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A Novel Group Decision Making Method Under Uncertain Multiplicative Linguistic Environment for Information System Selection

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ABSTRACT An effective method is proposed to solve the group decision-making problems with uncertain multiplicative linguistic information. First, to compare two uncertain multiplicative linguistic variables, a possibility degree formula is introduced in accordance with the operation laws. Some desirable properties of possibility degree are provided. Then, the uncertain multiplicative linguistic hybrid weighted geometric averaging (ULHWG) operator is developed to aggregate uncertain multiplicative linguistic variables into an overall preference value. Thus, a comprehensive algorithm for linguistic group decision-making is proposed based on the deviation measure and the ULHWG operator. Finally, a numerical example is provided to illustrate the practicality and validity of the proposed method.

INDEX TERMS Uncertain multiplicative linguistic information, possibility degree, deviation measure, decision making.

I. INTRODUCTION

Multiple attribute decision making has been receiving more and more attention over the last decades as an important research area in classical decision theory [1]–[8]. It is very natural and intuitional to evaluate the attributes by using some linguistic terms. For example, linguistic terms like ‘‘high’’, ‘‘very high’’ and ‘‘low’’ may be used when we evaluate the temperature, and linguistic terms like ‘‘fast’’, ‘‘ slow’’ and ‘‘extremely slow’’ may be used when we describe the running speed of a computer. There are many types of linguistic label sets have been proposed in the existing literature. The two main types linguistic label sets are introduced by Herrera and Xu called the additive linguistic label set [9], [10] and the multiplicative linguistic label set [11], respectively. Multiplicative linguistic label sets are very suitable for dealing with the practical problems of economic analysis[11], educational assessment [12] and information recovery [13]. To aggregate the decision information in the form of multiplicative linguistic variables, many aggregation operators are proposed based on the multiplicative linguistic label sets and their operational laws, such as the linguistic ordered weighted geometric averaging (LOWG) operator [10], the linguistic hybrid geometric averaging (LHG) operator [14], the generalized induced linguistic ordered weighted geometric averaging (GILOWG) operator [15], the uncertain linguistic weighted geometric averaging (ULWG) operator [11], the uncertain linguistic ordered weighted geometric averaging (ULOWG) operator [11], the induced uncertain linguistic ordered weighted geometric averaging (IULOWG) operator [11], and so on. Furthermore, Wei [16] proposed the uncertain linguistic hybrid geometric averaging (ULHG) operator combined the advantages of the ULWG operator and the ULOWG operator. The ULHG operator considers both the ordered position and importance of the given multiplicative linguistic variables. However, it fail in some essential properties of idempotent and boundary. Therefore, an effective hybrid aggregation operator is provided in this paper to overcome the above drawbacks of the ULHG operator.

Due to the increasing vagueness and complexity of the real world, it is less possible to consider all related aspects of decision making problems by a single decision maker. In fact, many decision making processes take place in group settings. A number of literatures have recently investigated group decision making under linguistic environment [17]–[24]. Xu and Da [25] proposed two methods named standard deviation method and mean deviation method to determine the optimal weighting vector. The method is based on the idea that the attribute with a smaller deviation value among alternatives should be assigned a small weight. Herrera *et al.* [9] developed a direct approach to group decision making using linguistic OWA operators. Wei [26] considered multiple attribute group decision-making problems with linguistic information of attribute values and weight values. Fan and Liu [27] gave a formula for transforming multi-granularity uncertain linguistic terms into trapezoidal fuzzy numbers. Chen and Ben-Arieh [28] provided a simple method to combine the information assessed in different linguistic sets. Zhang *et al.* [29] studied the incomplete 2-tuple fuzzy linguistic preference relations in multi-granular linguistic decision making with unknown weight information. Li *et al.* [30] developed a new decision-making method based on dominance degree and BWM with probabilistic hesitant fuzzy information. Luo *et al.* [31] proposed the group decision-making approach with probabilistic linguistic preference relations. Nevertheless, the group decision-making problems with uncertain multiplicative linguistic information are seldom discussed. In this paper, a novel method is proposed to solve the group decision making problem with uncertain multiplicative linguistic information. Firstly, a possibility degree formula is introduced in accordance with the operation laws to compare two uncertain multiplicative linguistic variables. Then, the uncertain multiplicative linguistic hybrid weighted geometric averaging (ULHWG) operator is developed. Some desirable properties of the ULHWG operator are provided. An effective algorithm for linguistic group decision making is proposed based on the deviation measure and ULHWG operator. Finally, the proposed method is applied to information system selection to illustrate its applicability and effectiveness.

The paper is organized as follows. Section 2 introduces some basic concepts of uncertain multiplicative linguistic variables. In section 3, a ranking formula of uncertain multiplicative linguistic variables is proposed. Section 4 introduced the uncertain multiplicative linguistic hybrid weighted geometric averaging operator. In Section 5, an approach to multiple attribute group decision making under uncertain multiplicative linguistic environment is developed. Section 6 provides a numerical example to demonstrate the practicality and validity of the proposed methods. The paper is concluded in Section 7.

II. BASIC CONCEPTS OF UNCERTAIN MULTIPLICATIVE LINGUISTIC VARIABLES

In the following, some basic concepts and operational laws on uncertain multiplicative linguistic variables are introduced.

Let $S = \{s_\alpha \mid \alpha = \frac{1}{t}, \cdots, \frac{1}{2}, 1, 2, \cdots, t\}$ be a multiplicative linguistic label set with odd cardinality, where s_α represents a possible value for a linguistic variables, and the multiplicative linguistic label set satisfy the following characteristics [11]:

(1) $s_\alpha < s_\beta$ if and only if $\alpha < \beta$.

(2) there is the reciprocal operator $rec(s_\alpha) = s_{\frac{1}{\alpha}}$, especially, $rec(s_1) = s_1$.

This multiplicative linguistic label set S is called the multiplicative linguistic scale. For example, S can be defined as follows [11]:

$$\begin{aligned} S = \{ & s_{\frac{1}{5}} = extremely\ poor, s_{\frac{1}{4}} = very\ poor, s_{\frac{1}{3}} = poor, \\ & s_{\frac{1}{2}} = slightly\ poor, s_1 = fair, s_2 = slightly\ good, \\ & s_3 = good, s_4 = very\ good, s_5 = extremely\ good\} \end{aligned}$$

In order to preserve all the information in the process of information aggregation, Xu [11] extended the discrete label set S to a continuous label set $\overline{S} = \{s_\alpha \mid \alpha \in [\frac{1}{t}, t]\}$. If $s_\alpha \in S$, then s_α is called an original linguistic term, if $s_\alpha \in \overline{S}$ and $s_\alpha \overline{\in} S$, then s_α is called a virtual linguistic term. Xu [11] pointed out that the original linguistic terms is utilized to evaluate alternatives by decision maker, and the virtual linguistic terms can only appear in the result of operations.

