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Regulated State Synchronization of Homogeneous Discrete-Time Multi-Agent Systems via Partial State Coupling in Presence of Unknown Communication Delays

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ABSTRACT This paper studies regulated state synchronization for homogeneous discrete-time multi-agent systems (MAS) in the presence of unknown nonuniform communication delays. We consider partial state coupling, i.e., agents are coupled through part of their states. A low gain-based protocol is designed which only requires rough knowledge of the communication network, such that the state synchronization for MAS is achieved where the required synchronized trajectory is only given to some of the agents.

INDEX TERMS Regulated state synchronization, discrete-time multi-agent systems, communication delays.

I. INTRODUCTION

The synchronization problem for multi-agent systems (MAS) has received substantial attention in the past decade, its objective is to secure an asymptotic agreement on common states (i.e., *state synchronization*) or output trajectories (*output synchronization*) through decentralized control protocols. Regulated synchronization problem, where we track a constant trajectory, has also attracted some attention due to its potential applications in cooperative control of micro-grids, platooning of autonomous vehicles, formation of satellites and others [1], [19]. For MAS with discrete-time agents, earlier work can be found in [3], [4], [9], [15], [18], and [24] for essentially first and second-order agents, and in [5], [7], [10], [28], [29], [31], [34], and [36] for higher-order agents. Most of these papers deal with homogeneous MAS (i.e., agents are identical), while [29] deals with heterogeneous MAS. However the latter deals with *introspective* agents (i.e., agents have access to part of their own state).

Many researchers have also focused on synchronization problems for MAS with time delays. In general, there are two types of delay in the study of MAS: *input delay* and *communication delay*. Input delay originates from computational limitations of an individual agent. There are many works dealing with input delay, for example, from single- and double-integrator agent dynamics (see [15], [22], [23], [30]) to more general agent dynamics (see [13], [14], [20], [25], [27], [33]).

Meanwhile, communication delay comes from limitations on the communication network between agents. Many studies use a protocol design based on introducing self-communication delay which has the same structure or value with the communication delay introduced by its neighbor agent, i.e.

$$\zeta_i(t) = \sum_{j=1}^{N} y_i(t-\tau_{ij}) - y_j(t-\tau_{ij}).$$

See, for instance, [6], [16], [21], and [35]. In this case, any trajectory can be synchronized for a MAS with communication delay.

On the other hand, if there is no such self-communication delay, the communication between agents becomes equal to

$$\zeta_i(t) = \sum_{j=1}^{N} y_i(t) - y_j(t-\tau_{ij})$$

In that case, it is obvious that for most time varying target trajectory it is not possible that $\zeta_i(t) \to 0$ as $t \to \infty$. In other words, the diffusive nature of the network (where asymptotically communication between agents converges to zero) can no longer be preserved. Currently, to preserve this diffusive nature of the network, the literature has focused on constant synchronized trajectories for this type of protocols, see some works in this area [2], [11], [22], [23], [32]. For discrete time MAS with communication delay, only few results consider (delayed) state synchronization by using a protocol design without self-communication delay but they impose special structure on the agents, see [8] (single integrator) or [12] (passivity).

In this paper, we study regulated state synchronization for MAS in discrete time for general linear systems. We assume that some of the agents have access to an, a *priori* given, constant trajectory. In our case, the communication network can have arbitrary large, unknown, nonuniform communication delays. We consider the case of an unknown, undirected communication network or a known, possibly directed, communication network. A low gain based dynamic protocol without self-communication delay is developed such that state synchronization is achieved among all agents where the synchronized output trajectory is equal to a given constant trajectory. The results show that synchronization can be achieved for a discrete-time network with arbitrarily large communication delay. Moreover, we find necessary and sufficient conditions whether a given constant synchronized trajectory can be achieved or not.

Notations and Definitions: Given a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} and A^* denote the transpose and conjugate transpose of A, respectively while $\|A\|$ denotes the induced 2-norm of A. A square matrix A is said to be Schur stable if all its eigenvalues are in the open unit disc. $A \otimes B$ depicts the Kronecker product of A and B. We will use I or 0 for the identity and zero matrix where the dimension is clear from the context.

A *weighted directed graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the weighting matrix. We have $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Moreover, we assume $a_{ii} = 0$. Each pair in $\mathcal{E}$ is called an *edge*. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A *directed tree* is a subgraph (subset of nodes and edges) in which every node has exactly one parent node except for one node, called the *root*, which has no parent node. A *directed spanning tree* is a subgraph which is a directed tree containing all the nodes of the original graph. If a directed spanning tree exists, the root has a directed path to every other node in the tree. A graph consisting of N nodes is called *undirected* if $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots N$. For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (a vector with all elements equal to 1).

II. PROBLEM DESCRIPTION

We will study a MAS consisting of N identical linear agents:

$$\begin{aligned} x_i(k+1) &= Ax_i(k) + Bu_i(k), \\ y_i(k) &= Cx_i(k), \end{aligned} \tag{1}$$

where $x_i(k) \in \mathbb{R}^n$, $u_i(k) \in \mathbb{R}^m$ and $y_i(k) \in \mathbb{R}^p$ are the state, input and output of agent $i = 1, \ldots, N$, respectively. We need the following assumption for the agents:

Assumption 1: We assume that

- *(A, B, C) is stabilizable and detectable.*
- *All eigenvalues of A are in the closed unit disc.*

The communication network provides each agent with a linear combination of its own output relative to that of other neighboring agents. In particular, each agent $i \in \{1, \ldots, N\}$ has access to the quantity,

$$\zeta_i(k) = \sum_{j=1}^{N} a_{ij}(y_i(k) - y_j(k - \kappa_{ij})), \tag{2}$$

where $\kappa_{ij} \in \mathbb{N}^+$ is an unknown constant communication delay from agent j to agent i for $i \neq j$. This communication delay means that it takes κ_{ij} seconds for agent j to transfer its state information to agent i. We set $\kappa_{ii} = 0$.

We will achieve state synchronization among agents, i.e.

$$\lim_{k\to\infty} (x_i(k) - x_j(k)) = 0$$

by tracking a constant reference trajectory y_r for the output of each agent. That is to say, the output of each agent should converge to this given trajectory, i.e.,

$$\lim_{k\to\infty} (y_i(k) - y_r) = 0. \tag{3}$$

Some of the agents have access to relative information about y_r. If agent i has information available about y_r then its measurement is modified as

$$\bar{\zeta}_i(k) = \zeta_i(k) + (y_i(k) - y_r).$$

On the other hand, if agent i has no direct information available about y_r, then the agent has the same information as before:

$$\bar{\zeta}_i(k) = \zeta_i(k).$$

We assume that a nonempty subset $\mathcal{S}$ of the agents have access to their own output relative to y_r. In particular, each agent i has access to the quantity

$$\psi_i(k) = \iota_i(y_i(k) - y_r), \quad \iota_i = \begin{cases} 1, & i \in \mathcal{S}, \\ 0, & i \notin \mathcal{S}. \end{cases} \tag{4}$$

The above can be combined and we obtain:

$$\bar{\zeta}_i(k) = \zeta_i(k) + \iota_i(y_i(k) - y_r) \tag{5}$$

for $i = 1, \dots, N$. To guarantee that each agent can achieve the required regulation, we need that there exists a path to each node starting with a node from the set $\mathcal{S}$. In other words, we need the following assumption on the network graph:

Assumption 2: Every node of the network graph $\mathcal{G}$ is a member of a directed tree which has its root contained in the set $\mathcal{S}$.

