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Mandelbrot and Julia Sets via Jungck–CR Iteration With *s*–Convexity

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ABSTRACT In today's world, fractals play an important role in many fields, e.g., image compression or encryption, biology, physics, and so on. One of the earliest studied fractal types was the Mandelbrot and Julia sets. These fractals have been generalized in many different ways. One of such generalizations is the use of various iteration processes from the fixed point theory. In this paper, we study the use of Jungck-CR iteration process, extended further by the use of *s*-convex combination. The Jungck-CR iteration process with *s*-convexity is an implicit three-step feedback iteration process. We prove new escape criteria for the generation of Mandelbrot and Julia sets through the proposed iteration process. Moreover, we present some graphical examples obtained by the use of escape time algorithm and the derived criteria.

INDEX TERMS Julia set, Jungck-CR iteration, Mandelbrot set, *s*-convex combination.

I. INTRODUCTION

In the 1970's Benoit Mandelbrot introduced to the world new field of mathematics. He named this field fractal geometry (*fractus* -- from Latin divided, fractional). Benoit Mandelbro extended the work of Gaston Julia and introduced the Mandelbrot set; a set of all connected Julia sets. Mandelbrot expanded the ideas of G. Julia, studied the Mandelbrot set by using the complex function $z^2 + c$ with using z as a complex function and c as a complex parameter. Fractal geometry breaks the way we see everything. It provides a new idea of modelling natural objects, such as clouds, plants, landscapes, galaxies. One of the fractal types studied by Mandelbrot were complex fractals, i.e, fractals defined in the complex plane. Mandelbrot and Julia sets are examples of those fractals. The fractal structure of Mandelbrot and Julia sets have been demonstrated for quadratic, cubic and higher degree polynomials, by using Picard orbit which is an application of one-step feedback process. Since then many different generalizations of those sets were proposed. One of the generalizations is the use of results from fixed point theory, namely the use of various iteration processes instead of the Picard one that is used in the generation of Mandelbrot and Julia sets.

In 2004, Rani and Kumar [1], [2] introduced superior Julia and Mandelbrot sets using Mann iteration scheme. Chauhan *et al.* [3] introduced the relative superior Julia sets using Ishikawa iteration scheme. Also, relative superior Julia sets, Mandelbrot sets and tricorn, multicorns by using the S-iteration scheme were presented in [4] and [5]. Recently, Ashish *et al.* [6] introduced Julia and Mandelbrot sets using the Noor iteration scheme, which is a three-step iterative procedure. The junction of a s-convex combination [7] and various iteration schemes was studied in many papers. Mishra *et al.* [8], [9] developed fixed point results in relative superior Julia sets, tricorn and multicorns by using the Ishikawa iteration with s-convexity. Kang *et al.* [10] introduced new fixed point results for fractal generation using the implicit Jungck-Noor orbit with s-convexity, whereas Nazeer *et al.* [11] used the Jungck-Mann and Jungck-Ishikawa iterations with s-convexity. The use of Noor iteration and s-convexity was shown in [12], whereas Gdawiec and Shahid [13] presented complex fractals generated by the S-iteration with s-convexity.

In this paper we study the use of Jungck-CR iteration with s-convexity in the generation of Mandelbrot and Julia sets. We prove an escape criterion for the function of the form $z^n - az + c$. Moreover, we present some graphical examples of Mandelbrot and Julia sets via the Jungck-CR iteration with s-convexity.

The rest of the paper is organized as follows. In Sec. II, we briefly introduce notions used in the paper. Next, in Sec. III, we extend the Jungck-CR iteration using the s-convex combination and we prove the escape criteria for the functions of the form $Q_c(z) = z^n - az + c$ using the Jungck-CR iteration with s-convexity. In Sec. IV we present some graphical examples of Mandelbrot and Julia sets obtained with the derived criteria. Finally, in Sec. V, we give some concluding remarks.

II. PRELIMINARIES

Definition 1 (Julia Set [14]): Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree ≥ 2. Let F_f be the set of points in $\mathbb{C}$ whose orbits do not converge to the point at infinity, i.e.,

$$F_f = \{z \in \mathbb{C} : \{|f^n(z)|\}_{n=0}^{\infty} \text{ is bounded}\}. \tag{1}$$

F_f is called as filled Julia set of the polynomial f. The boundary points of F_f are called the points of Julia set of the polynomial f or simply the Julia set.

Definition 2 (Mandelbrot Set [15]): The Mandelbrot set M consists of all parameters c for which the filled Julia set of $Q_c(z) = z^2 + c$ is connected, i.e.,

$$M = \{c \in \mathbb{C} : F_{Q_c} \text{ is connected}\}. \tag{2}$$

The Mandelbrot set M for the quadratic function $Q_c(z) = z^2 + c$ can be equivalently defined in the following way [16]:

$$M = \{c \in \mathbb{C} : \{Q_c^n(0)\} \text{ does not tend to } \infty \text{ as } n \to \infty\}, \tag{3}$$

We choose the initial point 0, because 0 is the only critical point of Q_c, i.e., $Q_c'(0) = 0$.

Definition 3 (Multicorn): Let $A_c(z) = \bar{z}^m + c$, where $c \in \mathbb{C}$. The multicorn $\mathcal{M}^*$ for A_c is defined as the collection of all $c \in \mathbb{C}$ for which the orbit of 0 under the action of A_c is bounded, i.e.,

$$\mathcal{M}^* = \{c \in \mathbb{C} : |A_c^n(0)| \not\to \infty \text{ as } n \to \infty\} \tag{4}$$

Multicorn for $m = 2$ is called the tricorn.

To generate visualizations of Mandelbrot and Julia sets we can use many different algorithms [17], [18], e.g., distance estimator, potential function, escape time etc. In this paper we use the escape time algorithm. The algorithm is based on the number of iterations necessary to determine whether the orbit sequence tends to infinity or not. To determine whether the orbit escapes or not we use the escape criterion. For instance, for the classical Mandelbrot and Julia sets, i.e., the sets defined by $Q_c(z) = z^2 + c$, the escape criterion is the following: if there exists $k \geq 0$ such that

$$|Q_c^k(z)| > \max\{|c|, 2\}, \tag{5}$$

then $Q_c^n(z) \to \infty$ as $n \to \infty$.