Due to the complexity and vagueness of knowledge, the evaluation of alternatives may between two linguistic terms. Therefore, Xu [11] developed the concept and operational laws of uncertain multiplicative linguistic variable as follows.

Definition 1: Let $s_\alpha, s_\beta \in \overline{S}$, and $s_\alpha \leq s_\beta$, then $\widetilde{s} = [s_\alpha, s_\beta]$ is called the multiplicative linguistic variable. For any there multiplicative linguistic variables $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$, and $\widetilde{s} = [s_\alpha, s_\beta]$ their operational laws are defined as follows:

(1) $\widetilde{s}_1 \bigotimes \widetilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \bigotimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}]$.

(2) $\widetilde{s}^\lambda = [s_{\alpha^\lambda}, s_{\beta^\lambda}]$, where $\lambda \in [0, 1]$.

(3) $\widetilde{s}_1 \bigotimes \widetilde{s}_2 = \widetilde{s}_2 \bigotimes \widetilde{s}_1$.

(4) $(\widetilde{s}_1 \bigotimes \widetilde{s}_2)^\lambda = \widetilde{s}_1^\lambda \bigotimes \widetilde{s}_2^\lambda$, where $\lambda \in [0, 1]$.

(5) $\widetilde{s}^{\lambda_1} \bigotimes \widetilde{s}^{\lambda_2} = \widetilde{s}^{\lambda_1+\lambda_2}$, where $\lambda_1, \lambda_2 \in [0, 1]$.

Let $\overline{\widetilde{S}}$ be the set of uncertain multiplicative linguistic variables. $\forall\, \widetilde{a} \in \overline{\widetilde{S}}$, then the lower indices [10], [11] of linguistic values corresponding to $\widetilde{a}$ is defined as $I(\widetilde{a}) = [I^-(\widetilde{a}), I^+(\widetilde{a})]$. For example, if $\widetilde{a} = [s_{\frac{1}{3}}, s_2]$, then we have $I^-(\widetilde{a}) = \frac{1}{3}$ and $I^+(\widetilde{a}) = 2$, respectively. Xu [11] developed the possibility formulae for the comparison between uncertain multiplicative linguistic variables as follows.

Definition 2: Let $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain multiplicative linguistic variables, and let $l(\widetilde{s_i}) = \beta_i - \alpha_i$ $(i = 1, 2)$, then the degree of possibility of $\widetilde{s_1} \geq \widetilde{s_2}$ is defined as

$$p(\widetilde{s_1} \geq \widetilde{s_2}) = \frac{\max\{0, l(\widetilde{s_1}) + l(\widetilde{s_2}) - \max(\beta_2 - \alpha_1, 0)\}}{l(\widetilde{s_1}) + l(\widetilde{s_2})} \quad (1)$$

It can be shown that $0 \leq p(\widetilde{s_1} \geq \widetilde{s_2}) \leq 1$, and if $p(\widetilde{s_1} \geq \widetilde{s_2}) \geq 0.5$ then $\widetilde{s_1}$ is said to be superior to $\widetilde{s_2}$.

It is important to aggregate uncertain multiplicative linguistic variables into an overall uncertain multiplicative linguistic variable in multiple attribute decision making. To do this, Xu [11] investigated the uncertain multiplicative linguistic weighted geometric mean (ULWG) operator and the uncertain multiplicative linguistic ordered weighted geometric mean (ULOWG) operator, shown as follows:

Definition 3: An ULWG operator of dimension n is a mapping ULWG: $\overline{\overline{S}}^n \to \overline{\overline{S}}$, according to the following formula:

$$ULWG_\lambda(\widetilde{s_1}, \widetilde{s_2}, \cdots, \widetilde{s_n}) = \widetilde{s_1}^{\lambda_1} \bigotimes \widetilde{s_2}^{\lambda_2} \bigotimes \cdots \bigotimes \widetilde{s_n}^{\lambda_n} \quad (2)$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is the weighting vector of the aggregated objects $\widetilde{s_1}, \widetilde{s_2}, \cdots, \widetilde{s_n}$, such that $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \in [0, 1]$.

Definition 4: An ULOWG operator of dimension n is a mapping ULOWG: $\overline{\overline{S}}^n \to \overline{\overline{S}}$, defined by an associated weighting vector $W = (w_1, w_2, \cdots, w_n)^T$, such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$, according to the following formula:

$$ULOWG_W(\widetilde{s_1}, \widetilde{s_2}, \cdots, \widetilde{s_n}) = \widetilde{r_1}^{w_1} \bigotimes \widetilde{r_2}^{w_2} \bigotimes \cdots \bigotimes \widetilde{r_n}^{w_n} \quad (3)$$

where $\widetilde{r_j}$ is the jth largest of the $\widetilde{s_1}, \widetilde{s_2}, \cdots, \widetilde{s_n}$, the ranking order of $\widetilde{s_i}$ $(i = 1, 2, \cdots, n)$ can be determined by using formula (1).

Remark 5: The associated weighting vector $W = (w_1, w_2, \cdots, w_n)^T$ can be obtained by using the following Basic unit-interval monotonic (BUM) function [32]:

$$w_i = Q(\frac{i}{n}) - Q(\frac{i-1}{n}), \quad i = 1, 2, \cdots, n. \quad (4)$$

where

$$Q(r) = \begin{cases} 0, & if\ r \le a \\ \frac{r-a}{b-a}, & if\ a \le r \le b \\ 1, & if\ r \ge b \end{cases} \quad (5)$$

with $a, b, r \in [0, 1]$, and (a, b) should be selected by decision maker with some fuzzy linguistic quantifiers, such as "at least half" with the pair (0,0.5), "more" with the pair (0.3,0.8), "as many as possible" with the pair (0.5,1), and so on.

III. RANKING OF UNCERTAIN MULTIPLICATIVE LINGUISTIC VARIABLES

In order to overcome some drawbacks of formula Eq. (1), in what follows, an effective formula for ranking uncertain multiplicative linguistic variables is proposed.

Definition 6: Let $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain multiplicative linguistic variables, then the degree of possibility of $\widetilde{s_1} \ge \widetilde{s_2}$ is defined as (6), as shown at the top of the next page If $p(\widetilde{s_1} \ge \widetilde{s_2}) \ge 0.5$, then $\widetilde{s_1}$ is said to be superior to $\widetilde{s_2}$. Obviously, the degree of possibility of $\widetilde{s_2} \ge \widetilde{s_1}$ can be further written as

$$p(\widetilde{s_1} \ge \widetilde{s_2}) = \frac{\max\{\ln\beta_1 - \ln\alpha_2, 0\} - \max\{\ln\alpha_1 - \ln\beta_2, 0\}}{\ln\beta_1 - \ln\alpha_1 + \ln\beta_2 - \ln\alpha_2} \quad (7)$$

Let $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$, and $\widetilde{s_3} = [s_{\alpha_3}, s_{\beta_3}]$ be three uncertain multiplicative linguistic variables, based on the above definition of degree of possibility, some desired properties are achieved.

