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Nonoverlapping Schwarz Waveform Relaxation and Numerical Recovery for Non-Fickian Problems With Time-Delay

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ABSTRACT In this paper, we study the nonoverlapping Schwarz waveform relaxation algorithm with Robin transmission conditions (TCs) and numerical recovery for non-Fickian diffusion equations with time-delay. We derive an ideal Robin parameter by technically solving a special min–max problem. We show that the free parameter in TCs has a significant effect on the convergence rate of the algorithm. Finally, we present several numerical results to confirm the effectiveness of the proposed algorithm.

INDEX TERMS Schwarz waveform relaxation, non-Fickian delay equations, parameter optimization, convergence analysis.

I. INTRODUCTION

Delay differential equations play an important role in scientific research and industrial production [22]–[24], [29], [31], [36], in areas such as ecology, the electronics industry, biology, medicine, and control systems. It is believed that this kind of equations can provide more precise descriptions of natural phenomena, where the rate of variation of state;and quantity depends not only on the present but also on history. In some real-world problems, the influence of space should be calculated, and this calculation leads to the following delay reaction-diffusion equations

$$\begin{aligned}\partial_t u(\mathbf{x}, t) &= D_1 \Delta u(\mathbf{x}, t) \\ &\quad + f(u(\mathbf{x}, t), u(\mathbf{x}, t-\tau), \mathbf{x}, t) \\ &\quad \times (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+, \end{aligned} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^d$ with $d = 1, 2$ or 3; $D_1 > 0$ is the rate of diffusion and Δ is the Laplacian operator. In past decades this type of delay partial differential equations(PDEs) have been investigated deeply and widely by many authors. The delay PDEs (1) are obtained by combing Fick's law for flux $J_1(\mathbf{x}, t)$-the Fickian flux

$$J_1(\mathbf{x}, t) = -D_1 \nabla u(\mathbf{x}, t), \tag{2}$$

and the mass conservation law

$$\partial_t u(\mathbf{x}, t) = -\nabla J_1(\mathbf{x}, t) + f(u(\mathbf{x}, t), u(\mathbf{x}, t-\tau)). \tag{3}$$

A rapidly developing area for this kind of reaction-diffusion equation research is the study of traveling wave solutions $u(x, t) = \phi(x - ct)$, which are characterized as solutions invariant in respect to transitional space [29], [31]. In this study, we found for the simple logistic reaction-diffusion equation, $\partial_t u(\mathbf{x}, t) = D_1 \Delta u(\mathbf{x}, t) + U_0 u(\mathbf{x}, t)(1 - u(\mathbf{x}, t))$, that the propagation speed c in the traveling wave solutions should satisfy $c \geq 2\sqrt{D_1 U_0}$ (see, e.g., [1]); consequently, if the reaction parameter U_0 goes to infinity, the propagation speed c also goes to infinity. This pathologic behavior is not observed in practical physical phenomena but is purely introduced by the mathematical model. To avoid this limitation in the context of reaction-diffusion phenomena, the following non-Fickian flux is introduced in [9] and [10]:

$$J_2(\mathbf{x}, t) = -\frac{D_2}{\delta} \int_0^t \exp\left(-\frac{t-s}{\delta}\right) \nabla u(\mathbf{x}, s) ds. \tag{4}$$

Hence, it is natural to consider in reaction-diffusion problems that the flux consists of two different components: one is the Fick's flux $J_1(\mathbf{x}, t)$ defined by (2), and the other is the non-Fickian flux $J_2(\mathbf{x}, t)$ defined by (4). Taking account of the mass conservation law (3) we then arrive at the following general delay reaction-diffusion equations:

$$\begin{aligned}\partial_t u(\mathbf{x}, t) &= D_1 \Delta u(\mathbf{x}, t) \\ &\quad + \frac{D_2}{\delta} \int_0^t \exp\left(-\frac{t-s}{\delta}\right) \Delta u(\mathbf{x}, s) ds \\ &\quad + f(u(\mathbf{x}, t), u(\mathbf{x}, t-\tau), \mathbf{x}, t), \end{aligned} \tag{5}$$

where $(\mathbf{x}, t) \in \Omega \times \mathbb{R}^+$. The non-Fickian model is very useful in many situations, such as biological research, polymers and medical applications, and in situations involving viscoelastic materials. We refer the interested reader to [12] for a survey of analysis and applications of this kind of models.

The goal of this paper is to solve (5) by using the Schwarz waveform relaxation (SWR) algorithm, which is iterative and combines the advantages of both the waveform relaxation methods [11], [17]–[20] and the domain decomposition method [28]. To use the SWR algorithm, we first decompose the spatial domain into several subdomains with small sizes; we then solve a series of time-dependent PDEs on these subdomains simultaneously or sequentially. These time-dependent PDEs are formed by using the same governing equation and the same initial value of the original PDE, but with carefully designed boundary conditions along the artificial boundaries. These boundary conditions are usually called *transmission conditions* (TCs), and they transmit information between neighboring subdomains via iterations. We refer to [13] and [14] for the original idea for the algorithms. The TCs have a significant influence on the convergence rate of the SWR algorithms and designing efficient TCs plays a central role in the research and application of this algorithm; see [2]–[4], [16] for systematic research on the issue (see also [8], [15], [25], [26], [32], [34], [35] for closely related work)). Among these studies, the Dirichlet and Robin type TCs attract considerable attention. The latter is more efficient than the former, but one needs to determine a free parameter. Such a parameter is key to the heuristic of the Robin TCs and therefore finding a good choice of this parameter is a top-priority matter. Determining a good Robin parameter is equivalent to solving a min-max problem, which varies for different time-dependent PDEs. Specifically, for the problems with non-Fickian flux and time-delay, the involved min-max problem is much more complicated than the one studied in [2]–[4], [6], [30], and [33], and finding the solution of this min-max problem is not covered in existing research literature.

The remainder of this paper is organized as follows. In Section 2, we introduce the SWR algorithm with the Robin TCs for linear non-Fickian reaction-diffusion equations with time delay. In Section 3, we present calculations for finding solutions for the min-max problem, which also involves determining the best paprameter for the Robin TCs. In Section 4, by adopting the linearization idea proposed recently by Cantano *et al.* [5], we generalize the results obtained in linear situations to nonlinear problems. Section 5 provides several numerical examples to verify our theoretical results, and we conclude this paper in Section 6.

