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Semi-Global Robust Stabilization for a Class of Nonlinear Systems With Uncertain Measurement Functions

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ABSTRACT This paper considers the problem of semi-global stabilization via linear feedback for a class of uncertain nonlinear systems. Different from the existing results, the systems considered here have uncertain measurement functions due to the sensors' property. By generalizing the notion of homogeneity with monotone degrees to the uncertain case, a measurement feedback controller is recursively constructed to semi-globally asymptotically stabilize the system by appropriately choosing a series of Lyapunov functions as well as the corresponding level sets. The proposed control scheme not only performs in a linear form for better implementation but also leads to robustness to different sensors. Finally, a numerical example is presented to demonstrate the effectiveness of the control law.

INDEX TERMS Linear controller, nonlinear systems, semi-global stabilization, uncertain measurement functions.

I. INTRODUCTION

In this paper, we will consider the following uncertain nonlinear systems

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) + \phi_1(x_1(t)),\\ &\vdots\\ \dot{x}_{n-1}(t) &= x_n(t) + \phi_{n-1}(x_1(t), \cdots, x_{n-1}(t)),\\ \dot{x}_n(t) &= u(t) + \phi_n(x_1(t), \cdots, x_n(t)),\\ y_i(t) &= x_i^{q_i}(t), \quad i = 1, \cdots, n, \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^T \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y_i(t)$ are the system states, the control input and the measurements of system states, respectively. For $i = 1, \cdots, n$, the unknown nonlinear terms $\phi_i(\cdot) : \mathbb{R}^i \to \mathbb{R}$ are continuous in system states and the unknown powers $q_i \in \mathbb{R}^+_{odd}$ satisfy $a_i \leq q_i \leq b_i$, with known constants $a_i, b_i > 0$.

Obviously, the uncertain system (1) is of a strict-feedback form whose control problem has attracted much attention from the nonlinear control community. However, it is common that the relationship between the sensor's output and the system state is uncertain, which can be mainly represented by the following three types. If the noise exists in feedback information obtained from the sensor, the measurement function can be written as $y = x + d$, with d denoting the uncertain noise. Instead of the traditional state-feedback controller, several investigations have been engaged in designing the measurement feedback controller, for example, [1]–[5] and the references therein. If the linear relationship holds or the first derivative of the output function is bounded, it can be represented by $y = dx$ for an constant d or $y = h(x)$ where the first derivative of the continuous function $h(x)$ is bounded. By constructing a state compensator and using the compensator states to design a controller, the stabilization result can be achieved in [6] based on the homogeneous domination approach, which has been further generalized to more complex systems [7], [8]. In addition, as shown in [9] that, the voltage output from the infrared distance sensor Sharp GP2D12 is a nonlinear function x^d where x is the real distance. For different products even from the same batch, the value of d may not be the same, i.e., the constant d is uncertain and varies from products to products. It has been proved in [9] that the designed robust controller is able to globally stabilize a family of nonlinear systems with different measurement drifts as long as the drifts vary within the assigned bounds. On this basis, the work [10] has solved the robust control problem for high-order nonlinear systems with more general conditions via measurement feedback.

Undeniably, the global stabilization result is perfect, while the controller is always in a nonlinear even nonsmooth form or the assumptions imposed on the nonlinearities are rigorous, which brings a lot of trouble for controller design and implementation. Therefore, a less ambitious control goal, *semi-global stabilization* is pursued. The work [11] has solved the semi-global output feedback stabilization problem for feedback linearizable systems. It is shown in [12] that uniform observability and global stabilizability by smooth state feedback cannot achieve global stabilizability by smooth output feedback, but can lead to semi-global result for nonlinear systems. Based on this conclusion, the semi-global stabilization via smooth output feedback has been achieved for a class of minimum-phase [13] and nonminimum-phase [14] nonlinear systems, respectively. By adopting the feedback domination method, the work [15] has constructed a linear output feedback controller to semi-globally stabilize the uncertain nonlinear systems under less restrictive conditions. For a class of nonuniformly observable and nonsmooth stabilizable nonlinear systems, semi-global stabilization has been achieved by nonsmooth output feedback in [16]. Recently, several new results have been proposed towards semi-global stabilization for different kinds of nonlinear systems, for example [17]–[19] and the references therein. However, the abovementioned results rely on that at least partial of the system states can be measured accurately and can be used to design the observer and controller. Otherwise, the designed controllers do not work anymore. For the first type measurement function, the work [20] has presented a unified framework of the semi-global stabilization for the uncertain nonlinear systems via measurement feedback. In [21], the semi-global output feedback stabilization problem has been solved for the upper-triangular nonlinear system whose output function is uncertain but its first derivative is bounded that can be described as the second type. Up till now, there is no controller design method for the measurement function described as $y = x^d$, i.e., the third type.

Motivated by the work [9], [10], this paper aims to solve the semi-global stabilization problem for the system (1) via linear measurement feedback. To this end, we first give the conditions on the unknown powers q_i's and the nonlinearities $\phi(\cdot)$'s based on the notion of the homogeneity with monotone degrees. Then, by subtly constructing the Lyapunov functions, as well as the associated level sets, a linear controller made up of the measurements is proposed to make the system (1) semi-globally asymptotically stable. The main contributions of this paper are as follows:

(i) The proposed controller only consists of the uncertain *measurements* rather than constructing any observer or compensator, which, to an extent, reduces the complexity of the nonlinear system.

(ii) Different from the existing nonlinear controllers, the designed *linear* controller has simple structure and is much easier to be implemented and therefore gains more value in the real systems;

(iii) Robustness can be achieved since only the known bounds for the uncertain q_i's are used in the controller design. It means that even though different sensors are used, the controller still works as long as their q_i's belong to the same interval.