Property 7: $0 \le p(\widetilde{s_1} \ge \widetilde{s_2}) \le 1$

Proof: It is straightforward and thus omitted.

Property 8: $p(\widetilde{s_1} \ge \widetilde{s_2}) + p(\widetilde{s_2} \ge \widetilde{s_1}) = 1$

Proof: It is straightforward and thus omitted.

Property 9: $p(\widetilde{s_1} \ge \widetilde{s_2}) \ge 0.5$ if and only if $\sqrt{I^-(\widetilde{s_1})I^+(\widetilde{s_1})} \ge \sqrt{I^-(\widetilde{s_2})I^+(\widetilde{s_2})}$, especially, $p(\widetilde{s_1} \ge \widetilde{s_2}) = 0.5$ if and only if $\sqrt{I^-(\widetilde{s_1})I^+(\widetilde{s_1})} = \sqrt{I^-(\widetilde{s_2})I^+(\widetilde{s_2})}$.

Proof: Since $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$ and $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$, we have $\alpha_1 = I^-(\widetilde{s_1})$, $\beta_1 = I^+(\widetilde{s_1})$, $\alpha_2 = I^-(\widetilde{s_2})$ and $\beta_1 = I^+(\widetilde{s_2})$. Assume $p(\widetilde{s_1} \ge \widetilde{s_2}) \ge 0.5$, we have $\alpha_2 < \beta_1$. Otherwise, if $\alpha_2 \ge \beta_1$, namely, $\beta_2 \ge \alpha_2 \ge \beta_1 \ge \alpha_1$, then $p(\widetilde{s_1} \ge \widetilde{s_2}) = 0 < 0.5$, which is not accord with the original assumption. After that, the situation $\alpha_2 < \beta_1$ is only considered.

If $\alpha_2 < \beta_1$ and $\beta_2 > \alpha_1$, it follows that

$$p(\widetilde{s_1} \ge \widetilde{s_2}) = \frac{\ln\beta_1 - \ln\alpha_2 - 0}{\ln\beta_1 - \ln\alpha_1 + \ln\beta_2 - \ln\alpha_2} \ge 0.5$$

we have $\frac{1}{2}(\ln\alpha_1 + \ln\beta_1) \ge \frac{1}{2}(\ln\alpha_2 + \ln\beta_2)$. Thus, $\sqrt{\alpha_1\beta_1} \ge \sqrt{\alpha_2\beta_2}$.

If $\alpha_2 < \beta_1$ and $\beta_2 \le \alpha_1$, it follows that

$$p(\widetilde{s_1} \ge \widetilde{s_2}) = \frac{\ln\beta_1 - \ln\alpha_2 - (\ln\alpha_1 - \ln\beta_2)}{\ln\beta_1 - \ln\alpha_1 + \ln\beta_2 - \ln\alpha_2} = 1 \ge 0.5$$

Here we have $\alpha_2 \le \beta_2 \le \alpha_1 \le \beta_1$, so we also get $\sqrt{\alpha_1\beta_1} \ge \sqrt{\alpha_2\beta_2}$.

In sum, the property is proved.

Property 10: If $p(\widetilde{s_1} \ge \widetilde{s_2}) \ge 0.5$ and $p(\widetilde{s_2} \ge \widetilde{s_3}) \ge 0.5$, then $p(\widetilde{s_1} \ge \widetilde{s_3}) \ge 0.5$.

Proof: Suppose $p(\widetilde{s_1} \ge \widetilde{s_2}) \ge 0.5$ and $p(\widetilde{s_2} \ge \widetilde{s_3}) \ge 0.5$, from Property 3, it follows that $\sqrt{I^-(\widetilde{s_1})I^+(\widetilde{s_1})} \ge \sqrt{I^-(\widetilde{s_2})I^+(\widetilde{s_2})}$ and $\sqrt{I^-(\widetilde{s_2})I^+(\widetilde{s_2})} \ge \sqrt{I^-(\widetilde{s_3})I^+(\widetilde{s_3})}$, so we have $\sqrt{I^-(\widetilde{s_1})I^+(\widetilde{s_1})} \ge \sqrt{I^-(\widetilde{s_3})I^+(\widetilde{s_3})}$, which can arrive $p(\widetilde{s_1} \ge \widetilde{s_3}) \ge 0.5$.

Therefore, the property is proved.

By Eq. (1), $p(\widetilde{s_1} \ge \widetilde{s_2}) = 0.5$ if and only if $\frac{\alpha_1+\beta_1}{2} = \frac{\alpha_2+\beta_2}{2}$. However, arithmetical mean of the end points is not in accordance with the operation laws of uncertain multiplicative linguistic variables. Property 9 shows that compare two uncertain multiplicative linguistic variables is equal to compare the geometric mean of the corresponding end points. With help of the Property 10, a complete ranking order for uncertain multiplicative linguistic variables is determined.

IV. THE UNCERTAIN MULTIPLICATIVE LINGUISTIC HYBRID WEIGHTED GEOMETRIC AVERAGING OPERATOR

Xu [33] developed the ULHG operator by combing the ULWG operator and the ULOWG operator, shown as follows:

Definition 11: An ULHG operator of dimension n is a mapping ULHG: $\overline{\overline{S}}^n \to \overline{\overline{S}}$, defined by an associated weighting vector $W = (w_1, w_2, \cdots, w_n)^T$, such that $\sum_{i=1}^{n} w_i = 1$ and

$$p(\widetilde{s}_1 \geq \widetilde{s}_2) = \frac{\max\{\ln(I^+(\widetilde{s}_1)) - \ln(I^-(\widetilde{s}_2)), 0\} - \max\{\ln(I^-(\widetilde{s}_1)) - \ln(I^+(\widetilde{s}_2)), 0\}}{\ln(I^+(\widetilde{s}_1)) - \ln(I^-(\widetilde{s}_1)) + \ln(I^+(\widetilde{s}_2)) - \ln(I^-(\widetilde{s}_2))} \tag{6}$$

$w_i \in [0, 1]$, according to the following formula:

$$ULHG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\beta_1}^{w_1} \bigotimes \widetilde{s}_{\beta_2}^{w_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\beta_n}^{w_n} \tag{8}$$

where $\widetilde{s}_{\beta_j}$ is the jth largest element of the weighted arguments $\widetilde{s}_i^{n\lambda_i}$ $(i = 1, 2, \cdots, n)$, and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is the weighting vector of the $\widetilde{s}_i$ $(i = 1, 2, \cdots, n)$, with $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \in [0, 1]$, and n is the balancing coefficient.