II. THE SWR ALGORITHM WITH ROBIN TRANSMISSION CONDITIONS

We first consider the non-Fickian reaction-diffusion equations with time delay in 1-D:

$$\partial_t u - D_1 \partial_{xx} u - \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx} u(x,s)ds + au + bu(x,t-\tau) = g, \quad (6)$$

where $(x,t) \in \mathbb{R}\times(0,\infty)$ and a and b are two constants. We assume that the solution $u(x,t)$ of (6) is bounded at infinity and satisfies $u(x,t) = u_0(x,t)$ for $(x,t)\in\mathbb{R}\times[-\tau,0]$ with some known function $u_0(x,t)$. To describe the SWR algorithm, we decompose the spatial domain $\Omega = \mathbb{R}$ into two overlapping subdomains $\Omega_1 = (-\infty, L]$ and $\Omega_2 = [0, +\infty)$, where $L \geq 0$ denotes the overlap size. Then, the SWR algorithm consists of solving iteratively sub problems on $\Omega_j \times \mathbb{R}^+, j = 1, 2$, using as a boundary condition at the interfaces $x = 0$ and $x = L$ the values obtained from the previous iteration. The scheme at the k-th iteration is thus given by

$$\begin{cases} \partial_t u_j^k - D_1\partial_{xx}u_j^k - \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx}u_j^k(x,s)ds \\ \quad + au_j^k + bu_j^k(x,t-\tau) = g, & t>0, \\ u_j^k(x,t) = u_0(x,t), & t\in[-\tau,0], \\ \mathcal{B}_j u_j^k(L_j,t) = \mathcal{B}_j u_{3-j}^{k-1}(L_j,t), & t>0, \end{cases} \quad (7)$$

where $x \in \Omega_j$ (with $j = 1, 2$), $L_1 = L$, $L_2 = 0$ and $\{\mathcal{B}_j\}_{j=1,2}$ are linear operators in space and time, possibly pseudodifferential. There are many choices of $\{\mathcal{B}_j\}_{j=1,2}$, and in this paper we consider the Robin condition:

$$\mathcal{B}_1 = \partial_x + p, \quad \mathcal{B}_2 = \partial_x - p, \quad (8)$$

where p is a free parameter. Let e_k^j be the errors on subdomain Ω_j at iteration $k \geq 0$, i.e.,

$$e_1^k = u|_{\Omega_1} - u_1^k, \quad e_2^k = u|_{\Omega_2} - u_2^k. \quad (9)$$

Then, the homogeneous error equations for the SWR iterations (7)-(8) are

$$\begin{cases} \partial_t e_j^k - D_1\partial_{xx}e_j^k - \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx}e_j^k(x,s)ds \\ \quad + ae_j^k + be_j^k(x,t-\tau) = 0, & t>0, \\ e_j^k(x,t) = 0, & t\in[-\tau,0], \\ \left(\partial_x + (-1)^{1+j}p\right)e_j^k(L_j,t) = \left(\partial_x + (-1)^{1+j}p\right)e_{3-j}^{k-1}(L_j,t), \\ & t>0. \end{cases} \quad (10)$$

Note that, the term $\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx}e_j^k(x,s)ds$ can be written as a convolution formula $\exp\left(-\frac{t}{\delta}\right) * e_j^k(x,t)$, where $j = 1, 2$ and we have extended $e_j^k(x,t) = 0$ for $t \leq -\tau$ and we denote the extension by e_j^k, too. For any function $V(t)$, we denote its Fourier transformation $\hat{V}(\omega) := \mathcal{F}(V(t))$ by

$$\mathcal{F}(V(t)) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} V(t)\exp(-i\omega t)dt.$$

It is easy to get

$$\mathcal{F}\left(\exp\left(-\frac{t}{\delta}\right) * e_j^k(x,t)\right) = \mathcal{F}\left(\exp\left(-\frac{t}{\delta}\right)\right)\mathcal{F}\left(e_j^k(x,t)\right) = \frac{1}{1/\delta + i\omega}\hat{e}_j^k(x,\omega), \quad j = 1, 2.$$

Performing the Fourier transform in time of the error equations (10), we arrive at

$$\begin{cases} \partial_{xx}\hat{e}_1^k(x,\omega) - \dfrac{a+i\omega+b\exp(-i\omega\tau)}{D_1+\frac{D_2}{i\omega\delta+1}}\hat{e}_k^1(x,\omega)=0, \\ (\partial_x+p)\,\hat{e}_k^1(L,\omega)=(\partial_x+p)\,\hat{e}_{k-1}^2(L,\omega), \end{cases}$$
$$\begin{cases} \partial_{xx}\hat{e}_2^k(x,\omega) - \dfrac{a+i\omega+b\exp(-i\omega\tau)}{D_1+\frac{D_2}{i\omega\delta+1}}\hat{e}_k^2(x,\omega)=0, \\ (\partial_x-p)\,\hat{e}_k^2(0,\omega)=(\partial_x-p)\,\hat{e}_{k-1}^1(0,\omega). \end{cases} \quad (11)$$

Thus, we need to solve an ordinary differential equation in each subdomain. The roots of the corresponding characteristic polynomial are

$$\lambda^+ = \sqrt{\frac{a+i\omega+b\exp(-i\omega\tau)}{D_1+\frac{D_2}{i\omega\delta+1}}},$$
$$\lambda^- = -\sqrt{\frac{a+i\omega+b\exp(-i\omega\tau)}{D_1+\frac{D_2}{i\omega\delta+1}}}.$$

Let

$$\begin{aligned} &A_1(\omega)=a+b\cos(\omega\tau),\ A_2(\omega)=D_2+D_1\left[1+(\omega\delta)^2\right], \\ &B_1(\omega)=\omega-b\sin(\omega\tau), B_2(\omega)=\omega\delta D_2, \\ &C(\omega)=(\omega\delta D_1)^2+(D_1+D_2)^2. \end{aligned} \quad (12a)$$

Then, routine calculations yield

$$\begin{aligned} \lambda^+ &= \frac{\sqrt{[A_1(\omega)+iB_1(\omega)][A_2(\omega)+iB_2(\omega)]}}{\sqrt{C(\omega)}} \\ &= \frac{\zeta(\omega)+i\sigma\psi(\omega)}{\sqrt{C(\omega)}}, \\ \lambda^- &= -\frac{\sqrt{[A_1(\omega)+iB_1(\omega)][A_2(\omega)+iB_2(\omega)]}}{\sqrt{C(\omega)}} \\ &= -\frac{\zeta(\omega)+i\sigma\psi(\omega)}{\sqrt{C(\omega)}}, \end{aligned} \quad (12b)$$

where

$$\begin{aligned} \zeta(\omega) &= \sqrt{\frac{\sqrt{\mathcal{A}^2(\omega)+\mathcal{B}^2(\omega)}+\mathcal{A}(\omega)}{2}}, \\ \psi(\omega) &= \sqrt{\frac{\sqrt{\mathcal{A}^2(\omega)+\mathcal{B}^2(\omega)}-\mathcal{A}(\omega)}{2}}, \\ \mathcal{A}(\omega) &= A_1(\omega)A_2(\omega)-B_1(\omega)B_2(\omega), \\ \mathcal{B}(\omega) &= A_1(\omega)B_2(\omega)+A_2(\omega)B_1(\omega), \quad \sigma=\text{sign}(\mathcal{B}(\omega)). \end{aligned} \quad (12c)$$

