alpha(B(u_i))$ and $w(A(u_i)) = w(B(u_i)) = 0$ for all $i = 1, \ldots, n$, that is, $A = B$ and $A, B \in FS(U)$.

(iii) By Lemma 34 (ii), $M_{IV}\big(d_{IV}(A(u_1), B(u_1)), \ldots, d_{IV}(A(u_n), B(u_n))\big) = 1_L$ if and only if $d_{IV}(A(u_i), B(u_i)) = 1_L$ for all $i = 1, \ldots, n$, which holds if and only if $\{K_\alpha(A(u_i)), K_\alpha(B(u_i))\} = \{0, 1\}$ and $M(w(K_\alpha(A(u_i))), w(K_\alpha(B(u_i)))) = 0$ for all $i = 1, \ldots, n$; that is, $\{A(u_i), B(u_i)\} = \{0_L, 1_L\}$ for all $i = 1, \ldots, n$. □

Example 38: Let $\alpha, \beta \in]0, 1[$ with $\beta \neq \alpha$. A function $D : IVFS(U) \times IVFS(U) \to L([0, 1])$, defined as in Corollary 37, is an IV distance measure w.r.t. $\leq_{\alpha,\beta}$, if, for instance, $M_1(x_1, \ldots, x_n) = M_2(x_1, \ldots, x_n) = \frac{x_1+\ldots+x_n}{n}$ for all $x_1, \ldots, x_n \in [0, 1]$; $d(x, y) = |x - y|$ and $M(x, y) = \max\{x, y\}$ for all $x, y \in [0, 1]$.

To see more clearly how our IV distance functions differ from those already considered in the literature, consider the function $Di_1 : IVFS(U) \times IVFS(U) \to [0, 1]$ given by:

$$Di_1(A, B) = \frac{1}{2n} \sum_{i=1}^{n} |\underline{A(u_i)} - \underline{B(u_i)}| + |\overline{A(u_i)} - \overline{B(u_i)}|$$

which is one of the most commonly used expression of distance for interval-valued fuzzy sets, see [20] and can be considered as a representative of distances for interval-valued fuzzy sets which provide a number in [0, 1] as result. If we consider the interval-valued fuzzy set A_1 where, for each $u \in U$ it holds that $A_1(u) = [0, 1]$, it holds that:

$$Di_1(A_1, A_1) = 0$$

so the uncertainty linked to the membership values [0, 1] is completely lost in the output. However, with the distance D in Example 38, it comes out that:

$$D(A_1, A_1) = [0, 1].$$

so uncertainty in the inputs is preserved in the output. In fact, note that, for any of the distances we have defined, as they preserve the width, we would have obtained this same result. Besides, there are not in the literature distances between interval-valued fuzzy sets which are interval-valued and make use of admissible orders, as it is the case of our definition.

C. WIDTH-BASED IV ENTROPIES

From now on, given $X \in L([0, 1])$, we denote by $\widetilde{X}$ the IVFS A in U such that $A(u) = X$ for all $u \in U$.

Definition 39 [8]: Let $\leq_{TL}$ be a total order in $L([0, 1])$. Let N be a strong IV negation with respect to $\leq_{TL}$ with an equilibrium point $\varepsilon \in L([0, 1])$. A function $E : IVFS(U) \to L([0, 1])$ is an IV entropy on $IVFS(U)$ with respect to the strong IV negation N if for all $A, B \in IVFS(U)$:

(E1) *$E(A) = 0_L$ if and only if A is crisp;*

(E2) *$E(\widetilde{\varepsilon}) = [1 - w(\varepsilon), 1]$;*

(E3) *$E(A) \leq_{TL} E(B)$, if $w(A(u)) = w(B(u))$ and $A(u) \leq_{TL} B(u) \leq_{TL} \varepsilon$ or $A(u) \geq_{TL} B(u) \geq_{TL} \varepsilon$ for all $u \in U$.*

The definition is taken from [8]. However, the third axiom is relaxed in a similar way as the fourth axiom in Definition 15 and the second axiom is adjusted in accordance with Remark 16.

Now we give a construction method of IV entropies in terms of normal E_N-functions.

