On the Inter-Departure Times in $M/\tilde{D}/1/B_{on}$ Queue With Queue-Length Dependent Service and Deterministic/Exponential Vacations

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ABSTRACT We derive the distribution of inter-departure times of a finite-buffer single-server queue with Poisson arrival process and queue-length-dependent service times, where the server goes to vacation if either the queue is emptied or a limited number (R_1) of packets are served, whichever occurs first, in the current busy period. We consider two types of vacation distributions: 1) deterministic and 2) exponential. Queue-length distribution at embedded points is derived first, then, the distribution and variance of inter-departure times are derived, for both types of vacations. The simulation results are in good agreement with the derived analytical results. The above framework would be useful at the receiver in modeling and analyzing the jitter and the waiting time of time-division multiplexing (TDM) emulated packets in TDM over packet-switched network (TDM over PSN) technology as a function of a buffer size.

INDEX TERMS Inter-departure times, variance, limited service discipline, queue-length dependent service, vacation queue.

I. INTRODUCTION

At least in the third world countries, the voice service currently supported by legacy (TDM based) carrier networks would be replaced by packet infrastructure as and when the legacy equipment fails. As an intermediate phase of this transition, one encounters a situation of TDM islands in a packet network, wherein the TDM services have to be emulated in a packet network for its end-to-end connectivity. This paper involves in statistical modeling and analysis of an intermediate node in the packet island and jitter buffer at the receiver Network-Network-Interface (NNI) in such networks.

The *requirements* of an intermediate router in emulating TDM in packet networks are: (a) QoS implemented through priority based multi-class queueing scheme (b) minimization of variance of inter-departure times (IDTs) (c) minimization of TEP loss and (d) service of contemporary TEPs. By 'contemporary' TEPs, we mean the voice samples of many caller-callee at a given TDM frame interval (The TEPs carrying voice samples in the next TDM frame could wait for the next round of the scheduler - explained below). Similarly, the requirements of the jitter buffer at the receiver NNI are items (b) through (d) above, apart from one important unique requirement (e), there should always be TDM frames to transmit (or video frames in VoD service or audio

packets in streaming audio to play). The modelling aspects mentioned in this paper are tailor made for the above requirements.

We now give the gist of our model and the related work, reserving the details and relevant justifications for Section II. We study a single server queueing system of finite-buffer (B_{on}) with Poisson input and queue-length dependent service along with multiple vacation policy and R_1−limited service vacation. The above queue-length dependent service times $\{s_j : 1 \leq j \leq B_{on}\}$ are dictated by algorithm-B [1] (see Section II(B) and Appendix A), which can be used for rate-jitter control in packet-switched networks. In the algorithm-B, service time of a head-of-line packet is calculated at its service initiation instant and is a linearly decreasing function of queue-length at that instant. In multiple vacation policy, a server immediately goes for another vacation if it finds the queue still empty at a vacation completion instant. We consider following distributions for vacation intervals: (i) deterministic and (ii) exponential.

We use the embedded Markov chain approach to derive the distributions of our interest. We first derive the queue-length distribution (QLD) at embedded points (time instants at which we observe the state of the system), then the distribution of IDTs is derived for both types of vacation.

Using this, we find expressions for the mean and variance of IDTs, for both types of vacation distributions.

The main goal of this paper is to derive the distribution and variance of IDTs of the queueing model under study and to demonstrate (by analysis & simulations) that the variance of IDTs (equivalently packet jitter) could be minimized by using a family of scheduling algorithm (algorithm-B) similar to algorithms developed by Yishay and Patt-Shamir [2] to control packet jitter.

A. LITERATURE REVIEW

Traditionally, the motivation for the study of departure process is to characterize the arrival process to the next node downstream. It has practical significance in analyzing queueing networks. But, in the context of TDM over PSN problem, it is of paramount importance to quantify the jitter of the departing TDM frames, both at an intermediate network node and in the output of jitter buffer at the Inter-Working Function (IWF) or NNI.

Burke has shown that the IDTs of an $M/M/c$ are IIDs with exponential distribution (with parameter related to Poisson mean arrival time) [3]. A review of output processes of a general queue can be found in [4]. An $M/G/1$ queue with exhaustive service discipline and both single and multiple vacation policies is studied in [5] and the Laplace-Stieltjes Transforms (LSTs) of occupation period, busy period and waiting time are derived. The generating function of number of packets in the system is also derived in [5]. The departure process of $M/G/1$ queue with exhaustive service discipline for both single and multiple vacation policies is derived in [6]. Matzka [7] has calculated the IDT distribution for single server $G/G/1$ queue and used this result for calculating approximated distribution of IDTs. Stanford *et al.* derived LSTs of IDTs of each class in a multi-class input queue, wherein arrivals in each class follow Poisson process and each class has its own generally distributed service times [8]. There has been considerable literature on IDTs in priority queues (both preemptive and non-preemptive) [9]–[11]. Kramer has derived QLD in $M/G/1$ queue with finite capacity and limited service system [12]. To derive QLD of the queueing system under study, we have followed the method proposed in [13].

The paper is organized as given below: In Section II, we discuss about our system model, assumptions and the outline of analysis. In Section III, we derive QLD using Embedded Markov Chain (EMC) analysis. In Sections IV and V, we derive distribution of IDTs when vacations are deterministic and exponential, respectively. Section VI discusses about simulation results and finally we conclude in Section VII.

II. OUR SYSTEM MODEL AND OUTLINE OF ANALYSIS

In this section, we discuss the relevant queueing system model of interest, the underlying assumptions, their justification and outline of the analysis. Though the queueing system described here is used in the TDM over PSN scenario, but its applications are not limited to the same.

A. NETWORK SCENARIO

Typically in a TDM over PSN network scenario, the network gives high priority to provide QoS to TEPs in the intermediate routers. This is implemented through scheduler, which ensures fixed bandwidth is allocated to the TEPs. At the core router, two *logical* queues (with a common single server) are maintained: one corresponding to the TEPs exclusively - the high priority queue, while the other queue being low priority corresponding to packets belonging to all other services. The transmission hardware (the single server) would transmit (serve) the TEPs according to the algorithm-B. The server *going on vacation* means the transmission hardware starts transmitting the packets in the other (low-priority) logical queue. Once the scheduler has transmitted all TEPs, it goes on to serve the low priority queue. The scheduler need not serve all the TEPs, but it is enough that it serves all the TEPs containing contemporary TDM frames. The more recently generated TEPs could wait for the next turn for transmission.

B. SYSTEM MODEL, ASSUMPTIONS AND JUSTIFICATIONS

We explain our system model, underlying assumptions and the corresponding justifications below:

(i) Packets arrive according to Poisson process with rate λ and service times being queue-length dependent, according to algorithm-B [1]. Queue has a capacity to hold B_{on} packets (including the one in service) and arrivals which see B_{on} packets will be dropped (this happens for arrivals until next departure). Server is allowed to serve a maximum of R_1 packets in each busy period. This queue model is denoted as $M/\widetilde{D}/1/B_{on}$ queue, with R_1−limited service.

Algorithm B: Algorithm B (see Appendix A) dictates the service interval for a packet, at its service initiation instant by looking at the queue length. The service intervals are linearly decreasing, deterministic function of queue length (to satisfy the requirements (b), (c) and (e) mentioned in Section I). The scheduling algorithm-B essentially does ensure that there is always customers (requirement (e)) on the one end, while on the other, reduces the TEP loss (by setting maximum service rate when buffer is full (or nearly full) - requirement (c)). The finite queue length states in the algorithm reduces the variance of the queue length which intuitively reduces the variance of IDI (or output jitter) - the requirement (b).

(ii) We use first-come first-serve (FCFS) discipline to serve packets. The R_1−limited service aspect of the model, justifies the transmission of *only contemporary* TDM frames (requirement (d) in previous section).

(iii) If R_1 packets are served in a busy period or if the queue is emptied (whichever occurs first), server takes a vacation of duration V_1. Upon returning from a vacation (at vacation completion instants), if the server finds that the queue is still empty, it takes one more vacation of duration V_2. This process continues (multiple vacations) until the server finds at least a packet in the queue at a vacation completion instant, as shown in Fig. 1. This attribute of our model justifies the time the scheduler serves the low priority queue.
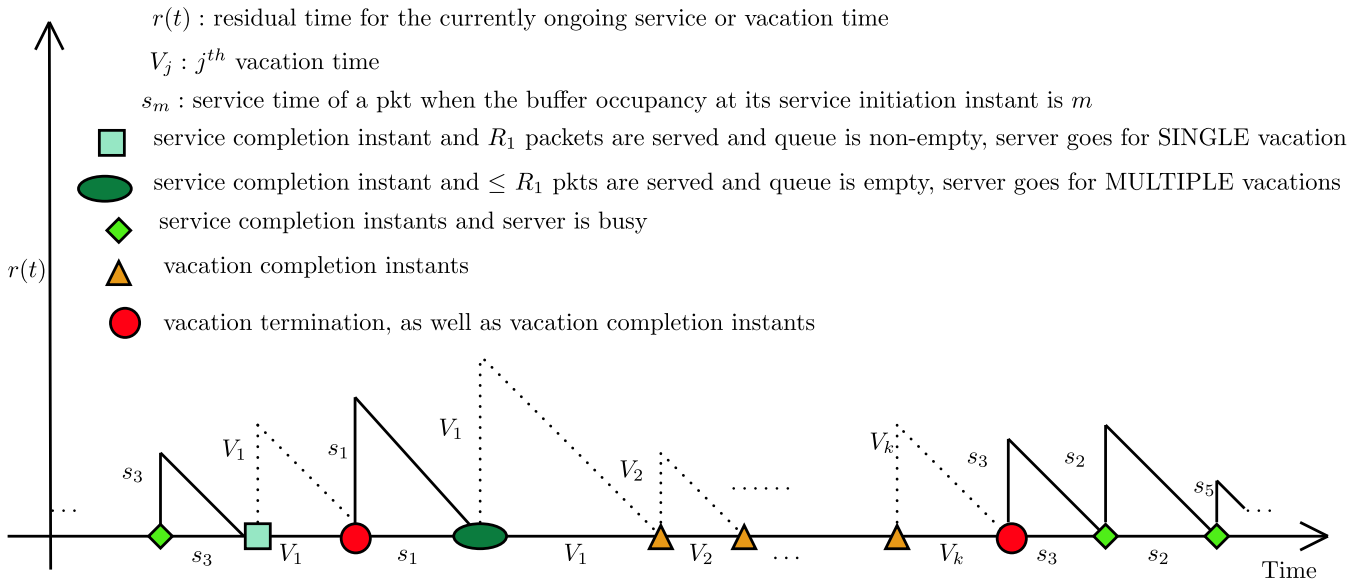
FIGURE 1. Embedded points for the $M/\widetilde{D}/1/B_{on}$ queue with multiple vacations and R_1-limited service.

