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Paradoxical Simulations to Enhance Education in Mathematics

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ABSTRACT The subject of probability and statistics is easily dismissed by students as assemblages of formulae to be rote-memorized. We propose here an integration of simulation-based activities with certain mathematical paradoxes using patchwork assessment to first-year undergraduates, such that they can better appreciate the real-world context of probability and statistics. The proposed examples alongside various facilitation skills for the instructor are discussed. We also provide an original spreadsheet simulation program in *Excel* and *Visual Basic for Applications* to reproduce the numerical experiments. This program is capable of running Monte Carlo simulations for all three seminal Parrondo's paradox variants, and can be easily used by students and instructors; moreover, the computed datasets and code are fully-transparent, thereby allowing interactive discussions, modifications and extensions, and further analyses. Our findings suggest that the proposed teaching strategy is useful, and we hope that this work will initiate the significant adoption of paradoxical simulations in teaching practice. The interactive program is freely available on open science framework.

INDEX TERMS Mathematics, interdisciplinary, education, paradoxical simulations, smart classroom, Parrondo's paradox, game design.

I. INTRODUCTION

The subject of probability and statistics is deeply entwined with a great plethora of real-life applications. Despite their ubiquity, probability and statistics are easily perceived by students as an assemblage of formulae to be rote-memorized, rather than a conceptual undertaking with wide-ranging applications [1]. More often than not, instructors do not equip students with the means to handle non-routine problems [2]; it is important to discuss general probabilistic principles and introduce more sophisticated strategies specific to practical problem solving [3], [4].

Many studies have investigated the use of mathematical paradoxes to motivate students in learning probability and statistics [2], [5]–[10]. Commonly used examples include the Monty Hall problem [11]–[13], which investigates the utility of re-selecting choices, and Simpson's paradox [14], [15], which elucidates ambiguity in the statistics of amalgamated groups. The introduction of paradoxes in the classroom encourages active learning, and forces students to confront conflicts between intuition and theory; when appropriately integrated into the curriculum, it is an effective tool to promote deeper conceptual learning and statistical literacy [9]. The advantages of using simulations as a pedagogical device [16] have also been discussed in literature [17]–[24], with recent success in the design of advanced virtual laboratories [25], [26] and the use of snake puzzles for learning structural bioinformatics [27]. Educators have found simulation-based approaches effective at enhancing the understanding of entwined concepts in complex problems [20]; and students have found such activities more interesting, intrinsically motivating, and closer to real-world experiences than other learning modes [28].

In this paper, we propose the integration of a simulation-based approach with certain mathematical paradoxes to enhance first-year undergraduate education in mathematics. The objective is to promote appreciation of the real-world applications of probability and statistics, so that students understand *why* they are learning the particular subjects. In modern times, computer simulations allow users to replicate results and extend established analyses with ease. While the theoretical frameworks may be too advanced to be discussed in their entirety, qualitative mathematical intuitions can still be gained through observing the simulation results. For instance, the Parrondo's paradox [29], Monty Hall problem [11]–[13], Simpson's paradox [14], [15], St. Petersburg paradox [30], and the birthday problem [31] can all be simulated, thereby allowing students to interactively explore and extend these problems. These paradox-based activities are not to replace the teaching of fundamental probability and statistics concepts, but rather to reinforce the base curriculum as a supplementary component.

Our key contribution is on showcasing the educational context of Parrondo's paradox [29] and how it can be introduced to students through simulations—starting from the necessary theoretical and technical scaffolds, to the final analysis of the simulation results. In addition to this novel educational perspective, we provide a comprehensive *Microsoft Excel* simulation suite, capable of running simulations for all three seminal variants of the Parrondo's paradox. Single-trial simulations are implemented entirely in native *Excel* formulae in a fully transparent fashion, so that students can pick up and extend the basic concepts easily. Multi-trial Monte Carlo simulations are implemented both in native formulae and in *Visual Basic for Applications (VBA)* macros, hence catering to students who are proficient in programming. This simulation-based approach brings students into close contact with key statistical concepts, such as the notions of randomness and determinism, and the law of large numbers, serving as an effective review platform outside of the classroom.

As a pilot programme, we have implemented the proposed paradoxical simulations in a first-year mathematics course for undergraduates enrolled in a Game Design degree (Spring 2017). A study was conducted to assess students' perception and receptiveness of the paradoxical simulations. The findings are generally positive, giving us the confidence to scale up such paradox-motivated simulation projects to larger class sizes.

II. THEORETICAL REVIEW & SIMULATION SUITE

An overview of the fundamentals of the Parrondo's paradox [29], [32], [33] is first presented, alongside the recommended conceptual scaffolding and possible teaching pointers. The proposed simulation suite is also demonstrated in a pedagogical context.