We call the right side of (5) the escape threshold. This threshold can be different for different functions Q_c and plays a very important role in the generation of Mandelbrot and Julia sets.

The escape time algorithms for the generation of Mandelbrot and Julia sets are presented in Algorithm 1 and 2, respectively.

Algorithm 1 Mandelbrot Set Generation

Input: $Q_c : \mathbb{C} \to \mathbb{C}$ – polynomial function, $A \subset \mathbb{C}$ – area, K – maximum number of iterations, $colourmap[0..C-1]$ – colourmap with C colours.
Output: Mandelbrot set for the area A.

for $c \in A$ **do**
  R = calculate the escape threshold
  $n = 0$
  z_0 = critical point of Q_c
  while $n \leq K$ **do**
    $z_{n+1} = Q_c(z_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour c with $colourmap[i]$

Algorithm 2 Julia Set Generation

Input: $Q_c : \mathbb{C} \to \mathbb{C}$ – polynomial function, $c \in \mathbb{C}$ – parameter, $A \subset \mathbb{C}$ – area, K – maximum number of iterations, $colourmap[0..C-1]$ – colourmap with C colours.
Output: Julia set for the area A.

R = calculate the escape threshold
for $z_0 \in A$ **do**
  $n = 0$
  while $n \leq K$ **do**
    $z_{n+1} = Q_c(z_n)$
    if $|z_{n+1}| > R$ **then**
      break
    $n = n + 1$
  $i = \lfloor (C-1)\frac{n}{K} \rfloor$
  colour z_0 with $colourmap[i]$

Definition 4 (Picard Iteration [15]): Let X be a nonempty set and $f : X \to X$. For any point $x_0 \in X$, the Picard's iteration is defined in the following way

$$x_{k+1} = f(x_k), \tag{6}$$

where $k = 0, 1, \ldots$.

Definition 5 (Jungck iteration [19]): Let $S, T : X \to X$ be the two maps such that S is injective. For any $x_0 \in X$ the

Jungck iteration is defined in the following way

$$S(x_{k+1}) = T(x_k), \tag{7}$$

where $k = 0, 1, \dots$.

Definition 6 (Jungck-Mann Iteration With s-Convexity [11]): Let $S, T : \mathbb{C} \to \mathbb{C}$ be the two complex maps such that T is a complex polynomial of degree greater than 1 and S is injective. For any $x_0 \in \mathbb{C}$ the Jungck-Mann iteration with s-convexity is defined in the following way

$$S(x_{k+1}) = (1-\alpha)^s S(x_k) + \alpha^s T(x_k), \tag{8}$$

where $\alpha, s \in (0, 1]$, $k = 0, 1, 2, \dots$.

Let us notice that for $\alpha = 1$ Jungck-Mann iteration with s-convexity reduces to the Jungck iteration.

Definition 7 (Jungck-Ishikawa Iteration With s-Convexity [11]): Let $S, T : \mathbb{C} \to \mathbb{C}$ be the two complex maps such that T is a complex polynomial of degree greater than 1 and S is injective. For any $x_0 \in \mathbb{C}$ the Jungck-Ishikawa iteration with s-convexity is defined in the following way

$$\begin{cases} S(x_{k+1}) = (1-\alpha)^s S(x_k) + \alpha^s T(y_k), \\ S(y_k) = (1-\beta)^s S(x_k) + \beta^s T(x_k), \end{cases} \tag{9}$$

where $\alpha, s \in (0, 1]$, $\beta \in [0, 1]$ and $k = 0, 1, 2, \dots$.

Let us notice that Jungck-Ishikawa iteration with s-convexity reduces to the:

Jungck-Mann

- iteration with s-convexity for $\beta = 0$,
- Jungck iteration for $\beta = 0$ and $\alpha = 1$.

Definition 8 (Jungck-Noor Iteration With s-Convexity [10]): Let $S, T : \mathbb{C} \to \mathbb{C}$ be the two complex maps such that T is a complex polynomial of degree greater than 1 and S is injective. For any $x_0 \in \mathbb{C}$ the Jungck-Noor iteration with s-convexity is defined in the following way

$$\begin{cases} S(x_{k+1}) = (1-\alpha)^s S(x_k) + \alpha^s T(y_k), \\ S(y_k) = (1-\beta)^s S(x_k) + \beta^s T(u_k), \\ S(u_k) = (1-\gamma)^s S(x_k) + \gamma^s T(x_k), \end{cases} \tag{10}$$

where $\alpha, s \in (0, 1]$, $\beta, \gamma \in [0, 1]$ and $k = 0, 1, 2, \dots$.

Let us notice that Jungck-Noor iteration with s-convexity reduces to the:

Jungck-Ishikawa

- iteration with s-convexity for $\gamma = 0$,

Jungck-Mann

- iteration with s-convexity for $\gamma = \beta = 0$,
- Jungck iteration for $\gamma = \beta = 0$ and $\alpha = 1$.

III. MAIN RESULT

Let us notice that the Picard iteration is the iteration used in the generation of complex fractals. In the literature we can find results of replacing Picard iteration with other iterations, e.g., with the ones presented in Sec. II [10], [11]. In this section we show how to extend the Jungck-CR iteration using the s-convex iteration, and next the use of the extended iteration in the generation of Mandelbrot and Julia sets.

Let us start with the definition of the Jungck-CR iteration.

Definition 9 (Jungck-CR iteration [19]): Let $S, T : \mathbb{C} \to \mathbb{C}$ be mappings, where S is injective, and let $x_0 \in \mathbb{C}$ be a starting point. The Jungck-CR iteration is defined as follows:

$$\begin{cases} S(x_{k+1}) = (1-\alpha)S(y_k) + \alpha T(y_k) \\ S(y_k) = (1-\beta)T(x_k) + \beta T(u_k), \\ S(u_k) = (1-\gamma)S(x_k) + \gamma T(x_k), \end{cases} \tag{11}$$

where $\alpha \in (0, 1]$, $\beta, \gamma \in [0, 1]$ and $k = 0, 1, 2, \dots$.