Given the set $\mathcal{S} \subseteq \{1, \dots, N\}$, we denote by $\mathbb{G}_{\mathcal{S}}^N$ the set of all graphs with N nodes which satisfy Assumption 2.

For any graph $\mathcal{G} \in \mathbb{G}_{\mathcal{S}}$, with the associated Laplacian matrix L, we define the expanded Laplacian matrix as

$$\bar{L} = L + \operatorname{diag}\{\iota_i\} = [\bar{\ell}_{ij}]_{N\times N}.$$

and we define

$$\bar{D} = I - \frac{1}{2 + D_{\text{in}}}\bar{L}. \tag{6}$$

where

$$D_{\text{in}} = \max_{i=1,\dots,N} \left(\sum_{j=1}^{N} a_{ij} \right)$$

It is easily verified that the matrix $\bar{D}$ is a matrix with all elements nonnegative and the sum of each row is less than or equal to 1.

Clearly, in order for all agents to follow the prescribed trajectory, we must have that for any agent i there exists an agent j which has access to the reference trajectory and is such that the associated network graph has a directed path from j to i. This property is equivalent to the condition in Assumption 2.

Lemma 1: The matrix $\bar{D}$ has all eigenvalues in the open unit disc if and only if Assumption 2 is satisfied.

Proof: We know that $\bar{L}$ is invertible if and only if Assumption 2 is satisfied.

Assume $\bar{L}$ is invertible. As noted in the proof of [18, Lemma 3.7] all eigenvalues of $\bar{D}$ are inside the unit circle or at 1. But if $\bar{D}$ has an eigenvalue at 1 then we immediately obtain that $\bar{L}$ is singular which gives a contradiction. Hence all eigenvalues of $\bar{D}$ are inside the unit disc.

On the other hand, if $\bar{D}$ has all eigenvalues in the open unit disc then it has no eigenvalue at 1. This immediately yields that $\bar{L}$ is invertible and hence Assumption 2 is satisfied. ■

Remark 1: If only one agent i has access to the reference trajectory then the matrix $\bar{D}$ is invertible if and only of the associated network graph contains a directed spanning tree with agent i as its root.

In this paper, we study state synchronization based on output regulation. The general result requires that the graph is undirected. If the graph is known then we can obtain a similar result even if the graph is directed.

Definition 1: Given a set $\mathcal{S}$ and a given real number $\beta \in (0, 1)$, let $\mathbb{G}_{\mathcal{S},\beta}^N$ be the subset of $\mathbb{G}_{\mathcal{S}}^N$ for which the corresponding matrix $\bar{D}$ has the property that $|\lambda_i| < \beta$ for $i = 1, \dots, N$.

The subset of $\mathbb{G}_{\mathcal{S},\beta}^N$ of undirected graphs is denoted by $\mathbb{G}_{\mathcal{S},\beta}^{u,N}$.

For the MAS (1), we formulate two state synchronization problems as follows.

Problem 1 (General Case): Consider a MAS described by (1) *and* (5) *with a given set of constant trajectories $\mathscr{C}_y \subseteq \mathbb{R}^p$.*

The problem of state synchronization with output regulation *given a set $\mathcal{S}$ and a set of graphs $\mathbb{G}_{\mathcal{S},\beta}^{u,N}$ with $\beta \in (0, 1)$ in the presence of unknown, nonuniform and arbitrarily large communication delays is to find a distributed linear dynamic protocol of the form,*

$$\begin{cases} x_{i,c}(k+1) = A_c x_{i,c}(k) + B_c \bar{\zeta}_i(k), \\ u_i(k) = C_c x_{i,c}(k), \end{cases} \tag{7}$$

for $i = 1, \dots, N$, such that

$$\lim_{k\to\infty} x_i(k) - x_j(k) = 0, \tag{8}$$

for all $i, j \in \{1, \dots, N\}$ while the output of each agent converges to y_r, i.e.,

$$\lim_{k\to\infty} y_i(k) - y_r = 0, \tag{9}$$

for all $i \in \{1, \dots, N\}$, for any $y_r \in \mathscr{C}_y$, for any graph $\mathcal{G} \in \mathbb{G}_{\mathcal{S},\beta}^N$, for all initial conditions and for any communication delay $\kappa_{ij} \in \mathbb{N}^+$.

Problem 2 (Known Communication Topology): Consider a MAS described by (1) *and* (5) *with a given set of constant trajectories $\mathscr{C}_y \subseteq \mathbb{R}^p$.*

The problem of state synchronization and output regulation *for a network associated to a given graph $\mathcal{G}$ and with unknown, nonuniform and arbitrarily large communication delays is to find a distributed linear dynamic protocol of the form* (7) *for each agent such that* (8) *is satisfied for all $i, j \in \{1, \dots, N\}$ and* (9) *holds for all $i \in \{1, \dots, N\}$, for any $y_r \in \mathscr{C}_y$, for all initial conditions and for any communication delay $\kappa_{ij} \in \mathbb{N}^+$,*

III. MAS VIA PARTIAL STATE COUPLING IN PRESENCE OF UNKNOWN COMMUNICATION DELAYS

We consider here the output synchronization problem for networks with partial-state coupling and unknown, nonuniform and arbitrarily large communication delays. We study this problem for the general case as defined in Problem 1 with undirected graphs.

In general, we have to restrict the choice of the given trajectory y_r. Let

$$\begin{aligned} \mathscr{C}_y &= \left\{ y \in \mathbb{R}^p \;\middle|\; \begin{pmatrix} 0 \\ y \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} A - I & B \\ C & 0 \end{pmatrix} \right\} \\ &= \left\{ y \in \mathbb{R}^p \;\middle|\; \exists\, x \in \mathbb{R}^n, u \in \mathbb{R}^m : Ax + Bu = x,\; Cx = y \right\}. \end{aligned} \tag{10}$$

It turns out that our problem is solvable if and only if $y_r \in \mathscr{C}_y$. Note that $\mathscr{C}_y = \mathbb{R}^p$ if (A, B, C) is right-invertible and has no invariant zeros in one.