(iv) If the server finds that the queue is non-empty upon returning from a vacation, it terminates the vacation and starts serving the head-of-the-line packet. This embedded point (vacation completion instant) is referred to as vacation termination instant. This attribute of our model, justifies the time the scheduler serves the low priority queue (holding packets of other services).

(v) The vacation times V_1, V_2, V_3, ... are assumed to be either: (a) deterministic or (b) exponential. The fact that the TEP streams are allocated a fixed time intervals, in a given allocation cycle (within a inter-state time interval in which the number of TDM payloads - say number of E1 lines - supported is constant) tells the remaining time in the cycle is allocated to packets of other services is also fixed. This justifies the deterministic vacation, in our model. In situations wherein the above 'fixed allocation cycle' is relaxed, the exponential vacation approximates the tail in the Gamma distribution arising out of sum of transmission times of lower priority packets.

C. OBJECTIVE AND OUTLINE OF THE ANALYSIS

Our aim is to calculate the distribution, mean and variance of IDTs (jitter in TDMoPSN perspective) of $M/\widetilde{D}/1/B_{on}$ queue, while the server is allowed to take R_1-limited service vacation (deterministic/exponentially distributed).

We first derive the QLD of $M/\widetilde{D}/1/B_{on}$ queue using EMC method. Then, we compute the IDT distribution for both cases of vacation distributions by using the derived QLD, state of server and queue-length. If the server is busy, IDTs follows service times and if the server is in vacation, IDT is a sum of total vacation duration and service time of first packet in the new busy period. Finally, the distribution of IDTs is computed by using law of total probability.

III. EMBEDDED MARKOV CHAIN ANALYSIS

We consider the state of the system at time instants t_k, $k \in \{0, 1, 2, ...\}$, as embedded points, which are vacation completion or service completion instants. In this section, we derive QLD at the embedded points using EMC method.

A. NOTATIONS

At an embedded point t_k, $k \in \{0, 1, 2, ...\}$, let

Q_k^E = number of packets in the queue, $0 \leq Q_k^E \leq B_{on}$
N_k = number of arrivals to queue during (t_{k-1}, t_k)

$$L_k^E = \begin{cases} 0, & \text{if the embedded point is a vacation completion instant} \\ 1, & \text{if the embedded point is a departure instant} \end{cases}$$

$$I_k = \begin{cases} \{0, 1, ..., R_1-1\}, & \text{number of service completions since the server started serving in the current busy period} \\ -1, & \text{when server is in vacation} \end{cases}$$

Queue-length Q_k^E satisfies the following recursive equation:

$$Q_k^E = min\{B_{on} - L_k^E, ([Q_{k-1}^E - L_k^E]^+ + N_k)\} \quad (1)$$

where,

$$[x]^+ = \begin{cases} x, & \text{when } x > 0 \\ 0, & \text{when } x \leq 0 \end{cases}$$

Claim 1: The bivariate process $\{U_k^E \overset{\triangle}{=} (I_k, Q_k^E), k \geq 0\}$ forms an EMC at embedded points $\{t_k\}$.

Proof: The arrival process is considered as Poisson, so N_k is independent of everything else in the system and from Equ. 1 it is clear that, Q_k^E depends only on Q_{k-1}^E and N_k, therefore, queue-length satisfies the one-step Markovian property
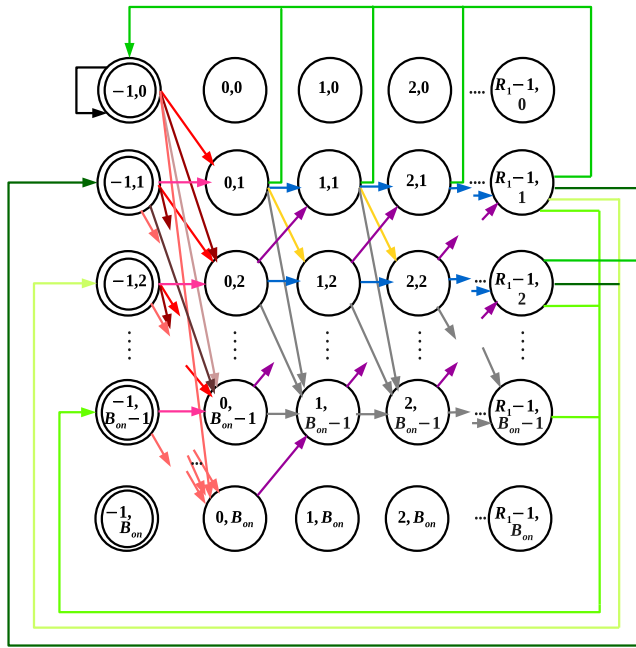
FIGURE 2. State transitions of EMC U_k^E.

FIGURE 3. Probabilities of state transitions shown in Fig. 2.
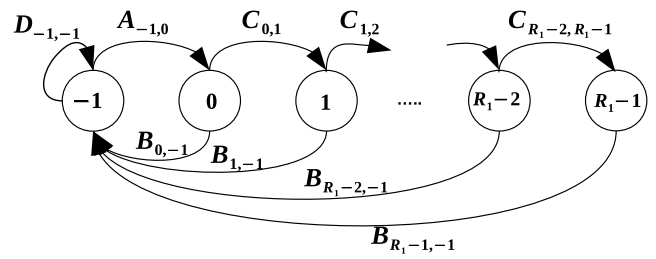
FIGURE 4. State transition diagram of one dimensional EMC formed by I_k.

at embedded points. According to the definition of I_k, it can be observed that I_k depends on I_{k-1} and Q_{k-1}. Therefore, the bivariate process $\{U_k^E = (I_k, Q_k^E), \ k \geq 0\}$ depends only on U_{k-1}^E and hence it forms an EMC with finite state space $\{-1, 0, 1, \ldots, R_1 - 1\} \times \{0, 1, 2, \ldots, B_{on}\}$. ∎

Fig. 4 shows possible state transitions of EMC formed by I_k. Fig. 2 shows the bivariate EMC U_k^E corresponding to the state $U^E = (i, m)$, where $i \in I_k$ and $m \in Q_k^E$. The probabilities corresponding to each state transition are given in Fig. 3.

Given the arrival process as Poisson with parameter λ, let $\alpha(n; t)$ be the probability of n arrivals to queue in the service interval $(0, t]$ and is given by,

$$\alpha(n; t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n \geq 0 \qquad (2)$$

Let $\beta(n)$ be the probability of $n \geq 0$ arrivals to queue in a *single* arbitrary vacation (V_i) as given below:

$$\beta(n) = \begin{cases} \dfrac{e^{-\lambda D}(\lambda D)^n}{n!}, \\ \quad \text{if vacation is Deterministic } (V_i = D) \\ \dfrac{\lambda^n \theta}{(\lambda + \theta)^{n+1}}, \\ \quad \text{if vacation is Exponential } (V_i \sim Exp(\theta)) \end{cases} \qquad (3)$$

If vacation distribution is deterministic with duration D, then $\beta(n)$ is equal to $\alpha(n; t)|_{t=D}$. If vacation is exponentially distributed with parameter θ, then $\beta(n)$ can be calculated as given in Appendix B. Let $\beta'(n)$ be the probability that

n packets arrive (and join the queue) during an idle period,[1] and is given by,

$$\beta'(n) = \sum_{l=0}^{\infty} (\beta(0))^l \beta(n) = \frac{\beta(n)}{1 - \beta(0)},$$

$$n = 1, 2, \ldots, B_{on} - 1$$

$$\beta'(B_{on}) = \sum_{l=0}^{\infty} (\beta(0))^l \beta^c(B_{on}) = \frac{\beta^c(B_{on})}{1 - \beta(0)} \qquad (4)$$

where $\beta^c(B_{on}) = \sum_{n=B_{on}}^{\infty} \beta(n)$.

Table 1 represents the state transition probability matrix $\mathbf{P}$, whose entries themselves are sub-matrices with each such sub-matrix representing similar transitions, constituting a type of transitions as in Fig. 4. Each sub-matrix of $\mathbf{P}$ is of size $(B_{on} + 1) \times (B_{on} + 1)$, and hence size of $\mathbf{P}$ is given by, $(R_1 + 1)(B_{on} + 1) \times (R_1 + 1)(B_{on} + 1)$.

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[1]The time interval between the instant at which the queue becomes empty and the instant at which the server starts serving again is referred to as an idle period.

TABLE 1. State transition probability matrix, P.