In each of the three steps of the Jungck-CR iteration we use a convex combination of two elements. In the literature we can find some generalizations of the convex combination. One of such generalizations is the s-convex combination.

Definition 10 (s-Convex Combination [7]): Let $z_1, z_2, \dots, z_n \in \mathbb{C}$ and $s \in (0, 1]$. The s-convex combination is defined in the following way:

$$\lambda_1^s z_1 + \lambda_2^s z_2 + \dots + \lambda_n^s z_n, \tag{12}$$

where $\lambda_k \geq 0$ for $k \in \{1, 2, \dots, n\}$ and $\sum_{k=1}^{n} \lambda_k = 1$.

Let us notice that the s-convex combination for $s = 1$ reduces to the standard convex combination. This type of combination was successfully used in the generation of Mandelbrot and Julia sets [10], [11]. Moreover, it was also used in the generation of other type of fractals generated in the complex plane, namely in the methods that use root finding of complex polynomials [20].

Now, we will replace the convex combination in the Jungck-CR iteration with the s-convex one.

Definition 11 (Jungck-CR Iteration With s-Convexity): Let $S, T : \mathbb{C} \to \mathbb{C}$ be mappings, where S is injective, and let $x_0 \in \mathbb{C}$ be a starting point. The Jungck-CR iteration with s-convexity is defined as follows:

$$\begin{cases} S(x_{k+1}) = (1-\alpha)^s S(y_k) + \alpha^s T(y_k), \\ S(y_k) = (1-\beta)^s T(x_k) + \beta^s T(u_k), \\ S(u_k) = (1-\gamma)^s S(x_k) + \gamma^s T(x_k), \end{cases} \tag{13}$$

where $\alpha, s \in (0, 1]$, $\beta, \gamma \in [0, 1]$ and $k = 0, 1, 2, \dots$.

Let us notice that the Jungck-CR iteration with s-convexity does not reduce to any of the iterations: Picard, Jungck-Mann with s-convexity, Jungck-Ishikawa with s-convexity, Jungck-Noor with s-convexity. Thus, using this iteration we will create completely new orbits and in consequence new fractal sets.

In Picard iteration we use only one mapping and in the Jungck-CR iteration with s-convexity we have two mappings. Thus, if we want to replace Picard iteration with the Jungck-CR iteration with s-convexity, then we need to handle the case of different number of mappings in the iterations. We handle this in a following way. Let $Q_c : \mathbb{C} \to \mathbb{C}$ be a polynomial function. We decompose Q_c into two mappings S, T in such a way that $Q_c = T - S$ and S is injective. In the case of multicorns we decompose $Q_c^*(z) = Q_c(\overline{z})$ in a following way: $Q_c^* = T - S$, where $T = A_c$ and S is injective. Of course this type of decomposition restricts the choice of the polynomial functions that can be used. Having the decomposition we also need to derive a new escape criterion for the mappings and (13).

In the following subsections we prove escape criteria for some classes of polynomials.

A. ESCAPE CRITERION FOR THE QUADRATIC COMPLEX POLYNOMIAL

Let $Q_c(z) = z^2 - az + c$, where $a, c \in \mathbb{C}$. We decompose Q_c in the following way: $T(z) = z^2 + c$ and $S(z) = az$.

Theorem 1: Assume that $|z| \geq |c| > \frac{2(1+|a|)}{s\alpha}$, $|z| \geq |c| > \frac{2(1+|a|)}{s\beta}$, $|z| \geq |c| > \frac{2(1+|a|)}{s\gamma}$ *where* $\alpha, \beta, \gamma, s \in (0, 1]$ *and define* $\{z_k\}_{k\in\mathbb{N}}$ *as follows*

$$\begin{cases} S(z_{k+1}) = (1-\alpha)^s S(y_k) + \alpha^s T(y_k) \\ S(y_k) = (1-\beta)^s T(z_k) + \beta^s T(u_k), \\ S(u_k) = (1-\gamma)^s S(z_k) + \gamma^s T(z_k) \quad k = 0, 1, 2, \ldots, \end{cases} \tag{14}$$

where $z_0 = z$. *Then* $|z_k| \to \infty$ *as* $k \to \infty$.

Proof: Because $T(z) = z^2 + c$, $S(z) = az$ and $z_0 = z$ we have

$$\begin{aligned} |S(u_0)| &= \left|(1-\gamma)^s S(z) + \gamma^s T(z)\right| \\ &= \left|(1-\gamma)^s az + (1-(1-\gamma))^s (z^2 + c)\right| \end{aligned}$$

Using binomials series up to linear terms of γ and $1-\gamma$, and condition $s < 1$ we get

$$\begin{aligned} |au_0| &\geq (1 - s(1-\gamma))|z^2 + c| - (1 - s\gamma)|az| \\ &\geq |(s - s(1-\gamma))(z^2 + c)| - |(1 - s\gamma)az|. \end{aligned}$$

Because $|z| \geq |c|$, $|a| \geq 1$ and $s\gamma < 1$ we obtain

$$\begin{aligned} |au_0| &\geq s\gamma \left| |z^2| - s\gamma|c| \right| - (1 - s\gamma)|az| \\ &= s\gamma \left| |z^2| - s\gamma|c| \right| - |az| + s\gamma|az| \\ &\geq s\gamma \left| |z^2| - |z| \right| - |a||z| \\ &= |z|\,(s\gamma\,|z| - (1 + |a|)). \end{aligned}$$