Consider a MAS described by (1) and (5). Let R be an injective matrix such that $\mathscr{C}_y = \operatorname{Im} R$. Choose Π and Γ such that:

$$\begin{pmatrix} 0 \\ R \end{pmatrix} = \begin{pmatrix} A-I & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \Pi \\ \Gamma \end{pmatrix} \tag{11}$$

and

$$\operatorname{rank} \begin{pmatrix} A-I & B\Gamma \\ C & 0 \end{pmatrix} = n + \operatorname{rank} \Gamma. \tag{12}$$

Choose a matrix Γ_1 such that $\operatorname{Im} \Gamma_1 = \operatorname{Im} \Gamma$, and then choose a matrix Γ_2 such that matrix

$$\bar{\Gamma} = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \tag{13}$$

is square and invertible. Define $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ according to

$$\tilde{A} = \begin{pmatrix} A & B\Gamma_1 \\ 0 & I \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B\Gamma_2 & 0 \\ 0 & I \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C & 0 \end{pmatrix}. \tag{14}$$

Moreover, choose K such that $\tilde{A} + K\tilde{C}$ is Schur stable. Let P_δ be the unique solution of the discrete-time algebraic Riccati equation

$$\tilde{A}^{\mathrm{T}} P_\delta \tilde{A} - P_\delta - \tilde{A}^{\mathrm{T}} P_\delta \tilde{B}(I + \tilde{B}^{\mathrm{T}} P_\delta \tilde{B})^{-1} \tilde{B}^{\mathrm{T}} P_\delta \tilde{A} + \delta I = 0, \tag{15}$$

where δ is a parameter to be chosen later on. We consider the following protocol:

$$\begin{cases} x_{i,c}(k+1) = \begin{pmatrix} \tilde{A} + K\tilde{C} & 0 \\ B_1 F_\delta & 0 \end{pmatrix} x_{i,c}(k) \\ \qquad - \dfrac{1}{2 + D_{\text{in}}} \begin{pmatrix} K \\ 0 \end{pmatrix} \bar{\zeta}_i(k), \\ u_i(k) = \begin{pmatrix} H_1 F_\delta & \Gamma_1 \end{pmatrix} x_{i,c}(k), \end{cases} \tag{16}$$

with

$$\begin{aligned} F_\delta &= -\frac{1}{1-\beta} (I + \tilde{B}^{\mathrm{T}} P_\delta \tilde{B})^{-1} \tilde{B}^{\mathrm{T}} P_\delta \tilde{A}, \\ B_1 &= \begin{pmatrix} 0 & I \end{pmatrix}, \\ H_1 &= \begin{pmatrix} \Gamma_2 & 0 \end{pmatrix}, \end{aligned} \tag{17}$$

Theorem 1: *Consider a MAS described by* (1) *and* (2) *with an associated undirected graph. Let a nonempty set* $\mathcal{S}$ *and a* $\beta \in (0, 1)$ *be given. Let* $\mathscr{C}_y$ *be defined by* (10).

In that case, Problem 1 is solvable if the agents are such that (A, B, C) *is stabilizable and detectable and all eigenvalues of* A *are in the closed unit disc.*

More specifically, there exists $\delta > 0$ *such that the linear protocol* (16) *achieves state synchronization and output regulation for any* $\mathcal{S}$*, for any undirected graph* $\mathcal{G} \in \mathbb{G}^{u,N}_{\mathcal{S},\beta}$*, for all initial conditions, for any communication delay* $\kappa_{ij} \in \mathbb{N}^+$ *and for any* $y_r \in \mathscr{C}_y$*.*

Remark 2: *We note that the results of Theorem 1 still hold if the graph is directed but balanced as long as the associated matrix* $\bar{D}$ *satisfies*

$$0 < \bar{D} + \bar{D}^{T} < 2\beta$$

Before we can prove this theorem, we need several preliminary lemmas:

Lemma 2: *Let* β *be an upper bound for the eigenvalues of a symmetric matrix* $\bar{D}$*. Then, for all communication delays* $\kappa_{ik} \in \mathbb{N}^+$ *for* $i, k = 1, \ldots, N$ *and for all* $\omega \in \mathbb{R}$*, all eigenvalues of* $\bar{D}_{j\omega}(\kappa)$ *are less than or equal to* β *where* $\bar{D}_{j\omega}(\kappa)$ *is such that*

$$[\bar{D}_{j\omega}(\kappa)]_{ik} = \begin{cases} \bar{d}_{ik} e^{-j\omega\kappa_{ik}} & \text{if } i \neq k \\ \bar{d}_{kk} & \text{if } i = k \end{cases} \tag{18}$$

Denote by κ *the matrix with* $[\kappa]_{ik} = \kappa_{ik}$ *where* $\kappa_{ii} = 0$*.*

Lemma 3: *Let* (A, B) *be stabilizable and* (A, C) *be detectable. Define* $\tilde{A}$*,* $\tilde{B}$ *and* $\tilde{C}$ *according to* (14)*. Let* K *be such that* $\tilde{A} + K\tilde{C}$ *is asymptotically stable while* F_δ *is defined by* (17)*. In that case, for any* $\beta \in (0, 1)$ *there exists a* $\delta^* > 0$ *such that for any* $\delta \in (0, \delta^*]$*, the matrix*

$$\begin{pmatrix} \tilde{A} & (1-\lambda)\tilde{B}F_\delta \\ \tilde{A} + K\tilde{C} & -K\tilde{C} \end{pmatrix}$$

is asymptotically stable for all $\lambda \in \mathbb{C}$ *with* $|\lambda| < \beta$*.*

The proof of Lemmas 2 and 3 can be found in Appendices A and B. The following lemma is a classical result. It is a minor modification of a result found in [26].

Lemma 4: *Consider a linear time-delay system*

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} A_i x(k - \kappa_i), \tag{19}$$

where $x(k) \in \mathbb{R}^n$ *and* $\kappa_i \in \mathbb{N}^+$*. Suppose* $A + \sum_{i=1}^m A_i$ *is Schur stable. Then,* (19) *is asymptotically stable if*

$$\det[e^{j\omega} I - A - \sum_{i=1}^{m} e^{-j\omega\kappa_i^r} A_i] \neq 0,$$

for all $\omega \in [-\pi, \pi]$ *and for all* $\kappa_i^r \in \mathbb{N}^+$ *with* $0 < \kappa_i^r \leqslant \kappa_i$ $(i = 1, \ldots, N)$*.*

Proof of Theorem 1: According to results from classical output regulation, an individual agent can track a constant reference signal y_r if and only if there exists an x_0 and a u_0 such that

$$\begin{pmatrix} A-I & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ y_r \end{pmatrix}. \tag{20}$$

Moreover, such x_0 and u_0 exist if and only if y_r is in the set $\mathscr{C}_y$.

In order to use protocol (16) we first need to show there exists Π and Γ such that (11) and (12) are satisfied.

Firstly, there exists an injective matrix R such that $\mathscr{C}_y = \operatorname{Im} R$. In that case, it is easily seen that we have a Π and a Γ satisfying (11).

To show that we can impose the rank condition (12), one can easy see that (A, C) detectable implies that the first n columns of (12) are linearly independent. If the rank condition (12) does not hold, then there must exist x and v such that

$$\begin{pmatrix} A-I & B\Gamma \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} = 0$$

with $v \perp \ker\Gamma$ and $v^{\mathrm{T}} v = 1$. We obtain

$$\begin{pmatrix} A-I & B \\ C & 0 \end{pmatrix}\begin{pmatrix} \Pi - xv^{\mathrm{T}} \\ \Gamma(I - vv^{\mathrm{T}}) \end{pmatrix} = 0.$$

which implies that $\bar{\Pi} = \Pi - xv^{\mathrm{T}}$ and $\bar{\Gamma} = \Gamma(I - vv^{\mathrm{T}})$ also satisfy the above result but with $\operatorname{rank}\bar{\Gamma} < \operatorname{rank}\Gamma$. Recursively, we can find a solution of (11) which also satisfies the extra rank condition (12).