II. MATHEMATICAL PRELIMINARIES

In this section, we will revisit some fundamental definitions including the homogeneous system theory and the semi-global asymptotic stabilization, and several useful inequalities, which will play an important role in the subsequent development.

Definition 1: For a fixed choice of coordinates $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and positive real numbers $(r_1, \cdots, r_n) \triangleq r$, a one-parameter family of dilation is a map $\Delta_\epsilon^r : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $\Delta_\epsilon^r x = (\epsilon^{r_1} x_1, \cdots, \epsilon^{r_n} x_n)$, $\forall \epsilon > 0$ with $r_i' s$ being the weights of the coordinates.

Definition 2: For a given dilation Δ_ϵ^r and a series of real monotone numbers $\tau_1 \geq \tau_2 \cdots \geq \tau_n$, a continuous vector field $f(x) = [f_1(x), \cdots, f_n(x)]^T$, $x \in \mathbb{R}^n$, is said to be homogeneous with monotone degrees (HWMD) $\tau_1, \cdots, \tau_n$, if $\forall x \in \mathbb{R}^n \backslash \{0\}$, $f_j(\Delta_\epsilon^r x) = \epsilon^{\tau_j + r_j} f_j(x)$, $j = 1, \cdots, n$.

When $\tau_1 = \tau_2 = \cdots = \tau_n = \tau$, the definition of homogeneity with monotone degrees reduces to the traditional homogeneity with homogeneous degree τ.

Definition 3 [22]: The problem of **semi-global asymptotic stabilization** (SGAS) by linear feedback for the nonlinear system means that given an upper bound $M > 0$, find, if possible, a linear controller $u = L_M x$ with the gain L_M depending on M, such that all the trajectories of the closed-loop system starting from the compact set $B_M \triangleq [-M, M]^n \subset \mathbb{R}^n$ converge uniformly to the origin.

Remark 4: From the definition above, it is known that the designed controller depending on the known bounds for initial values, can stabilize the system. Obviously, the bounds of the initial conditions are usually easy to be estimated and therefore, semi-global stabilization may be good enough in practical applications [23]–[25].

Lemma 5 [26]: For $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $p \geq 1$, the following inequalities hold:

$$|x + y|^p \leq 2^{p-1} |x^p + y^p|,$$
$$\left(|x| + |y|\right)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{\frac{p-1}{p}} \left(|x| + |y|\right)^{\frac{1}{p}}.$$

If $p \geq 1$ is an odd integer or a ratio of two odd integers,

$$|x - y|^p \leq 2^{p-1} |x^p - y^p|,$$
$$|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \leq 2^{1-\frac{1}{p}} |x - y|^{\frac{1}{p}},$$
$$\begin{aligned} |x^p - y^p| &\leq p|x - y|(x^{p-1} + y^{p-1}) \\ &\leq c|x - y||(x - y)^{p-1} + y^{p-1}|, \end{aligned}$$

with a constant $c > 0$.

Lemma 6 [26]: For any positive real numbers c, d and any real-valued function $\gamma(x, y) > 0$, the following inequality holds:

$$|x|^c |y|^d \leq \frac{c}{c+d} \gamma(x, y) |x|^{c+d} + \frac{d}{c+d} \gamma^{-\frac{c}{d}}(x, y) |y|^{c+d}.$$

III. MAIN RESULTS

In this section, a linear controller will be constructed iteratively to solve the SGAS problem for system (1). During the design procedure, considering that sensors cannot measure the system states accurately, we can only use the uncertain measurements and we still pursue its robustness. Therefore, the following two assumptions need to be imposed to guarantee the solvability of this problem.

Assumption 7: For $i = 2, \cdots, n-1$, $b_i \leq \frac{2a_{i-1}a_{i+1}}{a_{i-1}+a_{i+1}}$, $b_n \leq \frac{2a_{n-1}}{a_{n+1}+1}$ and $a_j \geq 1, j = 1, \cdots, n$.

Assumption 8: There are constants $\kappa_{nj} \geq b_j$, $1 \leq j \leq n$, and $\kappa_{ij} \geq \frac{b_j}{a_{i+1}}$, $i = 1, \cdots, n-1$, $1 \leq j \leq i$, such that

$$\phi_i(\cdot) \leq d(|x_1|^{\kappa_{i1}} + \cdots + |x_i|^{\kappa_{ii}}) \tag{2}$$

for a known constant $d \geq 0$.

Remark 9: The above two assumptions have given the restrictions on the uncertain powers q_i's and the nonlinear functions ϕ_i's. When the sensors are able to measure the system states accurately, $a_i = b_i = 1$ holds and the nonlinearities satisfy the linear growth condition, i.e., $\phi_i(\cdot) \leq \gamma_i(\cdot)(|x_1| + \cdots + |x_i|)$, with a $\mathcal{C}^1$ function $\gamma_i(\cdot)$. The SGAS result has been achieved in the work [13] by appropriately choosing the Lyapunov functions and the responding level sets.

Remark 10: Under Assumption 7, it can be concluded that the homogeneous degrees of the system (1) are nonincremental. Specifically, by choosing the homogeneous weights $r_i = \frac{1}{q_i}, i = 1, \cdots, n$ and $r_{n+1} = 1$, one has

$$\begin{aligned}\tau_i - \tau_{i+1} &= (r_{i+1} - r_i) - (r_{i+2} - r_{i+1}) \\ &= (\frac{1}{q_{i+1}} - \frac{1}{q_i}) - (\frac{1}{q_{i+2}} - \frac{1}{q_{i+1}}) \\ &\geq \frac{2}{b_{i+1}} - \frac{1}{a_i} - \frac{1}{a_{i+2}} \geq 0, \quad i = 1, \cdots, n-2,\end{aligned} \tag{3}$$

and $\tau_{n-1} - \tau_n = \frac{2}{q_n} - \frac{1}{q_{n-1}} - 1 \geq \frac{2}{b_n} - \frac{1}{a_{n-1}} - 1 \geq 0$. Therefore, the system (1) is said to be homogeneous with monotone degrees (HWMD).