By using $\Re(\lambda^+)\geq 0$ and $\Re(\lambda^-)\leq 0$, the solutions of (11), which are bounded at infinity, are

$$\begin{cases} \hat{e}_k^1(x,\omega)=\alpha_k(\omega)e^{\lambda^+(x-L)}, & \text{for } (x,\omega)\in(-\infty,L)\times\mathbb{R}, \\ \hat{e}_k^2(x,\omega)=\beta_k(\omega)e^{\lambda^- x}, & \text{for } (x,\omega)\in(0,+\infty)\times\mathbb{R}, \end{cases}$$

where $\alpha_k(\omega)$ and $\beta_k(\omega)$ will be computed with the boundary conditions on $x=L$ and $x=0$:

$$\alpha_k(\omega)=\frac{(\lambda_-+p)\,e^{\lambda_- L}}{\lambda_++p}\beta_{k-1}(\omega),$$
$$\beta_k(\omega)=\frac{(\lambda_+-p)\,e^{-\lambda_+ L}}{\lambda_--p}\alpha_{k-1}(\omega).$$

Hence, the errors $\hat{e}_k^j(x,\omega)$ $(j=1,2)$ satisfy $\hat{e}_k^j(x,\omega)=\frac{(\lambda_-+p)(\lambda_+-p)}{(\lambda_++p)(\lambda_--p)}e^{(\lambda_--\lambda_+)L}\hat{e}_{k-2}^j(x,\omega)$, where $j=1,2$. This relation, together with the well-known Parseval–Plancherel identity, gives

$$\|e_k^1\|_{L^2}\leq\rho(p)\|e_{k-2}^1\|_{L^2}, \quad \|e_k^2\|_{L^2}\leq\rho(p)\|e_{k-2}^2\|_{L^2}, \quad (13)$$

where $\rho(p)$ is the convergence factor of the SWR algorithms and will be defined later.

We note that in a numerical computation, a numerical grid in time with spacing Δt cannot carry arbitrary high frequencies, which implies that the quantity ω in λ_- and λ_+ cannot vary from $-\infty$ to $+\infty$. An estimate of the highest frequency is $\omega_{\max}=\frac{\pi}{\Delta t}$. Based on this consideration, we define the convergence factor $\rho(p)$ in (13) by

$$\rho(p)=\max_{\omega\in[-\omega_{\max},\omega_{\max}]}\left|\frac{(\lambda_-+p)\,(\lambda_+-p)}{(\lambda_++p)\,(\lambda_--p)}e^{(\lambda_--\lambda_+)L}\right|. \quad (14)$$

Hence, the best constant p involved in the transmission condition of the Robin type can be determined by the following min-max problem:

$$\begin{aligned} &\min_{p>0}\max_{\omega\in[-\omega_{\max},\omega_{\max}]}\left|\frac{(\lambda_-+p)\,(\lambda_+-p)}{(\lambda_++p)\,(\lambda_--p)}e^{(\lambda_--\lambda_+)L}\right| \\ &\qquad=\min_p\max_{\omega\in[-\omega_{\max},\omega_{\max}]}\left|\frac{(\lambda_+-p)^2}{(\lambda_++p)^2}e^{-2\lambda_+L}\right|. \end{aligned} \quad (15)$$

The negative values of p can be excluded, by noting that $\rho(p)\leq\rho(-p)$ if $p>0$.

Remark 1: The TCs in the error equations (10) may be equivalently written as

$$\left(\frac{\partial_x}{p}+1\right)e_1^k(L,t)=\left(\frac{\partial_x}{p}+1\right)e_2^{k-1}(L,t),$$
$$\left(\frac{\partial_x}{p}-1\right)e_2^k(0,t)=\left(\frac{\partial_x}{p}-1\right)e_1^{k-1}(0,t).$$

By letting $p\to\infty$ we have

$$e_1^k(L,t)=e_2^{k-1}(L,t), \quad e_2^k(0,t)=e_1^{k-1}(0,t),$$

which corresponds to the error equations for the SWR iteration (7)-(8) with Dirichlet transmission conditions $\mathcal{B}_1=\mathcal{B}_2=\mathcal{I}$, where $\mathcal{I}$ denotes the identity operator. Hence, by letting $p\to\infty$ in (14) we get the convergence factor of the SWR algorithms with Dirichlet transmission conditions $\rho_{Dir}:=\max_{\omega\in[-\omega_{\max},\omega_{\max}]}\left|e^{-2\lambda_+L}\right|=\max_{\omega\in[-\omega_{\max},\omega_{\max}]}\left|e^{-2\frac{\zeta(\omega)}{\sqrt{C(\omega)}}L}\right|$. Clearly, $\rho_{Dir}=1$ if $L=0$. This means that $L>0$ is a necessary condition to guarantee the convergence of the SWR algorithms with Dirichlet transmission conditions. However, as we will show in the next section, for $L=0$ the SWR algorithms with Robin transmission conditions converge with satisfactory convergence rates, provided the parameter p is properly chosen.

Remark 2: Let

$$\hat{\rho}(p,\omega)=\left|\frac{(\lambda_+-p)^2}{(\lambda_++p)^2}e^{-2\lambda_+L}\right|, \quad (16)$$

where λ_+ is a function of ω and is defined by (12a)-(12c). For regular reaction diffusion equations, i.e., $D_2 = 0$ and $b = 0$ in((6)), the term λ_+ becomes $\lambda_+ = \sqrt{(a+i\omega)/D_1}$. In this case, the calculations for finding the solution of the min-max problem (15) is relatively easy, because $\rho(p,\omega)$ has at most one local maximum for a given p (see Figure 1 on the top). The solution of the min-max problem (15) in the case $D_2 = 0$ and $b = 0$ is systematically addressed by Gander and his colleagues, see [2], [15]. However, if $D_2 \neq 0$ and $b \neq 0$, the function $\hat{\rho}(p,\omega)$ may have numerous maximums for a given p and this feature, as shown in Figure 1 on the bottom, makes the calculations for finding the solution of (15) extremely difficult and substantially different from existing solutions.