Next, choose an injective Γ_1 such that $\operatorname{Im}\Gamma = \operatorname{Im}\Gamma_1$ and choose Γ_2 such that (13) is a square and invertible matrix. We design a dynamic precompensator for each agent of the following form:

$$\begin{aligned} p_i(k+1) &= p_i(k) + \begin{pmatrix} 0 & I \end{pmatrix} v_i(k), \quad p_i(k) \in \mathbb{R}^{\nu} \\ u_i(k) &= \Gamma_1 p_i(k) + \begin{pmatrix} \Gamma_2 & 0 \end{pmatrix} v_i(k), \end{aligned} \tag{21}$$

where $p_i(k)$, $v_i(k)$, and $u_i(k)$ are state, input, and output of precompensator respectively, and $\nu = \operatorname{rank}\Gamma$. The interconnection of (1) and (21) is of the form,

$$\begin{cases} \tilde{x}_i(k+1) = \tilde{A}\tilde{x}_i(k) + \tilde{B}v_i(k), \\ y_i(k) = \tilde{C}\tilde{x}_i(k), \end{cases} \tag{22}$$

where

$$\tilde{x}_i(k) = \begin{pmatrix} x_i(k) \\ p_i(k) \end{pmatrix},$$

while $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are defined according to (14).

Next, we must verify the stabilizability and detectability of agents in combination with their precompensator as described by the triple $(\tilde{A}, \tilde{B}, \tilde{C})$. The stabilizability immediately from the invertibility of (13) and the stabilizability of (A, B).

For the detectability of (22), we verify the following condition

$$\operatorname{rank}\begin{pmatrix} zI - A & -B\Gamma_1 \\ 0 & (z-1)I \\ C & 0 \end{pmatrix} = n + \nu$$

for all z outside or on the unit circle, where ν is such that $\Gamma_1 \in \mathbb{R}^{n\times\nu}$. For $z \neq 1$, the above result can be obtained immediately from the detectability of (A, C). For $z = 1$, we have:

$$\begin{aligned} \operatorname{rank}\begin{pmatrix} I - A & -B\Gamma_1 \\ 0 & 0 \\ C & 0 \end{pmatrix} &= \operatorname{rank}\begin{pmatrix} I - A & -B\Gamma \\ C & 0 \end{pmatrix} \\ &= n + \operatorname{rank}\Gamma = n + \nu \end{aligned}$$

since $\operatorname{Im}\Gamma = \operatorname{Im}\Gamma_1$ and $\operatorname{rank}\Gamma_1 = \nu$ (since Γ_1 is injective). Thus, we can obtain the detectability of (22).

In the following, we design the following protocol

$$\begin{cases} \chi_i(k+1) = (\tilde{A} + K\tilde{C})\chi_i(k) - \dfrac{1}{2 + D_{\text{in}}} K\bar{\zeta}_i(k), \\ v_i(k) = F_\delta \chi_i(k), \end{cases} \tag{23}$$

for the multi-agent system whose agents are of the form (22), where $\chi_i(k)$ is the estimate for $\tilde{x}_i(k)$ in (22) while K is chosen such that $\tilde{A}+K\tilde{C}$ is Schur stable, and P_δ is the unique solution of the algebraic Riccati equation (15)

Next, we prove that the output of each agent converges to the constant trajectory y_r by using dynamical protocol (23). Firstly we show that there exists a $\tilde{\Pi}$ such that $\tilde{A}\tilde{\Pi} = \tilde{\Pi}$ and $\tilde{C}\tilde{\Pi} = R$. It is easily verified that a suitable $\tilde{\Pi}$ is given by:

$$\tilde{\Pi} = \begin{pmatrix} \Pi \\ V \end{pmatrix},$$

where V is such that $\Gamma_1 V = \Gamma$. For $i = 1, \dots, N$, we define $\bar{x}_i(k) = \tilde{x}_i(k) - \tilde{\Pi}x_r$ where x_r is such that $y_r = Rx_r$, and the output synchronization error $e_i(k) = y_i(k) - y_r$. Then, we get the error dynamics,

$$\begin{cases} \bar{x}_i(k+1) = \tilde{A}\bar{x}_i(k) + \tilde{B}v_i(k), \\ e_i(k) = \tilde{C}\bar{x}_i(k). \end{cases} \tag{24}$$

Moreover,

$$\frac{1}{2 + D_{\text{in}}}\bar{\zeta}_i(k) = \tilde{C}\bar{x}_i(k) - \sum_{j=1}^{N} \bar{d}_{ij}\tilde{C}\bar{x}_j(k - \kappa_{ij}).$$

Let

$$\bar{x}(k) = \begin{pmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \\ \vdots \\ \bar{x}_N(k) \end{pmatrix} \text{ and } \chi(k) = \begin{pmatrix} \chi_1(k) \\ \chi_2(k) \\ \vdots \\ \chi_N(k) \end{pmatrix}.$$

We find that the closed-loop system can be written in the frequency domain as

$$\begin{pmatrix} z\bar{x}(z) \\ z\chi(z) \end{pmatrix} = \begin{pmatrix} I_N \otimes \tilde{A} & I_N \otimes \tilde{B}F_\delta \\ -(I - \bar{D}_z(\kappa)) \otimes K\tilde{C} & I_N \otimes (\tilde{A} + K\tilde{C}) \end{pmatrix} \times \begin{pmatrix} \bar{x}(z) \\ \chi(z) \end{pmatrix}, \tag{25}$$

where $\bar{D}_z(\kappa)$ is the matrix defined by:

$$[\bar{D}_z(\kappa)]_{ij} = \begin{cases} \bar{d}_{ij} z^{-\kappa_{ij}} & \text{if } i \neq j \\ \bar{d}_{ii} & \text{if } i = j \end{cases}$$

Next, we prove the asymptotical stability of (25) for all communication delays $\kappa_{ij} \in \mathbb{N}^+$. We first prove the stability of (25) without communication delays and then prove the stability for the case that includes all communication delays κ_{ij}.