Now, the main result of this paper is summarized as follows:

Theorem 11: Under Assumptions 7 and 8, there exists a linear controller such that the closed-loop system is SGAS via measurement feedback.

Proof: According to Definition 3, we will design a linear controller such that the system states converge to the origin as long as the initial values starting from the compact set B_M.

Initial Step: Choose the first Lyapunov function $V_1(x_1) = \frac{r_1}{2-\tau_1} x_1^{\frac{2-\tau_1}{r_1}}$ and one has $\forall x \in B_M$,

$$V_1(x_1) \leq \frac{b_2}{2a_1a_2 - b_1 + a_2}(M^{2a_1+1-\frac{b_1}{a_2}} + M^{2b_1+1-\frac{a_1}{b_2}}). \tag{4}$$

Based on Assumption 8, the derivative of V_1 along the trajectory of system (1) is

$$\begin{aligned}\dot{V}_1 &\leq x_1^{\frac{2-\tau_1-r_1}{r_1}}(x_2 - x_2^*) + x_1^{\frac{2-\tau_1-r_1}{r_1}} x_2^* + d|x_1|^{\frac{2}{r_1}}|x_1|^{\kappa_{11}-\frac{r_2}{r_1}} \\ &= \xi_1^{2-r_2}(x_2 - x_2^*) + \xi_1^{2-r_2}x_2^* + \xi_1^2 h_{1,1}(x_1)\end{aligned} \tag{5}$$

with $\xi_1 = x_1^{\frac{1}{r_1}}$, $h_{1,1}(\cdot) = |x_1|^{\kappa_{11}-\frac{r_2}{r_1}}$ and a virtual controller x_2^* to be determined later.

For the first Lyapunov function $V_1(x_1)$, the corresponding level set is designed as $\Omega_1 \triangleq \{x \in \mathbb{R}^n | V_1(x_1) \leq N\}$, with

$$\begin{aligned}N = {} & \frac{b_2}{2a_1a_2 - b_1 + a_2}(M^{2a_1+1-\frac{b_1}{a_2}} + M^{2b_1+1-\frac{a_1}{b_2}}) \\ & + \sum_{i=2}^{n} 2^{1-\frac{1}{b_i}}\Big((\sum_{k=1}^{i} M^{a_k} + \sum_{k=1}^{i} M^{b_k})^{2+\frac{1}{a_i}-\frac{1}{b_{i+1}}} \\ & + (\sum_{k=1}^{i} M^{a_k} + \sum_{k=1}^{i} M^{b_k})^{2+\frac{1}{b_i}-\frac{1}{a_{i+1}}}\Big).\end{aligned} \tag{6}$$

This leads to that if $x \in B_M$, $V_1(x_1) \leq N$ holds naturally, i.e., $B_M \subset \Omega_1$.

Since $\kappa_{11} \geq \frac{b_1}{a_2} \geq \frac{r_2}{r_1}$, then $h_{1,1}(\cdot)$ is bounded on Ω_1, i.e., $h_{1,1}(\cdot) \leq M^{\kappa_{11}-\frac{b_1}{a_2}} + M^{\kappa_{11}-\frac{a_1}{b_2}} := \bar{h}_{1,1}$. By choosing the virtual controller

$$x_2^* = -\beta_1^{r_2}\xi_1^{r_2} = -\beta_1^{r_2}y_1^{r_2}, \quad \beta_1 \geq (\bar{h}_{1,1} + n)^{b_2} \tag{7}$$

the derivative of $V_1(x_1)$ becomes

$$\dot{V}_1(x_1)|_{\Omega_1} \leq -n\xi_1^2 + \xi_1^{2-r_2}(x_2 - x_2^*). \tag{8}$$

Obviously, the coefficient β_1 only involves the upper and lower bounds of q_1, q_2.

Step 2: By selecting $W_2(x_1, x_2) = (\frac{1}{\beta_1})^{2-\tau_2}\int_{x_2^*}^{x_2}(s^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}})^{2-r_3}ds$ and $\xi_2 = x_2^{\frac{1}{r_2}} - x_2^{*\frac{1}{r_2}}$, it can be calculated that $\forall x \in B_M$,

$$\begin{aligned}W_2(\cdot) &\leq (\frac{1}{\beta_1})^{2-\tau_2}2^{1-r_2}|\xi_2|^{2-\tau_2} \leq 2^{1-r_2}|x_2^{\frac{1}{r_2}} + x_1^{\frac{1}{r_1}}|^{2-\tau_2} \\ &\leq 2^{1-\frac{1}{b_2}}\Big((M^{a_1} + M^{b_1} + M^{a_2} + M^{b_2})^{2+\frac{1}{a_2}-\frac{1}{b_3}} \\ &\quad + (M^{a_1} + M^{b_1} + M^{a_2} + M^{b_2})^{2+\frac{1}{b_2}-\frac{1}{a_3}}\Big)\end{aligned} \tag{9}$$

with the lower and upper limits of q_1, q_2, q_3. Based on (9), the associated level set is defined as

$$\Omega_2 \triangleq \{x \in \mathbb{R}^n | V_2(x_1, x_2) = V_1(x_1) + W_2(x_1, x_2) \leq N\}, \tag{10}$$

which implies that

$$\forall x \in B_M \Rightarrow V_1(x_1) \leq V_2(x_1, x_2) \leq N \Rightarrow B_M \subset \Omega_2 \subset \Omega_1. \tag{11}$$