Proposition 40: Let $U = \{u_1, \ldots, u_n\}$ and let $N : L([0, 1]) \to L([0, 1])$ be a strong IV negation w.r.t. a total order $\leq_{TL}$. Let $M_{IV} : (L([0, 1]))^n \to L([0, 1])$ be an idempotent IV aggregation function w.r.t. $\leq_{TL}$ satisfying $M_{IV}(X_1, \ldots, X_n) = 0_L$ if and only if $X_1 = \ldots = X_n = 0_L$. Let $EN_{IV} : L([0, 1]) \to L([0, 1])$ be an IV normal E_N-function w.r.t. N (given by Definition 24). Then, the function $E : IVFS(U) \to L([0, 1])$, defined by:

$$E(A) = M_{IV}\big(EN_{IV}(A(u_1)), \ldots, EN_{IV}(A(u_n))\big)$$

for all $A \in IVFS(U)$, is an IV entropy on $IVFS(U)$ with respect to the strong IV negation N.

Proof: The proof is straightforward. □

We study now under which conditions the function EN_{IV} given by Equation (6) can be used in the previous proposition to obtain an IV entropy.

Corollary 41: Let $U = \{u_1, \ldots, u_n\}$ and $\alpha, \beta \in]0, 1[$ with $\beta \neq \alpha$. Let $M_{IV} : (L([0, 1]))^n \to L([0, 1])$ be an IV aggregation function w.r.t. $\leq_{\alpha,\beta}$ defined by two aggregation functions M_1, M_2, as in Proposition 33. Let $EN_{IV} : L([0, 1]) \to L([0, 1])$ be an IV normal E_N-function given in terms of a normal E_N-function E_N, as in Proposition 26, with N a strong IV negation w.r.t. $\leq_{\alpha,\beta}$ with an equilibrium point ε. Let $E : IVFS(U) \to L([0, 1])$ be a function defined by:

$$E(A) = M_{IV}\big(EN_{IV}(A(u_1)), \ldots, EN_{IV}(A(u_n))\big)$$

for all $A \in IVFS(U)$. Then

(i) *E satisfies axiom (E1), if*

$M_1(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$.

(ii) *E satisfies axiom (E2), if M_1 and M_2 are idempotent.*

(iii) *E satisfies axiom (E3) w.r.t.* $\leq_{\alpha,\beta}$.

(iv) *Let M_2 be idempotent. Then, for all $A \in IVFS(U)$, $w(A(u_1)) = \ldots = w(A(u_n))$ implies $w(E(A)) = w(A(u_1))$.*

Proof: The proof of (i) follows from Lemma 34 (i). The proof of (ii) is a consequence of Lemma 34 (iii), and the proof of (iii) and (iv) is straightforward. □

Example 42: Let $\alpha, \beta \in]0, 1[$ where $\beta \neq \alpha$. A function $E : IVFS(U) \to L([0, 1])$ defined as in Corollary 41, is an entropy for IVFSs w.r.t. $\leq_{\alpha,\beta}$, if, for instance,

$M_1(x_1, \ldots, x_n) = M_2(x_1, \ldots, x_n) = \frac{x_1+\ldots+x_n}{n}$ *for all* $x_1, \ldots, x_n \in [0, 1]$*; and* E_N *is any* E_N*-normal function.*

In order to see the difference between the notion of IV entropy that we have introduced and other that can be found in the literature, let us consider the expression of entropy proposed by Szmidt and Kacpryzk [37], given, for every $A \in IVFS(U)$, by:

$$\mathcal{E}_K(A) = \frac{1}{n}\sum_{i=1}^{n}(\overline{A(u_i)} - \underline{A(u_i)}).$$

Observe that the output of this entropy function is a real number. If we consider the entropy E obtained from Example 42 with the family of E_N functions in Example 30, and if we take $A(u) = [1/4, 3/4]$ for every $u \in U$, we see that:

$$E(A) = [1/2, 1]$$

whereas

$$\mathcal{E}_K(A) = 1/2.$$

That is, again, as we are getting interval-valued outputs, we are recovering the uncertainty reflected in the inputs. With respect to interval-valued entropies in the literature, again this is the first proposal using admissible orders, so no comparison is possible to other methods.