When there is no communication delay in the network, the stability of system (25) is equivalent to asymptotic stability of the matrix

$$\begin{pmatrix} \tilde{A} & \tilde{B}F_\delta \\ -(1 - \lambda_i)K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix}$$

for all $i \in \{1, \dots, N\}$, where $\lambda_1, \dots, \lambda_N$ are the eigenvalue of the matrix $\bar{D}$ which satisfy $|\lambda_i| \leqslant \beta$. Note that

$$\begin{aligned} &\begin{pmatrix} 1 - \lambda_i & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \tilde{A} & \tilde{B}F_\delta \\ -(1 - \lambda_i)K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix} \\ &\quad = \begin{pmatrix} \tilde{A} & (1 - \lambda_i)\tilde{B}F_\delta \\ -K\tilde{C} & \tilde{A} + K\tilde{C} \end{pmatrix}\begin{pmatrix} 1 - \lambda_i & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

and hence asymptotically stability of the closed loop system (25) without any communication delay is equivalent to the Schur stability of the matrix

$$\begin{pmatrix} \tilde{A} & (1-\lambda_i)\tilde{B}F_\delta \\ -K\tilde{C} & \tilde{A}+K\tilde{C} \end{pmatrix}$$

From Lemma 3, we then find that there exists a δ^* such that system (25) is asymptotically stable without communication delay for any $\delta \in (0, \delta^*]$.

In the case of communication delay, according to Lemma 4, the closed-loop system (25) is asymptotically stable for all communication delays $\kappa_{ij} \in \mathbb{N}^+$, if

$$\det\left[e^{j\omega}I - X_{j\omega}(\kappa^r)\right] \neq 0 \tag{26}$$

for all $\omega \in [-\pi, \pi]$ and any $\kappa_{ij}^r \in \mathbb{R}^+$ where

$$X_{j\omega}(\kappa^r) = \begin{pmatrix} I_N \otimes \tilde{A} & I_N \otimes \tilde{B}F_\delta \\ -(I-\bar{D}_{j\omega}(\kappa^r))\otimes K\tilde{C} & I_N \otimes (\tilde{A}+K\tilde{C}) \end{pmatrix}$$

while κ^r denotes a vector consisting of all κ_{ij}^r $(i \neq j)$ with $i, j \in \{1, \ldots, N\}$ and $\bar{D}_{j\omega}$ is defined in (18). The condition (26) holds if the matrix $X_{j\omega}(\kappa^r)$ has no eigenvalues on the unit circle for all $\omega \in [-\pi, \pi]$ and for all $\kappa_{ij}^r \in \mathbb{N}^+$.

Lemma 2 implies that all the eigenvalues of $\bar{D}_{j\omega}(\kappa^r)$ have amplitude less than β. Similarly as before, Lemma 3 implies that there exists a δ^* such that for any $\delta \in (0, \delta^*]$,

$$\begin{pmatrix} \tilde{A} & \tilde{B}F_\delta \\ -(1-\lambda)K\tilde{C} & \tilde{A}+K\tilde{C} \end{pmatrix}$$

is asymptotically stable for all λ with $|\lambda| < \beta$. It is then straightforward to show that the matrix $X_{j\omega}(\kappa^r)$ has no eigenvalues on the unit circle. Therefore, the closed-loop system (25) is asymptotically stable for any communication delay $\kappa_{ij} \in \mathbb{N}^+$.

Finally, by combining the pre-compensator (21) and protocol (23), we get the linear dynamic protocol (16), which achieves state synchronization and makes the output track the given trajectory y_r. ∎

IV. MAS WITH A KNOWN DIRECTED COMMUNICATION TOPOLOGY

In this section, we study state synchronization problem for a multi-agent system with a known graph $\mathcal{G}$. The main advantage of knowing the graph is that we can also handle directed graphs.

In this case, we still employ the protocol (16) to establish the state synchronization for one individual graph (instead of for a set), i.e. we assume the directed graph $\mathcal{G}$ is given.

We need a modified version of Lemma 2:

Lemma 5: Let β be an upper bound for the eigenvalues of a matrix $\bar{D}$. In that case there exists $\tilde{\beta} < 1$ such that, for all communication delays $\kappa_{ik} \in \mathbb{N}^+$ for $i, k = 1, \ldots, N$ and for all $\omega \in [-\pi, \pi]$, all eigenvalues of $\bar{D}_{j\omega}(\kappa)$ are less than or equal to $\tilde{\beta}$ where $\bar{D}_{j\omega}(\kappa)$ is defined by (18) and denote by κ the matrix with $[\kappa]_{ik} = \kappa_{ik}$ where $\kappa_{ii} = 0$.

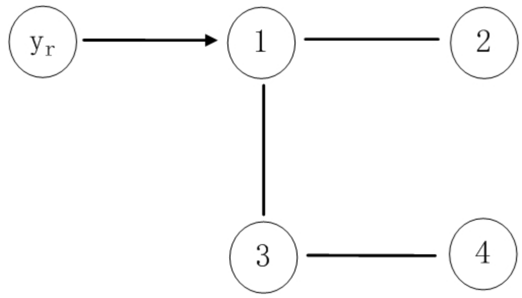

FIGURE 1. The undirected communication topology.

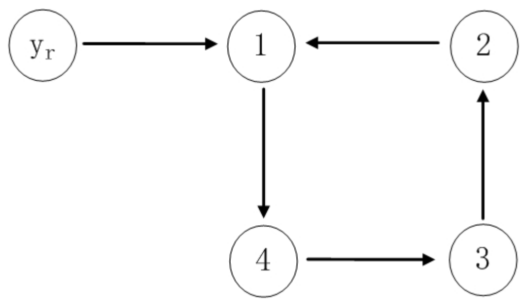

FIGURE 2. The known directed communication topology.

The proof of Lemma 5 can be found in Appendix C. The corresponding theorem is as follows.

Theorem 2: Consider a MAS described by (1) and (2) with an associated directed graph $\mathcal{G} \in \mathbb{G}_{\mathcal{S}}^N$ given a set $\mathcal{S}$.

Then Problem 2 is solvable if the agents are such that (A, B, C) is stabilizable and detectable and all eigenvalues of A are in the closed unit disc.

More specifically, given the directed graph $\mathcal{G}$, there exists $\delta > 0$ such that the linear protocol (16) achieves state synchronization and output regulation for any communication delays $\kappa_{ij} \in \mathbb{N}^+$, for all initial conditions and for any $y_r \in \mathscr{C}_y$.

Remark 3: If we have a finite set of possible graphs then we can still find a protocol that works for every graph in this finite set (use as an upper bound for δ, the maximum of the lower bounds for each individual graph in the set).

Proof of Theorem 2: We use Lemma 5 to obtain a bound $\tilde{\beta}$ for the eigenvalues of $\bar{D}_{j\omega}(\kappa)$. Except for using this new bound, the rest of the proof is identical to the proof of Theorem 1. ∎

V. EXAMPLES

In this section, we provide an example to verify our dynamic protocol design.