I_k	-1	0	1	2	$\ldots$	$\ldots$	$R_1 - 1$
-1	$\mathbf{D_{-1,-1}}$	$\mathbf{A_{-1,0}}$	0	0	$\ldots$	$\ldots$	0
0	$\mathbf{B_{0,-1}}$	0	$\mathbf{C_{0,1}}$	0	$\ldots$	$\ldots$	0
1	$\mathbf{B_{1,-1}}$	0	0	$\mathbf{C_{1,2}}$	0	$\ldots$	0
2	$\mathbf{B_{2,-1}}$	0	0	0	$\mathbf{C_{2,3}}$	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$R_1 - 2$	$\mathbf{B_{R_1-2,-1}}$	0	0	0	$\ldots$	$\ldots$	$\mathbf{C_{R_1-2,R_1-1}}$
$R_1 - 1$	$\mathbf{B_{R_1-1,-1}}$	0	0	0	$\ldots$	$\ldots$	0

B. ELEMENTS OF STATE TRANSITION PROBABILITY MATRIX P

As mentioned earlier, each state transition in Fig. 4 is an element in $\mathbf{P}$ and is a matrix itself. In Fig. 4, there are four types of state transitions and are given by:

(i) vacation state -1 to -1 itself, which is represented as $\mathbf{D_{-1,-1}}$ in $\mathbf{P}$

(ii) vacation state -1 to busy state 0, which is represented as $\mathbf{A_{-1,0}}$

(iii) busy state i to vacation state -1, which is represented as $\mathbf{B_{i,-1}}$, $i \in \{0, 1, \ldots, R_1 - 1\}$

(iv) busy state i to busy state $(i+1)$, which is represented as $\mathbf{C_{i,i+1}}$, $i \in \{0, 1, \ldots, R_1 - 2\}$ Each element of the blocks (A, B, C or D) of $\mathbf{P}$ is a conditional probability and are defined as,

$$D_{-1,-1}(m,n) = Pr(I_{k+1}=-1, Q^E_{k+1}=n | I_k = -1, Q^E_k = m)$$
$$A_{-1,0}(m,n) = Pr(I_{k+1}=0, Q^E_{k+1}=n | I_k = -1, Q^E_k = m)$$
$$B_{i,-1}(m,n) = Pr(I_{k+1}=-1, Q^E_{k+1}=n | I_k = i, Q^E_k = m)$$
$$C_{i,i+1}(m,n) = Pr(I_{k+1}=i+1, Q^E_{k+1}=n | I_k = i, Q^E_k = m)$$

Now, the elements of $\mathbf{D_{-1,-1}}$ are given by,

$$D_{-1,-1}(m,n) = \begin{cases} \beta(0), & \text{if } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

Elements of $\mathbf{A_{-1,0}}$ are given by,

$$A_{-1,0}(m,n)$$
$$= \begin{cases} 0, & \text{if } m = 0 \text{ and } n = 0 \\ \beta(n-m), & \text{if } n \geq m \text{ and } 0 \leq m \leq B_{on} - 1 \\ & \quad 1 \leq n \leq B_{on} - 1 \\ 1 - \sum_{i=0}^{B_{on}-1-m} \beta(i), & \text{if } n = B_{on} \text{ and } 0 \leq m \leq B_{on} - 1 \\ 0, & \text{otherwise} \end{cases}$$

Now, the elements of $\mathbf{B_{i,-1}}$, for $0 \leq i \leq R_1 - 2$ are given by,

$$B_{i,-1}(m,n) = \begin{cases} \alpha(0; s_1), & \text{if } m = 1 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

where s_m is the service time of a head-of-line packet when the queue-length at its service initiation instant is m.

Now, elements of $\mathbf{B_{R_1-1,-1}}$ are given by,

$$B_{R_1-1,-1}(m,n)$$
$$= \begin{cases} 0, & \text{if } m = 0 \text{ (or) } n = B_{on} \text{ (or)} \\ & \quad m = B_{on} \\ \alpha(n-m+1; s_m), & \text{if } n+1 \geq m, \ 1 \leq m \leq B_{on} - 1, \\ & \quad m-1 \leq n \leq B_{on} - 2 \\ 1 - \sum_{i=0}^{B_{on}-1-m} \alpha(i), & \text{if } 1 \leq m \leq B_{on} - 1, \ n = B_{on}-1 \\ 0, & \text{otherwise} \end{cases}$$

Now, the elements of $C_{0,1}$ are given by,

$$C_{0,1}(m,n)$$
$$= \begin{cases} 0, & \text{if } m = 0 \text{ (or) } n = 0 \text{ (or) } n = B_{on} \\ 0, & \text{if } m = B_{on} \text{ and } n \neq B_{on} - 1 \\ 1, & \text{if } m = B_{on} \text{ and } n = B_{on} - 1 \\ \alpha(n-m+1; s_m), & \text{if } n+1 \geq m, \ 1 \leq m \leq B_{on} - 1, \\ & \quad m-1 \leq n \leq B_{on} - 2 \\ 1 - \sum_{i=0}^{B_{on}-1-m} \alpha(i), & \text{if } 1 \leq m \leq B_{on} - 1, \ n = B_{on} - 1 \\ 0, & \text{otherwise} \end{cases}$$

Elements of $C_{i,i+1}$, for $1 \leq i \leq R_1 - 2$ are given by,

$$C_{i,i+1}(m,n)$$
$$= \begin{cases} 0, & \text{if } m = 0 \text{ (or) } m = B_{on} \\ & \quad \text{(or) } n = 0 \text{ (or) } n = B_{on} \\ \alpha(n-m+1; s_m), & \text{if } n+1 \geq m, \ 1 \leq m \leq B_{on} - 1, \\ & \quad m-1 \leq n \leq B_{on} - 2 \\ 1 - \sum_{i=0}^{B_{on}-1-m} \alpha(i), & \text{if } 1 \leq m \leq B_{on} - 1, \ n = B_{on} - 1 \\ 0, & \text{otherwise} \end{cases}$$

Let $\Pi^E = (\pi^E_{-1,0}, \pi^E_{-1,1}, \ldots, \pi^E_{-1,B_{on}}, \pi^E_{0,0}, \pi^E_{0,1}, \ldots, \pi^E_{0,B_{on}}, \pi^E_{1,0}, \pi^E_{1,1}, \ldots, \pi^E_{1,B_{on}}, \ldots \ldots, \pi^E_{R_1-1,0}, \pi^E_{R_1-1,1}, \ldots, \pi^E_{R_1-1,B_{on}})$ be the stationary probability vector at embedded

$$\pi^{SI}_{i,m} = \begin{cases} \pi^E_{i,m}, & \text{if } i \in (1, 2, \ldots, R_1 - 1), \ m \in (1, 2, \ldots, B_{on} - 1) \\ \pi^E_{0,m} + \pi^E_{-1,0}\beta'(m) + \sum_{k=1}^{m} \pi^E_{-1,k}\,\beta(m-k), & \text{if } i = 0 \text{ and } m \in (1, 2, \ldots, B_{on} - 1) \\ \pi^E_{0,B_{on}} + \pi^E_{-1,0}\beta'(B_{on}) + \sum_{k=1}^{B_{on}} \pi^E_{-1,k}\left(1 - \sum_{j=0}^{B_{on}-1-k}\beta(j)\right), & \text{if } i = 0 \text{ and } m = B_{on} \\ 0, & \text{if } i \in (1, 2, \ldots, R_1 - 1) \text{ and } m = B_{on} \end{cases} \quad (5)$$

points, where $\pi_{i,m}^E$ represents the steady state probability of the state (i, m) at embedded point.

Solve $\Pi^E = \Pi^E \mathbf{P}$ and $\sum_{m=0}^{B_{on}} \pi_{-1,m}^E + \sum_{i=0}^{R_1-1} \sum_{m=0}^{B_{on}} \pi_{i,m}^E = 1$ to get stationary probabilities of the EMC U_k at embedded points. Now, the steady state queue-length vector at embedded points can be obtained from above stationary probabilities as follows:

$$\pi_m^E = \pi_{-1,m}^E + \sum_{i=0}^{R_1-1} \pi_{i,m}^E \qquad (6)$$

for $0 \leq m \leq B_{on}$.

C. QLD AT SERVICE INITIATION INSTANTS
If the server is busy, state of the system at service initiation instant (U^{SI}) is same as the state of system at embedded point (U^E), otherwise they are different. The state space of U^{SI} is $\{0, 1, \ldots, R_1 - 1\} \times \{1, 2, \ldots, B_{on}\}$ and is related to U^E as

$$U^{SI} = \begin{cases} U^E, & \text{if } U^E = (i, m), \ i \in (1, 2, \ldots, R_1 - 1), \\ & m \in (1, 2, \ldots, B_{on} - 1) \\ (0, m), & \text{if } U^E = (-1, 0), \ (-1, m), \ (0, m), \\ & m \in (1, 2, \ldots, B_{on}) \end{cases}$$

Therefore, $\Pi^{SI} = Pr\left(U^{SI} = (i, m)\right)$ can be calculated as given in Equ. 5, where, $\pi_{i,m}^{SI}$ represents the stationary probability that the EMC will be in the state (i, m), $i \in \{0, 1, \ldots, R_1 - 1\}$, $m \in \{1, 2, \ldots, B_{on}\}$ at a service initiation instant. Finally, QLD at service initiation instants can be calculated as,

$$\pi_m^{SI} = \sum_{i=0}^{R_1-1} \pi_{i,m}^{SI} \qquad (7)$$

for $1 \leq m \leq B_{on}$.

Note here that, the QLD at embedded points π_m^E (and QLD at service initiation points π_m^{SI}) is different for different vacation distributions. We have to plug-in $\beta(n)$ corresponding to the type of vacation, while deriving π_m^E. In the following sections, we derive distribution of IDTs for two types of vacations, where we will use the corresponding QLD at embedded and service initiation instants.

IV. DISTRIBUTION OF IDTs WHEN VACATIONS ARE DETERMINISTIC
In this section, we derive the distribution of IDTs when the distribution of vacation duration is deterministic and using which we derive expressions for mean and variance of IDTs.

Let A be a random variable representing the status of the queue and server at an embedded point. It takes three values depending on the state of the queue and server:

$$A = \begin{cases} -1, & \text{if server is in vacation } and \text{ queue is empty} \\ 0, & \text{if server is in vacation } and \text{ queue is} \\ & \text{non-empty} \\ 1, & \text{if server is busy} \end{cases} \qquad (8)$$

The probability mass function (PMF) of A is given by,

$$p_A(a) = \begin{cases} p, & \text{if } a = -1 \\ 1 - \rho - p, & \text{if } a = 0 \\ \rho, & \text{if } a = 1 \end{cases} \qquad (9)$$

where, p is the probability of queue being empty, ρ is the probability of server being busy and $(1 - \rho - p)$ is the probability of server being in vacation and queue is non-empty. These probabilities can be calculated as given below:

$$p = \pi_{-1,0}^E + \sum_{j=0}^{R_1-1} \pi_{j,0}^E = \pi_{-1,0}^E$$

$$\rho = \sum_{j=0}^{R_1-1} \sum_{m=1}^{B_{on}} \pi_{j,m}^E = \pi_{0,B_{on}}^E + \sum_{j=0}^{R_1-1} \sum_{m=1}^{B_{on}-1} \pi_{j,m}^E$$

$$1 - \rho - p = \sum_{m=1}^{B_{on}} \pi_{-1,m}^E \qquad (10)$$

where, $\pi_{j,0}^E = 0$, $\forall j \in \{0, 1, \ldots, R_1 - 1\}$ and $\pi_{j,B_{on}}^E = 0$, $\forall j \in \{1, 2, \ldots, R_1 - 1\}$, can be observed from Fig. 2.