Therefore, it can be derived that

$$\begin{aligned}\dot{V}_2|_{\Omega_2} &= \dot{V}_1|_{\Omega_2} + (\frac{1}{\beta})^{2-\tau_2}\xi_2^{2-r_3}(x_3 + \phi_2) + \frac{\partial W_2}{\partial x_1}(x_2 + \phi_1) \\ &\leq -n\xi_1^2 + (\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}(x_3 - x_3^*) \\ &\quad + (\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}x_3^* + \xi_1^{2-r_2}(x_2 - x_2^*) \\ &\quad + (\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}\phi_2 + \frac{\partial W_2}{\partial x_1}(x_2 + \phi_1).\end{aligned} \tag{12}$$

Based on Lemmas 5 and 6, one can get

$$\xi_1^{2-r_2}(x_2-x_2^*)\le 2^{1-r_2}|\xi_1|^{2-r_2}|\xi_2|^{r_2}\le \frac{1}{3}\xi_1^2+c_2\xi_2^2, \tag{13}$$

where $c_2=\frac{1}{a_2 2^{1/b_2}}\left((6-\frac{3}{a_2})\frac{1}{2^{1/b_2}}\right)^{2b_2-1}$ is dependent of the upper and lower bounds of q_2.

According to Assumption 8, one has $\kappa_{2j}r_j\ge \frac{b_j}{a_3}r_j\ge r_3$, which leads to that

$$\begin{aligned}
&(\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}\phi_2\\
&\le(\frac{1}{\beta_1})^{2+\frac{1}{a_2}-\frac{1}{b_3}}d_1|\xi_2|^{2-r_3}(|\xi_1|^{\kappa_{21}r_1}+|\xi_2|^{\kappa_{22}r_2}\\
&\quad+\beta_1^{\kappa_{22}r_2}|\xi_1|^{\kappa_{22}r_2})\\
&\le(\frac{1}{\beta_1})^{2+\frac{1}{a_2}-\frac{1}{b_3}}d_1|\xi_2|^{2-r_3}(|\xi_1|^{r_3}+|\xi_2|^{r_3})\\
&\quad\times(|\xi_1|^{\kappa_{21}r_1-r_3}+|\xi_2|^{\kappa_{22}r_2-r_3}+\beta_1^{\kappa_{22}r_2}|\xi_1|^{\kappa_{22}r_2-r_3})\\
&\le\frac{1}{3}\xi_1^2+h_{2,1}(x_1,x_2)\xi_2^2,
\end{aligned}\tag{14}$$

where $d_1>0$ is a constant, $h_{2,1}(\cdot)>0$ is a continuous function of x_1,x_2 and only involves the known bounds of q_1,q_2,q_3.

In a similar way, the estimate for the last term in (12) can be given as

$$\begin{aligned}
\frac{\partial W_2}{\partial x_1}(x_2+\phi_1)&\le\frac{2-r_3}{r_1\beta_1^{1-\tau_2}}2^{1-r_2}|\xi_1|^{1-r_1}|\xi_2|^{1-\tau_2}\\
&\quad\times(|\xi_2|^{r_2}+\beta_1^{r_2}|\xi_1|^{r_2}+d|\xi_1|^{\kappa_{11}r_1})\\
&\le\frac{1}{3}\xi_1^2+h_{2,2}(x_1,x_2)\xi_2^2
\end{aligned}\tag{15}$$

with a continuous function $h_{2,2}(\cdot)>0$. Clearly, the last inequality holds owing to $\tau_1\ge\tau_2$.

Substituting (13)-(15) into (12) yields

$$\begin{aligned}
\dot V_2|_{\Omega_2}&\le-(n-1)\xi_1^2+(\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}(x_3-x_3^*)\\
&\quad+(\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}x_3^*+(c_2+h_{2,1}(\cdot)+h_{2,2}(\cdot))\xi_2^2.
\end{aligned}\tag{16}$$

Note that $\forall x\in B_M$, x_1,x_2 are bounded on Ω_2. As a result, the continuous functions $h_{2,1}(\cdot),h_{2,2}(\cdot)$ are bounded on the level set Ω_2, i.e., $h_{2,1}(\cdot)\le\bar h_{2,1}$, $h_{2,2}(\cdot)\le\bar h_{2,2}$.

By choosing the virtual controller $x_3^*=-\beta_2^{r_3}\xi_2^{r_3}$, $\beta_2\ge\beta_1^{2b_3+\frac{b_3}{a_2}-1}(\bar h_{2,1}+\bar h_{2,2}+c_2+n-1)^{b_3}$, one has

$$\dot V_2|_{\Omega_2}\le-(n-1)\xi_1^2-(n-1)\xi_2^2+(\frac{1}{\beta_1})^{2-\tau_2}\xi_2^{2-r_3}(x_3-x_3^*). \tag{17}$$

Step k: Suppose that at step $k-1$, there is a C^1 Lyapunov function $V_{k-1}:\mathbb{R}^{k-1}\to\mathbb{R}$ with the level sets $\Omega_{k-1}\triangleq\{x\in\mathbb{R}^n|V_{k-1}(x_1,\cdots,x_{k-1})\le N\}$ satisfying $\Omega_{k-1}\subset\cdots\subset\Omega_1$, and a set of virtual controllers x_i^* defined as

$$\begin{aligned}
x_1^*&=0,\xi_1=x_1^{1/r_1}-x_1^{*1/r_1},\\
x_i^*&=-\beta_{i-1}^{r_i}\xi_{i-1}^{r_i},\quad \xi_i=x_i^{1/r_i}-x_i^{*1/r_i},\ i=2,\cdots,k
\end{aligned}\tag{18}$$

with positive constants $\beta_1,\cdots,\beta_{k-1}$, such that

$$\begin{aligned}
\dot V_{k-1}|_{\Omega_{k-1}}&\le-(n-k+2)\sum_{i=1}^{k-1}\xi_i^2\\
&\quad+(\frac{1}{\beta_1\cdots\beta_{k-2}})^{2-\tau_{k-1}}\xi_{k-1}^{2-r_k}(x_k-x_k^*).
\end{aligned}\tag{19}$$