Example 1: Consider a MAS with 4 identical agents and a constant trajectory $y_r = 5$. The agent model is of the form of (1) with communication given by (5), where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}.$$

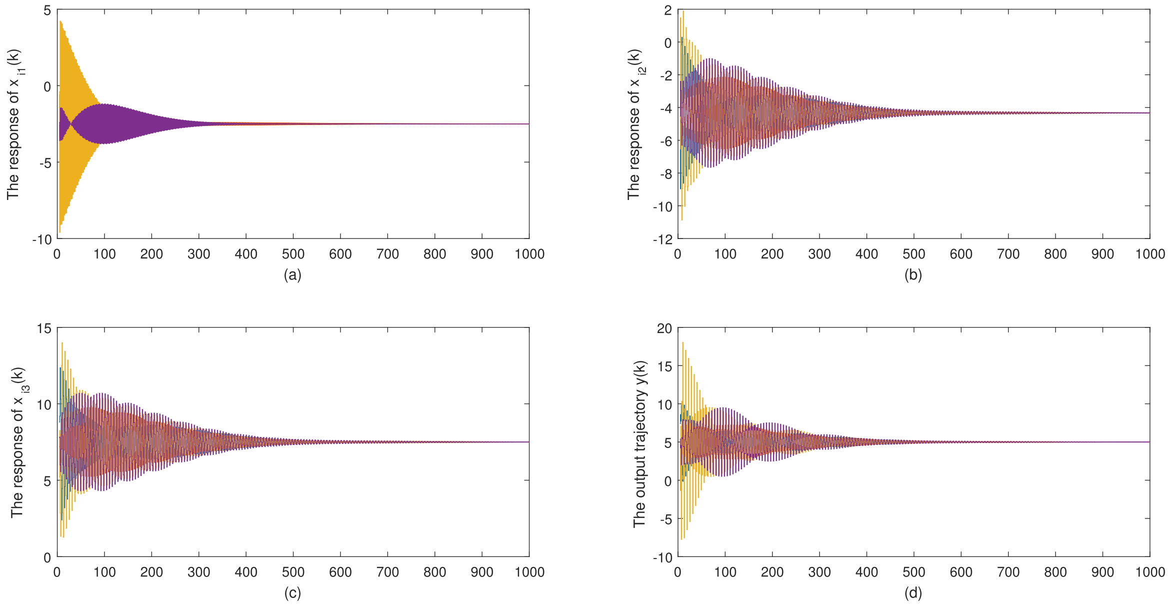

FIGURE 3. Synchronization of a discrete-time MAS with unknown undirected graph and uniform unknown communication delays.

We choose

$$R = 1, \quad \Pi = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix}, \; \Gamma = \Gamma_1 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \; \Gamma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which satisfy (11) *and* (12). *Delays are set as*

$$\begin{pmatrix} 0 & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ \kappa_{21} & 0 & \kappa_{23} & \kappa_{24} \\ \kappa_{31} & \kappa_{32} & 0 & \kappa_{34} \\ \kappa_{41} & \kappa_{42} & \kappa_{43} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ 2 & 3 & 1 & 0 \end{pmatrix}.$$

Case I (Unknown Undirected Graph): We define a set of networks $\mathbb{G}^{u,N}_{\mathcal{S},\beta}$ *with* $\beta = 0.9568$. *Here, we consider a network as shown in Figure 1 with the matrix*

$$\bar{D} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 0.75 & 0 & 0 \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.75 \end{pmatrix}$$

with $\iota_1 = 1$, $\iota_i = 0$ *for* $i = 2, 3, 4$, *and*

We can calculate $\delta^* = 1.5 * 10^{-5}$. *By choosing* $\delta = 2 * 10^{-6}$, *we obtain the following dynamic protocol,*

$$\begin{cases} x_{i,c}(k+1) = \hat{A}^I x_{i,c}(k) + \frac{1}{4}\begin{pmatrix} -0.1315 \\ -0.1205 \\ 0.7908 \\ 0.2128 \\ 0 \end{pmatrix} \bar{\zeta}_i(k), \\ u_i(k) = \hat{F}^I_\delta x_{i,c}(k), \end{cases} \tag{27}$$

with, $\hat{A}^I$, $\hat{F}^I_\delta$, *as shown at the bottom of the next page.*

The trajectory of the states of agents x_i *and the output trajectory of the MAS with communication delays are given in Figure 3. We see that all 4 agents achieve state synchronization, where (a)-(c) show the response of states* x_{i1}, x_{i2}, x_{i3} ($i = 1, \ldots, 4$), *and (d) shows the output trajectories y. That is, state* $x_i(k)$ *will synchronize to the constant vector*

$$\frac{1}{2}\begin{pmatrix} -5 \\ -5\sqrt{3} \\ 15 \end{pmatrix}. \tag{28}$$

Meanwhile, the output trajectory is shown in Figure 3-(d).

Case II (Known Directed Graph): We consider a known network as shown in Figure 2 with the matrix

$$\bar{D} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

with $\iota_1 = 1$, $\iota_i = 0$ *for* $i = 2, 3, 4$. *By choosing* $\delta = 2 * 10^{-6}$, *we obtain the dynamic protocol in the form of* (27) *with two different gains,* $\hat{A}^{II}$, $\hat{F}^{II}_\delta$, *as shown at the bottom of the next page, to substitute* $\hat{A}^I$ *and* $\hat{F}^I_\delta$. *Then, the trajectory of the states of agents* x_i *and the output trajectory of the MAS with communication delays are given in Figure 4. We see that all 4 agents achieve state synchronization, where (a)-(c) show the response of states* x_{i1}, x_{i2}, x_{i3} ($i = 1, \ldots, 4$), *and (d) shows the output trajectories y. That is, state* x_i *will synchronize to the constant vector* (28). *Meanwhile, the output trajectory is shown in Figure 4-(d). In this case, it is obvious that state synchronization with known directed graph need much more time to be achieved.*

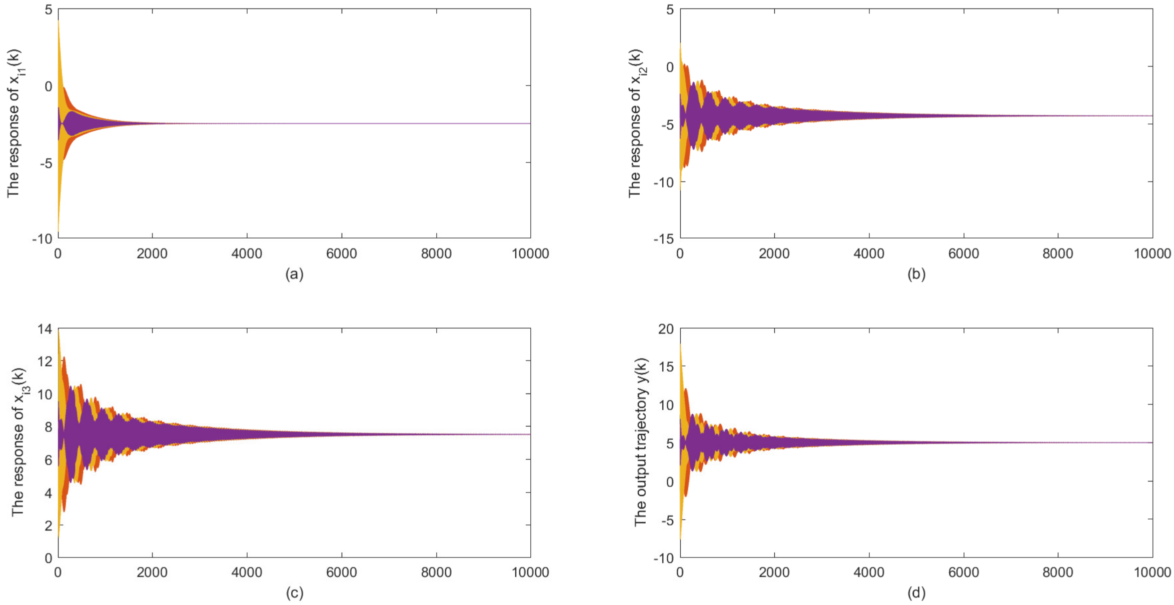

FIGURE 4. Synchronization of a discrete-time MAS with known directed graph and uniform unknown communication delays.