Let X be the random variable representing IDTs of $M/\tilde{D}/1/B_{on}$ queue, when vacations are deterministic D. In this case, let, p_d be the probability of queue being empty and ρ_d be the probability of server being busy, as shown in Fig. 5. Then, $p_d = p$ and $\rho_d = \rho$ with $\{\pi_m^E, \ 0 \leq m \leq B_{on}\}$ corresponds to QLD when vacations are deterministic.

The IDT (X) takes the form of the following conditional random variables, conditioning on the random variable A:

$X_E = $ conditional IDT when server is in vacation *and* queue is empty $(A = -1)$

$X_{NE} = $ conditional IDT when server is in vacation *and* queue is non-empty $(A = 0)$

$X_B = $ conditional IDT when server is busy $(A = 1)$ (11)

With the above notation, the law of total probability gives the unconditional distribution of X (can be observed from Fig. 5).

$$F_X(x) = p_d F_{X_E}(x) + (1 - \rho_d - p_d) F_{X_{NE}}(x) + \rho_d F_{X_B}(x) \qquad (12)$$

Now, we determine the distributions of random variables X_B, X_E and X_{NE}.

A. DISTRIBUTION OF X_B
When the server is busy, IDTs in a queue are equal to service times. Hence, $Pr(X_B) = Pr(X|A = 1) = Pr(S = x)$. For the service process considered in this paper, service times (denoted by random variable, S) distribution, $F_S(x)$ is equal to the QLD at service initiation instants, π_m^{SI}, $1 \leq m \leq B_{on}$ (for more details, see [1]). Therefore, the PMF of service
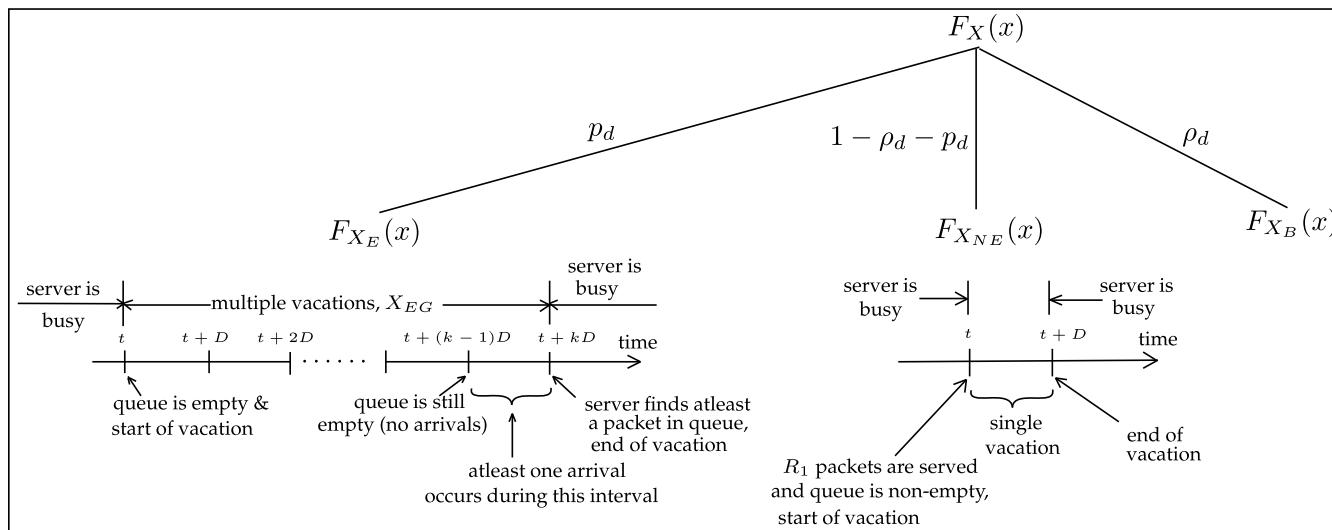
FIGURE 5. Distribution of IDTs when vacations are deterministic (D).

times (which is PMF of X_B also) is given by,

$$
p_{X_B}(x) = p_S(x) = \begin{cases} \pi_{1,d}^{SI}, & \text{if } x = s_1 \\ \pi_{2,d}^{SI}, & \text{if } x = s_2 \\ \vdots \\ \pi_{B_{on},d}^{SI}, & \text{if } x = s_{B_{on}} \end{cases} \quad (13)
$$

Note here that, $\pi_{m,d}^{SI}$ corresponds to QLD at service initiation instant, when vacations are deterministic.

1) MEAN AND VARIANCE OF X_B

The mean of random variable X_B can be calculated using its PMF (Equ. 13) as follows:

$$
E[X_B] = E[S] = \mu^{-1} = \sum_{m=1}^{B_{on}} s_m \, \pi_{m,d}^{SI} \quad (14)
$$

Now, the second moment of X_B is calculated as follows,

$$
E[X_B^2] = \sum_{m=1}^{B_{on}} s_m^2 \, \pi_{m,d}^{SI} \quad (15)
$$

Finally, the variance of X_B is given by,

$$
var(X_B) = var(S) = E[X_B^2] - E^2[X_B]
$$
$$
= \sum_{m=1}^{B_{on}} s_m^2 \, \pi_{m,d}^{SI} - \left(\mu^{-1}\right)^2 \quad (16)
$$

B. DISTRIBUTION OF X_E

When the queue is empty (number of service completions can be 1 or 2 or ... R_1), server goes for a vacation. Now, server checks the queue for an arrival at each vacation period end (since vacations are deterministic, server checks for every D time units) and if it finds at least one packet, it starts serving the head-of-line packet, as depicted in Fig. 5. So, the IDTs

X_E in this case is a sum of two random variables X_{EG} and S, where X_{EG} is total vacation duration and S is defined in Section IV-A.

In this case ($A = -1$), if the server sees that the queue is empty (after serving $\leq R_1$ packets) at a departure point, it terminates the busy period and goes on vacation. From this instant, the server polls the queue, every D time units, whether at least one packet has arrived or not. Since, the arrivals are Poisson (a pure Markov process), at every D, say, k^{th} interval D since the start of the vacation, the probability of at least one arrival occurs, is independent of k (due to memoryless property of Poisson process). This yields, the geometric distribution for random variable X_{EG} with parameter $(1 - q_d)$, where q_d is given by Equ. 3 by substituting $n = 0$, i.e., $q_d = \beta(0) = e^{-\lambda D}$ as given by,

$$
p_{X_{EG}}(x = kD) = q_d^{k-1}(1 - q_d), \quad k = 1, 2, 3, \ldots \quad (17)
$$

Now, the MGFs of X_{EG} and S are given by,

$$
M_{X_{EG}}(t) = E[e^{X_{EG}t}] = \sum_{k=1}^{\infty} q_d^{k-1}(1 - q_d)e^{kDt}
$$

$$
M_S(t) = \sum_{m=1}^{B_{on}} \pi_{m,d}^{SI} \, e^{s_m t}
$$

Finally, the MGF of X_E is given by,

$$
M_{X_E}(t) = M_{X_{EG}}(t) \, M_S(t)
$$
$$
= \sum_{k=1}^{\infty} \sum_{m=1}^{B_{on}} \pi_{m,d}^{SI} \, q_d^{k-1}(1 - q_d)e^{(kD + s_m)t}
$$

Now, by using the MGF $M_{X_E}(t)$, we can write,

$$
p_{X_E}(x) = P\left(X_E = (s_m + kD)\right) = \pi_{m,d}^{SI} \, q_d^{k-1}(1 - q_d) \quad (18)
$$

which is the PMF of X_E, for $1 \leq m \leq B_{on}$ and $k = 1, 2, 3, \ldots$.

1) CONDITIONAL MEAN OF X_E

We know that $X_E = X_{EG} + S$, so, $E[X_E] = E[X_{EG}] + E[S]$. First, we calculate the mean of X_{EG}.

$$E[X_{EG}] = \sum_{k=1}^{\infty} kD\, q_d^{k-1}(1 - q_d) = D\left(\frac{1}{1 - q_d}\right)$$

Therefore, the mean of X_E can be calculated as,

$$E[X_E] = E[X_{EG}] + E[S] = D\left(\frac{1}{1 - q_d}\right) + \mu^{-1} \quad (19)$$

2) CONDITIONAL VARIANCE OF X_E

The random variables X_{EG} and S are independent, so $cov(X_{EG}, S) = 0$. Hence the variance of X_E can be calculated as, $var(X_E) = var(X_{EG}) + var(S)$. Now,

$$E[X_{EG}^2] = \sum_{k=1}^{\infty} (kD)^2\, q_d^{k-1}(1 - q_d) = D^2\left(\frac{1 + q_d}{(1 - q_d)^2}\right)$$

$$var(X_{EG}) = E[X_{EG}^2] - E^2[X_{EG}] = D^2\left(\frac{q_d}{(1 - q_d)^2}\right)$$

Therefore, variance of X_E is given by,

$$var(X_E) = var(X_{EG}) + var(S)$$
$$= D^2\left(\frac{q_d}{(1 - q_d)^2}\right) + \sum_{m=1}^{B_{on}} s_m^2\, \pi_{m,d}^{SI} - \left(\mu^{-1}\right)^2 \quad (20)$$

C. CONDITIONAL DISTRIBUTION OF X_{NE}

When number of service completions are R_1 (and queue is still non-empty), server goes for a vacation. In this case, server takes a single vacation and returns to the queue and starts serving the packets, as depicted in Fig. 5. So, the IDTs (X_{NE}) in this case is a sum of D and service times S. Now, the PMF of X_{NE} is given by,

$$p_{X_{NE}}(x) = P\left(X_{NE} = (s_m + D)\right) = \pi_{m,d}^{SI} \quad (22)$$

1) CONDITIONAL MEAN OF X_{NE}

We know that $X_{NE} = D + S$, so mean of X_{NE} can be calculated as,

$$E[X_{NE}] = E[D] + E[S] = D + \mu^{-1} \quad (23)$$

2) CONDITIONAL VARIANCE OF X_{NE}

Since the random variables D and S are independent, their covariance is zero. Now, the variance of X_{NE} can be calculated as,

$$var(X_{NE}) = var(D) + var(S) = 0 + \sum_{m=1}^{B_{on}} s_m^2 \pi_{m,d}^{SI} - \left(\mu^{-1}\right)^2 \quad (24)$$

Note that $var(D) = 0$, because D is deterministic.

