IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received November 12, 2018, accepted December 6, 2018, date of publication December 10, 2018, date of current version January 7, 2019.

Digital Object Identifier 10.1109/ACCESS.2018.2885951

Stability Analysis of Boolean Networks With Stochastic Function Perturbations

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This work was supported in part by the National Natural Science Foundation of China under Grant 61873150, Grant 61503225, and Grant 61320106011, in part by the Natural Science Foundation of Shandong Province under Grant ZR2017MF068, in part by the Natural Science Fund for Distinguished Young Scholars of Shandong Province under Grant JQ201613, and in part by the Major International (Regional) Joint Research Project of the National Natural Science Foundation of China under Grant 61320106011.

ABSTRACT Using the semi-tensor product method, this paper studies the set stability of Boolean networks (BNs) with stochastic function perturbations. First, the definition of one-column function perturbation for BNs is defined, and two kinds of stochastic function perturbations are formulated. Second, by constructing a state transition matrix, a new criterion is proposed for the set stability of BNs with probabilistic function perturbation. Third, the set stability of BNs with Markov jump function perturbation is studied by calculating the state probability distribution. Finally, the obtained results are applied to D. melanogaster segmentation polarity gene network.

INDEX TERMS Boolean network, stability, stochastic function perturbation, semi-tensor product of matrices.

I. INTRODUCTION

Stability analysis is a fundamental issue in understanding the behavior of nonlinear dynamical systems [12], [40], [41]. Particularly, for gene regulatory networks (GRNs), it is shown that stability analysis can reveal the phenotype of a cell and explain some living phenomena [2], [8], [32], [33]. Recently, a semi-tensor product (STP) method [5], [7] has been established for Boolean networks (BNs) [14], [27], [29], [35], [37], [38], [42], which is an effective model of GRNs.

Cheng *et al.* [6] firstly studied the stability analysis problem of BNs based on STP. Then, the set stability analysis problem of BNs was proposed and solved in [10], [15], and [16]. The feedback stabilization problem of Boolean control networks was well studied by many scholars [4], [9], [17], [20]–[22], [25]. The Lyapunov function method has also been developed for the stability analysis of BNs in some recent works [18]. [31] investigated Markov jump switching and stability analysis combining the STP method with linear positive system theory. In [11], a new definition of stability in the distribution (SD) for probabilistic Boolean networks (PBNs) was proposed, based on which, SD problems of PBNs with Markov switching was considered. For other applications on STP, please refer to [19], [23], [24], and [28].

It should be pointed out that due to gene mutation, the function perturbation often occurs in the model of GRNs [33], [36]. The solvability of function perturbation analysis in GRNs can help us design therapeutic interventions that guide a GRN from some dangerous states to a healthy one [34]. Xiao and Dougherty [36] investigated the impact of one-bit function perturbation on the fixed point of BNs. Meng and Feng [30] further considered the impact of modifications of update schedule on the topological structure of BNs. The function perturbation impact on the transition matrix and topological structure of singular BNs was considered in [26] based on STP. Note that the existing results on function perturbation of BNs just considered the deterministic function perturbation [13], [39]. In practical GRNs, most of gene mutations are generated by some stochastic factors [3]. Hence, it is meaningful to study the impact of stochastic function perturbation on the topological structure of BNs. However, there exist fewer results on this topic.

In this paper, we investigate the set stability analysis of BNs with stochastic function perturbations based on STP, and present several new results. The main contributions of this paper are as follows:

(i) The stochastic function perturbation problem is firstly proposed in this paper, which is more practical than the existing deterministic function perturbation.
(ii) Based on STP, two new criteria are presented for the set stability of BNs with probabilistic function

perturbation and Markov jump function perturbation, respectively. These conditions are easily verified via MATLAB.

The paper is organized as follows. Section II presents preliminaries and problem formulation. In Section III, we study the stochastic function perturbation impact on the set stability of BNs, and present the main results. In Section IV, we give two illustrative examples. Section V is a brief conclusion.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. PRELIMINARIES

$\mathcal{D}$ is a set consisting of 0 and 1. A mapping $f : \mathcal{D}^n \to \mathcal{D}$ is called a logical function. $\Delta_k := \{\delta_k^i : i = 1, \cdots, k\}$, where δ_k^i denotes the i-th column of I_k. $\Delta_2 := \Delta$. $\mathbf{1}_n := [\underbrace{1 \ \cdots \ 1}_{n}]$.