Thus

$$|u_0| \geq |z|\,(\frac{s\gamma\,|z|}{1 + |a|} - 1).$$

In the second step of the iteration we have

$$\begin{aligned} |S(y_0)| &= \left|(1-\beta)^s T(z) + \beta^s T(u_0)\right| \\ &\geq \left|(1 - s\beta)(z^2 + c) + (1 - s(1-\beta))(u_0^2 + c)\right| \\ &\geq \left|(1 - s\beta)(z^2 + c) + (s - s(1-\beta))(u_0^2 + c)\right| \\ &\geq s\beta|u_0^2| - s\beta|c| - |c| + s\beta|c| + (1 - s\beta)|z^2| \\ &\geq s\beta|u_0^2| - |z| (\text{because } |1 - s\beta| \geq 0, |z| \geq |c|). \end{aligned}$$

Since $|z| > \frac{2(1+|a|)}{s\gamma}$, which implies $|z|^2 \left(\frac{s\gamma|z|}{1+|a|} - 1\right)^2 > |z|^2$. Hence $|u_0|^2 > |z|^2 \left(\frac{s\gamma|z|}{1+|a|} - 1\right)^2 > |z|^2 > s\gamma\,|z|^2$, and $1 + |a| \geq 1$. We get

$$ay_0 \geq s^2\beta\gamma|z^2| - (1 + |a|)\,|z|.$$

Thus

$$|y_0| \geq |z| \left(\frac{s^2\beta\gamma\,|z|}{1 + |a|} - 1\right).$$

In the third step of the iteration we have

$$\begin{aligned} |S(z_1)| &= \left|(1-\alpha)^s S(y_0) + \alpha^s T(y_0)\right| \\ &\geq \left|(1 - s\alpha)ay_0 + s\alpha(y_0^2 + c)\right| \\ &\geq s\alpha|y_0^2 + c| - (1 - s\alpha)|ay_0|. \end{aligned}$$

Since $|y_0| \geq |z| \left(\frac{s^2\beta\gamma|z|}{1+|a|} - 1\right)$, which implies $|y_0^2| \geq s^2\beta\gamma|z^2|$. We get

$$\begin{aligned} |az_1| &\geq s^3\alpha\beta\gamma \left|z^2\right| - s\alpha|c| - |a||z| \\ &\geq s^3\alpha\beta\gamma \left|z^2\right| - |z| - |a||z| \\ &= s^3\alpha\beta\gamma \left|z^2\right| - (1 + |a|)\,|z| \\ &= |z|\,(s^3\alpha\beta\gamma\,|z| - (1 + |a|)). \end{aligned}$$

Hence

$$|z_1| \geq |z| \left(\frac{s^3\alpha\beta\gamma\,|z|}{1 + |a|} - 1\right).$$

Since $|z| > \frac{2(1+|a|)}{s\alpha}$, $|z| > \frac{2(1+|a|)}{s\beta}$, $|z| > \frac{2(1+|a|)}{s\gamma}$, so $|z| > \frac{2(1+|a|)}{s^3\alpha\beta\gamma}$ and in consequence $\frac{s^3\alpha\beta\gamma|z|}{1+|a|} - 1 > 1$. Therefore there exist $\lambda > 0$ such that $\frac{s^3\alpha\beta\gamma|z|}{1+|a|} - 1 > 1+\lambda$. Consequently $|z_1| > (1+\lambda)|z|$. In particular $|z_1| > |z|$. So we may apply the same argument repeatedly to find $|z_k| > (1+\lambda)^k|z|$. Thus, the orbit of z tends to infinity and this completes the proof. □

Corollary 1: Suppose that

$$|c| > \frac{2(1 + |a|)}{s\alpha}, |c| > \frac{2(1 + |a|)}{s\beta} \text{ and } |c| > \frac{2(1 + |a|)}{s\gamma}, \tag{15}$$

then the Jungck-CR iteration with s-convexity escapes to infinity.

In the proof of theorem we used the facts that $|z| \geq |c| > \frac{2(1+|a|)}{s\alpha}$, $|z| \geq |c| > \frac{2(1+|a|)}{s\beta}$ and $|z| \geq |c| > \frac{2(1+|a|)}{s\gamma}$. Hence the following corollary is the refinement of the escape criterion discussed in the theorem.

Corollary 2 (Escape Criterion): Let $\alpha, \beta, \gamma, s \in (0, 1]$ *and*

$$|z| > \max\left\{|c|, \frac{2(1 + |a|)}{s\alpha}, \frac{2(1 + |a|)}{s\beta}, \frac{2(1 + |a|)}{s\gamma}\right\}, \tag{16}$$

then there exist $\lambda > 0$ *such that* $|z_k| > (1+\lambda)^k|z|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

Corollary 3: Suppose that

$$|z_m| > \max\left\{|c|, \frac{2(1 + |a|)}{s\alpha}, \frac{2(1 + |a|)}{s\beta}, \frac{2(1 + |a|)}{s\gamma}\right\} \tag{17}$$

for some $m \geq 0$. *Then there exist* $\lambda > 0$ *such that* $|z_{m+k}| > (1 + \lambda)^k|z_m|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

B. ESCAPE CRITERION FOR THE CUBIC COMPLEX POLYNOMIAL

Let $Q_c(z) = z^3 - az + c$, where $a, c \in \mathbb{C}$. We decompose Q_c in the following way: $T(z) = z^3 + c$ and $S(z) = az$.

Theorem 2: Assume that $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{2}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{2}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{2}}$ *where* $\alpha, \beta, \gamma, s \in (0, 1]$ *and define* $\{z_k\}_{k\in\mathbb{N}}$ *as follows*

$$\begin{cases} S(z_{k+1}) = (1-\alpha)^s S(y_k) + \alpha^s T(y_k) \\ S(y_k) = (1-\beta)^s T(z_k) + \beta^s T(u_k), \\ S(u_k) = (1-\gamma)^s S(z_k) + \gamma^s T(z_k) \quad k = 0, 1, 2, \dots, \end{cases} \tag{18}$$

where $z_0 = z$. *Then* $|z_k| \to \infty$ *as* $k \to \infty$.