V. IV-ENTROPY FOR IMAGE THRESHOLDING

Image segmentation is a process where an image is partitioned into regions that represent the objects in it [19]. In order to segment an image, all the pixels are assigned a label representing the object to whom they belong. Pixels with the same properties share the same label. The number of labels assigned to an image depend on the level of detail we want.

One of the most commonly used technique, known as thresholding [36], assigns only two labels, based on the analysis of the grey levels of the image. The image is analysed as if there were only two regions, the object and the background. The process consists in obtaining the best value of the grey level intensity; that is, the value that best separates the two regions of the image.

We present an illustrative example for image thresholding where the best threshold to segment the image is selected as the result of applying our IV-entropy. In our work, an image is represented as a matrix with dimensions $D = X \times Y = \{1, ..., w\} \times \{1, ..., h\}$, where w represents the width, *i.e.*, the number of the columns of the image; and h, the height, *i.e.*, the number of rows of the image. Every element (pixel) of the matrix can take values in a set of values $L = \{0, ..., 255\}$.

For the sake of the experiment, we use an adapted version of the algorithm (Algorithm 1) presented by Huang and Wang [21], where we build a series of fuzzy sets from different membership functions to represent the image and obtain the corresponding set of entropies. The algorithm consists in building an IVFS for each grey level and calculate the corresponding IV entropies in order to choose as threshold the graylevel associated to the lowest value of the IV entropy.

Algorithm 1 Algorithm for Thresholding an Image Using Entropy

Input: Image I with L intensity values.
Output: Image threshold t.

for each level of intensity t, $(t = 0, t = 1, \ldots, L-1)$ **do**
  Build k fuzzy sets $Q_t^1 \ldots Q_t^k$;
  Build an IVFS $\tilde{Q}_t$ from the fuzzy sets $Q_t^1 \ldots Q_t^k$;
  for each $q \in \{0, \ldots, L-1\}$ **do**
    $\mu_{\tilde{Q}_t}(q) = \left[T\left(\mu_{Q_t^1}(q), \ldots, \mu_{Q_t^k}(q)\right), S\left(\mu_{Q_t^1}(q), \ldots, \mu_{Q_t^k}(q)\right)\right]$, with T a t-norm and S a t-conorm.
  end for
  Compute the entropy of each of the L interval valued fuzzy sets $\tilde{Q}_t$;
end for
Select the threshold t with the smallest entropy.;

In this algorithm we can select a variety of fuzzy sets to represent the background and the object of the image for generating the sets $Q_t^1 \ldots Q_t^k$ representing the image.

In order to construct the fuzzy sets, we consider different membership functions for which two maxima exist, in order to be able to represent both the background and the object of the image. The following expressions are used to build the membership functions:

- *REF*-base membership Functions: if we use restricted equivalence functions (REF, see [10]) to build the membership functions, the output will be greater when the difference between the value of the intensity of a pixel and the mean of the intensities of the pixels which belong to the object or the background is smaller. In addition, we take a function $F : [0, 1] \to [0.5, 1]$ to scale the membership function to ensure that the minimum entropy is obtained when the membership degree is 1.
  Then, given an image I and a threshold value t, we build the membership function of each set Q_t as follows:

$$\mu_{Q_t}(q) = \begin{cases} F(REF(q, m_b(t))), & \text{if } q \leq t. \\ F(REF(q, m_o(t))), & \text{if } q > t. \end{cases} \quad (12)$$

  where $m_b(t)$ and $m_o(t)$ are the mean of the intensities of the pixels which are assumed to belong to the background and the mean of the intensities of the pixels which are assumed to belong to the object, respectively:

$$m_o(t) = \frac{\sum_{q=0}^{t} q \cdot h(q)}{\sum_{q=0}^{t} h(q)}, \quad m_b(t) = \frac{\sum_{q=t+1}^{L-1} q \cdot h(q)}{\sum_{q=t+1}^{L-1} h(q)} \quad (13)$$

  with $h(q)$ representing the number of pixels with intensity q of the image.
  The different REF used for the construction of the membership functions are given in Table 1.