In order to overcome some flaws of the ULHG operator, a new uncertain multiplicative linguistic hybrid aggregation operator is proposed as follows.

Definition 12: An uncertain multiplicative linguistic hybrid weighted geometric averaging (ULHWG) operator of dimension n is a mapping ULHWG: $\widetilde{\overline{S}}^n \to \widetilde{\overline{S}}$, defined by an associated weighting vector $W = (w_1, w_2, \cdots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$, such that

$$ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{\theta_1} \bigotimes \widetilde{s}_{\sigma(2)}^{\theta_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{\theta_n} \tag{9}$$

where $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ being the permutation such that $\widetilde{s}_{\sigma(i)} > \widetilde{s}_{\sigma(i+1)}$ $(i = 1, 2, \cdots, n-1)$, and $\theta_i = \frac{\lambda_{\sigma(i)} w_i}{\sum_{i=1}^{n} \lambda_{\sigma(i)} w_i}$. $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is the weighting vector of the $\widetilde{s}_i$ $(i = 1, 2, \cdots, n)$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \in [0, 1]$.

Theorem 13: The ULWG operator is a special case of the ULHWG operator.

Proof: When $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then $\theta_i = \frac{\lambda_{\sigma(i)} w_i}{\sum_{i=1}^{n} \lambda_{\sigma(i)} w_i} = \frac{\lambda_{\sigma(i)} \frac{1}{n}}{\sum_{i=1}^{n} \lambda_{\sigma(i)} \frac{1}{n}} = \lambda_{\sigma(i)}$, $i = 1, 2, \cdots, n$.

So we get $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{\lambda_{\sigma(1)}} \bigotimes \widetilde{s}_{\sigma(2)}^{\lambda_{\sigma(2)}} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{\lambda_{\sigma(n)}} = \widetilde{s}_1^{\lambda_1} \bigotimes \widetilde{s}_2^{\lambda_2} \bigotimes \cdots \bigotimes \widetilde{s}_n^{\lambda_n} = ULWG_\lambda (\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n)$

Therefore, the theorem is proved.

Theorem 14: The ULOWG operator is a special case of the ULHWG operator.

Proof: When $\lambda = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then $\theta_i = \frac{\lambda_{\sigma(i)} w_i}{\sum_{i=1}^{n} \lambda_{\sigma(i)} w_i} = \frac{\frac{1}{n} w_i}{\sum_{i=1}^{n} \frac{1}{n} w_i} = w_i$, $i = 1, 2, \cdots, n$.

So we get $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{w_1} \bigotimes \widetilde{s}_{\sigma(2)}^{w_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{w_n} = ULOWG_\lambda(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n)$

Therefore, the theorem is proved.

It is must be pointed out that the ULHWG operator extended both the ULWG operator and ULOWG operator, and has the following desired properties.

Theorem 15 (Boundary): Let $\widetilde{s}_i = [s_{\alpha_i}, s_{\beta_i}]$ $(i = 1, 2, \cdots, n)$ be n uncertain multiplicative linguistic variables, then

$$\min_i\{s_{\alpha_i}\} \leq ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) \leq \max_i\{s_{\beta_i}\}$$

Proof: Let $\min_i\{s_{\alpha_i}\} = \dot{s}$, $\max_i\{s_{\beta_i}\} = \ddot{s}$, then $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{\theta_1} \bigotimes \widetilde{s}_{\sigma(2)}^{\theta_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{\theta_n}$ $\geq \dot{s}^{\theta_1} \bigotimes \dot{s}^{\theta_2} \bigotimes \cdots \bigotimes \dot{s}^{\theta_n} = \dot{s}^{\sum_{i=1}^{n}\theta_i} = \dot{s}$

as such, $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{\theta_1} \bigotimes \widetilde{s}_{\sigma(2)}^{\theta_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{\theta_n} \leq \ddot{s}^{\theta_1} \bigotimes \ddot{s}^{\theta_2} \bigotimes \cdots \bigotimes \ddot{s}^{\theta_n} = \ddot{s}^{\sum_{i=1}^{n}\theta_i} = \ddot{s}$

In sum, the proof of theorem is complete.

Theorem 16 (Monotonicity): $\forall\, \widetilde{a}_i, \widetilde{b}_i \in \widetilde{\overline{S}}$ $(i = 1, 2, \cdots, n)$ with $\widetilde{a}_i \leq \widetilde{b}_i$. If $\exists$ a permutation $\sigma : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ such that $\widetilde{a}_{\sigma(i)} > \widetilde{a}_{\sigma(i+1)}$ and $\widetilde{b}_{\sigma(i)} > \widetilde{b}_{\sigma(i+1)}$ $(i = 1, 2, \cdots, n-1)$, then $ULHWG_{\lambda,W}(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \leq ULHWG_{\lambda,W}(\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_n)$.

Proof: Since $\widetilde{a}_i \leq \widetilde{b}_i$ $(i = 1, 2, \cdots, n)$, $\widetilde{a}_{\sigma(i)} > \widetilde{a}_{\sigma(i+1)}$ and $\widetilde{b}_{\sigma(i)} > \widetilde{b}_{\sigma(i+1)}$ for all $i = 1, 2, \cdots, n-1$, from Eq. (9), we have $ULHWG_{\lambda,W}(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) = (\widetilde{a}_{\sigma(1)})^{\frac{\lambda_{\sigma(1)} w_1}{\sum_{i=1}^{n} \lambda_{\sigma(1)} w_1}} \bigotimes (\widetilde{a}_{\sigma(2)})^{\frac{\lambda_{\sigma(2)} w_2}{\sum_{i=1}^{n} \lambda_{\sigma(2)} w_2}} \bigotimes \cdots \bigotimes (\widetilde{a}_{\sigma(n)})^{\frac{\lambda_{\sigma(n)} w_n}{\sum_{i=1}^{n} \lambda_{\sigma(n)} w_n}}$ $\leq (\widetilde{b}_{\sigma(1)})^{\frac{\lambda_{\sigma(1)} w_1}{\sum_{i=1}^{n} \lambda_{\sigma(1)} w_1}} \bigotimes (\widetilde{b}_{\sigma(2)})^{\frac{\lambda_{\sigma(2)} w_2}{\sum_{i=1}^{n} \lambda_{\sigma(2)} w_2}} \bigotimes \cdots \bigotimes (\widetilde{b}_{\sigma(n)})^{\frac{\lambda_{\sigma(n)} w_n}{\sum_{i=1}^{n} \lambda_{\sigma(n)} w_n}} = ULHWG_{\lambda,W}(\widetilde{b}_1, \widetilde{b}_2, \cdots, \widetilde{b}_n)$.