In what follows, we will prove that (19) also holds at step k. Construct the kth Lyapunov function

$$\begin{aligned}
V_k(x_1,\cdots,x_k)&=V_{k-1}(\cdot)+W_k(\cdot)\\
&=V_{k-1}(\cdot)+(\frac{1}{\beta_1\cdots\beta_{k-1}})^{2-\tau_k}\int_{x_k^*}^{x_k}\\
&\quad\times(s^{\frac{1}{r_k}}-x_k^{*\frac{1}{r_k}})^{2-r_{k+1}}ds
\end{aligned}\tag{20}$$

and the corresponding level set $\Omega_k=\{x\in\mathbb{R}^n|V_k(x_1,\cdots,x_k)\le N\}$. Then, it can be calculated that $\forall x\in B_M$,

$$\begin{aligned}
W_k&\le(\frac{1}{\beta_1\cdots\beta_{k-1}})^{2-\tau_k}2^{1-r_k}|\xi_k|^{2-\tau_k}\\
&\le 2^{1-\frac{1}{b_i}}|x_1^{\frac{1}{r_1}}+x_2^{\frac{1}{r_2}}+\cdots+x_k^{\frac{1}{r_k}}|^{2-\tau_k}\\
&\le 2^{1-\frac{1}{b_i}}\Big((\sum_{i=1}^k M^{a_i}+\sum_{i=1}^k M^{b_i})^{2+\frac{1}{a_k}-\frac{1}{b_{k+1}}}\\
&\quad+(\sum_{i=1}^k M^{a_i}+\sum_{i=1}^k M^{b_i})^{2+\frac{1}{b_k}-\frac{1}{a_{k+1}}}\Big),
\end{aligned}\tag{21}$$

which indicates that $\forall x\in B_M$, $V_{k-1}(\cdot)\le V_k(\cdot)\le N$, i.e., $B_M\subset\Omega_k\subset\Omega_{k-1}$ holds. Therefore, the derivative of V_k arrives at

$$\begin{aligned}
\dot V_k|_{\Omega_k}&\le-(n-k+2)\sum_{i=1}^{k-1}\xi_i^2+(\frac{1}{\beta_1\cdots\beta_{k-1}})^{2-\tau_k}\xi_k^{2-r_{k+1}}\\
&\quad\times(x_{k+1}-x_{k+1}^*)+(\frac{1}{\beta_1\cdots\beta_{k-1}})^{2-\tau_k}\xi_k^{2-r_{k+1}}x_{k+1}^*\\
&\quad+(\frac{1}{\beta_1\cdots\beta_{k-2}})^{2-\tau_{k-1}}\xi_{k-1}^{2-r_k}(x_k-x_k^*)\\
&\quad+(\frac{1}{\beta_1\cdots\beta_{k-1}})^{2-\tau_k}\xi_k^{2-r_{k+1}}\phi_k\\
&\quad+\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(x_{i+1}+\phi_i).
\end{aligned}\tag{22}$$

With the definition of ξ_k, the following inequalities can be easily achieved based on Lemmas 5 and 6,

$$\begin{aligned}
&(\frac{1}{\beta_1\cdots\beta_{k-2}})^{2-\tau_{k-1}}\xi_{k-1}^{2-r_k}(x_k-x_k^*)\\
&\le(\frac{1}{\beta_1\cdots\beta_{k-2}})^{2-\tau_{k-1}}2^{1-r_k}|\xi_{k-1}|^{2-r_k}|\xi_k|^{r_k}\\
&\le\frac{1}{3}\xi_{k-1}^2+c_k\xi_k^2,
\end{aligned}\tag{23}$$

where c_k is a positive constant dependent of a_i's and b_i's, $i = 1, \cdots, k$.

Similar to (14) and (15), the estimates for the last two terms in the right-hand side of (22) will be given in the following propositions whose proofs are included in Appendix.

Proposition 12: There exists a continuous function $h_{k,1}(x_1, \cdots, x_k) \geq 0$, such that

$$(\frac{1}{\beta_1 \cdots \beta_{k-1}})^{2-\tau_k} \xi_k^{2-r_{k+1}} \phi_k \leq \frac{1}{3} \sum_{i=1}^{k-1} \xi_i^2 + h_{k,1}(\cdot)\xi_k^2. \quad (24)$$

Proposition 13: There is a continuous function $h_{k,2}(x_1, \cdots, x_k) \geq 0$, satisfying

$$\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}(x_{i+1} + \phi_i) \leq \frac{1}{3} \sum_{i=1}^{k-1} \xi_i^2 + h_{k,2}(\cdot)\xi_k^2. \quad (25)$$

Substituting (23)-(25) into (22), one has

$$\begin{aligned} \dot{V}_k|_{\Omega_k} \leq& -(n-k+1)\sum_{i=1}^{k-1} \xi_i^2 + (\frac{1}{\beta_1 \cdots \beta_{k-1}})^{2-\tau_k} \xi_k^{2-r_{k+1}} \\ &\times (x_{k+1} - x_{k+1}^*) + (\frac{1}{\beta_1 \cdots \beta_{k-1}})^{2-\tau_k} \xi_k^{2-r_{k+1}} x_{k+1}^* \\ &+ (c_k + h_{k,1}(\cdot) + h_{k,2}(\cdot))\xi_k^2. \end{aligned} \quad (26)$$

Thus, we can construct the virtual controller $x_{k+1}^* = -\beta_k^{r_{k+1}} \xi_k^{r_{k+1}}$, $\beta_k \geq (n-k+1+c_k+\bar{h}_{k,1}+\bar{h}_{k,2})^{b_{k+1}}(\beta_1 \cdots \beta_{k-1})^{2b_{k+1}+\frac{b_{k+1}}{a_k}-1}$ with the bounds of $h_{k,1}(\cdot)$, $h_{k,2}(\cdot)$ on the level set Ω_k, such that

$$\begin{aligned} \dot{V}_k|_{\Omega_k} \leq& -(n-k+1)\sum_{i=1}^{k-1} \xi_i^2 \\ &+ (\frac{1}{\beta_1 \cdots \beta_{k-1}})^{2-\tau_k} \xi_k^{2-r_{k+1}} (x_{k+1} - x_{k+1}^*). \end{aligned} \quad (27)$$

This completes the inductive proof.