D. DISTRIBUTION OF X

We have PMFs of the random variables X_B, X_E and X_{NE} given by Equs. 13, 18 and 22, respectively, using these equations, we can calculate the PMF of IDT (X) using Equ. 12, and is given by,

$$P_X(x) = \begin{cases} \rho_d\, \pi_{m,d}^{SI}, & \text{if } x = s_m \\ (1 - \rho_d - p_d)\, \pi_{m,d}^{SI}, & \text{if } x = s_m + D \\ p_d\, q_d^{k-1}(1 - q_d)\, \pi_{m,d}^{SI}, & \text{if } x = s_m + kD \end{cases} \quad (25)$$

where, $1 \leq m \leq B_{on}$ and $k = 2, 3, 4, \ldots$. Note that IDTs (X) is a discrete random variable, when vacations are deterministic, because both service times and vacation durations are discrete.

1) MEAN OF IDTs (X)

We can calculate the mean of X by applying expectation operator on both sides of Equ. 12,

$$E[X] = p_d E[X_E] + (1 - \rho_d - p_d)E[X_{NE}] + \rho_d E[X_B]$$

Now, substitute Equs. 14, 19 and 23 in the above equation, we get

$$E[X] = D\left(1 - \rho_d + p_d\frac{q_d}{1 - q_d}\right) + \mu^{-1} \quad (26)$$

2) VARIANCE OF IDTs (X)

The variance of IDTs can be derived using *law of conditional variances* [14]. It is defined as follows:

$$var(X) = E[var(X|A)] + var(E[X|A]) \quad (27)$$

Note that $var(X|A)$ and $E[X|A]$ are random variables. Now, $var(X|A)$ is given by,

$$var(X|A) = \begin{cases} var(X|A = -1) = var(X_E), & w.p\ p_d \\ var(X|A = 0) = var(X_{NE}), & w.p\ 1 - \rho_d - p_d \\ var(X|A = 1) = var(X_B), & w.p\ \rho_d \end{cases}$$

with notation *w.p* denoting "with probability". Therefore, $E[var(X|A)]$ can be calculated as follows:

$$E[var(X|A)] = p_d\, var(X_E) + (1 - \rho_d - p_d)\, var(X_{NE}) + \rho_d\, var(X_B) \quad (28)$$

Now, substituting Equs. 16, 20 and 24 in above equation, we get

$$E[var(X|A)] = D^2 p_d\left(\frac{q_d}{(1 - q_d)^2}\right) + \sum_{m=1}^{B_{on}} s_m^2\, \pi_{m,d}^{SI} - \left(\mu^{-1}\right)^2 \quad (29)$$

Now, $E[X|A]$ is given by,

$$E[X|A] = \begin{cases} E[X|A = -1] = E[X_E], & w.p\ p_d \\ E[X|A = 0] = E[X_{NE}], & w.p\ 1 - \rho_d - p_d \\ E[X|A = 1] = E[X_B], & w.p\ \rho_d \end{cases}$$

$$var(X) = D^2 \left[\frac{p_d q_d}{(1-q_d)^2} + (p_d + \rho_d)(1 - \rho_d - p_d) - \frac{2p_d(1 - \rho_d - p_d)}{1 - q_d} + \frac{p_d(1 - p_d)}{(1-q_d)^2} \right] + \sum_{m=1}^{B_{on}} s_m^2\, \pi_{m,d}^{SI} - \left(\mu^{-1}\right)^2 \quad (21)$$

Note here that, $E[E[X|A]] = E[X]$. Now, $var(E[X|A])$ can be calculated as follows:

$$var(E[X|A]) = \sum_{a \in \{-1,0,1\}} \left(E[X|A = a] - E[X] \right)^2 P(A = a)$$

where, $a \in \{-1, 0, 1\}$ and PMF of random variable A is given in Equ. 9. Now,

$$var(E[X|A]) = \left(E[X_E] - E[X] \right)^2 P(A = -1)$$
$$+ \left(E[X_{NE}] - E[X] \right)^2 P(A = 0)$$
$$+ \left(E[X_B] - E[X] \right)^2 P(A = 1) \quad (30)$$

Now, substitute Equs. 9, 14, 19, 23 and 26 in above equation, we get,

$$var(E[X|A]) = D^2 \left[(p_d + \rho_d)(1 - \rho_d - p_d) \right.$$
$$\left. - \frac{2p_d(1 - \rho_d - p_d)}{1 - q_d} + \frac{p_d(1 - p_d)}{(1-q_d)^2} \right] \quad (31)$$

Therefore, the variance of IDTs in case of deterministic vacation, can be obtained by adding Equs. 29 and 31, it is given in Equ. 21.

V. DISTRIBUTION OF IDTS WHEN VACATIONS ARE EXPONENTIALLY DISTRIBUTED

In this section, we derive distribution of IDTs when vacations are exponentially distributed and we also derive mean and variance of IDTs using the derived distribution of IDTs.

Let Y be the random variable representing IDTs of $M/\tilde{D}/1/B_{on}$ queue, when vacations are exponentially distributed (represented by a random variable W) with parameter θ. We use A (as before) for the random variable denoting the status of the queue and server (Equ. 8 and Equ. 9 gives the definition and PMF of A, respectively). In this case, let, p_e be the probability of queue being empty and ρ_e be the probability of server being busy. Then, $p_e = p$ and $\rho_e = \rho$ with $\{\pi_m^E, 0 \le m \le B_{on}\}$ corresponds to QLD when vacations are exponentially distributed.

The IDT (Y) takes the form of the following conditional random variables, conditioning on the random variable A:

Y_E = conditional IDT when server is in vacation *and*
         queue is empty ($A = -1$)

Y_{NE} = conditional IDT when server is in vacation *and*
         queue is non-empty ($A = 0$)

Y_B = conditional IDT when server is busy ($A = 1$) (32)

Now, the distribution of Y can be written using Fig. 6 as follows,

$$F_Y(y) = p_e F_{Y_E}(y) + (1 - \rho_e - p_e) F_{Y_{NE}}(y) + \rho_e F_{Y_B}(y) \quad (33)$$

Now, we determine the distributions of random variables Y_B, Y_E and Y_{NE}.

A. DISTRIBUTION OF Y_B

When the server is busy, IDTs in a queue are equal to service times. Hence, the distribution of IDTs, $F_{Y_B}(y)$, in this case is same as service time distribution. Therefore, the PMF of service times (which is PMF of Y_B also) is given by,

$$p_{Y_B}(y) = p_S(y) = \begin{cases} \pi_{1,e}^{SI}, & \text{if } y = s_1 \\ \pi_{2,e}^{SI}, & \text{if } y = s_2 \\ \vdots \\ \pi_{B_{on},e}^{SI}, & \text{if } y = s_{B_{on}} \end{cases} \quad (34)$$

Note here that, $\pi_{m,e}^{SI}$ corresponds to QLD at service initiation instant, when vacations are exponentially distributed.

1) MEAN AND VARIANCE OF Y_B

The mean of random variable Y_B can be calculated using its PMF (Equ. 34) as follows:

$$E[Y_B] = E[S] = \gamma^{-1} = \sum_{m=1}^{B_{on}} s_m\, \pi_{m,e}^{SI} \quad (35)$$

Now, the second moment of X_B is calculated as follows,

$$E[Y_B^2] = \sum_{m=1}^{B_{on}} s_m^2\, \pi_{m,e}^{SI} \quad (36)$$

Finally, the variance of Y_B is given by,

$$var(Y_B) = var(S) = E[Y_B^2] - E^2[Y_B]$$
$$= \sum_{m=1}^{B_{on}} s_m^2\, \pi_{m,e}^{SI} - \left(\gamma^{-1}\right)^2 \quad (37)$$

B. DISTRIBUTION OF Y_E

When the queue is empty (number of service completions can be 1 or 2 or ... R_1), server goes for multiple vacations, as depicted in Fig. 6. The IDTs Y_E in this case is a sum of two random variables Y_{EE} and S, where Y_{EE} is a random sum of random variables and S is random variable representing service times. In other words, Y_{EE} records the total vacation duration until a non-zero number of arrivals to the queue and S represents the length of first service time in the new busy cycle. As explained for X_E, number of vacation periods (here it is represented by a random variable N) required for the
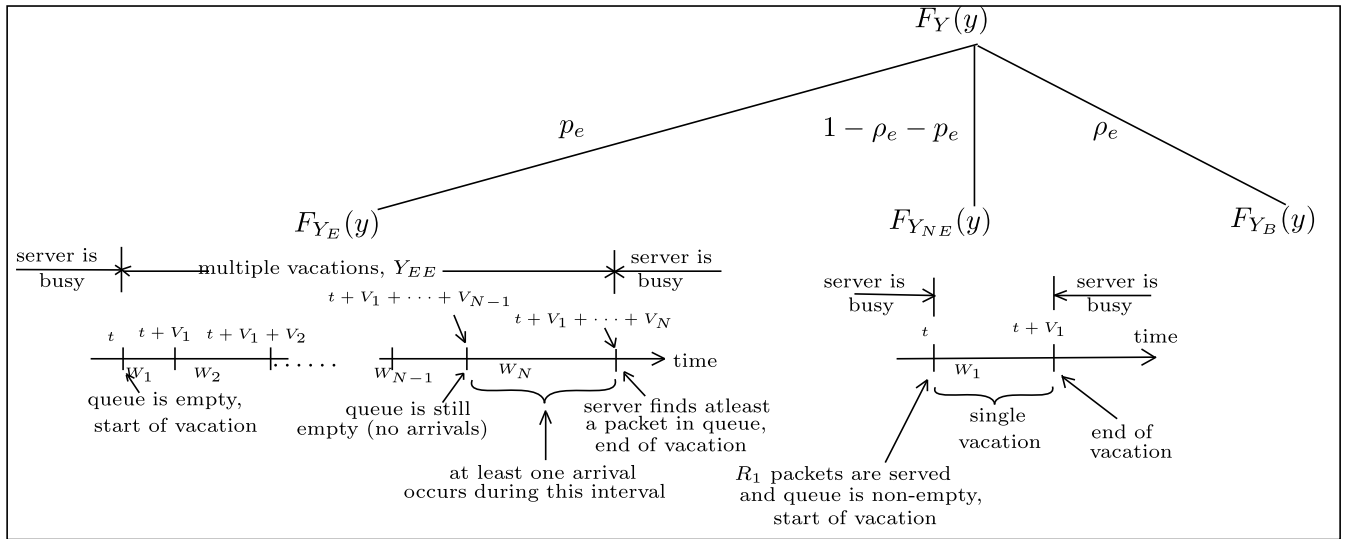
FIGURE 6. Distribution of IDTs when vacations are exponentially distributed.