Two games, termed Games A and B, are considered, each producing a losing outcome when individually played. Playing these losing games in a random or periodic order may counter-intuitively produce winning outcomes. The Parrondo's paradox is a mathematical abstraction of the flashing Brownian ratchet, a phenomenon in which a Brownian particle undergoes systematic drift in the presence of a switching potential [34]–[36]. The paradox has wide-ranging implications across engineering [32], [33], life science [37]–[44], medicine [45], [46], and the physical sciences [47]–[49], including applications to quantum-physical systems [50]–[55], computational optimization techniques [56], and control theory [57]–[59]. This wide range of connections to real-world physical and biological phenomena enables meaningful engagement with students across diverse disciplines.

There exist three seminal variants of Parrondo's paradox—the capital-dependent variant [29], the history-dependent variant [60], and the cooperative variant [61]. In all variants, the dynamics of the *capital* of an autonomous player or ensemble of players is analyzed. These three variants are discussed in Sections II-A, II-B and II-C.

Our provided spreadsheet simulation suite encompasses all three variants. The approach of introducing Parrondo's paradox via spreadsheets is not new [62], albeit for the capital-dependent variant only; and existing programs written outside of *Excel*, likewise, typically supports only the capital-dependent variant only. As such, our presented simulation suite is a significant extension to these existing works, whilst preserving the transparency and readability not typically available on non-spreadsheet platforms. Instructors may adopt this simulation suite for their classes or use it to design customized teaching materials.

A. CAPITAL-DEPENDENT PARRONDO'S PARADOX

In the capital-dependent formulation [29], Games A and B are played by a player $\mathcal{P}$ in either a deterministic fashion (fixed sequence) or a random fashion. In the latter, a mixing parameter γ is defined, such that Games A and B are played with probabilities γ and $(1-\gamma)$ respectively at each round.

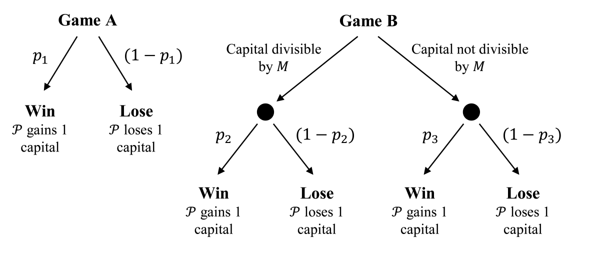


FIGURE 1. Game structure of the capital-dependent variant. Game A is a simplistic coin-flipping game, but Game B presents branching dependent on the divisibility of the player's capital by a constant M. A total of three coins are used for the games, with winning probabilities p_1, p_2, and p_3. The capital of the player is incremented by a unit if the round is a win, and decremented by a unit if the round is a loss; the player starts with zero capital.

The game rules of the capital-dependent variant are illustrated in Figure 1, with pseudocode available in Section C of the Supplementary Information. A constant ϵ introduces stochastic bias, such that $\epsilon > 0$ reflects a tendency towards losing. We take $p_1 = 1/2 - \epsilon$, $p_2 = 1/10 - \epsilon$, $p_3 = 3/4 - \epsilon$ and $M = 3$ for purposes of illustration [29].