Theorem 17 (Idempotency): Let $\widetilde{s}_i, \widetilde{s} \in \widetilde{\overline{S}}$, if $\widetilde{s}_i = \widetilde{s}$ $(i = 1, 2, \cdots, n)$, then

$$ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}$$

Proof: Suppose $\widetilde{s}_i = \widetilde{s}$ $(i = 1, 2, \cdots, n)$, we have $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \cdots, \widetilde{s}_n) = \widetilde{s}_{\sigma(1)}^{\theta_1} \bigotimes \widetilde{s}_{\sigma(2)}^{\theta_2} \bigotimes \cdots \bigotimes \widetilde{s}_{\sigma(n)}^{\theta_n} = \widetilde{s}^{\theta_1} \bigotimes \widetilde{s}^{\theta_2} \bigotimes \cdots \bigotimes \widetilde{s}^{\theta_n} = \widetilde{s}^{\sum_{i=1}^{n}\theta_i} = \widetilde{s}$.

The ULHG operator combined the advantages of the ULWG operator and the ULOWG operator. However, it fail in some desirable properties. For example, assume the multiplicative linguistic scale is selected as $S = \{s_{\frac{1}{4}}, s_{\frac{1}{3}}, s_{\frac{1}{2}}, s_1, s_2, s_3, s_4\}$, aggregated objects $(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) = ([s_{\frac{1}{4}}, s_{\frac{1}{3}}], [s_{\frac{1}{2}}, s_1], [s_2, s_3])$, and $W = \lambda = (0, 0, 1)^T$, then $ULHG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) = \widetilde{s}_3^{\,3} = [s_8, s_{27}]$. In this situation, the ULHG operator does not satisfy boundary. By using the ULHWG operator, we have $ULHWG_{\lambda,W}(\widetilde{s}_1, \widetilde{s}_2, \widetilde{s}_3) = [s_2, s_3]$, which is logical and is consistent with the intuitive judgment.

V. AN APPROACH TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING UNDER UNCERTAIN MULTIPLICATIVE LINGUISTIC ENVIRONMENT

With respect to multiple attribute group decision making problems with uncertain multiplicative linguistic information. Let $G = \{g_1, g_2, \cdots, g_m\}$ be a finite set of alternatives, and $U = \{u_1, u_2, \cdots, u_n\}$ be the set of attributes, the weight vector of U is $\rho = (\rho_1, \rho_2, \cdots, \rho_n)^T$ with $\rho_i \in [0, 1]$ and $\sum_{i=1}^{N} \rho_i = 1$. Let $E = \{e_1, e_2, \cdots, e_l\}$ be the set of experts, and suppose $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{m\times n}$ is the uncertain multiplicative linguistic decision matrix given by the expert e_k, where $\widetilde{a}_{ij}^{(k)} = [a_{ij}^{(k)-}, a_{ij}^{(k)+}] \in \widetilde{\overline{S}}$ is an uncertain multiplicative linguistic preference value for alternative g_i with respect to attribute u_j $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n; k = 1, 2, \cdots, l)$.

A. DEVIATION MEASURE OF UNCERTAIN MULTIPLICATIVE LINGUISTIC DECISION MATRICES

Definition 18: Let $\widetilde{s_1} = [s_{\alpha_1}, s_{\beta_1}]$, $\widetilde{s_2} = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain multiplicative linguistic variables, then the deviation degree between $\widetilde{s_1}$ and $\widetilde{s_2}$ is defined as follows (10), as shown at the top of the next page:

where $2t - 1$ is the number of linguistic terms in the set S. The parameter δ can be regarded as the attitude of expertąŕs preference attitude towards risk, $\delta \in [0, 1]$.

The Eq. (10) can be further expressed as

$$D(\widetilde{s_1}, \widetilde{s_2}) = \frac{(1-\delta)|\ln\alpha_1 - \ln\alpha_2| + \delta|\ln\beta_1 - \ln\beta_2|}{2\ln t} \quad (11)$$

Theorem 19: Let $\widetilde{s_1}, \widetilde{s_2} \in \widetilde{\overline{S}}$, then $0 \le D(\widetilde{s_1}, \widetilde{s_2}) \le 1$. Especially, $D(\widetilde{s_1}, \widetilde{s_2}) = 0$ if and only if $\widetilde{s_1} = \widetilde{s_2}$.

Proof: Since $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [\frac{1}{t}, t]$, we can get $0 \le |\ln\alpha_1 - \ln\alpha_2| \le 2\ln t$, $0 \le |\ln\beta_1 - \ln\beta_2| \le 2\ln t$, so $0 \le (1-\delta)|\ln\alpha_1 - \ln\alpha_2| + \delta|\ln\beta_1 - \ln\beta_2| \le 2\ln t$ which can be further written as $0 \le \frac{(1-\delta)|\ln\alpha_1 - \ln\alpha_2| + \delta|\ln\beta_1 - \ln\beta_2|}{2\ln t} \le 1$ we have $0 \le D(\widetilde{s_1}, \widetilde{s_2}) \le 1$. Thus, the proof of theorem is complete.

Definition 20: Let $\widetilde{A} = (\widetilde{a}_{ij})_{m\times n}$ and $\widetilde{B} = (\widetilde{b}_{ij})_{m\times n}$ be two uncertain multiplicative linguistic decision matrices, where $\widetilde{a}_{ij} = [a_{ij}^-, a_{ij}^+]$, $\widetilde{b}_{ij} = [b_{ij}^-, b_{ij}^+] \in \widetilde{\overline{S}}$, then deviation degree between $\widetilde{A}$ and $\widetilde{B}$ is defined as follows:

$$D(\widetilde{A}, \widetilde{B}) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} D(\widetilde{a}_{ij}, \widetilde{b}_{ij}) \quad (12)$$

which can be further expressed as (13), as shown at the top of the next page Eq.(13) can also be equivalently written as

$$D(\widetilde{A}, \widetilde{B}) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{b}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{b}_{ij}))|) \quad (14)$$

Remark 21: If $t = 1$, then $S = \{s_1\}$. For any two uncertain multiplicative linguistic decision matrices $\widetilde{A}$ and $\widetilde{B}$, we always have $D(\widetilde{A}, \widetilde{B}) = 0$. Thus, in what follows, we only consider the situation that $t \ge 2$.

Theorem 22: Let $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are any three uncertain multiplicative linguistic decision matrices, then

(1) $D(\widetilde{A}, \widetilde{B}) = D(\widetilde{B}, \widetilde{A})$

(2) $0 \le D(\widetilde{A}, \widetilde{B}) \le 1$, especially, $D(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$.