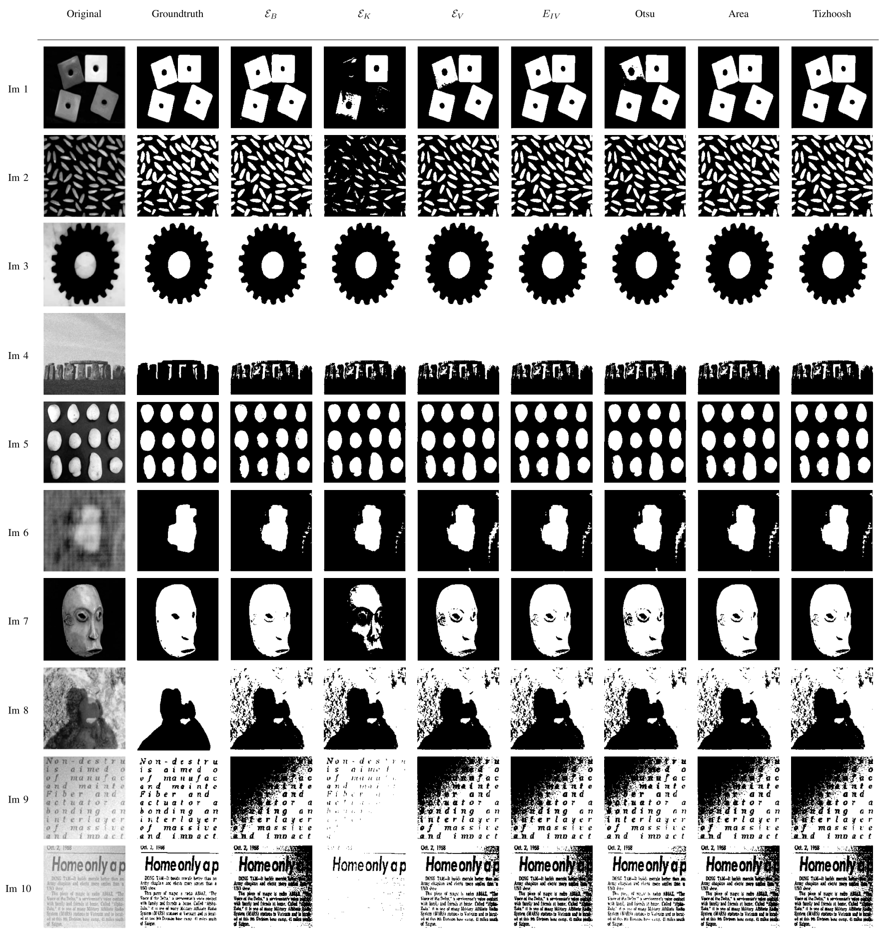

FIGURE 1. Comparison of the resulting images obtained with the thresholds from the $\mathcal{E}_B$, $\mathcal{E}_K$, $\mathcal{E}_V$, E_{IV} entropy expressions with the ideal groundtruth provided by the dataset and those images applying thresholds obtained with the Otsu, Area-based and Tizhoosh methods. Thresholded images with our method are obtained from the *REF* membership construction (*C*1).

TABLE 1. Membership functions generated from different *REF* and *F* functions used in Eq.(12).

μ_1	$REF(x,y) = 1 - \lvert x-y \rvert$	$F(x) = 0.5(1+x)$
μ_2	$REF(x,y) = (1 - \lvert x-y \rvert)^2$	
μ_3	$REF(x,y) = 1 - \lvert x-y \rvert^2$	
μ_4	$REF(x,y) = (1 - \lvert x-y \rvert)^{0.5}$	
μ_5	$REF(x,y) = 1 - \lvert x-y \rvert$	$F(x) = \frac{1}{(2-x)}$
μ_6	$REF(x,y) = (1 - \lvert x-y \rvert)^2$	
μ_7	$REF(x,y) = 1 - \lvert x-y \rvert^2$	
μ_8	$REF(x,y) = (1 - \lvert x-y \rvert)^{0.5}$	

- SZ-Function: as in [31], we use the S and Z functions to represent the *brightness* and *darkness* of the image, respectively. The S function is defined with the following expression:

$$S(q,a,b,c) = \begin{cases} 0, & q \leq a \\ 2\left(\dfrac{q-a}{c-a}\right)^2, & a \leq q \leq b \\ 1 - 2\left(\dfrac{q-c}{c-a}\right)^2, & b \leq q \leq c \\ 1, & q \geq c. \end{cases} \tag{14}$$

where $a = t$, $b = a + \frac{60}{2}$ and $c = a + 60$.
Then the Z function is given by the negation of the S function. As in [39] the union of the two functions represents both the background and the object having the following membership function:

$$\mu_{Q_t}(q) = \begin{cases} S(q,a,b,c), & \text{if } q < t. \\ 1 - S(q,a,b,c), & \text{if } q \geq t. \end{cases} \tag{15}$$