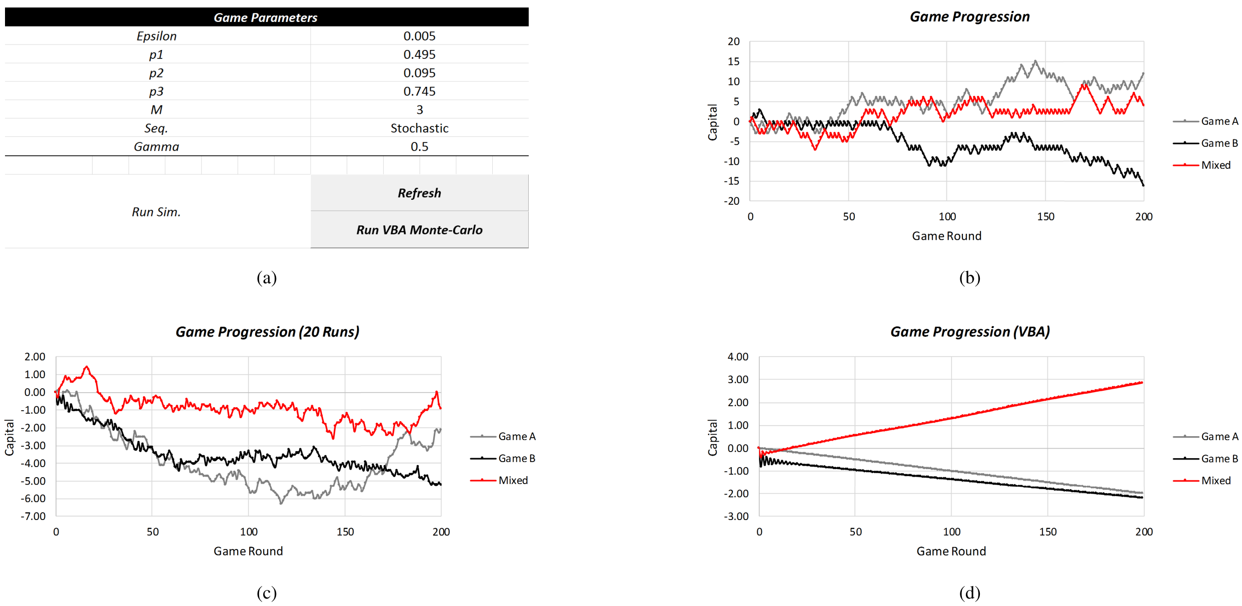


FIGURE 2. **(a) A screenshot of the parameter input interface for the capital-dependent paradox variant, where the user may freely modify the game parameters; (b) single-trial simulation results obtained from the spreadsheet simulation suite; (c) multi-trial averaged simulation results with $N = 20$; and (d) multi-trial averaged simulation results with $N = 200000$. Simulations shown were run with random game sequence for mixed games.**

With $\epsilon > 0$, both games are losing when played individually. The mathematics and educational value of the paradox are explored further in Sections II-A.1 to II-A.3.

The *Excel* simulation suite is fully capable of simulating the capital-dependent paradox variant, with all parameters freely modifiable by the user. The default parameters are set at the aforementioned values, with $\epsilon = 0.005$, $\gamma = 0.5$, and number of game rounds $n = 200$; these values are freely modifiable by the user (Figure 2a). Single-trial simulations will produce highly variable results (Figure 2b), indicative of the stochastic nature of the games; but averaged Monte Carlo simulation sets with large number of trials N (Figure 2d) will produce consistent results. Instructors may introduce the Monte Carlo simulation technique by highlighting the improving consistency of the results as N is increased, a natural consequence of the *law of large numbers*. The single-trial and multi-trial simulations up to $N = 20$ are implemented with native *Excel* formulae, allowing excellent code readability by students; multi-trial sets extendable to arbitrarily large n and N are implemented in *VBA*, and may be of interest to students proficient in programming, or instructors seeking to extend the framework.

It is notable that the capital-dependent paradox variant, while originating as a mathematical abstraction of flashing Brownian ratchets [34]–[36], has also been linked to the financial dynamics of stock markets [63]; and plausible connections to gene transcription and DNA replication dynamics [64] have also been established. The mechanism of the paradox has also been applied to improve computational optimization techniques, at present applied to electron-optical instruments [56], and to the control of chaotic systems [59]. Instructors may discuss these applications to provide relevant interdisciplinary context and to engage students of different disciplines.

1) GAME A

Game A is winning, fair, and losing when $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$ respectively. In particular, the expected capital of the player is $C(u) = (2p_1 - 1)u = -2\epsilon u$ at game round u.

2) GAME B

Game B provides an excellent opportunity to review potential statistical fallacies with students. When asked to evaluate the fairness of Game B, students may very likely make the following calculation [32]:

$$P_B(\text{win}) = \frac{1}{3} \cdot p_2 + \frac{2}{3} \cdot p_3 = \frac{8}{15} - \epsilon \tag{1}$$

This suggests that Game B can be winning for $\epsilon < 1/30$. This is, however, a flawed result. The assumption that Game B will enter the p_2 branch $1/3$ of the time, and the p_3 branch the remaining $2/3$ of the time, is incorrect. One-third of all integers are divisible by $M = 3$, seemingly supporting the naïve assumption; the crucial observation is that consecutive rounds of Game B are not independent of each other, and therefore the ratio of p_2 and p_3 coin flips cannot be deduced from this divisibility condition alone. A proper analysis of the winning probability necessitates the use of discrete-time Markov chains, the detailed mathematical derivation of which is presented in Section B of the Supplementary Information. The correct winning probability of Game B as derived

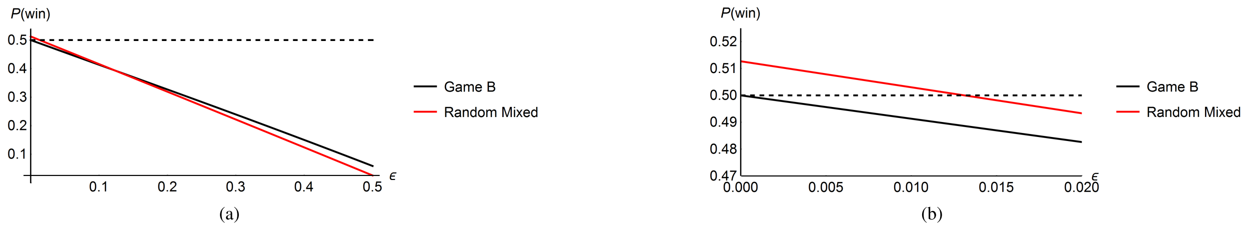


FIGURE 3. (a) A plot of winning probabilities P_B(win) and P_R(win) against ϵ, as obtained from the discrete-time Markov chain analysis in Eq. (2) and (3), and (b) a zoomed-in version of the same graph for $0 \leq \epsilon \leq 0.02$. Default values of game parameters are used. It is clear that Game B is losing for $0 < \epsilon \leq 0.5$, but random mixed games can be winning for ϵ below a threshold.