server to find at least an arrival in the queue is geometrically distributed with parameter $(1-q_e)$, where q_e is the probability of zero arrivals during a single vacation period, when vacations are exponentially distributed. So, q_e can be obtained from Equ. 3 and is given by, $q_e = \beta(0) = \dfrac{\theta}{\lambda + \theta}$.

Let $W_1, W_2, \ldots$ is a sequence of independent and identically distributed (i.i.d.) exponential random variables with parameter θ and common generating function $M_W(t)$ and N is geometrically distributed, which is independent of W_i and has a generating function $M_N(t)$.

For exponential vacations, total vacation duration Y_{EE} is a sum of N i.i.d. exponential random variables as shown in Fig. 6, where $V_1, V_2, \ldots$ are realizations of exponential random variables $W_1, W_2, \ldots$, respectively. Therefore,

$$Y_{EE} = \sum_{i=1}^{N} W_i \qquad (38)$$

Here, each exponential random variable W_i corresponds to one vacation duration.

1) MGF OF Y_{EE}
The MGF of exponential random variable W_i is given by,

$$M_{W_i}(t) = \frac{\theta}{\theta - t}$$

and MGF of geometric random variable N is given by,

$$M_N(t) = \frac{(1 - q_e)e^t}{1 - q_e e^t} \qquad (39)$$

Now, we can find the MGF of Y_{EE} as follows:

$$M_{Y_{EE}}(t) = E[e^{Y_{EE}t}]$$
$$= E[E[e^{Y_{EE}t}|N]] \quad \text{(law of conditional expectations)}$$

For a given $N = n$, Y_{EE} is a sum of n (i.i.d.) exponential random variables, therefore,

$$E[e^{Y_{EE}t}|N=n] = \left(\frac{\theta}{\theta - t}\right)^n$$

and

$$E[e^{Y_{EE}t}|N] = \left(\frac{\theta}{\theta - t}\right)^N$$

Now,

$$M_{Y_{EE}}(t) = E[E[e^{Y_{EE}t}|N]] = E\left[\left(\frac{\theta}{\theta - t}\right)^N\right]$$
$$= E\left[e^{ln\left(\frac{\theta}{\theta - t}\right)^N}\right] = E\left[e^{Nln\left(\frac{\theta}{\theta - t}\right)}\right]$$

where, N is geometrically distributed and using Equ. 39, we can write $E\left[e^{Nln\left(\frac{\theta}{\theta - t}\right)}\right]$ as,

$$M_{Y_{EE}}(t) = E\left[e^{Nln\left(\frac{\theta}{\theta - t}\right)}\right] = \frac{(1 - q_e)e^{ln\left(\frac{\theta}{\theta - t}\right)}}{1 - q_e e^{ln\left(\frac{\theta}{\theta - t}\right)}}$$
$$= \frac{(1 - q_e)\dfrac{\theta}{\theta - t}}{1 - q_e\dfrac{\theta}{\theta - t}} = \frac{(1 - q_e)\theta}{(1 - q_e)\theta - t}$$

By comparing the MGFs $M_{W_i}(t)$ and $M_{Y_{EE}}(t)$, we can say that the random variable Y_{EE} is exponentially distributed with parameter $(1 - q_e)\theta$. Therefore, the pdf of Y_{EE} are given by,

$$f_{Y_{EE}}(y) = (1 - q_e)\theta \, e^{-(1-q_e)\theta y} \qquad (41)$$

$$F_Y(y) = \begin{cases} 0, & \text{if } y < s_{B_{on}} \\ (1 - \rho_e) - \sum_{m=1}^{B_{on}} \pi_{m,e}^{SI} e^{-\theta(y-s_m)}[(1 - \rho_e - p_e) + p_e e^{q_e\theta(y-s_m)}]u(y - s_m) + \rho_e \sum_{j=0}^{k} \pi_{B_{on}-j,e}^{SI}, \\ \quad \text{if } s_{B_{on}-k} \leq y < s_{B_{on}-k-1} \text{ and } 0 \leq k \leq B_{on} - 2 \\ 1 - \sum_{m=1}^{B_{on}} \pi_{m,e}^{SI} e^{-\theta(y-s_m)}[(1 - \rho_e - p_e) + p_e e^{q_e\theta(y-s_m)}]u(y - s_m), & \text{if } y \geq s_1 \end{cases} \quad (40)$$

As mentioned earlier, $Y_E = Y_{EE} + S$, so we can write the pdf of Y_E as,

$$f_{Y_E}(y) = \sum_{m=1}^{B_{on}} \pi_{m,e}^{SI} e^{(1-q_e)\theta s_m}(1 - q_e)\theta \, e^{-(1-q_e)\theta y} \, u(y - s_m) \quad (42)$$

for $y \geq s_{B_{on}}$. The term $e^{(1-q_e)\theta s_m}(1 - q_e)\theta \, e^{-(1-q_e)\theta y}$ represents the pdf of shifted exponential random variable, whose parameter is $(1 - q_e)\theta$, shifted by s_m and the probability of this shift is $\pi_{m,e}^{SI}$.

2) MEAN AND VARIANCE OF Y_E

We know that, $Y_E = Y_{EE} + S$, so, $E[Y_E] = E[Y_{EE}] + E[S]$. Since, Y_{EE} is exponentially distributed with parameter $(1 - q_e)\theta$,

$$E[Y_{EE}] = \frac{1}{(1 - q_e)\theta} = \frac{\lambda + \theta}{\lambda\theta}$$

and mean of S is given by Equ. 35. Therefore, mean of Y_E can be calculated as,

$$E[Y_E] = \frac{\lambda + \theta}{\lambda\theta} + \gamma^{-1} \quad (43)$$

Now, variance of Y_{EE} can be calculated as,

$$var(Y_{EE}) = \frac{1}{(1 - q_e)^2 \theta^2} = \frac{(\lambda + \theta)^2}{\lambda^2\theta^2}$$

Now, the variance of Y_E can be calculated as, $var(Y_E) = var(Y_{EE}) + var(S)$, since Y_{EE} and S are independent, $cov(Y_{EE}, S) = 0$. Now,

$$var(Y_E) = var(Y_{EE}) + var(S)$$
$$= \frac{(\lambda + \theta)^2}{\lambda^2\theta^2} + \sum_{m=1}^{B_{on}} s_m^2 \pi_{m,e}^{SI} - \left(\gamma^{-1}\right)^2 \quad (44)$$

C. DISTRIBUTION OF Y_{NE}

When the number of service completions are R_1 (and queue is non-empty), server goes for a single vacation, returns to the queue after one vacation duration and starts serving the packets, as depicted in Fig. 6. So, the IDTs (Y_{NE}) in this case is a sum of exponential random variable W and service time S. Now, the pdf of Y_{NE} is given by,

$$f_{Y_{NE}}(y) = \sum_{m=1}^{B_{on}} \pi_{m,e}^{SI} e^{\theta s_m}\theta \, e^{-\theta y} u(y - s_m), \quad y \geq s_{B_{on}} \quad (45)$$

where, $e^{\theta s_m}\theta \, e^{-\theta y}$ is the pdf of shifted exponential random variable, whose parameter is θ, with a shift of s_m and the probability of this shift is $\pi_{m,e}^{SI}$.

1) MEAN AND VARIANCE OF Y_{NE}

We know that $Y_{NE} = W + S$, so mean of X_{NE} can be calculated as,

$$E[Y_{NE}] = E[W] + E[S] = \frac{1}{\theta} + \gamma^{-1} \quad (46)$$

Now, the variance of Y_{NE} can be calculated as,

$$var(Y_{NE}) = var(W) + var(S)$$
$$= \frac{1}{\theta^2} + \sum_{m=1}^{B_{on}} s_m^2 \pi_{m,e}^{SI} - \left(\gamma^{-1}\right)^2 \quad (47)$$

Note that, W and S are independent, so, $cov(W, S) = 0$.

D. DISTRIBUTION OF Y

Now, using the density functions of Y_B, Y_E and Y_{NE} given in Equs. 34, 42 and 45, respectively, we can calculate the CDF of IDTs $F_Y(y)$ and is given in Equ. 40. Note that IDTs (Y) is a mixed random variable, when vacations are exponentially distributed, because service times are discrete and vacation durations are continuous.