Given $Q \in \mathbb{R}^{m\times n}$, $Col_\alpha(Q)$, $Row_\alpha(Q)$ and $(Q)_{\alpha,\beta}$ denote the α-th column, α-th row and (α, β)-th entry of Q, respectively. Given $Q \in \mathbb{R}^{n\times t}$, if $Col(Q) \subseteq \Delta_n$, then $Q = [\delta_n^{i_1} \ \cdots \ \delta_n^{i_t}] = \delta_n[i_1 \ \cdots \ i_t]$ is called the logical matrix. The set containing all $n\times t$ logical matrices is denoted by $\mathcal{L}_{n\times t}$. $Q \in \mathbb{R}^{n\times t}$ satisfying $(Q)_{\alpha,\beta} \geq 0$ and $\sum_{\alpha=1}^{n}(Q)_{\alpha,\beta} = 1, \forall\ \beta = 1, \cdots, t$ is called the stochastic matrix. The set of all $n \times t$ stochastic matrices is denoted as $\Upsilon_{n\times t}$.

The main tool of this paper is STP, denoted by "$\ltimes$". For its definition and properties, please refer to [5].

B. PROBLEM FORMULATION

The considered BN is given as follows:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \quad x_2(t), \cdots, x_n(t)), \\ \qquad \vdots \\ x_n(t+1) = f_n(x_1(t), \quad x_2(t), \cdots, x_n(t)), \end{cases} \tag{1}$$

where $X(t) = (x_1(t), \cdots, x_n(t)) \in \mathcal{D}^n$ and $f_i : \mathcal{D}^n \mapsto \mathcal{D}$, $i = 1, \cdots, n$ are the state variables and logical functions, respectively.

Identify logical variables as the vector form and set $x(t) = \ltimes_{i=1}^{n} x_i(t)$. Then, system (1) can obtain the following equivalent algebraic form of system (1):

$$x(t+1) = Lx(t), \tag{2}$$

where $L \in \mathcal{L}_{2^n\times 2^n}$ is called the state transition matrix. Denote

$$L = \delta_{2^n}[i_1 \ i_2 \ \cdots \ i_{2^n}].$$

The definition of one-column function perturbation for system (1) is given as follows.

Definition 1: A one-column perturbation for system (1) occurs if some column of L changes.

In practical GRNs, gene mutation always occurs stochastically [3]. Thus, in this paper, we consider the following two kinds of stochastic function perturbations:

(i) Probabilistic function perturbation: the j-th column of L changes from i_j to $k \in \{1, 2, \cdots, 2^n\}$ with $\mathbb{P}\{i_j = k\} = p_k \geq 0$. Obviously, $\sum_{k=1}^{2^n} p_k = 1$.

(ii) Markov jump function perturbation: the j-th column of L changes according to a Markov chain, that is, $\mathbb{P}\{i_j(t+1) = s \mid i_j(t) = l\} = p_{sl}$, and $\sum_{s=1}^{2^n} p_{sl} = 1$, $l = 1, \cdots, 2^n$.

Now, we give the concepts of set stability with probability one and with positive probability for system (1) with stochastic function perturbations, respectively.

Given a nonempty set $A = \{\delta_{2^n}^{\alpha_1}, \cdots, \delta_{2^n}^{\alpha_r}\} \subseteq \Delta_{2^n}$ and one of the above two kinds of stochastic function perturbations.

Definition 2: System (1) is said to be stable at the set A with probability one under one of the above two kinds of stochastic function perturbations, if $\exists\ \tau \in \mathbb{Z}_+$ such that under the considered stochastic function perturbation,

$$\mathbb{P}\{x(t) \in A \mid x(0) = x_0\} = 1$$

holds for $\forall\ x_0 \in \Delta_{2^n}$ and $\forall\ t \geq \tau, t \in \mathbb{N}$.

Definition 3: System (1) is said to be stable at the set A with positive probability under one of the above two kinds of stochastic function perturbations, if $\exists\ \tau \in \mathbb{Z}_+$ such that under the considered stochastic function perturbation,

$$\mathbb{P}\{x(t) \in A \mid x(0) = x_0\} > 0$$

holds for $\forall\ x_0 \in \Delta_{2^n}$ and $\forall\ t \geq \tau, t \in \mathbb{N}$.

III. MAIN RESULTS

A. PROBABILISTIC FUNCTION PERTURBATION

This part considers the set stability problem of system (1) with probabilistic function perturbation.

Setting

$$L_k := \delta_{2^n}[i_1 \ \cdots \ i_{j-1} \ k \ i_{j+1} \ \cdots \ i_{2^n}], \tag{3}$$

we obtain the following equivalent stochastic system of system (1) with probabilistic function perturbation:

$$x(t+1) = \widetilde{L}x(t), \tag{4}$$

where $\widetilde{L} \in \mathcal{L}_{2^n\times 2^n}$ and $\mathbb{P}\{\widetilde{L} = L_k\} = p_k$.