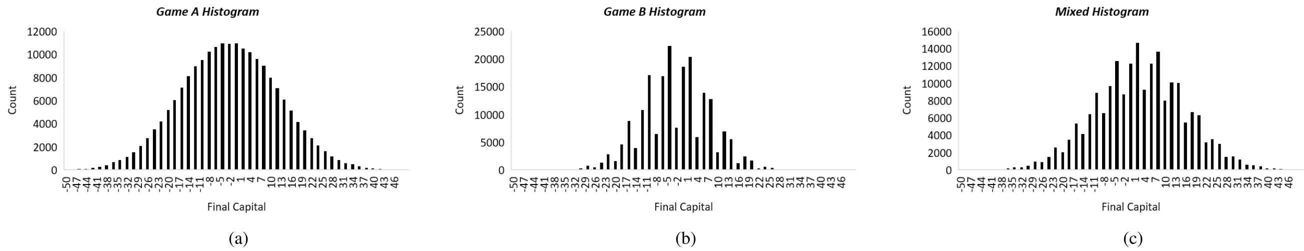


FIGURE 4. Capital distribution histograms for the capital-dependent variant, for (a) pure Game A, (b) pure Game B, and (c) mixed games with random game sequence. Capital at the final game round ($n = 200$) is plotted, collected over a total of $N = 200000$ trials using the automated functionality of the spreadsheet simulation suite.

through this method is

$$P_B(\text{win}) = \frac{p_2 + 2p_3 - 2p_2p_3 - p_3^2 + 3p_2p_3^2}{3 - p_2 - 2p_3 + 2p_2p_3 + p_3^2}. \tag{2}$$

A fair game is achieved at $\epsilon = 0$, and a losing game ($P_B(\text{win}) < 1/2$) results whenever $0 < \epsilon \leq 1/2$, as shown in Figure 3.

3) MIXED GAMES

Games A and B can be mixed in either a deterministic or random order to produce paradoxical winning outcomes. It is crucial that students understand the difference between a *deterministic* process and a *random* process. The former involves no probabilistic quantities, and hence will always give the same result with the same initial conditions; the latter is stochastic in nature, but this does not imply unpredictability. The average outcome of a stochastic process can be statistically predicted, and it is this averaged behaviour that is described in our mathematical analyses.

The simulation suite offers flexibility in specifying the order of the games (Figure 2a). With the stochastic sequence selected, the user may freely choose the value of mixing parameter γ. Numerous periodic deterministic sequences are also available, for instance ABAB, ABAAB, and ABABB, the last of which produces especially prominent paradoxical winning outcomes. These game sequences are implemented via a record sheet, which can be freely modified by students and instructors. Instructors may engage students to find the optimal deterministic sequence for maximum capital gain, a task that demands an excellent mathematical understanding of the paradox.

Why does the paradox occur? An explanation can be given in terms of the localization tendency of the player's capital [32]. In Game B, the capital of a player tends to drift towards the $Mk-1$ and Mk states ($k \in \mathbb{Z}$), since $p_2 < 1/2$ and $p_3 > 1/2$. At $Mk-1$, there is a high probability p_3 of winning into Mk; and at Mk, there is a high probability $(1 - p_2)$ of losing into $Mk - 1$. The capital, thus, localizes between these two states, and a winning outcome is unachievable. By introducing Game A into the game sequence, however, perturbations are provided. When the capital declines to the $Mk - 1$ state, recovery back to Mk is likely; and when the capital reaches $Mk + 1$, there is a high probability p_3 of inflating onto $Mk + 2$. In other words, the presence of Game A enables the exploitation of the inherent asymmetry in Game B [32], [33]. This is known as the *agitation-ratcheting* mechanism. This localization phenomenon can be observed directly in the sawtooth-like capital distributions of Game B and mixed games (Figure 4), readily generated and visualized in the simulation suite.

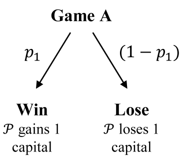
A rigorous analysis of mixed games can be performed via discrete-time Markov chains—this is detailed in Section B of the Supplementary Information. Defining $r_i = \gamma p_1 + (1-\gamma)p_i, i \in \{2, 3\}$, the winning probability of stochastically mixed games is

$$P_R(\text{win}) = \frac{r_2 + 2r_3 - 2r_2r_3 - r_3^2 + 3r_2r_3^2}{3 - r_2 - 2r_3 + 2r_2r_3 + r_3^2}. \tag{3}$$

The above is analogous to Eq. (2) under the transformation $p_i \rightarrow r_i$. This analytical result indeed reflects a winning outcome (Figure 3), so long as ϵ do not exceed a threshold. Instructors may wish to discuss such an analysis for advanced classes; a further challenge can be to determine the threshold of ϵ beyond which paradoxical outcomes are no longer possible. Specific solutions have already been found [32], with the bounding condition being $320\epsilon^3 - 16\epsilon^2 + 299\epsilon - 3 < 0$ for $M = 3$ and $\gamma = 1/2$.