- Triangular-based membership function: we build a membership function joining two triangular functions centered on the mean of the intensities of the pixels of the object ($m_o(t)$) and the mean of the pixels of the background of the object ($m_b(t)$). To define these functions, we take as width of the base of the triangle the value 100.

In order to choose the IVFS that represents the best threshold for the image we take the one with the lowest IV entropy. In this step, we use the expression from Proposition 40. In addition, and for comparison purposes, we obtain the threshold for an image using the following entropy expressions:

- Sambuc's indetermination index [33]:

$$\mathcal{E}_B(\tilde{Q}) = \frac{1}{N}\sum(\overline{\mu}_{\tilde{Q}}(u) - \underline{\mu}_{\tilde{Q}}(u)) \quad (16)$$

- Kacpryzk and Smidzt's expression [37]:

$$\mathcal{E}_K(\tilde{Q}) = \frac{1}{N}\sum \frac{1-\vee\left(1-\overline{\mu}_{\tilde{Q}}(u), \underline{\mu}_{\tilde{Q}}(u)\right)}{1-\wedge\left(1-\overline{\mu}_{\tilde{Q}}(u), \underline{\mu}_{\tilde{Q}}(u)\right)} \quad (17)$$

- Vlachos and Sergiadis' expression [41]:

$$\mathcal{E}_V(\tilde{Q}) = \frac{1}{N}\sum \frac{2\underline{\mu}_{\tilde{Q}}(u)(1-\overline{\mu}_{\tilde{Q}}(u)) + (\overline{\mu}_{\tilde{Q}}(u) - \underline{\mu}_{\tilde{Q}}(u))^2}{\underline{\mu}_{\tilde{Q}}(u)^2 + (1-\overline{\mu}_{\tilde{Q}}(u))^2 + (\overline{\mu}_{\tilde{Q}}(u) - \underline{\mu}_{\tilde{Q}}(u))^2} \quad (18)$$

Our experiments consist in constructing different fuzzy sets with combinations of the considered membership functions and comparing the results obtained with the new expression of IV entropy proposed in this work with the results obtained with the ones in the literature. We perform three experiments: the first one, considering only *REF* membership functions ($C1$); the second one, combining *REF* and triangular- based membership functions ($C2$); and the third one, combining the three types of membership functions, *REF*, triangular and $S-Z$-based ($C3$).

Moreover, we compare our method with some of the well-known algorithms from the literature. To show the performance of our method we take our best results for each image and compare them to the results obtained with the following methods:

- *Otsu* [30];
- *Area* [9] with $\varphi_1(x) = x^2$ and $\varphi_2(x) = x$;
- *Tizhoosh* [40] with $\varphi(x) = x$ and $\alpha = 2$.

As we can see in Figure 1, the result of applying the different entropy expressions is quite similar. When using the $\mathcal{E}_K$ expression, the threshold obtained for some of the images is not acceptable, and removes some important parts of the image, like in *Im 1*, where two of the dices disappear in the background, or in *Im 7*, where part of the mask is removed. In the case of the last two images, this expression is capable of removing completely the shade that remains in the rest of the cases. With the literature methods, we can also see that the results are similar except in the case of Otsu's method, where in image *Im1* an important part of a dice disappear, but, *e.g.*, *Im 10*, it is one of the best.

A deeper analysis is done in Tables 2-4, where we show the obtained threshold value for each image with each entropy expression along with the percentage of pixels correctly thresholded according to the ideal images provided by the dataset. In Table 5 we also show the thresholds and results obtained with our best performers compared to the literature methods.

In Table 2 we can see how our new entropy expression is comparable to the other expression, obtaining better results in four of the images. It is worth mentioning that, as seen in Figure 1, the last two images are not correctly thresholded and offer poor quantitative results, except in the case of the $\mathcal{E}_K$ expression.