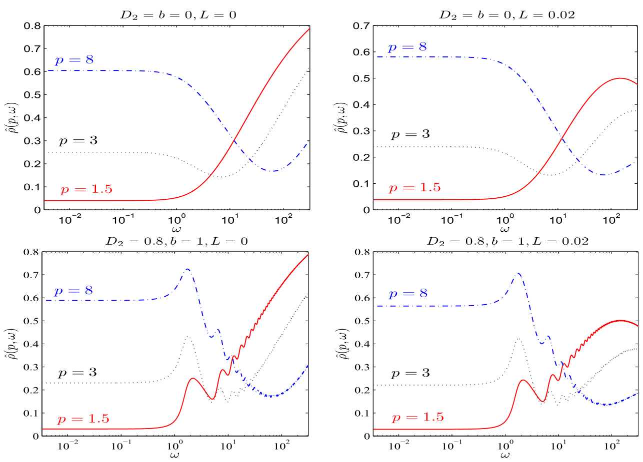

FIGURE 1. Comparison of $\hat{\rho}(p,\omega)$ between two definitions of λ_+: $\lambda_+ = \sqrt{(a+i\omega)/D_1}$ corresponds to regular reaction diffusion equation (top) and $\lambda^+ = (\zeta(\omega) + i\sigma\psi(\omega))/\sqrt{C(\omega)}$ corresponds to the non-Fickian problem with time-delay (bottom). For the latter, a complete formula of λ_+ is given by (12b)-(12c). Here, $D_1 = 1$, $\Delta t = 0.01$, $\delta = 0.8$, $a = 1$ and $\tau = 1.3$.

III. AN APPROXIMATE SOLUTION OF THE MIN-MAX PROBLEM

It is difficult to solve the min-max problem (15) for $L > 0$, because the exponential term $e^{-2\lambda_+ L}$ adds additional complexity for calculating the maximums of $\left|\frac{(\lambda_+-p)^2}{(\lambda_++p)^2}e^{-2\lambda_+ L}\right|$ for $|\omega| \leq \omega_{\max}$. For $L = 0$, which corresponds to the nonoverlapping case, it is much easier to analyze the min-max problem (15), but it is still difficult to derive a closed formula for the optimal parameter p_{opt}, because, as shown in the bottom-left panel of Figure 1, the function $\left|\frac{(\lambda_+-p)^2}{(\lambda_++p)^2}\right|$ has many local maximums in the relevant interval $|\omega| \in [0, \omega_{\max}]$. In what follows, we try to derive an approximate solution of the min-max problem (15) with $L = 0$. Our numerical results, in Section 5, indicate that the derived approximate solution predicts the best choice that one can make in the fully discretized algorithm. The idea supporting this goal consists of two parts: constructing a sharp upper function of $\rho(p)$ and then minimizing such an upper function instead of $\rho(p)$.

Lemma 1: Let

$$\alpha = \min_{\omega\in[0,\omega_{\max}]} \frac{(a+b\cos(\omega\tau))\left(D_2+D_1\left[1+(\omega\delta)^2\right]\right)-\omega\delta D_2[\omega-b\sin(\omega\tau)]}{(\omega\delta D_1)^2+(D_1+D_2)^2},$$

$$\eta_0 = \min_{\omega\in[0,\omega_{\max}]} \frac{\zeta(\omega)}{\sqrt{C(\omega)}},$$

$$\eta_1 = \max_{\omega\in[0,\omega_{\max}]} \frac{\zeta(\omega)}{\sqrt{C(\omega)}}$$

and $\mathcal{R}(\eta,p,\alpha) = \frac{(\eta-p)^2+\eta^2-\alpha}{(\eta+p)^2+\eta^2-\alpha}$. Then, for $L = 0$, it holds that

$$\rho(p) \leq \max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta,p,\alpha), \quad \forall p > 0. \tag{17}$$

Proof: For $L = 0$, the convergence factor $\rho(p)$ defined by (14) can be rewritten as

$$\begin{aligned}\rho(p) &= \max_{\omega\in[-\omega_{\max},\omega_{\max}]} \left|\frac{\left[\frac{\zeta(\omega)}{\sqrt{C(\omega)}} + i\sigma\frac{\psi(\omega)}{\sqrt{C(\omega)}} - p\right]^2}{\left[\frac{\zeta(\omega)}{\sqrt{C(\omega)}} + i\sigma\frac{\psi(\omega)}{\sqrt{C(\omega)}} + p\right]^2}\right| \\ &= \max_{\omega\in[-\omega_{\max},\omega_{\max}]} \frac{\left(\frac{\zeta(\omega)}{\sqrt{C(\omega)}} - p\right)^2 + \frac{\psi^2(\omega)}{C(\omega)}}{\left(\frac{\zeta(\omega)}{\sqrt{C(\omega)}} + p\right)^2 + \frac{\psi^2(\omega)}{C(\omega)}},\end{aligned} \tag{18}$$

where $\zeta(\omega)$ and $\psi(\omega)$ are defined by (12c). Since $\frac{\psi^2(\omega)}{C(\omega)} = \frac{\zeta^2(\omega)}{C(\omega)} - \frac{\mathcal{A}(\omega)}{C(\omega)}$, from (18) we have

$$\rho(p) = \max_{\omega\in[-\omega_{\max},\omega_{\max}]} \frac{\left(\frac{\zeta(\omega)}{\sqrt{C(\omega)}} - p\right)^2 + \frac{\zeta^2(\omega)}{C(\omega)} - \frac{\mathcal{A}(\omega)}{C(\omega)}}{\left(\frac{\zeta(\omega)}{\sqrt{C(\omega)}} + p\right)^2 + \frac{\zeta^2(\omega)}{C(\omega)} - \frac{\mathcal{A}(\omega)}{C(\omega)}}.$$

Let $\alpha_0 = \min_{\omega\in[-\omega_{\max},\omega_{\max}]} \frac{\mathcal{A}(\omega)}{C(\omega)}$ and $\alpha_1 = \max_{\omega\in[-\omega_{\max},\omega_{\max}]} \frac{\mathcal{A}(\omega)}{C(\omega)}$. Then it is easy to get $\alpha_0 = \alpha$ and

$$\rho(p) \leq \bar{\rho}(p), \tag{19}$$

where

$$\bar{\rho}(p) = \max_{\eta\in[\eta_0,\eta_1], s\in[\alpha_0,\alpha_1]} \frac{(\eta-p)^2+\eta^2-s}{(\eta+p)^2+\eta^2-s}. \tag{20}$$

For any $p > 0$ and $\eta \geq \eta_0 \geq 0$, it holds that $\partial_s\left(\frac{(\eta-p)^2+\eta^2-s}{(\eta+p)^2+\eta^2-s}\right) = \frac{-4\eta p}{[(\eta+p)^2+\eta^2-s]^2} \leq 0$, and this implies

$$\begin{aligned}&\max_{\eta\in[\eta_0,\eta_1], s\in[\alpha_0,\alpha_1]} \frac{(\eta-p)^2+\eta^2-s}{(\eta+p)^2+\eta^2-s} \\ &\leq \max_{\eta\in[\eta_0,\eta_1]} \frac{(\eta-p)^2+\eta^2-\alpha_0}{(\eta+p)^2+\eta^2-\alpha_0} \\ &= \max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta,p,\alpha).\end{aligned} \tag{21}$$

Combining (19) and (21) gives (17). □

In what follows, we try to solve the following min-max problem

$$\min_{p>0} \max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta,p,\alpha), \tag{22}$$

and the solution can be regarded as an approximate solution (15).