Now, we derive expressions for mean and variance of IDTs in this case as follows:

1) MEAN AND VARIANCE OF Y

We can calculate the mean of Y by applying expectation operator on both sides of Equ. 33,

$$E[Y] = p_e E[Y_E] + (1 - \rho_e - p_e)E[Y_{NE}] + \rho_e E[Y_B]$$

Now, substitute Equs. 35, 43 and 46 in the above equation, we get

$$E[Y] = \frac{1 - \rho_e}{\theta} + \frac{p_e}{\lambda} + \gamma^{-1} \quad (48)$$

The variance of IDTs can be derived using law of conditional variances. It is defined as follows:

$$var(Y) = E[var(Y|A)] + var(E[Y|A])$$

Note that $var(Y|A)$ and $E[Y|A]$ are random variables. Now, $var(Y|A)$ is given by,

$$var(Y|A) = \begin{cases} var(Y|A = -1) = var(Y_E), & w.p \ p_e \\ var(Y|A = 0) = var(Y_{NE}), & w.p \ 1 - \rho_e - p_e \\ var(Y|A = 1) = var(Y_B), & w.p \ \rho_e \end{cases}$$

Therefore, $E[var(Y|A)]$ can be calculated as follows:

$$E[var(Y|A)] = p_e \, var(Y_E) + (1 - \rho_e - p_e) \, var(Y_{NE}) \\ + \rho_e \, var(Y_B) \quad (49)$$

Now, substituting Equs. 37, 44 and 47 in above equation, we get

$$E[var(Y|A)] = p_e \left(\frac{1}{\lambda^2} + \frac{2}{\lambda\theta} \right) \\ + \frac{1 - \rho_e}{\theta^2} + \sum_{m=1}^{B_{on}} s_m^2 \, \pi_{m,e}^{SI} - \left(\gamma^{-1} \right)^2 \quad (50)$$

Now, $E[Y|A]$ is given by,

$$E[Y|A] = \begin{cases} E[Y|A = -1] = E[Y_E], & w.p \ p_e \\ E[Y|A = 0] = E[Y_{NE}], & w.p \ 1 - \rho_e - p_e \\ E[Y|A = 1] = E[Y_B], & w.p \ \rho_e \end{cases}$$

Note here that, $E[E[Y|A]] = E[Y]$. Now, $var(E[Y|A])$ can be calculated as follows:

$$var(E[Y|A]) = \sum_{a \in \{-1,0,1\}} \left(E[Y|A = a] - E[Y] \right)^2 P(A = a)$$

where, $a \in \{-1, 0, 1\}$ and PMF of random variable A is given in Equ. 9. Now,

$$var(E[Y|A]) = \left(E[Y_E] - E[Y] \right)^2 P(A = -1) \\ + \left(E[Y_{NE}] - E[Y] \right)^2 P(A = 0) \\ + \left(E[Y_B] - E[Y] \right)^2 P(A = 1) \quad (51)$$

Now, substitute Equs. 9, 35, 43, 46 and 48 in above equation, we get,

$$var(E[Y|A]) = \frac{2p_e\rho_e}{\lambda\theta} + \frac{\rho_e(1 - \rho_e)}{\theta^2} + \frac{p_e(1 - p_e)}{\lambda^2} \quad (52)$$

Therefore, the variance of IDTs in case of exponential vacations, can be obtained by adding Equs. 50 and 52, it is given by,

$$var(Y) = \frac{1 - \rho_e^2}{\theta^2} + \frac{p_e(2 - p_e)}{\lambda^2} + \frac{2p_e(1 + \rho_e)}{\lambda\theta} \\ + \sum_{m=1}^{B_{on}} s_m^2 \, \pi_{m,e}^{SI} - \left(\gamma^{-1} \right)^2 \quad (53)$$

VI. SIMULATION RESULTS AND DISCUSSIONS

In this section, we first discuss the simulation methodology used to simulate the queueing system under study, as a culmination of application of our theory developed, to TDM over PSN problem. This means that to consider the typical numerical values in a practical TDM over PSN problem, to simulate the scenario and to evaluate its performance. We compare the derived analytical results with the above simulated results, for both types of vacation distributions. Towards this end, we then compare the variance of IDTs with the bounds defined in standard [15].

A. SIMULATION METHODOLOGY

We consider an E1 line[2] for emulating TDM signals within in a PSN. We consider the following simulation scenario: At the transmitter, eight E1 frames are aggregated to create an IP packet (TEPs) with 32 bytes header. These TEPs are transported across the PSN which provides guaranteed QoS, through an unacknowledged connection oriented service.[3] While the multi-class queueing (with a single server) is the most common way of providing QoS in a PSN, we replicate the same in our TDM over PSN scenario. A two-class *logical* queueing model is used for the core router (as mentioned in Section II). But, for jitter buffer in the receiving IWF (at NNI), the above queueing model reduces to single class consisting of only TEPs obtained by setting deterministic vacation being zero. This makes sense, as at the receiver IWF, the TEPs are extracted from the background traffic, which are accommodated in a separate, independent (hardware) queue. It is to be noted that the single server FIFO jitter buffer queue is operated with high utilization.

B. CHOICE OF SIMULATION PARAMETERS

E1 line data rate is 2.048 *Mbps*[4] For carrying out the simulations, we assume that each TDM over PSN packet carries eight consecutive E1 frames, so there are $8*32*8 = 2048$ bits per packet (apart from 32 bytes IP header). Hence, the average arrival rate λ is 1000 TEPs per second, which means that one TEP arrives for every 1 *ms*, i.e., the average inter-arrival time (IAT) is 1 *ms*. The parameter h of algorithm-B is set to 2, R_1 is set to $B + h$ and buffer size $B_{on} = 2B + h$.

C. SIMULATION RESULTS AND DISCUSSIONS

Figs. 7 and 10 shows both analytical and simulated QLDs of queue with deterministic and exponential distributed vacations. It is intuitive to expect the distribution is unimodal, which it turns out to be.

Fig. 8 (Fig. 11) compares the analytical and simulated distribution of IDTs of $M/\widetilde{D}/1/B_{on}$ queue, when vaca-

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[2]It is an European digital standard for transmitting simultaneous telephone calls using TDM.

[3]The unacknowledged connection oriented service reduces delay per hop - by avoiding the unneccessary retransmissions - at DLL level

[4]Each E1 line can carry 30 voice channels plus 2 system channels. Every voice channel is sampled by 8000 times per second and 8 bits are required to encode each sample, hence E1 line data rate is $32 * 8000 * 8 = 2.048$ *Mbps*.
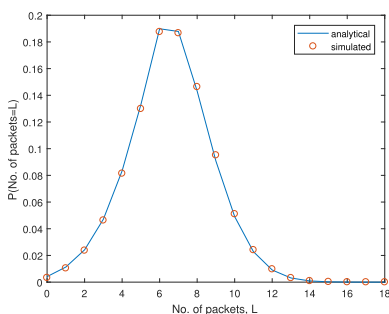
FIGURE 7. Comparison of simulated QLD with derived analytical QLD for $M/\tilde{D}/1/B_{on}$ queue with deterministic vacations and $\lambda = 1000$, $B_{on} = 18$, $D = 10^{-3}$.
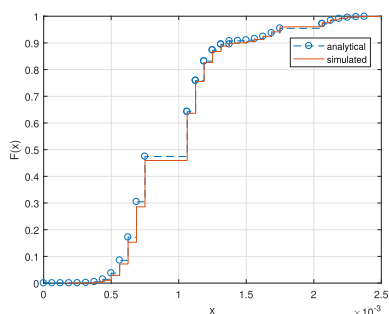

FIGURE 8. CDF of IDTs of $M/\tilde{D}/1/B_{on}$ queue with deterministic vacations and $\lambda = 1000$, $B_{on} = 18$, $D = 10^{-3}$.
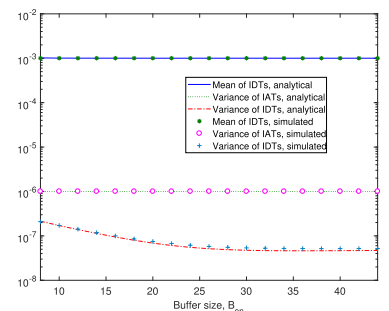

FIGURE 9. Mean and variance of IDTs of $M/\tilde{D}/1/B_{on}$ queue with deterministic vacations and $\lambda = 1000$, $D = 10^{-3}$.
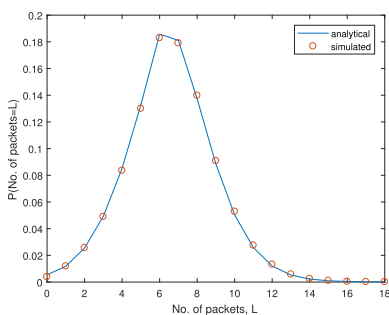

FIGURE 10. Comparison of simulated QLD with derived analytical QLD for $M/\tilde{D}/1/B_{on}$ queue with exponential vacations and $\lambda = 1000$, $B_{on} = 18$, $\theta = 1000$.
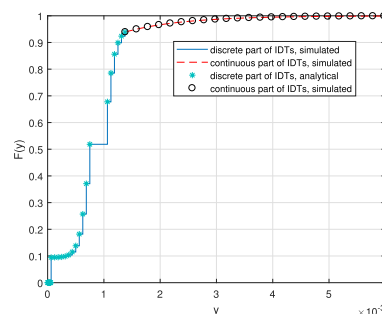

FIGURE 11. CDF of IDTs of $M/\tilde{D}/1/B_{on}$ queue with exponential vacations and $\lambda = 1000$, $B_{on} = 18$, $\theta = 1000$.
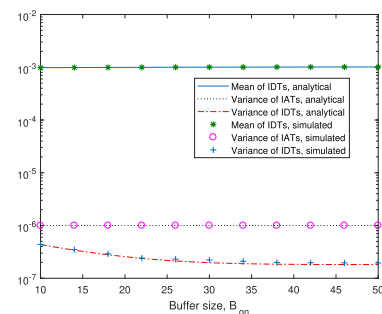

FIGURE 12. Mean and variance of IDTs of $M/\tilde{D}/1/B_{on}$ queue with exponential vacations and $\lambda = 1000$, $\theta = 1000$.

tions are deterministic (exponential). The CDF of IDTs X in Fig. 8 is a staircase function, because X is a discrete random variable. In this case, the vacations are deterministic

(discrete in nature) and the service times dictated by algorithm-B are also discrete in nature. Hence the IDT X is a discrete random variable, irrespective of whether the server is busy or is in vacation, which can be observed from Fig. 5 also. Finally, we observe from Fig. 8, the simulation results matches reasonably well with the analytical results.