TABLE 2. Results using different entropy expressions with REF only membership function ($C1$).

	$\mathcal{E}_B$		$\mathcal{E}_K$		$\mathcal{E}_V$		E_{IV}	
	t	%	t	%	t	%	t	%
Im 1	15	96.740	131	78.880	67	96.758	45	**97.391**
Im 2	39	92.700	148	76.709	72	92.274	51	**92.761**
Im 3	52	**98.362**	111	97.854	106	97.972	80	98.357
Im 4	135	95.728	135	95.728	136	**95.828**	136	**95.828**
Im 5	167	94.490	124	95.776	130	**95.807**	147	95.766
Im 6	143	96.952	144	**97.023**	135	95.791	141	96.783
Im 7	36	**96.908**	137	79.810	69	96.116	50	96.672
Im 8	122	**89.360**	125	88.526	124	88.796	122	**89.360**
Im 9	207	55.535	125	**88.017**	197	66.529	204	58.935
Im 10	205	70.070	144	**83.309**	192	78.218	202	72.662

TABLE 3. Results using different entropy expressions with *REF* and triangular membership function ($C2$).

	$\mathcal{E}_B$		$\mathcal{E}_K$		$\mathcal{E}_V$		E_{IV}	
	t	%	t	%	t	%	t	%
Im 1	13	96.674	140	75.207	41	**97.419**	33	97.357
Im 2	44	**92.802**	186	66.845	71	92.274	45	92.766
Im 3	52	**98.362**	183	88.358	109	97.902	81	98.349
Im 4	133	95.475	136	**95.828**	133	95.475	133	95.475
Im 5	177	93.376	123	95.750	129	**95.817**	149	95.661
Im 6	139	96.564	138	96.409	134	**95.641**	139	96.564
Im 7	29	**96.954**	163	72.431	49	96.721	53	96.649
Im 8	116	**90.684**	131	86.718	117	90.461	120	89.855
Im 9	207	55.535	165	**86.799**	200	63.629	202	61.317
Im 10	205	70.070	165	**82.843**	194	77.400	203	71.885

TABLE 4. Results using different entropy expressions with *REF*, triangular and $S-Z$ membership function ($C3$).

	$\mathcal{E}_B$		$\mathcal{E}_K$		$\mathcal{E}_V$		E_{IV}	
	t	%	t	%	t	%	t	%
Im 1	55	**97.556**	140	75.207	48	97.345	55	**97.556**
Im 2	55	**92.738**	108	88.608	71	92.274	55	**92.738**
Im 3	54	98.362	170	91.944	109	97.902	75	**98.392**
Im 4	133	95.475	136	**95.828**	133	95.475	133	95.475
Im 5	166	94.595	108	**95.916**	129	95.817	157	95.248
Im 6	139	**96.564**	136	95.983	136	95.983	139	**96.564**
Im 7	54	**96.739**	110	87.524	54	**96.739**	54	**96.739**
Im 8	116	**90.684**	122	89.360	117	90.461	120	89.855
Im 9	163	87.245	160	**87.847**	166	86.799	164	87.245
Im 10	163	**82.994**	165	82.843	165	82.843	163	**82.994**

In the case of Table 3, when adding triangular-based membership functions to the combination of membership functions, the results obtained with our IV entropy expression are comparable to the ones from the other expressions, in particular to $\mathcal{E}_B$, but remaining just under it, except for the last two images, where the results are improved. In this second approach, the last two images remain better thresholded with the $\mathcal{E}_K$ expression,

In the third round of experiments, when combining *REF*, triangular, and $S-Z$ membership functions, the results obtained by our new expression are the best ones, remaining in some cases the same as with $\mathcal{E}_B$. In this particular exper-