B. HISTORY-DEPENDENT PARRONDO'S PARADOX

The history-dependent Parrondo's paradox [60] replaces the former memory-less Game B with one of behaviour dependent on the past participatory outcomes of the player—the game history of the player hence becomes relevant. The game rules are illustrated in Figure 5. Dependence upon the previous two rounds is considered; there are therefore four possible outcome combinations, namely {L, L}, {L, W}, {W, L} and {W, W}, where W and L denote a win and a loss respectively.



Game B

$u-2$	$u-1$	*Coin*	*Win Prob.*	*Lose Prob.*
L	L	B_1	p_1	$1-p_1$
L	W	B_2	p_2	$1-p_2$
W	L	B_3	p_3	$1-p_3$
W	W	B_4	p_4	$1-p_4$

FIGURE 5. Game structure of the history-dependent variant. Branching in Game B at round u is dependent on the past two game outcomes of the player, at rounds $u-2$ and $u-1$. W and L denote a winning and losing outcome respectively. The past outcomes determine which coin is used to play Game B, all four of which may in general have different winning probabilities.

The simulation suite provides full flexibility in the specification of parameters and game sequences (Figure 6a), in an identical fashion to the capital-dependent variant. There are four possible initial conditions for the game, each selectable by the user; an option to average results across all initial conditions is also offered. Consistent with the original formulation [60], defaults of $p = 1/2 - \epsilon$, $p_1 = 9/10 - \epsilon$, $p_2 = p_3 = 1/4 - \epsilon$, $p_4 = 7/10 - \epsilon$, and $\epsilon = 0.003$ are adopted. The suite includes a worksheet preprogrammed to run single-trial simulations of $n = 200$, freely extendable by the user (Figure 6b). A second worksheet executes averaged Monte Carlo simulation sets up to $N = 20$ with native *Excel* formulae, or to arbitrarily large n and N with *VBA* implementation (Figure 6c and Figure 6d).

Games A and B are both individually losing, but deterministic and stochastic mixed games can be winning. A similar agitation-ratcheting mechanism explains the occurrence of this history-dependent paradox variant. In particular, there is a high probability $p_1 > 1/2$ of winning from the {L, L} to {L, W} state. There is then a high probability $1 - p_2 > 1/2$ to lose into {W, L}, followed by a second loss with probability $1 - p_3 > 1/2$ back to {L, L}. These three states form a localization cycle, in a similar fashion as the capital-dependent variant. The player loses twice whilst winning only once in each cycle, hence suggesting a losing tendency. The introduction of Game A, however, induces perturbations that enable the {W, W} state to be accessed more frequently. There is then a high chance $p_4 > 1/2$ of winning from {W, W}, effectively creating a second competing localization cycle that leads to sustained capital gain. Instructors may find a transition diagram (Figure 7) helpful for lecture delivery.

Likewise, a rigorous analysis can be performed via discrete-time Markov chains, detailed in Section B of the Supplementary Information. With $r_i = \gamma p + (1 - \gamma)p_i$, the winning probabilities of Game B and stochastically mixed games can be derived as

$$\begin{aligned} P_B(\text{win}) &= \frac{p_1(1 + p_2 - p_4)}{p_1 p_2 + (1 + 2p_1 - p_3)(1 - p_4)} = \frac{9}{9 + 10\epsilon} - \frac{1}{2}, \\ P_R(\text{win}) &= \frac{r_1(1 + r_2 - r_4)}{r_1 r_2 + (1 + 2r_1 - r_3)(1 - r_4)} = \frac{217 - 310\epsilon}{429 + 220\epsilon}. \end{aligned} \tag{4}$$

Game B and stochastically mixed games hence are losing and winning respectively, as long as ϵ is small; with conditions $P_B(\text{win}) < 1/2$ and $P_R(\text{win}) > 1/2$, paradoxical outcomes are achievable for $0 < \epsilon < 1/168$. Instructors may extend similar analyses to deterministic game sequences, or for general γ. This history-dependent paradox variant is applicable in population genetics, evolution, and economics, in which the time dynamics are oftentimes coupled to some combination of past parameter values [60].

C. COOPERATIVE PARRONDO'S PARADOX

In addition to the capital-dependent and history-dependent paradox variants, the *Excel* simulation suite also supports the cooperative Parrondo's paradox [61]. Details on this variant and the related capabilities of the simulation suite can be found in Section A of the Supplementary Information. The multi-agent, graph-theoretic nature of this variant makes it suitable as a higher-order thinking question for students, and had also enabled close connections with the analysis and control of networked systems, encompassing spatio-temporal noise suppression and topological effects on information propagation [65]–[70].