Next, the CDF of IDTs Y in Fig. 11 contains two parts: discrete and continuous. This is because of the fact that Y is a mixed random variable. From Fig. 6, we can observe that when the server is busy, the IDTs are same as service times, which are discrete in nature. But, when the server is in vacation, IDT is a sum of exponential vacation (continuous in nature) and service time (discrete). Hence the IDT Y is a mixed random variable when the vacations are exponentially distributed. At the end we observe that from Fig. 11, the derived CDF of IDTs with exponential vacation agrees well with the simulated values.

Figs. 9 and 12 compare the analytical and simulated curves of mean and variance of IDTs, for deterministic and exponential vacation cases respectively. Since the derived analytical CDF closely matches with the simulated CDF (in Figs. 8 and Fig. 11), the analytical mean and variance also matches with their simulated counterparts, which can be observed in Figs. 9 and 12. The variance of IDTs at $B_{on} = 36$ from Fig. 9 is 46.14 ns or equivalently the standard deviation is 0.215 ms, which gives a maximum IDT of $\frac{(1+0.215) \, 10^{-3}}{2048} = 593.3$ ns and minimum IDT of $\frac{(1-0.215) \, 10^{-3}}{2048} = 383.3$ ns. Note here that the average IAT (and average IDT) is 1 ms
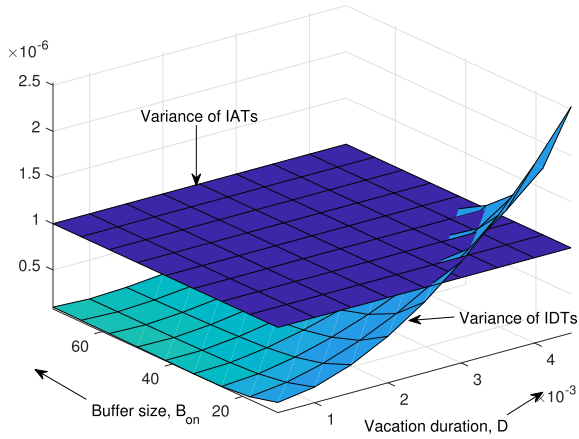
FIGURE 13. Variance of IDTs and variance of IATs: $M/\tilde{D}/1/B_{On}$ queue with deterministic vacations and $\lambda = 1000$ for different values of B_{On} and D.
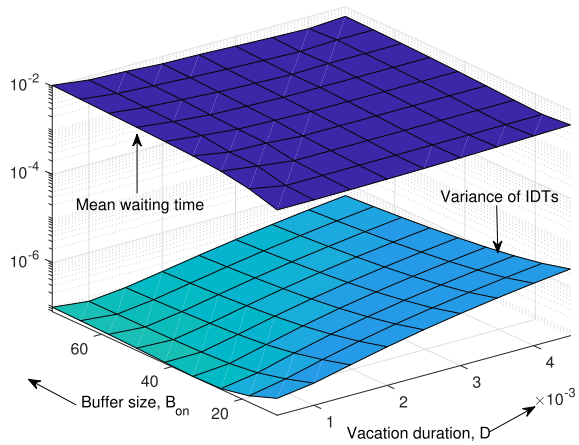

FIGURE 14. Variance of IDTs and MWT: $M/\tilde{D}/1/B_{On}$ queue with deterministic vacations and $\lambda = 1000$ for different values of B_{On} and D.
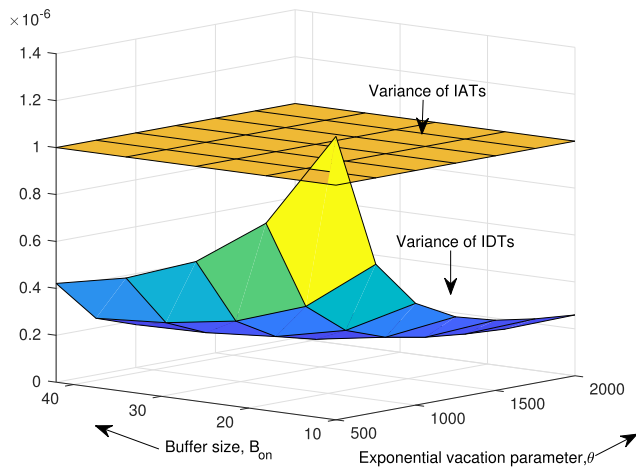

FIGURE 15. Variance of IDTs and variance of IATs: $M/\tilde{D}/1/B_{On}$ queue with exponential vacations and $\lambda = 1000$ for different values of B_{On} and θ.

and $1\ UI_{pp} = \frac{1}{2.048*10^6} = 488\ ns$. Therefore the jitter is $\frac{593.3-383.3}{488} = 0.43\ UI_{pp}$, which is less than $0.5\ UI_{pp}$ bound defined in the standard [15]. This shows the efficiency of algorithm-B in reducing the jitter.
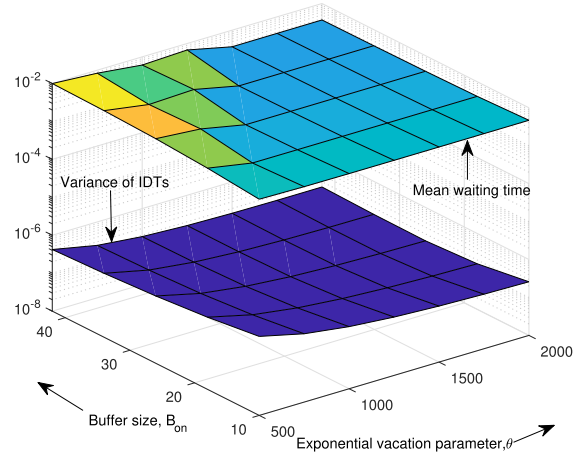

FIGURE 16. Variance of IDTs and MWT: $M/\tilde{D}/1/B_{On}$ queue with exponential vacations and $\lambda = 1000$ for different values of B_{On} and θ.

Fig. 13 shows the variance of IDTs and variance of inter-arrival times (IATs) for different values of buffer size B_{on} and vacation duration D. Fig. 14 shows the variance of IDTs and mean waiting time (MWT) as a function of buffer size B_{on} and vacation duration D. The analytical MWT is computed by using the method described in [16] and the analytical value of variance of IATs is $\frac{1}{\lambda^2} = \frac{1}{(10^3)^2} = 10^{-6}$. From Fig. 14, it can be observed that as the vacation duration D increases the variance of IDTs increases and MWT decreases. Also, for a particular value of D, the variance of IDTs decreases and MWT increases with Buffer size B_{on}. If D increases beyond some value, the variance of IDTs becomes greater than variance of IATs, as shown in Fig. 13. Similar analogy can be observed for exponential vacation plots in shown in Figs. 15 and 16. We can select proper values for B_{on} and D (θ) such that the variance of IDTs is less than the variance of IATs, and at the same time satisfying the given constraints on delay (MWT) and jitter (variance of IDTs).

VII. CONCLUSIONS

In this paper, an attempt was made to explore the statistical modelling and performance analysis of an intermediate node and the jitter buffer at the receiver NNI in the PSN through which TDM over PSN is emulated. Accordingly, both the intermediate node and the jitter buffer at the receiver node at NNI were modelled as a finite-buffer vacation queue with Poisson input, queue-length dependent service times and suitable vacation distributions.

We derived the distributions for queue-length at embedded points and later the variance of IDT which gave an idea of jitter. We can observe that all the derived analytical results closely matches with the corresponding simulated results in Figs. 7-12, which shows the correctness of mathematical analysis in the paper.

Using the method illustrated and the analysis carried out in this paper, one could choose the proper values for buffer size and vacation duration. The above parameters could be chosen such that the variance of IDTs is less than the variance of IATs, for a given arrival rate, complying to the constraints

Algorithm 1 Algorithm-B

1) Calculate $\delta \overset{\triangle}{=} \left(\dfrac{B_{on} + 1 - m}{2B} \right) X_a$
2) **if** $\delta > I_{min} + \dfrac{X_a}{B}$ and $m \leq B - h$ **then**
   $\quad s_m \leftarrow \delta + \dfrac{hX_a}{B}$
3) **else if** $\delta > I_{min} + \dfrac{X_a}{B}$ and $m > B - h$ **then**
   $\quad s_m \leftarrow \delta$
4) **else if** $\delta < I_{min} + \dfrac{X_a}{B}$ and $m > B - h$ **then**
   $\quad s_m \leftarrow \delta + I_{min}$
5) **end if**

on delay (MWT) and jitter (variance of IDTs) for applications of TDM over PSN. The delay and jitter could be matched to the allowable end-to-end delay and jitter tolerance limit as specified by the TDM standards [15].

A rigorous formulation of optimization framework for dictating the buffer size and the vacation distribution parameter for a given TDM line through the PSN, is currently being undertaken. Moreover, extending this work to the non-Poissonian arrival process would be another direction for future work.

APPENDIX

A. BRIEF DESCRIPTION OF ALGORITHM-B

In this scheduling algorithm, the service time of a head-of-the line packet is a linearly decreasing function of queue-length at its service initiation instant. If m represents the number of packets at the service initiation instant of a packet then the service time s_m of that packet is calculated as follows:

where X_a is the average inter-arrival time (IAT) of input process, B and B_{on} are the buffer sizes of offline and online algorithms, respectively, such that $B_{on} = 2B + h$, $1 \leq h < B$ and I_{min} is the minimum bound on the IDT of an offline algorithm. Reference [1] for more details on these parameters.

B. CALCULATION OF $\beta(n)$ FOR EXPONENTIALLY DISTRIBUTED VACATIONS

As described earlier, $\beta(n)$ represents probability of n arrivals to the queue during single vacation period. Let W be the random variable representing exponentially distributed (with parameter θ) vacation periods. Now, we calculate $\beta(n)$, when vacations are exponentially distributed and arrivals are according to a Poisson process with parameter λ, as follows:

$$
\begin{aligned}
\beta(n) &= \int_{t=0}^{\infty} \alpha(n; t) f_W(t) \, dt \\
&= \int_{t=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \, \theta e^{-\theta t} \, dt \\
&= \frac{\lambda^n \theta}{(\lambda + \theta)^{n+1}}
\end{aligned}
\tag{54}
$$

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