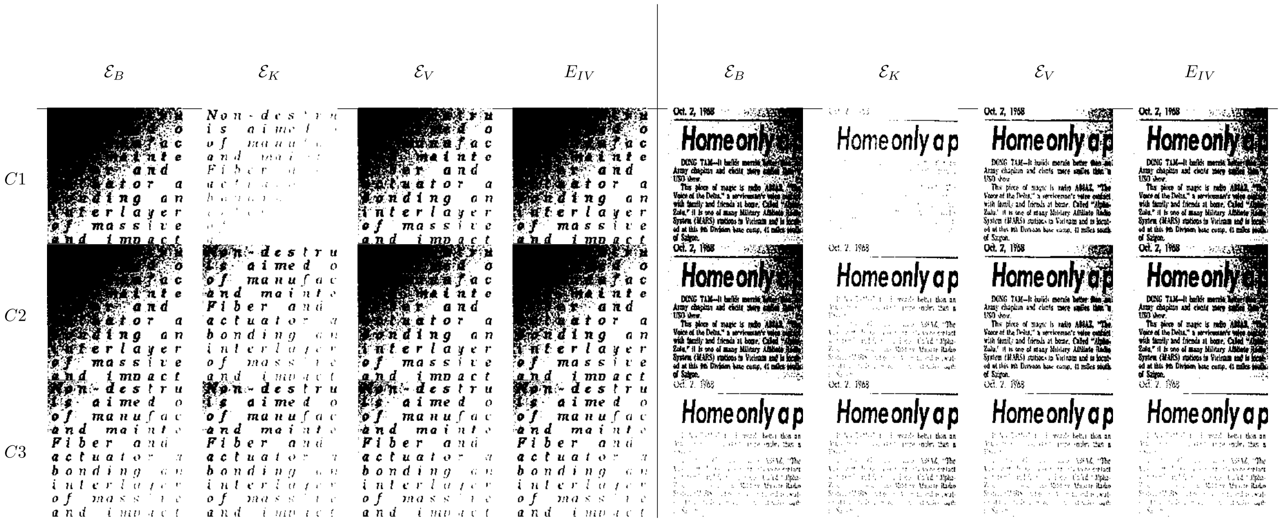

FIGURE 2. **Comparison of the resulting images *Im 9* and *Im 10* obtained with the thresholds from the $\mathcal{E}_B$, $\mathcal{E}_K$, $\mathcal{E}_V$, E_{IV} entropy expressions, with the three combinations of membership functions ($C1$, $C2$, $C3$).**