VI. CONCLUSION

The regulated state synchronization problem for homogeneous discrete-time MAS has been studied in this paper, where the agents is with unknown nonuniform communication delays. Based on the low gain, a dynamic protocol has been designed to achieve the state synchronization and its output track a given constant trajectory. Meanwhile, the protocol design only needs the rough knowledge of the network topology which belongs to a set of undirected or balanced networks. It is also confirmed that synchronization can be achieved for a MAS with arbitrary communication delays.

APPENDIX A
PROOF OF LEMMA 2

All eigenvalues of $\bar{D}_{j\omega}(\kappa)$ are in the set

$$\left\{ v^* \bar{D}_{j\omega}(\kappa) v \mid v \in \mathbb{C}^N, \|v\| = 1 \right\}.$$

Therefore, it is sufficient that all elements in this set have amplitude less than or equal to β.

Since $\bar{D}$ is symmetric and β is an upper bound for the amplitude of eigenvalues of $\bar{D}$, we find that $v^*\bar{D}v$ is less than or equal to β, provided $\|v\| = 1$.

Next, for an arbitrary vector $v \in \mathbb{C}^N$, we have

$$v^* \bar{D}_{j\omega}(\kappa) v = \sum_{i=1}^{N} \sum_{m=1}^{N} v_i^* v_m \bar{d}_{im} e^{-\kappa_{im} j\omega}.$$

Since $\bar{d}_{im}$ are all nonnegative, we get

$$\begin{aligned} |v^* \bar{D}_{j\omega}(\kappa) v| &\leqslant \sum_{i=1}^{N} \sum_{m=1}^{N} |v_i^* v_m| \bar{d}_{im} \\ &= \begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix}^{\mathrm{T}} \bar{D} \begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix} \leqslant \beta, \end{aligned}$$

which completes the proof.

$$\hat{A}^{\mathrm{I}} = \begin{pmatrix} -0.8685 & 0 & 0.1315 & 1.0000 & 0 \\ 0.1205 & 0.5000 & 0.9865 & -1.7321 & 0 \\ -0.7908 & -0.8660 & -0.2908 & 0 & 0 \\ -0.2128 & 0 & -0.2128 & 1.0000 & 0 \\ -0.0326 & -0.0399 & 0.0002 & -0.0858 & 0 \end{pmatrix}$$

$$\hat{F}_\delta^{I} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1.0000 \\ -1.3724 * 10^{-7} & -0.0115 & -0.0200 & 0.0202 & -1.7321 \end{pmatrix}$$

$$\hat{A}^{\mathrm{II}} = \begin{pmatrix} -0.8685 & 0 & 0.1315 & 1.0000 & 0 \\ 0.1205 & 0.5000 & 0.9865 & -1.7321 & 0 \\ -0.7908 & -0.8660 & -0.2908 & 0 & 0 \\ -0.2128 & 0 & -0.2128 & 1.0000 & 0 \\ -0.0319 & -0.0391 & 0.0002 & -0.0840 & 0 \end{pmatrix}$$

$$\hat{F}_\delta^{II} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1.0000 \\ -1.3447 * 10^{-7} & -0.0113 & -0.0196 & 0.0198 & -1.7321 \end{pmatrix}$$

APPENDIX B
PROOF OF LEMMA 3

In the proof of Theorem 1 it was established that $(\tilde{A}, \tilde{B})$ is stabilizable and $(\tilde{C}, \tilde{A})$ is detectable. This guarantees the existence of K such that $\tilde{A}+K\tilde{C}$ is asymptotically stable and also yields that F_δ is well-defined.

We basically need to prove that the system

$$\begin{cases} x(k+1) = \tilde{A}x(k) + (1-\lambda)\tilde{B}F_\delta \chi(k), \\ \chi(k+1) = (\tilde{A}+K\tilde{C})\chi(k) - K\tilde{C}x(k), \end{cases} \tag{29}$$

is asymptotically stable for any λ that satisfies $|\lambda| < \beta$.

Define $e = x - \chi$. Then, we have

$$\begin{cases} x(k+1) = [\tilde{A} + (1-\lambda)\tilde{B}F_\delta]x(k) - (1-\lambda)\tilde{B}F_\delta e(k) \\ e(k+1) = [\tilde{A}+K\tilde{C} - (1-\lambda)\tilde{B}F_\delta]e(k) \\ \qquad\qquad + (1-\lambda)\tilde{B}F_\delta x(k). \end{cases}$$

Let Q be the positive definite solution of the Lyapunov equation,

$$(\tilde{A}+K\tilde{C})^{\mathrm{T}}Q(\tilde{A}+K\tilde{C}) - Q + 4I = 0.$$

Since $F_\delta \to 0$ as $\delta \to 0$, there exists a δ_1 such that for a $\delta \in (0, \delta_1]$,

$$(\tilde{A}+K\tilde{C}-(1-\lambda)\tilde{B}F_\delta)^* Q(\tilde{A}+K\tilde{C}-(1-\lambda)\tilde{B}F_\delta) - Q + 3I \leqslant 0.$$

Consider $V_1(k) = e(k)^*Qe(k)$. Let $\mu(k) = F_\delta x(k)$. Here we omit the time label (k) for ease of presentation. We have

$$\begin{aligned} &V_1(k+1) - V_1(k) \\ &\quad \leqslant -3\|e\|^2 + |1-\lambda|^2 \mu^* \tilde{B}^{\mathrm{T}} Q \tilde{B}\mu \\ &\qquad + 2\left|(1-\lambda)^* \mu^* \tilde{B}^{\mathrm{T}} Q[\tilde{A}+K\tilde{C} - (1-\lambda)\tilde{B}F_\delta]e\right| \\ &\quad \leqslant -3\|e\|^2 + M_1\|\mu\|\|e\| + M_2\|\mu\|^2, \end{aligned} \tag{30}$$

where

$$M_1 = 4\|\tilde{B}^{\mathrm{T}}Q\|\|\tilde{A}+K\tilde{C}\| + 8\|\tilde{B}^{\mathrm{T}}Q\| \max_{\delta\in[0,1]}\{\|\tilde{B}F_\delta\|\},$$

$$M_2 = 4\|\tilde{B}^{\mathrm{T}}Q\tilde{B}\|.$$

It should be noted that M_1 and M_2 are independent of δ and λ provided that $|\lambda| < \beta$.