(3) $D(\widetilde{A}, \widetilde{C}) \le D(\widetilde{A}, \widetilde{B}) + D(\widetilde{B}, \widetilde{C})$

Proof: (1) According to Eq.(13), we get $D(\widetilde{A}, \widetilde{B}) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{b}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{b}_{ij}))|) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{b}_{ij})) - \ln(I^-(\widetilde{a}_{ij}))| + \delta|\ln(I^+(\widetilde{b}_{ij})) - \ln(I^+(\widetilde{a}_{ij}))|) = D(\widetilde{B}, \widetilde{A})$

(2) By definition 18, we have $0 \le D(\widetilde{a}_{ij}, \widetilde{b}_{ij}) \le 1$, then

$$0 \le \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} D(\widetilde{a}_{ij}, \widetilde{b}_{ij}) \le \sum_{i=1}^{m}\sum_{j=1}^{n} 1 = 1$$

Obviously, it follows that $0 \le D(\widetilde{A}, \widetilde{B}) \le 1$.

Especially, if $D(\widetilde{A}, \widetilde{B}) = 0$, then we have

$$D(\widetilde{a}_{ij}, \widetilde{b}_{ij}) = 0, \quad \forall i = 1, 2, \cdots, m; j = 1, 2, \cdots, n.$$

From Eq.(10), we get

$$\widetilde{a}_{ij} = \widetilde{b}_{ij}, \quad \forall i = 1, 2, \cdots, m; j = 1, 2, \cdots, n.$$

Therefore, $\widetilde{A} = \widetilde{B}$.

(3) By Eq.(13), it follows that $D(\widetilde{A}, \widetilde{C}) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{c}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{c}_{ij}))|) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{b}_{ij})) + \ln(I^-(\widetilde{b}_{ij})) - \ln(I^-(\widetilde{c}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{b}_{ij})) + \ln(I^+(\widetilde{b}_{ij})) - \ln(I^+(\widetilde{c}_{ij}))|) \le \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{b}_{ij}))| + (1-\delta)|\ln(I^-(\widetilde{b}_{ij})) - \ln(I^-(\widetilde{c}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{b}_{ij}))| + \delta|\ln(I^+(\widetilde{b}_{ij})) - \ln(I^+(\widetilde{c}_{ij}))|) = \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij})) - \ln(I^-(\widetilde{b}_{ij}))| + \delta|\ln(I^+(\widetilde{a}_{ij})) - \ln(I^+(\widetilde{b}_{ij}))|) + \frac{1}{2mn\ln t}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{b}_{ij})) - \ln(I^-(\widetilde{c}_{ij}))| + \delta|\ln(I^+(\widetilde{b}_{ij})) - \ln(I^+(\widetilde{c}_{ij}))|) = D(\widetilde{A}, \widetilde{B}) + D(\widetilde{B}, \widetilde{C})$ In sum, the theorem 22 is proved.

B. A PROCEDURE TO OBTAIN THE WEIGHTS OF EXPERTS

It is very important to get the expert weights in multiple attribute group decision making, in the following, some simple and efficient formulas are developed to obtain the weighting vector of experts.

Definition 23: Suppose $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{m\times n}$ is an uncertain multiplicative linguistic decision matrix, then the overall expect deviation degree of $\widetilde{A}^{(k)}$ compared with other decision matrices $\widetilde{A}^{(p)}$ $(p = 1, 2, \cdots, l$ and $p \ne k)$ is defined as

$$D(\widetilde{A}^{(k)}) = \frac{1}{l-1}\sum_{\substack{p=1\\p\ne k}}^{l} D(\widetilde{A}^{(k)}, \widetilde{A}^{(p)}) \quad (15)$$

$$D(\widetilde{s}_1,\widetilde{s}_2)=\frac{(1-\delta)|\ln(I^-(\widetilde{s}_1))-\ln(I^-(\widetilde{s}_2))|+\delta|\ln(I^+(\widetilde{s}_1))-\ln(I^+(\widetilde{s}_2))|}{2\ln t} \tag{10}$$

$$D(\widetilde{A},\widetilde{B})=\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(1-\delta)|\ln(I^-(\widetilde{a}_{ij}))-\ln(I^-(\widetilde{b}_{ij}))|+\delta|\ln(I^+(\widetilde{a}_{ij}))-\ln(I^+(\widetilde{b}_{ij}))|}{2\ln t} \tag{13}$$

Since $D(\widetilde{A}^{(k)},\widetilde{A}^{(k)})=0$, Eq.(15) can be equivalently written as

$D(\widetilde{A}^{(k)})=\frac{1}{l-1}\sum_{p=1}^{l}D(\widetilde{A}^{(k)},\widetilde{A}^{(p)})=\frac{1}{2(l-1)mn\ln t}\sum_{p=1}^{l}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij}^{(k)}))-\ln(I^-(\widetilde{a}_{ij}^{(p)}))|+\delta|\ln(I^+(\widetilde{a}_{ij}^{(k)}))-\ln(I^+(\widetilde{a}_{ij}^{(p)}))|)$

Theorem 24: $0\le D(\widetilde{A}^{(k)})\le 1,\ k=1,2,\cdots,l$. Especially, $D(\widetilde{A}^{(k)})=0$ if and only if $\widetilde{A}^{(1)}=\widetilde{A}^{(2)}=\cdots=\widetilde{A}^{(l)}$.

Proof: According to theorem 22, we have

$$0\le D(\widetilde{A}^{(k)},D(\widetilde{A}^{(p)})\le 1,\ k,p=1,2,\cdots,l.$$

Thus, we get

$$0\le\sum_{\substack{p=1\\p\neq k}}^{l}D(\widetilde{A}^{(k)},\widetilde{A}^{(p)})\le l-1$$

which can be equivalently written as

$$0\le\frac{1}{l-1}\sum_{\substack{p=1\\p\neq k}}^{l}D(\widetilde{A}^{(k)},\widetilde{A}^{(p)})\le 1$$

then from Eq.(15), we get $0\le D(\widetilde{A}^{(k)})\le 1$.

Especially, if $D(\widetilde{A}^{(k)})=0$, we get

$$D(\widetilde{A}^{(k)},\widetilde{A}^{(p)})=0,\quad p=1,2,\cdots,l.$$

It follows that $\widetilde{A}^{(k)}=\widetilde{A}^{(p)}, p=1,2,\cdots,l$. Therefore, we get $\widetilde{A}^{(1)}=\widetilde{A}^{(2)}=\cdots=\widetilde{A}^{(l)}$ (17), as shown at the top of the next page.

The overall expect deviation degree $D(\widetilde{A}^{(k)})$ reflects the difference between $D(\widetilde{A}^{(k)})$ and other decision matrices $D(\widetilde{A}^{(p)})$ $(p=1,2,\cdots,l)$. In general, the majority opinion should be emphasized. It is clear that the smaller the value of overall expect deviation degree $D(\widetilde{A}^{(k)})$, the greater weight the kth expert has. So the weights of experts can be defined as

$$\lambda_k=\frac{1-D(\widetilde{A}^{(k)})}{\sum_{k=1}^{l}(1-D(\widetilde{A}^{(k)}))},\quad k=1,2,\cdots,l. \tag{16}$$

Obviously, we have $\lambda_k\in[0,1]$ and $\sum_{k=1}^{l}\lambda_k=1$. By Eq.(14), Eq.(16) can also be further expressed as Eq.(17).