Proof: Because $T(z) = z^3 + c$, $S(z) = az$ and $z_0 = z$ we have

$$\begin{aligned} |S(u_0)| &= \left|(1-\gamma)^s S(z) + \gamma^s T(z)\right| \\ &= \left|(1-\gamma)^s az + (1-(1-\gamma))^s (z^3 + c)\right| \end{aligned}$$

Using binomials series up to linear terms of γ and $1-\gamma$, and condition $s < 1$ we get

$$\begin{aligned} |au_0| &\geq (1 - s(1-\gamma))|z^3 + c| - (1 - s\gamma)|az| \\ &\geq |(s - s(1-\gamma))(z^3 + c)| - |(1 - s\gamma)az| . \end{aligned}$$

Because $|z| \geq |c|$, $|a| \geq 1$ and $s\gamma < 1$ we obtain

$$\begin{aligned} |au_0| &\geq s\gamma \left|z^3| - s\gamma|c\right| - (1 - s\gamma)|az| \\ &= s\gamma \left|z^3| - s\gamma|c\right| - |az| + s\gamma|az| \\ &\geq s\gamma \left|z^3| - |z\right| - |a||z| \\ &= |z| \left(s\gamma \left|z^2\right| - (1 + |a|)\right). \end{aligned}$$

Thus

$$|u_0| \geq |z| \left(\frac{s\gamma \left|z^2\right|}{1 + |a|} - 1\right).$$

In the second step of the iteration we have

$$\begin{aligned} |S(y_0)| &= \left|(1-\beta)^s T(z) + \beta^s T(u_0)\right| \\ &\geq \left|(1 - s\beta)(z^3 + c) + (1 - s(1-\beta))(u_0^3 + c)\right| \\ &\geq \left|(1 - s\beta)(z^3 + c) + (s - s(1-\beta))(u_0^3 + c)\right| \\ &\geq s\beta|u_0^3| - s\beta|c| - |c| + s\beta|c| + (1 - s\beta)|z^3| \\ &\geq s\beta|u_0^3| - |z| (\text{because } |1 - s\beta| \geq 0, |z| \geq |c|). \end{aligned}$$

Since $|z| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{2}}$, which implies $|z|^3 \left(\frac{s\gamma|z|^2}{1+|a|} - 1\right)^3 > |z|^3$. Hence $|u_0|^3 > |z|^3 \left(\frac{s\gamma|z|^2}{1+|a|} - 1\right)^3 > |z|^3 > s\gamma\, |z|^3$, and $1 + |a| \geq 1$. We get

$$ay_0 \geq s^2\beta\gamma|z^3| - (1 + |a|)\, |z| .$$

Thus

$$|y_0| \geq |z| \left(\frac{s^2\beta\gamma\, |z|^2}{1 + |a|} - 1\right).$$

In the third step of the iteration we have

$$\begin{aligned} |S(z_1)| &= \left|(1-\alpha)^s S(y_0) + \alpha^s T(y_0)\right| \\ &\geq \left|(1 - s\alpha)ay_0 + s\alpha(y_0^3 + c)\right| \\ &\geq s\alpha|y_0^3 + c| - (1 - s\alpha)|ay_0|. \end{aligned}$$

Since $|y_0| \geq |z| \left(\frac{s^2\beta\gamma|z|^2}{1+|a|} - 1\right)$, which implies $|y_0^3| \geq s^2\beta\gamma|z^3|$. We get

$$\begin{aligned} |az_1| &\geq s^3\alpha\beta\gamma \left|z^3\right| - s\alpha|c| - |a||z| \\ &\geq s^3\alpha\beta\gamma \left|z^3\right| - |z| - |a||z| \\ &= s^3\alpha\beta\gamma \left|z^3\right| - (1 + |a|)\, |z| \\ &= |z| \left(s^3\alpha\beta\gamma \left|z^2\right| - (1 + |a|)\right). \end{aligned}$$

Hence

$$|z_1| \geq |z| \left(\frac{s^3\alpha\beta\gamma \left|z^2\right|}{1 + |a|} - 1\right).$$

Since $|z| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{2}}$, $|z| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{2}}$, $|z| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{2}}$, so $|z|^2 > \frac{2(1+|a|)}{s^3\alpha\beta\gamma}$ and in consequence $\frac{s^3\alpha\beta\gamma|z|^2}{1+|a|} - 1 > 1$. Therefore there exist $\lambda > 0$ such that $\frac{s^3\alpha\beta\gamma|z|^2}{1+|a|} - 1 > 1 + \lambda$. Consequently $|z_1| > (1 + \lambda)|z|$. In particular $|z_1| > |z|$. So we may apply the same argument repeatedly to find $|z_k| > (1 + \lambda)^k|z|$. Thus, the orbit of z tends to infinity and this completes the proof. □

Corollary 4: Suppose that

$$|c| > \left(\frac{2(1 + |a|)}{s\alpha}\right)^{\frac{1}{2}}, \quad |c| > \left(\frac{2(1 + |a|)}{s\beta}\right)^{\frac{1}{2}} \quad \textit{and} \quad |c| > \left(\frac{2(1 + |a|)}{s\gamma}\right)^{\frac{1}{2}}, \tag{19}$$

then the Jungck-CR iteration with s-convexity escapes to infinity.

In the proof of theorem we used the facts that $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{2}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{2}}$ and $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{2}}$. Hence the following corollary is the refinement of the escape criterion discussed in the theorem.

Corollary 5 (Escape Criterion): Let $\alpha, \beta, \gamma, s \in (0, 1]$ *and*

$$|z| > \max\left\{|c|, \left(\frac{2(1 + |a|)}{s\alpha}\right)^{\frac{1}{2}}, \left(\frac{2(1 + |a|)}{s\beta}\right)^{\frac{1}{2}}, \left(\frac{2(1 + |a|)}{s\gamma}\right)^{\frac{1}{2}}\right\}, \tag{20}$$

then there exist $\lambda > 0$ *such that* $|z_k| > (1 + \lambda)^k|z|$ *and* $|z_k| \to \infty$ *as* $k \to \infty$.

Corollary 6: Suppose that

$$|z_m| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{2}}, \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{2}}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{2}}\right\}, \quad (21)$$

for some $m \geq 0$. Then there exist $\lambda > 0$ such that $|z_{m+k}| > (1+\lambda)^k |z_m|$ and $|z_k| \to \infty$ as $k \to \infty$.