III. CASE-STUDY

To investigate the effectiveness of paradox-motivated simulation class activities, a Parrondo's paradox simulation-based assignment was introduced to a pilot class after their mid-term examinations. In this section, we describe the teaching strategies and observed learning outcomes for the sample class as a case study, alongside facilitation recommendations for instructors.

In the assignment given to the pilot class, three progressive milestones were stipulated in a patchwork assessment model. Students had to submit an initial group report reviewing the paradox in their own words after the background reading (first milestone), after which feedback was given by the instructor. They then implemented simulations of their own design to investigate the paradoxical effects (second milestone). Queries pertaining to the simulations were addressed as a class. At the end of the course, students were required to submit their simulation work and a final report that documents their analysis and learning process (third milestone). They were also required to write individual reflections on

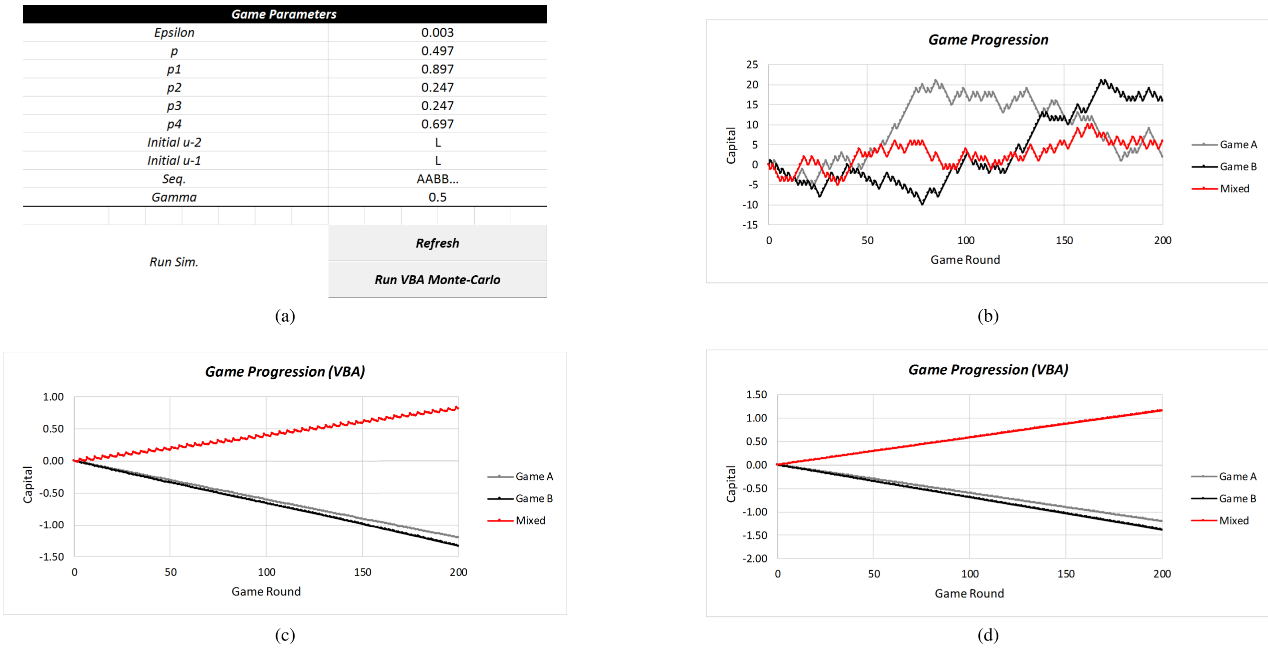


FIGURE 6. **(a) A screenshot of the parameter input interface for the history-dependent paradox variant, where the user may freely modify the game parameters; (b) single-trial simulation results obtained from the spreadsheet simulation suite; (c) multi-trial averaged simulation results with $N = 500000$ and periodic *ABAB* deterministic game sequence; and (d) multi-trial averaged simulation results with $N = 500000$ and random game sequence. Simulations in (c) and (d) were averaged across all four possible initial conditions, using the built-in option within the simulation suite.**

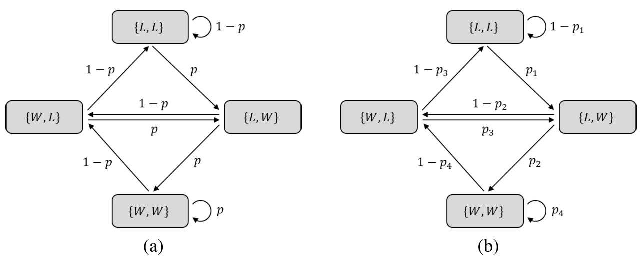


FIGURE 7. **State transition flow charts for the history-dependent paradox variant, (a) Game A and (b) Game B.**

their learning journey. The simulation task was intentionally designed as a group project, to encourage collaboration between students with different backgrounds and expertise.