Algorithm 1

Step1. Expert e_k evaluates all attributes in U with alternatives $g_i\in G$ $(i=1,2,\cdots,m)$, then uncertain multiplicative linguistic decision matrix $\widetilde{A}^{(k)}=(\widetilde{a}_{ij}^{(k)})_{m\times n}$ is established, $k=1,2,\cdots,l$.

Step2. Utilize the uncertain multiplicative linguistic information in the matrix $\widetilde{A}^{(k)}$, the individual overall preference values is derived by ULWG operator as

$$\widetilde{V}_i^{(k)}=ULWG_\rho(\widetilde{a}_{i1}^{(k)},\widetilde{a}_{i2}^{(k)},\cdots,\widetilde{a}_{in}^{(k)}),\quad i=1,2,\cdots,m.$$

Step3. According to expert′s preference attitude, the parameter δ is selected. By Eqs.(16) and (17), the weighting vector $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)^T$ of experts is obtained.

Step4. Select a fuzzy linguistic quantifier in accordance with the preference of decision maker, then the associated weighting vector $W=(w_1,w_2,\cdots,w_l)^T$ is calculated by Eq.(4) and Eq.(5).

Step5. Use the ULHWG operator, the collective overall preference values $\widetilde{V}_i$ is determined as

$$\widetilde{V}_i=ULHWG_{\lambda,W}(\widetilde{V}_i^{(1)},\widetilde{V}_i^{(2)},\cdots,\widetilde{V}_i^{(l)}),\quad i=1,2,\cdots,m.$$

Step6. Ranking all the collective overall preference values $\widetilde{V}_i$ $(i=1,2,\cdots,m)$ by using Eq.(6).

Step7. Ranking alternatives $g_1,g_2,\cdots,g_m$ and select the best one in accordance with the ranking of $\widetilde{V}_1,\widetilde{V}_2,\cdots,\widetilde{V}_m$.

Step8. End.

C. A COMPREHENSIVE ALGORITHM FOR LINGUISTIC GROUP DECISION MAKING

Based on the above analysis, in the following, we shall utilize the deviation measure and aggregation operators to develop a valid algorithm to multiple attribute group decision making.

In next section, a practical example will be utilized to illustrate the application of the developed method.

VI. THE APPLICATION OF PROPOSED METHOD TO INFORMATION SYSTEM SELECTION

Let us suppose there is a company, which wants to purchase an information system in the best option. There is a panel with four possible information systems to be selected: g_1,g_2,g_3,g_4. The company must take a decision according to the following five attributes: u_1 (Price), u_2 (Operability), u_3 (Stability), u_4 (Intellectual property right), u_5 (Interface), the weighting vector of $U=\{u_1,u_2,\cdots,u_5\}$ is $\rho=(0.1,0.25,0.15,0.3,0.2)^T$.

$$\lambda_k = \frac{2(l-1)mn\ln t - \sum_{p=1}^{l}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij}^{(k)})) - \ln(I^-(\widetilde{a}_{ij}^{(p)}))| + \delta|\ln(I^+(\widetilde{a}_{ij}^{(k)})) - \ln(I^+(\widetilde{a}_{ij}^{(p)}))|)}{2l(l-1)mn\ln t - \sum_{k=1}^{l}\sum_{p=1}^{l}\sum_{i=1}^{m}\sum_{j=1}^{n}((1-\delta)|\ln(I^-(\widetilde{a}_{ij}^{(k)})) - \ln(I^-(\widetilde{a}_{ij}^{(p)}))| + \delta|\ln(I^+(\widetilde{a}_{ij}^{(k)})) - \ln(I^+(\widetilde{a}_{ij}^{(p)}))|)} \tag{17}$$

Step 1: Four alternative information systems g_1, g_2, g_3, g_4 are to be evaluated using the label set:

$S = \{s_{\frac{1}{5}} = extremely\ poor, s_{\frac{1}{4}} = very\ poor, s_{\frac{1}{3}} = poor, s_{\frac{1}{2}} = slightly\ poor, s_1 = fair, s_2 = slightly\ good, s_3 = good, s_4 = very\ good, s_5 = extremely\ good\}$ by three experts e_k $(k = 1, 2, 3)$ under the above five attributes, shown as follows:

$$\widetilde{A}^{(1)} = \begin{bmatrix} [s_{\frac{1}{2}}, s_1] & [s_1, s_3] & [s_{\frac{1}{3}}, s_{\frac{1}{2}}] & [s_3, s_5] & [s_1, s_2] \\ [s_{\frac{1}{5}}, s_{\frac{1}{3}}] & [s_2, s_4] & [s_{\frac{1}{2}}, s_1] & [s_1, s_2] & [s_{\frac{1}{4}}, s_{\frac{1}{3}}] \\ [s_2, s_3] & [s_{\frac{1}{2}}, s_1] & [s_1, s_3] & [s_2, s_3] & [s_{\frac{1}{3}}, s_1] \\ [s_1, s_2] & [s_{\frac{1}{3}}, s_1] & [s_2, s_3] & [s_{\frac{1}{2}}, s_1] & [s_{\frac{1}{2}}, s_1] \end{bmatrix}$$

$$\widetilde{A}^{(2)} = \begin{bmatrix} [s_1, s_2] & [s_{\frac{1}{2}}, s_2] & [s_{\frac{1}{3}}, s_{\frac{1}{2}}] & [s_2, s_4] & [s_1, s_3] \\ [s_{\frac{1}{4}}, s_{\frac{1}{2}}] & [s_3, s_5] & [s_1, s_2] & [s_1, s_3] & [s_{\frac{1}{3}}, s_{\frac{1}{2}}] \\ [s_1, s_3] & [s_1, s_2] & [s_{\frac{1}{2}}, s_2] & [s_2, s_3] & [s_{\frac{1}{2}}, s_1] \\ [s_1, s_2] & [s_{\frac{1}{2}}, s_1] & [s_1, s_2] & [s_1, s_2] & [s_{\frac{1}{2}}, s_1] \end{bmatrix}$$

$$\widetilde{A}^{(3)} = \begin{bmatrix} [s_{\frac{1}{2}}, s_2] & [s_{\frac{1}{2}}, s_1] & [s_{\frac{1}{2}}, s_1] & [s_2, s_3] & [s_1, s_2] \\ [s_{\frac{1}{3}}, s_1] & [s_1, s_3] & [s_1, s_2] & [s_1, s_3] & [s_{\frac{1}{3}}, s_1] \\ [s_1, s_3] & [s_{\frac{1}{2}}, s_1] & [s_{\frac{1}{2}}, s_1] & [s_1, s_3] & [s_{\frac{1}{3}}, s_{\frac{1}{2}}] \\ [s_1, s_3] & [s_{\frac{1}{2}}, s_2] & [s_1, s_2] & [s_1, s_2] & [s_{\frac{1}{2}}, s_1] \end{bmatrix}$$