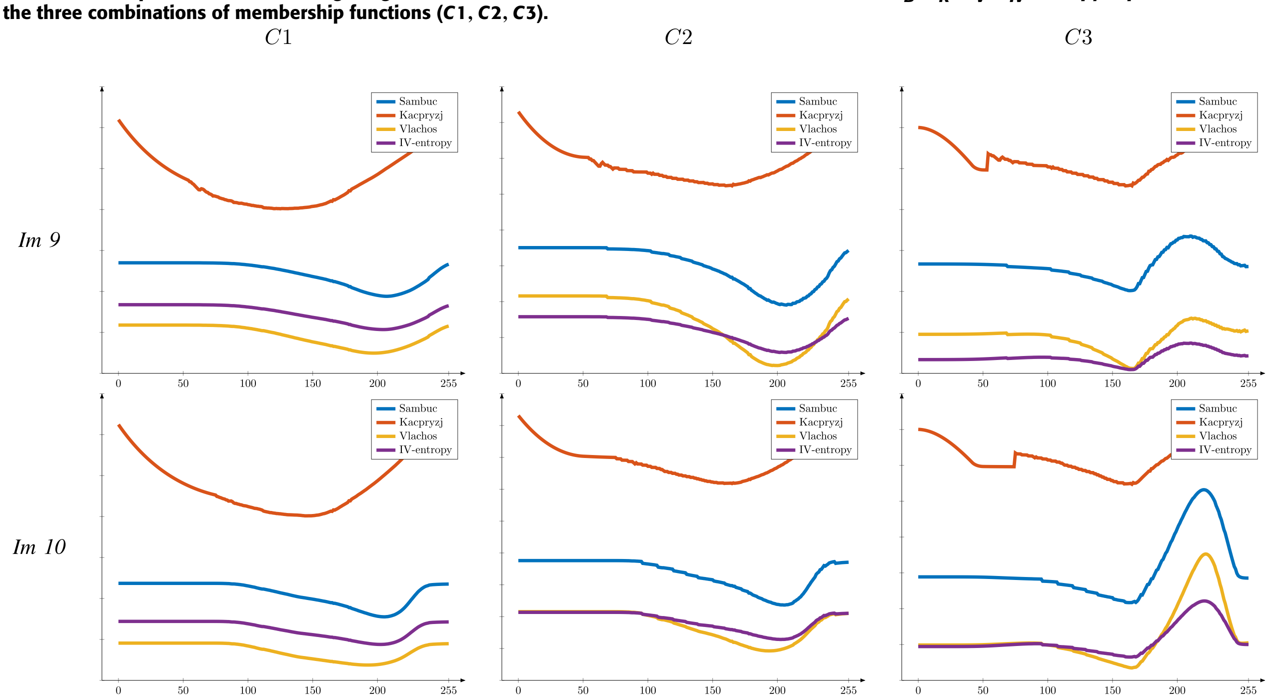

FIGURE 3. **Comparison of the resulting entropy for *Im9* and *Im 10* on each of the three experiment combination ($C1$, $C2$, $C3$).**

iment, our expression obtains good results for the last two images, getting the best results in the case of *Im 10*

If we analyse in a more detailed way the particular cases of *Im 9* and *Im 10* (Figure 2), we can see how the results obtained with $C1$ and $C2$ are visually similar and quite bad, not removing the shade on *Im 9*, except when using $\mathcal{E}_K$, but loosing some of the letters in the lower-right part of the image. In *Im 10*, almost the same happens, but the best result is obtained with $\mathcal{E}_V$. When using the $C3$ combination, all the entropy expressions obtain similar results, and all the shadows are removed in the case of *Im 9*. Concerning *Im10*, the shadow is removed, but the bottom part of the text is almost removed, too, losing important information of the image.

The behaviour of the entropy in the case of the last two images can bee seen in Figure 3, where we show the entropy values along the different grey levels of the image for each expression and with each membership function combination. We can clearly see that when using the $S-Z$ function ($C3$), the entropy draws a peak, easing the threshold selection, and therefore obtaining better results in the segmentation. In the case of $C1$ and $C3$, the entropy does not present any abrupt peak and is smoother, being more difficult to find the suitable threshold as seen in the visual example (Figure 2).

As it can be deduced from the results shown in the experiments, the new entropy expression is suitable for finding the best threshold to segment images, when using *REF* only and *REF*, triangular and $S-Z$ -based functions combined.

TABLE 5. Results using classical methods in the literature as Otsu, Area and Tizhoosh methods.

	Otsu		Area		Tizhoosh		E_{IV} Best	
	t	%	t	%	t	%	t	%
Im 1	79	93.661	39	97.424	43	97.421	33	**97.556**
Im 2	74	92.223	50	92.758	52	**92.766**	45	**92.766**
Im 3	104	98.015	78	98.367	80	98.357	75	**98.392**
Im 4	136	**95.828**	135	95.728	136	**95.828**	136	**95.828**
Im 5	127	**95.847**	148	95.722	148	95.722	147	95.766
Im 6	134	95.641	139	96.564	140	96.679	141	**96.783**
Im 7	72	95.975	49	96.721	50	96.672	54	**96.739**
Im 8	123	89.093	121	89.573	122	89.360	120	**89.855**
Im 9	193	70.003	203	60.127	204	58.935	164	**87.245**
Im 10	187	79.941	201	73.464	202	72.662	163	**82.994**

Moreover, as shown in Table 5 our method outperforms the classical ones in the majority of the images. Only one image, *Im 5*, obtains better results with Otsu's method and similar performance is obtained with Tizhoosh's method for *Im 2* and with Tizhoosh's and Otsu's ones in the case of *Im 4*. With these results in mind and the ones comparing the different entropies, we can state that our method is better than the ones in the literature, since the use of interval-valued fuzzy sets and our notion of IV entropy enables us to get a better representation of the uncertainty linked to the representation of the image.

VI. CONCLUSIONS

We have defined and studied interval-valued restricted dissimilarity functions and interval-valued normal E_N-functions. For the first time in the literature, both concepts have been defined with respect to a total order between intervals and considering the width of the inputs. This has allowed us to construct distance measures between interval-valued fuzzy sets and entropies for interval-valued fuzzy sets. The utility of these constructions is illustrated by an example in image thresholding using an expression of the proposed entropy of interval-valued fuzzy sets.

In future works we intend to consider the use of these new functions in different image processing, decision making and classification problems where fuzzy sets and interval-valued fuzzy sets have shown themselves useful (see [12]).

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