Remark 3: For regular reaction diffusion equations without time-delay, i.e., $D_2 = b = 0$ in (6), it is easy to get $\eta_0 = \sqrt{a/D_1}$ and $\alpha = \eta_0^2$. In this situation, the min-max problem (22) becomes $\min_{p>0} \max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta, p, \eta_0^2)$, which is already solved in [15]. However, for the non-Fickian reaction diffusion equations with time-delay, we have $\alpha \leq \eta_0^2$. This can be seen by noticing that $\frac{\zeta^2(\omega)}{C(\omega)} - \frac{\mathcal{A}(\omega)}{C(\omega)} = \frac{\psi^2(\omega)}{C(\omega)} \geq 0$, which gives $\min_{\omega\in[0,\omega_{\max}]} \frac{\zeta^2(\omega)}{C(\omega)} \geq \min_{\omega\in[0,\omega_{\max}]} \frac{\mathcal{A}(\omega)}{C(\omega)}$, i.e., $\eta_0^2 \geq \alpha$. This implies that (22) is a more general min-max problem than the one studied in [15].

The following theorem gives the solution of the min-max problem (22), which can be used to determine a good Robin parameter for the SWR algorithm.

Theorem 1: For $L = 0$ and $\eta_0 > 0$, the solution of the min-max problem (22) is

$$p^* = \sqrt{2\eta_0\eta_1 + \alpha}, \tag{23}$$

if $\eta_0^2 - \eta_0\eta_1 - \alpha \leq 0$ and $\eta_1^2 - \eta_0\eta_1 - \alpha \geq 0$; otherwise,

$$p^* = \begin{cases} \sqrt{2\eta_0^2 - \alpha}, \\ \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) \geq \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right), \\ \sqrt{2\eta_1^2 - \alpha}, \\ \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) < \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right), \end{cases} \tag{24}$$

where η_0, η_1, α and $\mathcal{R}$ are given by Lemma 1. With p^*, the convergence factor of the SWR algorithm (7)-(8) applied to the non-Fickian delay equations (6) can be bounded by

$$\rho(p^*) \leq \begin{cases} \frac{(\eta_0 - p^*)^2 + \eta_0^2 - \alpha}{(\eta_0 + p^*)^2 + \eta_0^2 - \alpha}, \\ \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) \geq \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right) \\ \text{and } \eta_0^2 - \eta_0\eta_1 - \alpha > 0 \text{ or } \eta_1^2 - \eta_0\eta_1 - \alpha < 0, \\ \frac{(\eta_1 - p^*)^2 + \eta_1^2 - \alpha}{(\eta_1 + p^*)^2 + \eta_1^2 - \alpha}, \quad \text{otherwise}. \end{cases} \tag{25}$$

Proof: For any $\eta \in [\eta_0, \eta_1]$, we have

$$\frac{\partial\mathcal{R}(\eta, p, \alpha)}{\partial p} = 4\eta\frac{p^2 - 2\eta^2 + \alpha}{[(\eta + p)^2 + \eta^2 - \alpha]^2}. \tag{26}$$

Hence, the best parameter p^* shall satisfy $\sqrt{2\eta_0^2 - \alpha} \leq p^* \leq \sqrt{2\eta_1^2 - \alpha}$. Otherwise, if $p \in \left(0, \sqrt{2\eta_0^2 - \alpha}\right)$, we have $\partial_p\mathcal{R}(\eta, p, \alpha) < 0$ and therefore increasing p can uniformly decrease $\mathcal{R}$. Similarly, if $p > \sqrt{2\eta_1^2 - \alpha}$, we have $\partial_p\mathcal{R}(\eta, p, \alpha) > 0$ and thus decreasing p can also uniformly decrease $\mathcal{R}$. Now, for any $p \in \left[\sqrt{2\eta_0^2 - \alpha}, \sqrt{2\eta_1^2 - \alpha}\right]$ we state the following three points:

(a) *$\mathcal{R}$ does not have a local maximum for $\eta \in [\eta_0, \eta_1]$.* By the contrary, we suppose there exists some $\eta^* \in (\eta_0, \eta_1)$ such that $\frac{\partial\mathcal{R}(\eta^*,p,\alpha)}{\partial\eta} = 0$, i.e., $\eta^* = \sqrt{\frac{p^2-\alpha}{2}}$, since $\frac{\partial\mathcal{R}(\eta,p,\alpha)}{\partial\eta} = 4p\frac{2\eta^2-p^2+\alpha}{[(\eta+p)^2+\eta^2-\alpha]^2}$. A routine calculation yields $\frac{\partial^2\mathcal{R}(\eta,p,\alpha)}{\partial\eta^2} = 16p\frac{\mathcal{D}(\eta)}{[(\eta+p)^2+\eta^2-\alpha]^3}$, where

$$\mathcal{D}(\eta) = -2\eta^3 + 3(p^2 - \alpha)\eta + p(p^2 - \alpha).$$

It is easy to get

$$\begin{aligned} \mathcal{D}(\eta^*) &= -2\left(\sqrt{\frac{p^2-\alpha}{2}}\right)^3 + 3(p^2 - \alpha)\left(\sqrt{\frac{p^2-\alpha}{2}}\right) \\ &\quad + p(p^2 - \alpha) \\ &= (p^2 - \alpha)\left(2\sqrt{\frac{p^2-\alpha}{2}} + p\right). \end{aligned} \tag{27}$$

Moreover, by using the assumption $\eta^* \in (\eta_0, \eta_1)$ we have $\sqrt{\frac{p^2-\alpha}{2}} > \eta_0$, i.e., $p^2 > 2\eta_0^2 + \alpha$, which gives $p^2 - \alpha > 0$ since $\eta_0 > 0$. By (27) we get $\mathcal{D}(\eta^*) > 0$ and this implies $\frac{\partial^2\mathcal{R}(\eta,p,\alpha)}{\partial\eta^2}\Big|_{\eta=\eta^*} > 0$. Hence, $\eta = \eta^*$ is not a local maximum of $\mathcal{R}$, but a local minimum. Therefore,

$$\min_{p>0}\left(\max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta, p, \alpha)\right) = \min_{p>0}\{\mathcal{R}(\eta_0, p, \alpha), \mathcal{R}(\eta_1, p, \alpha)\}.$$

(b) *$\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ have a unique intersection point at $p = \bar{p}^* = \sqrt{2\eta_0\eta_1 + \alpha}$, if $2\eta_0\eta_1 + \alpha \geq 2\eta_0^2 - \alpha$, i.e., $\eta_0^2 - \eta_0\eta_1 - \alpha \leq 0$.* With (26), this result can be verified directly.