C. ESCAPE CRITERION FOR HIGHER DEGREE COMPLEX POLYNOMIALS

Let $Q_c(z) = z^n - az + c$, where $a, c \in \mathbb{C}$. We decompose Q_c in the following way: $T(z) = z^n + c$ and $S(z) = az$.

Theorem 3: Assume that $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}$ where $\alpha, \beta, \gamma, s \in (0, 1]$ and define $\{z_k\}_{k\in\mathbb{N}}$ as follows

$$\begin{cases} S(z_{k+1}) = (1-\alpha)^s S(y_k) + \alpha^s T(y_k) \\ S(y_k) = (1-\beta)^s T(z_k) + \beta^s T(u_k), \\ S(u_k) = (1-\gamma)^s S(z_k) + \gamma^s T(z_k) \quad k = 0, 1, 2, \ldots, \end{cases} \quad (22)$$

where $z_0 = z$. Then $|z_k| \to \infty$ as $k \to \infty$.

Proof: Because $T(z) = z^n + c$, $S(z) = az$ and $z_0 = z$ we have

$$\begin{aligned} |S(u_0)| &= \left|(1-\gamma)^s S(z) + \gamma^s T(z)\right| \\ &= \left|(1-\gamma)^s az + (1-(1-\gamma))^s (z^n + c)\right| \end{aligned}$$

Using binomials series up to linear terms of γ and $1-\gamma$, and condition $s < 1$ we get

$$\begin{aligned} |au_0| &\geq (1 - s(1-\gamma))|z^n + c| - (1 - s\gamma)|az| \\ &\geq |(s - s(1-\gamma))(z^n + c)| - |(1 - s\gamma)az| . \end{aligned}$$

Because $|z| \geq |c|$, $|a| \geq 1$ and $s\gamma < 1$ we obtain

$$\begin{aligned} |au_0| &\geq s\gamma\left|z^n\right| - s\gamma|c| - (1 - s\gamma)|az| \\ &= s\gamma\left|z^n\right| - s\gamma|c| - |az| + s\gamma|az| \\ &\geq s\gamma\left|z^n\right| - |z| - |a||z| \\ &= |z|\,(s\gamma\left|z^{n-1}\right| - (1+|a|)). \end{aligned}$$

Thus

$$|u_0| \geq |z|\,(\frac{s\gamma\left|z^{n-1}\right|}{1+|a|} - 1).$$

In the second step of the iteration we have

$$\begin{aligned} |S(y_0)| &= \left|(1-\beta)^s T(z) + \beta^s T(u_0)\right| \\ &\geq \left|(1 - s\beta)(z^n + c) + (1 - s(1-\beta))(u_0^n + c)\right| \\ &\geq \left|(1 - s\beta)(z^n + c) + (s - s(1-\beta))(u_0^n + c)\right| \\ &\geq s\beta|u_0^n| - s\beta|c| - |c| + s\beta|c| + (1 - s\beta)|z^n| \\ &\geq s\beta|u_0^n| - |z|(\text{because } |1 - s\beta| \geq 0, |z| \geq |c|). \end{aligned}$$

Since $|z| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}$, which implies $|z|^n\left(\frac{s\gamma|z|^{n-1}}{1+|a|} - 1\right)^n > |z|^n$. Hence $|u_0|^n > |z|^n\left(\frac{s\gamma|z|^{n-1}}{1+|a|} - 1\right)^n > |z|^n > s\gamma\,|z|^n$, and $1 + |a| \geq 1$. We get

$$ay_0 \geq s^2\beta\gamma|z^n| - (1+|a|)\,|z|\,.$$

Thus

$$|y_0| \geq |z|\left(\frac{s^2\beta\gamma\,|z|^{n-1}}{1+|a|} - 1\right).$$

In the third step of the iteration we have

$$\begin{aligned} |S(z_1)| &= \left|(1-\alpha)^s S(y_0) + \alpha^s T(y_0)\right| \\ &\geq \left|(1 - s\alpha)ay_0 + s\alpha(y_0^n + c)\right| \\ &\geq s\alpha|y_0^n + c| - (1 - s\alpha)|ay_0|. \end{aligned}$$

Since $|y_0| \geq |z|\left(\frac{s^2\beta\gamma|z|^{n-1}}{1+|a|} - 1\right)$, which implies $|y_0^n| \geq s^2\beta\gamma|z^n|$. We get

$$\begin{aligned} |az_1| &\geq s^3\alpha\beta\gamma\left|z^n\right| - s\alpha|c| - |a||z| \\ &\geq s^3\alpha\beta\gamma\left|z^n\right| - |z| - |a||z| \\ &= s^3\alpha\beta\gamma\left|z^n\right| - (1+|a|)\,|z| \\ &= |z|\,(s^3\alpha\beta\gamma\left|z^{n-1}\right| - (1+|a|)). \end{aligned}$$

Hence

$$|z_1| \geq |z|\left(\frac{s^3\alpha\beta\gamma\left|z^{n-1}\right|}{1+|a|} - 1\right). \quad (23)$$

Since $|z| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}$, $|z| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}$, $|z| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}$, so $|z|^{n-1} > \frac{2(1+|a|)}{s^3\alpha\beta\gamma}$ and in consequence $\frac{s^3\alpha\beta\gamma|z|^{n-1}}{1+|a|} - 1 > 1$. Therefore there exist $\lambda > 0$ such that $\frac{s^3\alpha\beta\gamma|z|^{n-1}}{1+|a|} - 1 > 1 + \lambda$. Consequently $|z_1| > (1+\lambda)|z|$. In particular $|z_1| > |z|$. So we may apply the same argument repeatedly to find $|z_k| > (1+\lambda)^k|z|$. Thus, the orbit of z tends to infinity and this completes the proof. □

Corollary 7: Suppose that

$$|c| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}, \quad |c| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}} \quad \textit{and}$$

$$|c| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}, \quad (24)$$

then the Jungck-CR iteration with s-convexity escapes to infinity.