Consider $V_2(k) = x^*(k)P_\delta x(k)$. We have

$$\begin{aligned} &V_2(k+1) - V_2(k) \\ &\quad \leqslant -\delta\|x\|^2 - \frac{1}{2}(1-\beta)^2\|\mu\|^2 \\ &\qquad -2\,\mathrm{Re}\Big((1-\lambda)^* e^* F_\delta^{\mathrm{T}} \tilde{B}^{\mathrm{T}} P_\delta[\tilde{A} + (1-\lambda)\tilde{B}F_\delta]x\Big) \\ &\qquad + |1-\lambda|^2 e^* F_\delta^{\mathrm{T}} \tilde{B}^{\mathrm{T}} P_\delta \tilde{B} F_\delta e. \end{aligned}$$

where we used that

$$\begin{aligned} &(1-\beta)^2 x^* F_\delta^{\mathrm{T}}(I + \tilde{B}^{\mathrm{T}}P_\delta\tilde{B})F_\delta x \\ &\quad -2\,\mathrm{Re}(1-\lambda)(1-\beta)x^*F_\delta^{\mathrm{T}}(I+\tilde{B}^{\mathrm{T}}P_\delta\tilde{B})F_\delta x \\ &\quad + |1-\lambda|^2\, x^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}F_\delta x \\ &\leqslant (1-\beta)^2\, x^*F_\delta^{\mathrm{T}}(I+\tilde{B}^{\mathrm{T}}P_\delta\tilde{B})F_\delta x \\ &\quad -2(1-\beta)^2\, x^*F_\delta^{\mathrm{T}}(I+\tilde{B}^{\mathrm{T}}P_\delta\tilde{B})F_\delta x \\ &\quad + (1+\beta)^2\, x^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}F_\delta x \\ &\leqslant -(1-\beta)^2\, x^*F_\delta^{\mathrm{T}}F_\delta x + (1+\beta)^2 x^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}F_\delta x \\ &\leqslant -\frac{1}{2}(1-\beta)^2\, x^*F_\delta^{\mathrm{T}}F_\delta x \end{aligned}$$

provided δ is small enough such that

$$\tilde{B}^{\mathrm{T}}P_\delta\tilde{B} \leqslant \frac{(1-\beta)^2}{2(1+\beta)^2} \tag{31}$$

Note that

$$\begin{aligned} &e^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta[\tilde{A}+(1-\lambda)\tilde{B}F_\delta]x \\ &\quad = e^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{A}x + (1-\lambda)e^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}\mu \\ &\quad = -e^*F_\delta^{\mathrm{T}}(\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}+I)\mu + (1-\lambda)e^*F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}\mu \\ &\quad = -e^*[F_\delta^{\mathrm{T}} + \lambda F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}]\mu, \end{aligned}$$

and hence

$$\begin{aligned} &V_2(k+1) - V_2(k) \\ &\quad \leqslant -\delta\|x\|^2 - \frac{1}{2}(1-\beta)^2\|\mu\|^2 + \theta_1(\delta)\|e\|\|\mu\| + \theta_2(\delta)\|e\|^2, \end{aligned} \tag{32}$$

where

$$\theta_1(\delta) = 4(\|F_\delta\| + \|F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}\|),$$
$$\theta_2(\delta) = 4\|F_\delta^{\mathrm{T}}\tilde{B}^{\mathrm{T}}P_\delta\tilde{B}F_\delta\|.$$

Consider a Lyapunov candidate $V(k) = V_1(k) + \alpha V_2(k)$ with

$$\alpha = \frac{2(M_2 + M_1^2)}{(1-\beta)^2}.$$

In view of (30) and (32), we get

$$\begin{aligned} &V(k+1) - V(k) \\ &\quad \leqslant -\delta\alpha\|x\|^2 - M_1^2\|\mu\|^2 \\ &\qquad -[3-\alpha\theta_2(\delta)]\|e\|^2 + [M_1 + \alpha\theta_1(\delta)]\|\mu\|\|e\|. \end{aligned}$$

There exists a $\delta^* \leqslant \delta_1$ such that for $\delta \in (0, \delta^*]$ we have (31) and

$$3 - \alpha\theta_2(\delta) \geqslant 2, \quad M_1 + \alpha\theta_1(\delta) \leqslant 2M_1.$$

This yields

$$V(k+1) - V(k) \leqslant -\delta\alpha\|x\|^2 - \|e\|^2 - (\|e\| - M_1\|\mu\|)^2.$$

Therefore, for $\delta \in (0, \delta^*]$, the system (29) is globally asymptotically stable for any λ that satisfies $|\lambda| < \beta$.

APPENDIX C
PROOF OF LEMMA 5

We know the matrix $\bar{D}$ has all its eigenvalues inside the unit circle.

By [17, Th. 4.36], if the associated graph is irreducible and all eigenvalues of $\bar{D}$ are strictly less than 1, there exists a positive diagonal matrix P and $\tilde{\beta} < 1$ such that

$$\bar{D}^{\mathrm{T}}P\bar{D} < \tilde{\beta}P$$

This implies

$$\left(P^{1/2}\bar{D}_{j\omega}(\kappa)P^{-1/2}\right)^{\mathrm{T}}\left(P^{1/2}\bar{D}_{j\omega}(\kappa)P^{-1/2}\right) < \tilde{\beta} I$$

We find:

$$v^* P^{1/2}\bar{D}P^{-1/2}v \leqslant \|P^{1/2}\bar{D}P^{-1/2}\| v^* v \leqslant \tilde{\beta}^{1/2} v^* v$$

for all $v \in \mathbb{C}^N$. We have:

$$v^* P^{1/2}\bar{D}_{j\omega}(\kappa)P^{-1/2}v = \sum_{i=1}^{N}\sum_{m=1}^{N} v_i^* v_m p_i^{1/2} p_m^{-1/2}\bar{d}_{im} e^{-\kappa_{im} j\omega}.$$

where $p_1, \ldots, p_N$ are the diagonal elements of P. Since $\bar{d}_{im}$ are all nonnegative, we get

$$\begin{aligned} &|v^* P^{1/2}\bar{D}_{j\omega}(\kappa)P^{-1/2}v| \\ &\leqslant \sum_{i=1}^{N}\sum_{m=1}^{N} |v_i^* v_m| p_i^{1/2} p_m^{-1/2}\bar{d}_{im} \\ &= \begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix}^{\mathrm{T}} P^{1/2}\bar{D}P^{-1/2}\begin{pmatrix} |v_1| \\ \vdots \\ |v_N| \end{pmatrix} \leqslant \tilde{\beta}^{1/2}, \end{aligned}$$

and hence all eigenvalues of $\bar{D}_{j\omega}(\kappa)$ which are equal to the eigenvalues of $P^{1/2}\bar{D}_{j\omega}(\kappa)P^{-1/2}$ are less than $\tilde{\beta}$ in magnitude. If the graph is not irreducible we can obtain the same result using the strongly connected components. After all if $\bar{D}$ has a block triangular structure then $\bar{D}_{j\omega}(\kappa)$ has the same block triangular structure and the eigenvalues of the whole matrix are the union of the eigenvalues of the blocks on the diagonal. We can guarantee that the blocks on the diagonal are irreducible and hence the previous argument applies.

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