(c) *$\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ are respectively increasing and decreasing functions of p.* This is also deduced from (26), together with some routine calculations.

Based on the conclusions (a)-(c) stated above, we next consider the following three cases.

Case 1: $\eta_0^2 - \eta_0\eta_1 - \alpha > 0$. In this case, $\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ do not intersect. Hence, for $p \in \left[\sqrt{2\eta_0^2 - \alpha}\sqrt{2\eta_1^2 - \alpha}\right]$ by using the monotonicity of $\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ we have

$$\begin{cases} \mathcal{R}(\eta_0, p, \alpha) \leq \mathcal{R}(\eta_1, p, \alpha), \\ \quad \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) < \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right), \\ \mathcal{R}(\eta_0, p, \alpha) \geq \mathcal{R}(\eta_1, p, \alpha), \\ \quad \text{otherwise}. \end{cases}$$

Therefore, for any $p \in \left[\sqrt{2\eta_0^2 - \alpha}, \sqrt{2\eta_1^2 - \alpha}\right]$ it holds that

$$\max_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta, p, \alpha) = \begin{cases} \mathcal{R}(\eta_1, p, \alpha), & \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) \\ & \quad < \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right), \\ \mathcal{R}(\eta_0, p, \alpha), & \text{otherwise}. \end{cases}$$

This implies that the solution of the min-max problem (22) is

$$p^* = \begin{cases} \sqrt{2\eta_1^2-\alpha}, & \text{if } \mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) \\ & \quad < \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right), \\ \sqrt{2\eta_0^2-\alpha}, & \text{otherwise.} \end{cases}$$

Case 2: $\eta_0^2 - \eta_0\eta_1 - \alpha \le 0$ *and* $\bar{p}^* > \sqrt{2\eta_1^2-\alpha}$ *(i.e.,* $\eta_1^2 - \eta_0\eta_1 - \alpha < 0$*).* In this case, $\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ intersect at $p = \bar{p}^*$, which satisfies $\bar{p}^* > \sqrt{2\eta_1^2-\alpha}$. Therefore, there are only two situations to be considered depending on $\mathcal{R}\left(\eta_0, \sqrt{2\eta_0^2-\alpha}, \alpha\right) < \mathcal{R}\left(\eta_1, \sqrt{2\eta_0^2-\alpha}, \alpha\right)$ or not. The calculation is therefore similar to *Case 1*, and we omit it.

Case 3: $\eta_0^2 - \eta_0\eta_1 - \alpha \le 0$ *and* $\bar{p}^* \le \sqrt{2\eta_1^2-\alpha}$ *(i.e.,* $\eta_1^2 - \eta_0\eta_1 - \alpha \ge 0$ *).* In this case, the unique intersection point $\bar{p}^*$ satisfies $\bar{p}^* \in \left[\sqrt{2\eta_0^2-\alpha}, \sqrt{2\eta_1^2-\alpha}\right]$. Hence, it is clear that the solution of the min-max problem (22) is the point where $\mathcal{R}(\eta_0, p, \alpha)$ and $\mathcal{R}(\eta_1, p, \alpha)$ are balanced. Moreover, we have $\max\limits_{\eta\in[\eta_0,\eta_1]} \mathcal{R}(\eta, p^*, \alpha) = \mathcal{R}(\eta_0, p^*, \alpha) = \mathcal{R}(\eta_1, p^*, \alpha)$.

Through an analysis of *Case 3*, we get (23), which together with *Cases 1* and *2* gives (24). The bound given by (25) can be deduced from the above analysis. □

IV. APPLICATION TO NONLINEAR NON-FICKIAN DELAY PROBLEMS

For the linear problem (6), we can denote the Robin parameter by

$$p = \mathcal{P}(D_1, D_2, \delta, a, b, \tau, \Delta t), \tag{28}$$

where the formula $\mathcal{P}$ is given by (23)-(24). The derivation of this formula essentially depends on the linear form of the underlying equations, which permits us to perform a Fourier analysis for the error equations of the SWR algorithm. For nonlinear problems, such as

$$\partial_t u = D_1 \partial_{xx} u + \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx} u(x,s)ds + f(u, u(x,t-\tau), x, t), \tag{29}$$

a Fourier analysis is not applicable and therefore we cannot get a parameter formula as given by (28). However, the parameter formula (28) can be used *adaptively* in the nonlinear situation. The idea lies in a *linearization* procedure for the error equations, which is proposed by Caetano *et al.* [5] in the study of CO_2 geological storage simulation by the SWR algorithm.

Precisely, similar to (10) the error equations of the SWR algorithm applied to (29) are

$$\partial_t e_j^k - D_1\partial_{xx} e_j^k - \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx} e_j^k(x,s)ds - \left[f(u_j, u_j(x,t-\tau), x, t) - f\left(u_j^k, u_j^k(x,t-\tau), x, t\right)\right] = 0.$$

Linearizing the nonlinear term $f(u(x,t), u(x,t-\tau), x, t)$ around $u_j^k(x,t)$ and $u_j^k(x,t-\tau)$ gives

$$\begin{aligned} &f(u_j(x,t), u_j(x,t-\tau), x, t) - f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right) \\ &\simeq e_j^k(x,t)\partial_1 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right) \\ &\quad + e_j^k(x,t-\tau)\partial_2 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right), \end{aligned}$$

where for any u and v the partial derivatives $\partial_1 f(u,v,x,t)$ and $\partial_2 f(u,v,x,t)$ are defined by

$$\partial_1 f(u,v,x,t) = \frac{\partial f(u,v,x,t)}{\partial u}, \quad \partial_2 f(u,v,x,t) = \frac{\partial f(u,v,x,t)}{\partial v}.$$