Last Step: Based on the inductive arguments, it can be proved that (27) holds for $k = n$. As a matter of fact, we can choose the Lyapunov function $V_n = V_{n-1} + (\frac{1}{\beta_1 \cdots \beta_{n-1}})^{2-\tau_n} \int_{x_n^*}^{x_n} (s^{\frac{1}{r_n}} - x_n^{*\frac{1}{r_n}}) ds$, and the controller

$$u = x_{n+1}^* = -\beta_n\Big(y_n + \beta_{n-1}\big(y_{n-1} + \cdots + \beta_2(y_2 + \beta_1 y_1)\big)\Big) \quad (28)$$

such that $\dot{V}_n|_{\Omega_n} \leq -\sum_{i=1}^n \xi_i^2$, with the level set $\Omega_n \triangleq \{x \in \mathbb{R}^n | V_n(x_1, \cdots, x_n) \leq N\}$ satisfying $B_M \subset \Omega_n \subset \cdots \subset \Omega_1$ and $\beta_n \geq 0$.

In conclusion, according to Definition 3, we can claim that for any system states starting from the compact set B_M, the linear controller (28) with the controller gains β_i's depending on M will make the system states converge to the origin asymptotically, i.e., the system (1) is SGAS by the linear controller (28).

Remark 14: Letting $p_i = 1$ in the work [10] where high-order nonlinear systems are investigated, a *nonsmooth* measurement feedback controller can be designed to make the system (1) globally asymptotically stable. However, linear controllers are easy to be implemented. By taking a tradeoff, the less ambitious control goal, SGAS rather than GAS may be better in practical applications.

IV. AN ILLUSTRATIVE EXAMPLE

Consider the following nonlinear system

$$\dot{x}_1 = x_2 + 0.1x_1^2, \quad \dot{x}_2 = u, \ y_1 = x_1^{q_1}, \ y_2 = x_2, \quad (29)$$

where q_1 is an uncertain constant but belongs to the interval $[1, 2]$, $|x_1(0)| \leq 1.5$ and $|x_2(0)| \leq 1.5$. By simple calculation, Assumptions 7 and 8 hold naturally with $b_2 \leq \frac{2a_1}{a_1+1} = 1$ and $\kappa_{11} \geq \frac{b_1}{a_2} = 2$. Therefore, according to Theorem 11, a linear controller can be designed to make the system (29) SGAS.

Specifically, choose $V_1 = \frac{1}{1+q_1} x_1^{1+q_1}$ and $\Omega_1 = \{x \in \mathbb{R}^2 | V_1(x_1) \leq 16.6\}$. The derivative of V_1 is

$$\begin{aligned} \dot{V}_1|_{\Omega_1} &= x_1^{q_1}(x_2 - x_2^*) + x_1^{q_1} x_2^* + 0.1x_1^{q_1} x_1^2 \\ &\leq \xi_1(x_2 - x_2^*) + \xi_1 x_2^* + 0.1\xi_1^2(1 + |\xi_1|) \\ &\leq \xi_1(x_2 - x_2^*) + \xi_1 x_2^* + 0.475\xi_1^2 \end{aligned} \quad (30)$$

for $\xi_1 = x_1^{q_1} = y_1$. With the virtual controller $x_2^* = -1.475\xi_1$, (30) becomes

$$\dot{V}_1|_{\Omega_1} \leq -\xi_1^2 + \xi_1(x_2 - x_2^*). \quad (31)$$

Defining $V_2 = V_1 + 0.2298\xi_2^2$ with $\xi_2 = x_2 - x_2^*$ and $\Omega_2 = \{x \in \mathbb{R}^2 | V_2(x_1, x_2) \leq 16.6\}$, one has $[-1.5, 1.5]^2 \subset \Omega_2 \subset \Omega_1$. It can be deduced that

$$\begin{aligned} \dot{V}_2|_{\Omega_2} &= \dot{V}_1|_{\Omega_2} + 0.4596\xi_2(u + 1.475q_1 x_1^{q_1-1}(x_2 + 0.1x_1^2)) \\ &\leq -\xi_1^2 + \xi_1\xi_2 + 0.4596\xi_2 u + 0.1356\xi_1^{1+\frac{1}{q_1}}|\xi_2| \\ &\quad + 1.3559\xi_1^{1-\frac{1}{q_1}}|\xi_2|(|\xi_2| + \beta_1|\xi_1|) \\ &\leq -\frac{1}{4}\xi_1^2 + 0.4596\xi_2 u + 17.1632\xi_2^2. \end{aligned} \quad (32)$$

Therefore, one can construct the controller

$$u = -37.3438\xi_2 = -37.3438(y_2 + 1.475y_1) \quad (33)$$

such that $\dot{V}_2|_{\Omega_2} \leq -\frac{1}{4}\xi_1^2 - \frac{1}{4}\xi_2^2$.