Corollary 8 (Escape Criterion): Let $\alpha, \beta, \gamma, s \in (0, 1]$ and

$$|z| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}\right\}, \quad (25)$$

then there exist $\lambda > 0$ such that $|z_k| > (1+\lambda)^k|z|$ and $|z_k| \to \infty$ as $k \to \infty$.

Corollary 9: Suppose that

$$|z_m| > \max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}\right\}, \tag{26}$$

for some $m \geq 0$. Then there exist $\lambda > 0$ such that $|z_{m+k}| > (1+\lambda)^k|z_m|$ and $|z_k| \to \infty$ as $k \to \infty$.

Theorem 4: Suppose that $\{z_k\}_{k\in\mathbb{N}}$ be the sequence of Jungck-CR iteration with s-convexity and $|z_k| \to \infty$ as $k \to \infty$, then $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}$, $|z| \geq |c| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}$ where $\alpha, \beta, \gamma, s \in (0, 1]$.

Proof: Since $\{z_k\}_{k\in\mathbb{N}}$ be the sequence of Jungck-CR iteration with s-convexity and $|z_k| \to \infty$ as $k \to \infty$, then it must be hold

$$|z_k| \geq (1+\lambda)^k|z|.$$

For $k = 1$, we have

$$|z_1| \geq (1+\lambda)|z|. \tag{27}$$

But, since $\{z_k\}_{k\in\mathbb{N}}$ be the sequence of Jungck-CR iteration with s-convexity, then from (23)

$$|z_1| \geq |z|\left(\frac{s^3\alpha\beta\gamma\left|z^{n-1}\right|}{1+|a|} - 1\right). \tag{28}$$

Comparing (27) and (28), we have

$$\frac{s^3\alpha\beta\gamma\left|z^{n-1}\right|}{1+|a|} - 1 = 1+\lambda$$
$$\frac{s^3\alpha\beta\gamma\left|z^{n-1}\right|}{1+|a|} - 1 > 1,$$

because $\lambda > 0$. This yields

$$|z| > \left(\frac{2(1+|a|)}{s^3\alpha\beta\gamma}\right)^{\frac{1}{n-1}}.$$

Consequently, we have $|z| > \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}$, $|z| > \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}$ and $|z| > \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}$ where $n \geq 2$ and $\alpha, \beta, \gamma, s \in (0, 1]$ and for complex fractal generation $|z|$ must be greater and equal to $|c|$, because for any given point $|z| < |c|$, we have to compute the Jungck-CR orbit with s-convexity of z. If for some k, $|z_k|$ lies outside the circle of radius $\max\left\{|c|, \left(\frac{2(1+|a|)}{s\alpha}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\beta}\right)^{\frac{1}{n-1}}, \left(\frac{2(1+|a|)}{s\gamma}\right)^{\frac{1}{n-1}}\right\}$, we guarantee that the proposed orbit escapes. Hence, z is not in the Julia sets and also, is not in Mandelbrot sets. On the other hand, if $|z_k|$ never exceeds this bound (i.e. $|z| \geq |c|$ holds), then by definition of the Julia sets, Mandelbrot sets and Multicorns, $|z_k|$ lies in Julia sets, Mandelbrot sets and Multicorns. This completes the proof. □

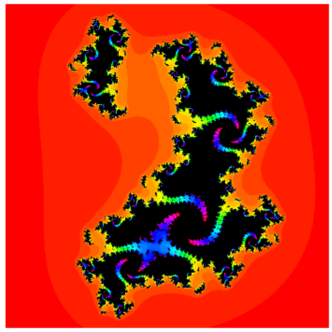

FIGURE 1. Example of Julia set for a quadratic function generated using Jungck-CR iteration with *s*-convexity.

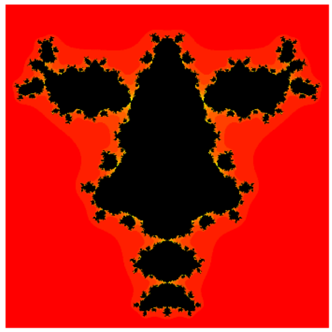

FIGURE 2. Example of Julia set for a cubic function generated using Jungck-CR iteration with *s*-convexity.

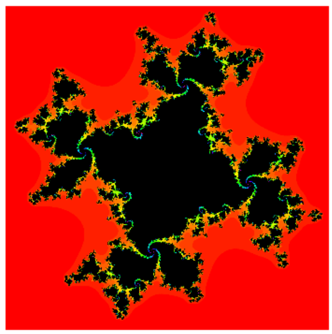

FIGURE 3. Example of Julia set for a quartic function generated using Jungck-CR iteration with *s*-convexity.

IV. GRAPHICAL EXAMPLES

In this section we present some graphical examples of Mandelbrot and Julia sets obtained with the Jungck-CR iteration with s-convexity. To generate the images we used the escape time algorithm with the escape criteria derived in Sec. III. The The algorithms were implemented in Mathematica.

In the first example we present some Julia sets generated by using the Jungck-CR iteration with s-convexity. In all the examples the same maximum number of iterations was preformed, and it was equal to 50. The resulting images are presented in Figs. 1–3 and the parameters used to generate them were the following:

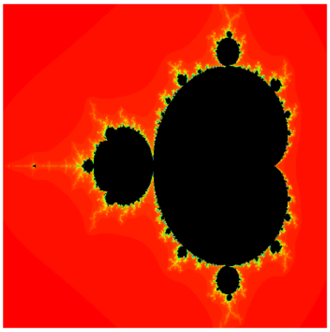

FIGURE 4. Example of Mandelbrot set for a quadratic function generated using Jungck-CR iteration with *s*-convexity.

FIGURE 5. Example of Mandelbrot set for a cubic function generated using Jungck-CR iteration with *s*-convexity.