We therefore get a linear approximation of the error equation:

$$\begin{aligned} &\partial_t e_j^k - D_1\partial_{xx} e_j^k - \frac{D_2}{\delta}\int_0^t \exp\left(-\frac{t-s}{\delta}\right)\partial_{xx} e_j^k(x,s)ds \\ &\quad - e_j^k\partial_1 f\left(u_j^k, u_j^k(x,t-\tau), x, t\right) \\ &\quad - e_j^k(x,t-\tau)\partial_2 f\left(u_j^k, u_j^k(x,t-\tau), x, t\right) \simeq 0. \end{aligned}$$

From this, we see that the parameter p in the Robin transmission condition can be determined by substituting $a = -\partial_1 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right)$ and $b = -\partial_2 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right)$ in the parameter formula (28) derived for the linear model, that is

$$\begin{aligned} p = \mathcal{P}\Big(&D_1, D_2, \delta, -\partial_1 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right), \\ &- \partial_2 f\left(u_j^k(x,t), u_j^k(x,t-\tau), x, t\right), \tau, \Delta t\Big). \end{aligned} \tag{30}$$

The parameter formula (30) depends on the iteration index k and the space point x and the time point t. Therefore, the Robin parameter for the SWR algorithm is determined adaptively along the artificial boundaries for each time point.

V. NUMERICAL RESULTS

In this section, we do numerical experiments to check the performance of the SWR algorithms using the Robin parameter derived in Sections 3 and 4. We also show numerical results to compare the convergence rate of the SWR algorithm using the Dirichlet and Robin transmission conditions respectively. To this end, throughout this section we use a minimal overlap size, $L = \Delta x$, for the SWR algorithm with Dirichlet transmission conditions, because this algorithm does not converge without overlap (see Remark 1). For the SWR algorithm with Robin transmission conditions, we decompose the spatial domain without overlap, i.e., $L = 0$. For discretizations, we use a centered finite difference method in space and a backward Euler method in time. The fully discretized version of the SWR algorithm (7)-(7) is

$$\begin{aligned} u^k_{j,n+1,m} &= u^k_{j,n,m} + \Delta t D_1 \nabla^2_x u^k_{j,n+1,m} \\ &\quad + (\Delta t)^2 \frac{D_2}{\delta} \sum_{l=1}^{n+1} \exp\left(-\frac{t_{n+1}-t_l}{\delta}\right) \nabla^2_x u^k_{j,l,m} \\ &\quad + \Delta t f\left(u^k_{j,n+1,m}, u^k_{j,n+1-n_0,m}, x_m, t_{n+1}\right), \\ &\qquad j = 1, 2, \end{aligned} \tag{31}$$

where j denotes the index of the subdomain, $t_n = n\Delta t$, $x_m = m\Delta x$, $n_0 = \frac{\tau}{\Delta t}$ is a positive integer and ∇^2_x denotes the central finite difference operator

$$\nabla^2_x u^k_{j,l,m} = \frac{u^k_{j,l,m-1} - 2u^k_{j,l,m} + u^k_{j,l,m+1}}{(\Delta x)^2}.$$

For all the experiments in this section, the SWR algorithm starts with a random initial iterate.

Example 1 (Linear Model Problem): In the first example, we apply the SWR algorithms to the linear model problem (6) for $(x, t) \in (0, 2) \times (0, 5)$ with homogeneous initial and boundary conditions $u(0, t) = u(2, t) = 0$ and $u_0(x, t) = 0$ and source function $g(x, t) = 100 \sin\left(x^3 t^{10}\right)$. The space and time discretization parameters are chosen as $\Delta t = \Delta x = 0.02$. We choose for the problem parameters $a = 4$, $b = -2$, $D_1 = 4$, $D_2 = 1$, $\delta = 0.1$ and $\tau = 2$.

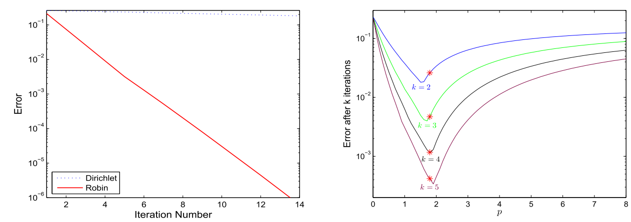

FIGURE 2. **Left: comparison between the convergence rates of the SWR algorithms with the two types of transmission conditions. Right: the errors obtained running the algorithm with Robin transmission conditions after k iterations and various choices of the free parameters p, and indicated by a star for the choice $p = p^*$ from Theorem 1.**

In Fig.2 on the left, we show the convergence rate of the SWR algorithms with the two types of transmission conditions, Dirichlet and Robin. The convergence rate is measured by the error between the domain decomposition solution and the numerical solution computed in the global domain by using the same discretization. We see clearly in the left subfigure that the parameter p^* given by Theorem 1 can drastically improve the performance of the algorithms. In Fig. 2 on the right, we verify to what degree the parameter $p = p^*$ corresponds to the best choice that one can make in the fully discretized implementation. we precisely show the error obtained after running the SWR algorithm after k iterations using various values for the Robin parameter p in the transmission condition. The choice $p = p^*$ is indicated by a star. One can see in this subfigure that the parameter $p = p^*$ given by Theorem 1 successfully predicts the best-one.

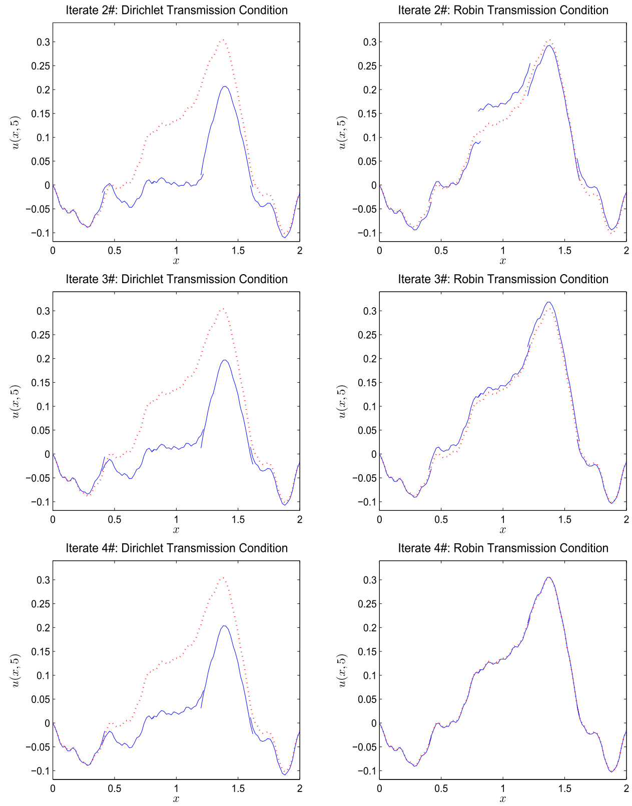

FIGURE 3. **From top to bottom, the second, third and fourth iterates $\{u^k_j(x, T)\}^5_{j=1}$ of the SWR algorithm (in the 5 subdomain case) at the end of the time interval $T = 5$ (solid lines), together with the exact solution (dot). Left column: the Dirichlet transmission condition; Right column: the Robin transmission condition.**