While there were no restrictions on the platform for the simulations, the adoption of *Microsoft Excel* was encouraged, as it is readily accessible by students of all backgrounds. In the present study, a majority of students expressed skepticism on the feasibility of implementing numerical simulations in *Excel*, and a preliminary spreadsheet program was therefore demonstrated by the instructor to serve as a starting point. Guidance on the programming specifics of the simulation were intentionally relaxed, because the class had prior programming experience on the *C#* language. Nonetheless, comprehensive initial guidance, possibly with the dissemination of reference works, is recommended when teaching classes of limited programming exposure.

The provided simulation suite is sufficiently comprehensive to facilitate the conduct of lectures and assignments spanning all three seminal Parrondo's paradox variants; indeed, all numerical simulation results presented in this paper was obtained using the in-house suite. This suite has been made publicly available, and is written in *Excel* with *VBA* macros for extended functionality, therefore catering both to students of low and high programming exposure. Instructors may adopt the suite as-is for their classes, or modify the framework for customized teaching materials; alternatively students may be tasked to perform their own simulations and investigations, as was done in the present study. The presented flowcharts (Figures 1 and 5) may be useful in helping students understand the game implementations, and the relevant pseudocode as provided in the Supplementary Information may also be discussed for a finer-grained analysis. The key is to maintain an active learning environment by giving students space for independent hands-on exploration.

It was observed in this study that students tend to carry out single-trial simulations using a large number of games rounds ($n \geq 100$) in an attempt to obtain satisfactory results; but neglect the need to repeat the simulations over a large number of trials. There is a crucial need to distinguish between the number of game rounds n and the number of trials N. This is an excellent opportunity to discuss *the law of large numbers*, a fundamental topic in introductory probability and statistics courses, for students can interactively observe the concept manifest when running their simulations. The key learning point is that the averaged repetition of a stochastic process provides an increasing good indication of its expected behaviour as the number of trials increases. When the number of trials is too low, highly variable results will be observed. Heavily utilized in the stochastic simulations

TABLE 1. Survey questions and results as evaluated by the pilot cohort. The voluntary anonymous survey implements a 5-point Likert scale rating across 6 questions, where a score of 5 represents a stance of "strongly agree", and a score of 1 represents "strongly disagree". Response rate was $32/37 \approx 86.5\%$.

	Survey Question	Score Count					Mean	95% C.I.	
		5	4	3	2	1		Lower	Upper
Q1	The Parrondo's paradox assignment has stimulated my interest in learning probability/statistics.	4	14	6	6	2	3.38	2.97	3.78
Q2	The Parrondo's paradox assignment has strengthened my concepts in probability/statistics.	4	16	6	4	2	3.50	3.11	3.89
Q3	After the Parrondo's paradox assignment, I am now able to use simulation to model random events, such that simulated outcomes closely match real-world outcomes.	4	11	13	4	0	3.47	3.15	3.79
Q4	I have a better understanding of the 'Law of Large Numbers' after the Parrondo's paradox assignment.	6	16	7	3	0	3.78	3.47	4.09
Q5	My involvement in the Parrondo's paradox assignment has given me the confidence to apply what I have learned in this module to solve/approximate the solution to other practical problems.	4	13	9	6	0	3.47	3.13	3.81
Q6	Through the Parrondo's paradox assignment, I can collaborate effectively with my team members.	5	13	10	2	2	3.53	3.15	3.91

is pseudo-random number generation—instructors may optionally discuss their mechanisms and statistical limitations [71]–[73].

The scaffolding for the pilot class in our study was conducted over 3 sessions of 30 minutes each, covering much of the theoretical content presented in Section II, with the Markov-chain analyses briefly mentioned for independent reading by interested students. The conceptual resolution of the paradox was expected to appeal to the curiosity of students; and the associated opportunities to review key statistical enabled the assignment to serve as an interactive end-of-course review. A voluntary anonymous evaluation survey implementing a five-point Likert scale rating and a free-response section was carried out at the end of the project to evaluate the students' perceptions of these activities.

As a pilot programme, we have implemented the proposed paradoxical simulation in a first-year mathematics course for undergraduates enrolled in a Game Design degree in Spring 2017. A study was conducted to assess students' perception of the effectiveness of the activities, as well as their overall reception response. The cohort comprised students of diverse mathematical aptitude, therefore making this a good preliminary study on the utilization of paradox-motivated activities in probability and statistics courses. The survey results of the students are presented in Table 1.