According to Theorem 11, the designed linear controller (33) not only has simple structure, but also is robust to $q_1 \in [1, 2]$. In order to show the effectiveness, we choose $q_1 = \frac{5}{3}$ and $q_1 = \frac{13}{7}$, respectively, with the same initial value $(x_1(0), x_2(0)) = (1, -1.5)$ to conduct the simulation. As illustrated in FIGURE 1 and FIGURE 2, it can be concluded that the system states starting from the set $[-1.5, 1.5]^2$ can converge to the origin in both situations. Therefore, it implies that as long as $q_1 \in [1, 2]$, the controller u can semi-globally asymptotically stabilize the closed-loop system (29)-(33) under different kinds of sensors, which is consistent with the theoretical analysis in the controller design procedure.

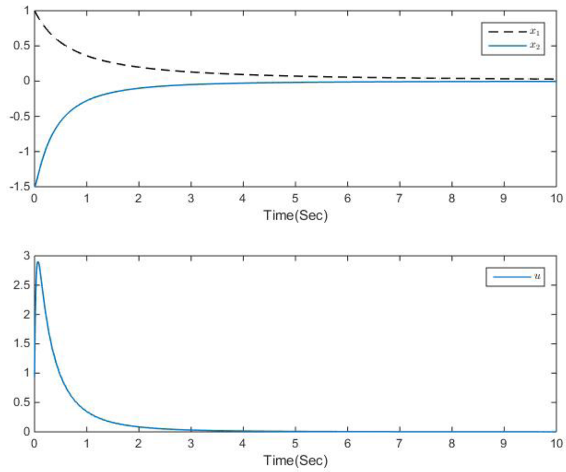


FIGURE 1. **The responses of the closed-loop system (29) and (33) with $q_1 = \frac{5}{3}$.**

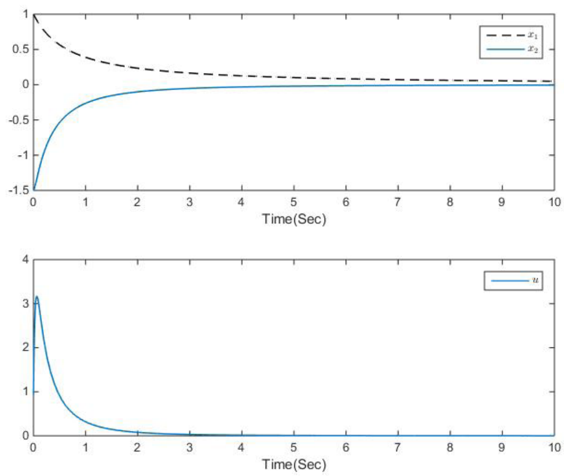


FIGURE 2. **The responses of the closed-loop system (29) and (33) with $q_1 = \frac{13}{7}$.**

V. CONCLUSION

In this paper, we have developed a systematic controller design scheme for a class of nonlinear systems with uncertain measurement functions by adopting the notion of HWMD. With respect to any prescribed bounds for the initial values, the designed linear controller depending on the known bounds can make the trajectories of the system states converge to the origin asymptotically, i.e., the closed-loop system is SGAS. Moreover, since only the uncertain measurements and the known bounds are used, the robustness of the controller holds for different sensors as long as the the unknown powers q_i's belong to certain intervals.

APPENDIX

This part contains the technical details of the proofs. For convenience, we introduce some generic functions $g_i(x_1, \cdots, x_i) \geq 0$ to represent any continuous functions only dependent of the known bounds of q_i's and may be implicitly changed in various places.

Proof of Proposition 12: According to Assumption 8 and the definition of ξ_i's, one has

$$\begin{aligned}\phi_k &\leq d\Big(|\xi_1|^{\kappa_{k1} r_1} + \sum_{i=2}^{k} |\xi_i + \beta_{i-1}\xi_{i-1}|^{\kappa_{ki} r_i}\Big)\\ &\leq g_k(\cdot)(|\xi_1|^{r_{k+1}} + \cdots + |\xi_k|^{r_{k+1}}),\end{aligned} \tag{34}$$

under which

$$\begin{aligned}&\Big(\frac{1}{\beta_1 \cdots \beta_{k-1}}\Big)^{2-\tau_k} \xi_k^{2-r_{k+1}} \phi_k\\ &\qquad \leq \frac{1}{3} \sum_{i=1}^{k-1} \xi_i^2 + h_{k,1}(x_1, \cdots, x_k)\xi_k^2\end{aligned} \tag{35}$$

with a continuous function $h_{k,1}(\cdot) \geq 0$ dependent of the known bounds q_i's, $i = 1, \cdots, k$.

Proof of Proposition 13: According to (34), Lemmas 5 and 6, one can get

$$\begin{aligned}&\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}(x_{i+1} + \phi_i)\\ &\leq \sum_{i=1}^{k-1} g_i(\cdot)|\xi_k|^{1-r_{k+1}}|\xi_k|^{r_k}|x_i|^{\frac{1-r_i}{r_i}}\\ &\quad \times (|\xi_1|^{r_{i+1}} + \cdots + |\xi_{i+1}|^{r_{i+1}})\\ &\leq \sum_{i=1}^{k-1} g_i(\cdot)|\xi_k|^{1-\tau_k}(|\xi_i|^{1-r_i} + |\xi_{i-1}|^{1-r_i})\\ &\quad \times (|\xi_1|^{r_{i+1}} + \cdots + |\xi_{i+1}|^{r_{i+1}}).\end{aligned} \tag{36}$$

From Remark 10, we know that the homogeneous degrees satisfy $\tau_1 \geq \tau_2 \cdots \geq \tau_n$, and then $1 - \tau_k + 1 - r_i + r_{i+1} = 2 - \tau_k - \tau_i \geq 0$, $\forall i = 1, \cdots, k-1$. Therefore, Proposition 13 holds naturally, with a continuous function $h_{k,2}(x_1, \cdots, x_k) \geq 0$ and Lemma 6.

REFERENCES

[1] Z.-P. Jiang, I. Mareels, and D. Hill, "Robust control of uncertain nonlinear systems via measurement feedback," *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 807–812, Apr. 1999.