Step 2: Utilize the uncertain multiplicative linguistic information in the matrix $\widetilde{A}^{(1)}, \widetilde{A}^{(2)}$, and $\widetilde{A}^{(3)}$, the individual overall preference values is derived by ULWG operator as

$$\begin{aligned} \widetilde{V}_1^{(1)} &= [s_{1.10}, s_{2.21}], & \widetilde{V}_2^{(1)} &= [s_{0.69}, s_{1.25}], \\ \widetilde{V}_3^{(1)} &= [s_{0.89}, s_{1.83}], & \widetilde{V}_4^{(1)} &= [s_{0.60}, s_{1.26}]; \\ \widetilde{V}_1^{(2)} &= [s_{0.88}, s_{2.17}], & \widetilde{V}_2^{(2)} &= [s_{0.92}, s_{1.87}], \\ \widetilde{V}_3^{(2)} &= [s_{0.97}, s_{2.05}], & \widetilde{V}_4^{(2)} &= [s_{0.73}, s_{1.46}]; \\ \widetilde{V}_1^{(3)} &= [s_{0.87}, s_{1.71}], & \widetilde{V}_2^{(3)} &= [s_{0.72}, s_{2.03}], \\ \widetilde{V}_3^{(3)} &= [s_{0.61}, s_{1.35}], & \widetilde{V}_4^{(3)} &= [s_{0.73}, s_{1.81}]. \end{aligned}$$

Step 3: Assume that experts are neutral to risk, then the attitude parameter $\delta = 0.5$ is selected. Form Eq.(15), we get

$$D(\widetilde{A}^{(1)}) = 0.451, \quad D(\widetilde{A}^{(2)}) = 0.229, \quad D(\widetilde{A}^{(3)}) = 0.111$$

According to Eqs.(16) and (17), the weighting vector of experts is calculated as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3)^T = (0.249, 0.348, 0.403)^T$$

Step 4: Select the fuzzy linguistic quantifier "more" in accordance with the preference of decision maker, then the pair $(a, b) = (0.3, 0.8)$. By Eq.(4) and Eq.(5), the associated weighting vector

$$W = (w_1, w_2, w_3)^T = (0.066, 0.668, 0.266)^T$$

Step 5: Use the ULHWG operator, the collective overall preference values is determined as

$$\begin{aligned} \widetilde{V}_1 &= ULHWG_{\lambda,W}(\widetilde{V}_1^{(1)}, \widetilde{V}_1^{(2)}, \widetilde{V}_1^{(3)}) \\ &= ULHWG_{\lambda,W}([s_{1.10}, s_{2.21}], [s_{0.88}, s_{2.17}], [s_{0.87}, s_{1.71}]) \\ &= [s_{0.89}, s_{2.02}]; \\ \widetilde{V}_2 &= ULHWG_{\lambda,W}(\widetilde{V}_2^{(1)}, \widetilde{V}_2^{(2)}, \widetilde{V}_2^{(3)}) \\ &= ULHWG_{\lambda,W}([s_{0.69}, s_{1.25}], [s_{0.92}, s_{1.87}], [s_{0.72}, s_{2.03}]) \\ &= [s_{0.73}, s_{1.85}]; \\ \widetilde{V}_3 &= ULHWG_{\lambda,W}(\widetilde{V}_3^{(1)}, \widetilde{V}_3^{(2)}, \widetilde{V}_3^{(3)}) \\ &= ULHWG_{\lambda,W}([s_{0.89}, s_{1.83}], [s_{0.97}, s_{2.05}], [s_{0.61}, s_{1.35}]) \\ &= [s_{0.72}, s_{1.54}]; \\ \widetilde{V}_4 &= ULHWG_{\lambda,W}(\widetilde{V}_4^{(1)}, \widetilde{V}_4^{(2)}, \widetilde{V}_4^{(3)}) \\ &= ULHWG_{\lambda,W}([s_{0.60}, s_{1.26}], [s_{0.73}, s_{1.46}], [s_{0.73}, s_{1.81}]) \\ &= [s_{0.85}, s_{1.73}]. \end{aligned}$$

Step 6: Ranking all the collective overall preference values $\widetilde{V}_i$ $(i = 1, 2, 3, 4)$ by using Eq.(6), we have $p(\widetilde{V}_1 \geq \widetilde{V}_4) = 0.566$; $p(\widetilde{V}_4 \geq \widetilde{V}_2) = 0.526$; $p(\widetilde{V}_2 \geq \widetilde{V}_3) = 0.558$.

Therefore, it follows that $\widetilde{V}_1 \overset{56.6\%}{\succ} \widetilde{V}_4 \overset{52.6\%}{\succ} \widetilde{V}_2 \overset{55.8\%}{\succ} \widetilde{V}_3$.

Step 7: The ranking of the priority of applicants is $g_1 \succ g_4 \succ g_2 \succ g_3$. Namely, the most desirable information system is g_1.

VII. CONCLUSIONS

This paper introduced a new uncertain multiplicative linguistic hybrid weighted geometric averaging (ULHWG) operator and a possibility degree formula to overcome the drawback in the existed reference. Based on the proposed operators and deviation measure, a valid algorithm for linguistic group decision making is proposed. To illustrate the application of the proposed method, an practical example is employed to information system selection. The main advantages of this paper are shown as follows:

(1) The ULHWG operator considers both the ordered position and importance of uncertain multiplicative linguistic variables, and helps us to overcome some drawbacks of the existing operators.

(2) A possibility degree formula for ranking uncertain multiplicative linguistic variables is proposed based on geometric mean of the end points, which is consistent with the characteristics of multiplicative linguistic label set.

(3) The weights of experts are obtained based on deviation measure of uncertain multiplicative linguistic variables.

(4) A comprehensive algorithm is developed to solve the group decision making problem with linguistic setting.

In the future, we shall continue working in the extension and application of the developed method to other domains, such as traffic control, energy evaluation and environmental assessment.

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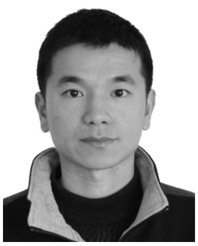

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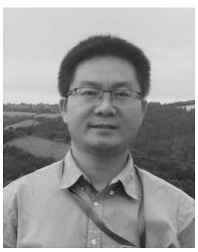

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