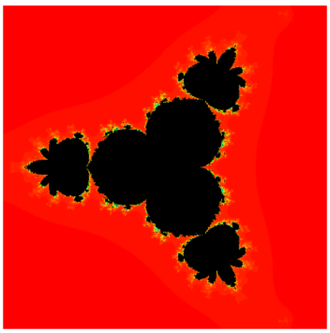

FIGURE 6. Example of Mandelbrot set for a quartic function generated using Jungck-CR iteration with *s*-convexity.

- Fig. 1: $Q_c(z) = z^2 + 2\,z + c$, $c = 0.95 + 0.5\mathbf{i}$, $A = [-3.5, 3.5]^2$, $\alpha = 0.5$, $\beta = 0.7$, $\gamma = 0.8$, $s = 0.6$,
- Fig. 2: $Q_c(z) = z^3 + \frac{3}{2}z + c$, $c = -2.295\mathbf{i}$, $A = [-3.2, 3.2] \times [-3.4, 3.4]$, $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $s = 0.7$,
- Fig. 3: $Q_c(z) = z^4 + \mathbf{i}z + c$, $c = -0.03 + 0.81\mathbf{i}$, $A = [-2.7, 2.7]^2$, $\alpha = 0.9$, $\beta = 0.1$, $\gamma = 0.8$, $s = 0.9$.

In the second example we present some Mandelbrot sets generated by using the Jungck-CR iteration with s-convexity. In all the examples the same maximum number of iterations was preformed, and it was equal to 100. The resulting images

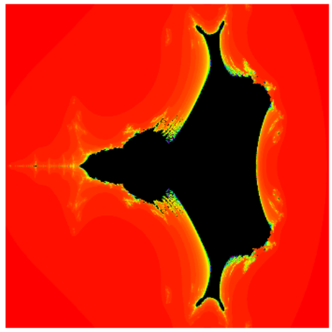

FIGURE 7. Example of multicorn for a quadratic function generated using Jungck-CR iteration with *s*-convexity.

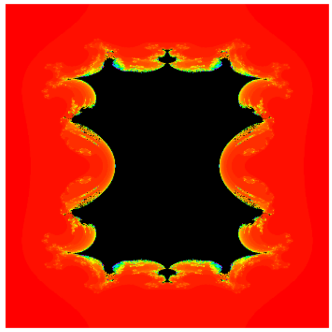

FIGURE 8. Example of multicorn for a cubic function generated using Jungck-CR iteration with *s*-convexity.

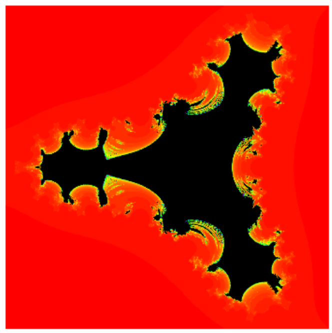

FIGURE 9. Example of multicorn for a quartic function generated using Jungck-CR iteration with *s*-convexity.

are presented in Figs. 4–6 and the parameters used to generate them were the following:

- Fig. 4: $Q_c(z) = z^2 + 2\,z + c$, $A = [-35, 12] \times [-12, 12]$, $\alpha = 0.9$, $\beta = 0.9$, $\gamma = 0.9$, $s = 0.8$,
- Fig. 5: $Q_c(z) = z^3 + \frac{3}{2}z + c$, $A = [-4, 4] \times [-8, 8]$, $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $s = 0.7$,
- Fig. 6: $Q_c(z) = z^4 + \mathbf{i}z + c$, $A = [-4, 4]^2$, $\alpha = 0.9$, $\beta = 0.1$, $\gamma = 0.8$, $s = 0.9$.

In the last example examples of multicorns generated via the Jungck-CR iteration with s-convexity are presented. Similar to the example with the Mandelbrot sets the maximum number of iteration is the same for all examples, and it is equal to 100. The generated images of multicorns are presented in Fig. 7–9 and the parameters used to generate them were the following:

- Fig. 7: $Q_c(z) = z^2 + 2\,z + c$, $A = [-35, 14] \times [-9, 9]$, $\alpha = 0.9$, $\beta = 0.9$, $\gamma = 0.9$, $s = 0.8$,
- Fig. 8: $Q_c(z) = z^3 + \frac{3}{2}z + c$, $A = [-4, 4] \times [-6, 6]$, $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $s = 0.7$,
- Fig. 9: $Q_c(z) = z^4 + \mathbf{i}z + c$, $A = [-3.5, 3.5]^2$, $\alpha = 0.9$, $\beta = 0.1$, $\gamma = 0.8$, $s = 0.9$.

V. CONCLUSIONS

In this paper, we study the Jungck-CR iteration with s-convexity in the generation of fractals (i.e. Julia sets, Mandelbrot sets, tricorn and Multicorns) for non-linear dynamics. We proved escape criterion to generate Mandelbrot sets, Julia sets, tricorn and multicorns using this type of iteration for complex quadratic, cubic and nth degree complex polynomials. We presented the use of iteration other than the Picard one in the generation of Mandelbrot and Julia sets. Moreover, we presented some graphical examples which showed that with the use of the Jungck-CR iteration with s-convexity we are able to obtain many diverse shapes of fractal sets. The colors of figures depend upon the number of iterations and input parameters. The input parameters are different for each figure, so color difference appeared.

The Jungck-CR iteration with s-convexity does not reduce to neither Picard, nor Jungck-Mann, nor Jungck-Ishikawa, nor Jungck-Noor, nor any other iteration studied in the literature on the generation of complex fractals, so the results of this paper open new class of complex fractals. Moreover, the obtained complex fractals could further extend the capabilities of the algorithms that use Mandelbrot and Julia sets, e.g., they can expand the domain dictionary used in fractal image compression [21] or broaden the space for the initial keys used in image encryption [22].

In our further work we will try to derive the escape criteria in the Jungck-CR iteration with s-convexity for functions of other classes than the polynomial one, e.g., trigonometric. Moreover, in the fixed point literature we can find many different iteration methods that can be used in the study of Julia and Mandelbrot sets. A review of the explicit iterations and their dependencies can be found in the paper by Gdawiec and Kotarski [23].

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