[2] A. A. Ball and H. K. Khalil, "Analysis of a nonlinear high-gain observer in the presence of measurement noise," in *Proc. ACC*, San Francisco, CA, USA, Jun. 2011, pp. 2584–2589.

[3] J. H. Ahrens and H. K. Khalil, "High-gain observers in the presence of measurement noise: A switched-gain approach," *Automatica*, vol. 45, no. 4, pp. 936–943, Apr. 2009.

[4] H.-W. Jo, H.-L. Choi, and J.-T. Lim, "Measurement feedback control for a class of feedforward nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 23, no. 12, pp. 1405–1418, Apr. 2012.

[5] Z. Chen, "A remark on sensor disturbance rejection of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2206–2210, Sep. 2009.

[6] C. J. Qian and J. Y. Zhai, "Global control of nonlinear systems with uncertain output function using homogeneous domination approach," *Int. J. Robust Nonlinear Control*, vol. 22, no. 14, pp. 1543–1562, Sep. 2012.

[7] W. Q. Ai, J.-Y. Zhai, S.-M. Fei, and W.-T. Zha, "Decentralized global output feedback stabilization for a class of uncertain nonlinear systems," *Trans. Inst. Meas. Control*, vol. 35, no. 6, pp. 777–787, Aug. 2013.

[8] W. T. Zha, J. Y. Zhai, and S. M. Fei, "Global adaptive control for a class of uncertain stochastic nonlinear systems with unknown output gain," *Int. J. Control Autom. Syst.*, vol. 15, no. 3, pp. 1125–1133, Jun. 2017.

[9] W. Zha, C. Qian, J. Zhai, and S. Fei, "Robust control for a class of nonlinear systems with unknown measurement drifts," *Automatica*, vol. 71, pp. 33–37, Sep. 2016.

[10] W. Zha, C. Qian, J. Zhai, and S. Fei, "Robust control for a class of high-order uncertain nonlinear systems via measurement feedback," *Int. J. Control*, to be published. Accessed: Nov. 21, 2017, doi: 10.1080/00207179.2017.1396361.

[11] Z. Lin and A. Saberi, "Robust semiglobal stabilization of minimum-phase input-output linearizable systems via partial state and output feedback," *IEEE Trans. Autom. Control*, vol. 40, no. 6, pp. 1029–1041, Jun. 1995.

[12] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Syst. Control. Lett.*, vol. 22, no. 5, pp. 313–325, May 1994.

[13] A. Teel and L. Praly, "Tools for semiglobal stabilization by partial state and output feedback," *SIAM J. Control Optim.*, vol. 33, no. 5, pp. 1443–1488, 1995.

[14] A. Isidori, "A tool for semi-global stabilization of uncertain non-minimum-phase nonlinear systems via output feedback," *IEEE Trans. Autom. Control*, vol. 45, no. 10, pp. 1817–1827, Oct. 2000.

[15] C. Qian, "Semi-global stabilization of a class of uncertain nonlinear systems by linear output feedback," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 4, pp. 218–222, Apr. 2005.

[16] B. Yang and W. Lin, "Semi-global stabilization of nonlinear systems by nonsmooth output feedback," *Int. J. Robust Nonlinear Control*, vol. 24, no. 16, pp. 2522–2545, May 2013.

[17] A. M. Boker and H. K. Khalil, "Semi-global output feedback stabilization of non-minimum phase nonlinear systems," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4005–4010, Aug. 2017.

[18] F. Gao, Y. Wu, and F. Yuan, "Semi-global finite-time stabilization of uncertain nonholonomic systems via output feedback," *Trans. Inst. Meas. Control*, vol. 37, no. 1, pp. 122–130, Jan. 2015.

[19] C. Zhang, R. Jia, C. Qian, and S. Li, "Semi-global stabilization via linear sampled-data output feedback for a class of uncertain nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 25, no. 13, pp. 2041–2061, 2015.

[20] S. Battilotti, "A unifying framework for the semiglobal stabilization of nonlinear uncertain systems via measurement feedback," *IEEE Trans. Autom. Control*, vol. 46, no. 1, pp. 3–16, Jan. 2001.

[21] B. Yang and W. Lin, "What can linear state feedback accomplish for nonlinear systems?" in *Proc. IEEE CDC*, Cancún, Mexico, Dec. 2008, pp. 1593–1598.

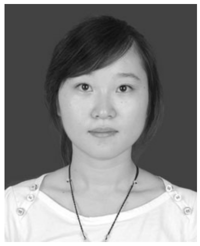
[22] X. Jia, W. Chen, and Z. Liu, "Semi-global output feedback stabilization of upper-triangular systems with uncertain output function," in *Proc. CCDC*, Yinchuan, China, May 2016, pp. 6479–6484.

[23] H. Du and S. Li, "Semi-global finite-time attitude stabilization by output feedback for a rigid spacecraft," *Proc. Inst. Mech. Eng., G, J. Aerosp. Eng.*, vol. 227, no. 12, pp. 1881–1891, 2013.

[24] C. Zhang, Y. Yan, A. Narayan, and H. Yu, "Practically oriented finite-time control design and implementation: Application to a series elastic actuator," *IEEE Trans. Ind. Electron.*, vol. 65, no. 5, pp. 4166–4176, May 2018.

[25] C. Zhang, Y. Yan, C. Wen, J. Yang, and H. Yu, "A nonsmooth composite control design framework for nonlinear systems with mismatched disturbances: Algorithms and experimental tests," *IEEE Trans. Ind. Electron.*, vol. 65, no. 11, pp. 8828–8839, Nov. 2018.

[26] C. Qian and W. Lin, "Non-Lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization," *Syst. Control Lett.*, vol. 42, no. 3, pp. 185–200, 2001.



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