Example 2 (Experiments With Many Subdomains): We now show experiments which indicate that our results for two subdomains are also useful in the case of multiple subdomains. Using the same model problem as in 1, we now decompose the domain into 5 subdomains. In Figure 3 on the left and right, we show the second, third and fourth iterates at the end of the time interval of the Schwarz waveform relaxation algorithm with Dirichlet and Robin TCs. For the Robin TCs, the parameter p is given by Theorem 1. This clearly shows how important the transmission conditions are in the multiple subdomain case.

In Figure 4 on the left, we show the measured convergence rates of the SWR algorithm using the two types of transmission conditions, where we see clearly that the parameter analyzed in the 2 subdomain case is also very efficient in the case of multiple subdomains. In the right subfigure, similar to the two-subdomain case, we show the error obtained after running the SWR algorithm after k iterations using various values for the Robin parameter p in the transmission condition. The choice $p = p^*$ is indicated by a star and again, one can see that the parameter $p = p^*$ given by Theorem 1 successfully predicts the best one.

Example 3 (Nonlinear Model): At the end of this section, we present numerical experiments for nonlinear non-Fickian

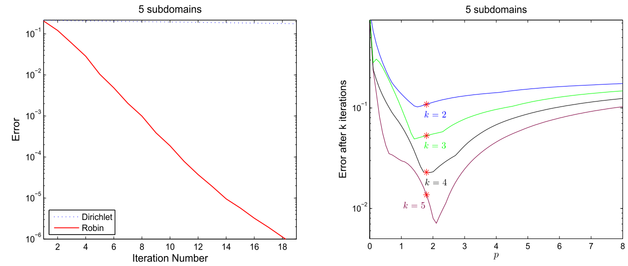

FIGURE 4. Left: comparison of the convergence rates of the SWR algorithm using two types of transmission conditions in the case of 5 subdomains. Right: the errors obtained running the SWR algorithm after *k* iterations and various choices of the free Robin parameters *p* in the transmission conditions, and indicated by a star the choice $p = p^*$ from Theorem 1.

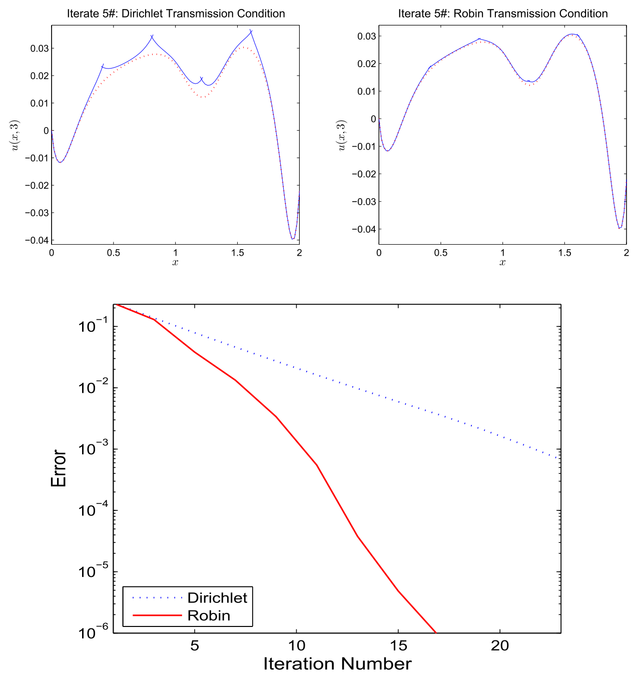

FIGURE 5. Top: the fifth iterate $\{u_j^5(x,T)\}_{j=1}^5$ (solid lines) of the SWR algorithm using the two types of transmission conditions at the end of the time interval $T = 3$, together with the exact solution (dot). Bottom: comparison of the convergence rates of the algorithm using these two types of transmission conditions.

Nicholson's blowflies equation

$$\partial_t u = D_1 \partial_{xx} u + \frac{D_2}{\delta} \int_0^t \exp\left(-\frac{t-s}{\delta}\right) \partial_{xx} u(x,s) ds \\ -au + bu(x, t-\tau)e^{-cu(x,t-\tau)}, \quad (32)$$

with $(x,t) \in (0,2) \times (0,3)$ and homogeneous boundary conditions $u(0,t) = u(2,t) = 0$. The initial function is chosen as

$$u(x,t) = \begin{cases} \exp\left(\cos\left(t^3\right)\sin\left(x^2\right)\right), & (x,t) \in (0,1] \times [-\tau, 0], \\ \exp\left(\sin\left(x^3\right)\cos\left(t^2\right)\right), & (x,t) \in (1,2) \times [-\tau, 0]. \end{cases}$$

We decompose the whole spatial space $\Omega = [0,2]$ into 5 overlapping subdomains and choose for problem parameters $a = 1.5$, $b = \exp(-3.5)$, $c = 0.55$, $D_1 = 0.01$, $D_2 = 0.02$, $\delta = 10$ and $\tau = 1$. Then, we plot in Figure 5 on the top the fifth iterate of the SWR algorithm with Dirichlet (left subfigure) and Robin (right subfigure), transmission conditions. The fifth iterate and the converged solution are denoted by a solid line and a dotted line, respectively. A complete comparison for the convergence rate of the algorithm using these two types of transmission conditions is shown in Figure 5 on the bottom, and we see that by using the parameter determined by (30) for the Robin transmission condition, the SWR algorithm converges much faster.

VI. CONCLUSION

We have studied the convergence behavior of the SWR algorithm with Robin transmission conditions for non-Fickian reaction-diffusion equations with time-delay. The Robin parameter p contained in the transmission conditions has a significant influence on the convergence rate of the SWR algorithm and we determine it by solving a special min-max problem. This min-max problem is different from the existing ones studied in the research literature and the calculations to find a solution are more complicated. With the parameter formula calculated for a linear model, the Robin parameter used for the nonlinear model can be fixed adaptively by using the *linearization* idea proposed recently by Caetano *et al.* [5], Li and Zhang [21]. Numerical results were provided, and, these results validate the Robin parameter prediction for a successful fully discretized implementation.

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