IV. DISCUSSION

The evaluation survey results (Table 1) indicate that the student cohort is in general supportive of the inclusion of the Parrondo's paradox simulation assignment as part of coursework. Student responses to all six survey questions are positive, with average scores and 95% confidence intervals approximately > 3.0 throughout. In particular, survey question Q4 stands out with a Likert rating of 3.78, alongside Q2 with a rating of 3.50; this reflects that students had perceived the hands-on simulation-based activity to be beneficial to their understanding of statistical concepts. The assignment also appears to have stimulated interest in the subject of probability and statistics (Likert rating 3.38). The narrow confidence intervals indicate that the the student population perceives this activity relatively consistently, despite their diverse backgrounds and experiences. Based on the limited student number, our analysis also reveals that the survey results are internally consistent.

Unique to hands-on activities of this type is the opportunity for students to apply their knowledge in problems closely resembling those of the real-world. Because restrictions on the allowed tools and resources were largely removed, students are able to leverage their full potential in their work. The survey results affirms these positive pedagogical traits, as student's perceptions reflect that they are now more confident in applying their knowledge to real-world problems (Likert rating 3.47), and that they assess themselves to have gained proficiency in practical simulation skills (Likert rating 3.47). In direct support of the ratings, the free-response section of the survey reflects that the paradox simulation activity was well-received. As a way of illustration, we quote from a few students:

- "Most importantly…I have learnt to better apply theories to real-life problems";
- "…it surprises me what *Excel* can do. The functionalities possible are unexpected";
- "…the most striking take-away from this project is…this contradicting theory that makes sense upon looking deeper into its reasoning and mathematics";
- "…we often fail to see the applications of the concepts we are taught, especially for mathematics. I felt that this assignment has shed light on the underlying concepts, and has allowed us to apply our knowledge outside of the typical context of practice questions";
- "…has provided me with a different take on mathematics, which often a times is an individual and mundane activity, by providing a group-work setting. It has guided me to understand the module better";
- "As mentioned in class, evolution may be a case of Parrondo's paradox. This opens up the possibility for discussion as to what other matters in nature that can be explained by such concepts…";
- "I have learnt that mathematics is much closer to us in our daily lives than we have thought".

A student had also wrote that "the Parrondo's paradox has changed our ideology of mathematics, from a theoretical textbook-only perspective into practical science"—the appreciation of the practical application of statistics is precisely the goal of the proposed simulation-based activities.

More importantly, the study had also revealed numerous pedagogical considerations that have to be intricately balanced, thus paving a route for improved future implementations. Firstly, the assessment mode of the course ought to be fundamentally compatible with open-ended project work. The pilot class in our study had raised concerns that their performance for the end-of-course written examinations would be affected by the time and effort required for the simulation-based activities. While mathematical education ought to be sufficiently holistic and applicable in a wide range of real-world problems, and exposure to practical problem-solving and hands-on statistical analyses of the type proposed will prepare students better in this respect than a traditional pen-and-paper approach, the time constraints that students inexorably face have to be kept in mind; instructors are therefore advised to adapt course assessment policies to maintain a balance of weightage between traditional examinations and such types of project work. In our pilot programme, the implemented simulation assignment was kept at 10% of the final class grade; but this may not be truly reflective of the effort required. An alternative to modifying assessment components is to collaborate with other mathematical or computing courses to implement a separate simulation-based 'practical' course, though the scope of the activities will have to be vastly expanded.

It was also observed that the learning experience of students can be significantly impaired if constructive guidance is not provided at times of extended difficulties, especially if the cohort is inexperienced in programming; indeed, the free-response results suggest that students are hugely appreciative of timely interventions by instructors to clear misconceptions and technical difficulties. While the sample size in this pilot study is limited, planned future scaling-up of the proposed pedagogical approach will yield more comprehensive statistical assessments.

V. CONCLUSION

By integrating simulation-based activities with mathematical paradoxes in courses covering probability and statistics, the interest of students in exploring and internalizing concepts can be stimulated. It is our hope that instructors can make use of our exposition in introducing paradox-motivated, hands-on activities in their courses, thereby reinforcing taught concepts and enabling students to appreciate the real-world applicability of the subject. To this end, we have demonstrated in close detail how the necessary theoretical and technical scaffolding can be carried out in the classroom, focusing on the educational context of Parrondo's paradox as a way of example; we also provide, in addition, an original simulation suite written in *Microsoft Excel* with *VBA* implementations for extended functionality, capable of executing simulations for all three seminal paradox variants. The plethora of applications of the Parrondo's paradox, including quantum physics, information thermodynamics, eco-evolutionary modelling, computational optimization, and network analysis, enables it to be an interesting topic for students across engineering and non-engineering disciplines; furthermore, a benefit of a simulation-based pedagogical approach is the possibility of remote laboratories where students may enjoy effective learning at home or at venues of convenience, as had been proposed in several other educational contexts [74]–[76]. The *Excel* simulation suite is available on Open Science Framework at: `https://goo.gl/5fXpgS`.

COMPETING INTERESTS

The authors declare no competing interests.

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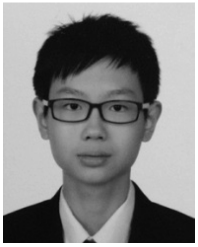

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