Define

$$M = \sum_{k=1}^{2^n} p_k L_k \in \Upsilon_{2^n\times 2^n} \tag{5}$$

and

$$B = \{\alpha_1, \cdots, \alpha_r\}. \tag{6}$$

The necessary and sufficient condition for the set stability with probability one of system (1) with probabilistic function perturbation can be obtained as follows.

Theorem 1: System (1) with probabilistic function perturbation is stable at the set A with probability one, iff $\exists\ \tau \leq 2^n$, $\tau \in \mathbb{Z}_+$ such that

$$\sum_{i\in B} Row_i(M^\tau) = \mathbf{1}_{2^n}. \tag{7}$$

Proof: For $\forall\ t \in \mathbb{N}$ and $\forall\ x(t) \in \Delta_{2^n}$, we can gain

$$\mathbb{P}\{x(t+1) = L_k x(t)\} = p_k,$$

which implies that

$$\mathbb{P}\{x(t+1)=\delta_{2^n}^{\alpha} \mid x(t)=\delta_{2^n}^{\beta}\}=\Big(\sum_{k=1}^{2^n}p_kL_k\Big)_{\alpha,\beta}=(M)_{\alpha,\beta}.$$

Thus,

$$\mathbb{P}\{x(t)=\delta_{2^n}^{\alpha} \mid x(0)=\delta_{2^n}^{\beta}\}=(M^t)_{\alpha,\beta} \tag{8}$$

holds for $\forall\, t\in\mathbb{N}$ and $\forall\,\alpha,\beta\in\{1,2,\cdots,2^n\}$.

(Necessity) Suppose that system (1) with probabilistic function perturbation is stable at the set A with probability one. Then, from (8) and Definition 2, there exists a $\tau\in\mathbb{Z}_+$ such that

$$\begin{aligned}\mathbb{P}\{x(t)\in A \mid x(0)=\delta_{2^n}^{\beta}\}&=\sum_{i\in B}\mathbb{P}\{x(t)=\delta_{2^n}^{i} \mid x(0)=\delta_{2^n}^{\beta}\}\\&=\sum_{i\in B}(M^t)_{i,\beta}=1\end{aligned} \tag{9}$$

holds for $\forall\ t\ \geq\ \tau$ and $\forall\ \beta\ \in\ \{1,2,\cdots,2^n\}$. Hence, $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $\forall\, t\geq\tau$.

Let τ be the smallest integer such that

$$\sum_{i\in B}Row_i(M^{\tau})=\mathbf{1}_{2^n}.$$

If $\tau>2^n$, then there exist two positive integers $p\leq 2^n, p\notin B$ and $q\leq 2^n$ such that $(M^{2^n})_{p,q}>0$, that is,

$$\mathbb{P}\{x(2^n)=\delta_{2^n}^{p} \mid x(0)=\delta_{2^n}^{q}\}>0.$$

Since $\{x(t):t\in\{0,1,\cdots,2^n\}\}$ has 2^n different elements, there exist $t_1,t_2\leq 2^n$ $(t_1<t_2, t_1,t_2\in\mathbb{N})$ such that

$$x(t_1)=x(t_2):=\delta_{2^n}^{\gamma}$$

and

$$\mathbb{P}\{x(t_2-t_1)=\delta_{2^n}^{\gamma} \mid \delta_{2^n}^{\gamma}\}>0.$$

Then,

$$\begin{aligned}&\mathbb{P}\{x(2^n+s(t_2-t_1))=\delta_{2^n}^{p} \mid x(0)=\delta_{2^n}^{q}\}\\&\geq\Big(\mathbb{P}\{x(t_2-t_1)=\delta_{2^n}^{\gamma} \mid \delta_{2^n}^{\gamma}\}\Big)^s\\&\quad\times\mathbb{P}\{x(2^n)=\delta_{2^n}^{p} \mid x(0)=\delta_{2^n}^{q}\}>0\end{aligned}$$

holds for $\forall\ s>0, s\in\mathbb{Z}_+$. Therefore, there exists $s\in\mathbb{Z}_+$ satisfying $2^n+s(t_2-t_1)>\tau$, which contradicts to the minimality of τ. Hence, $\tau\leq 2^n$.

(Sufficiency) Presume that (7) holds. By induction, we prove that $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $\forall\, t\geq\tau, t\in\mathbb{N}$ at first. Obviously, $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $t=\tau$. Provided that $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $t=\xi>\tau$, then

$$\begin{aligned}&\sum_{i\in B}Row_i(M^{\xi+1})\\&=\sum_{i\in B}Row_i(M^{\xi}M)\\&=\sum_{i\in B}[Row_i(M^{\xi})M]=\Big[\sum_{i\in B}Row_i(M^{\xi})\Big]M\\&=\mathbf{1}_{2^n}M=\mathbf{1}_{2^n},\end{aligned}$$

which implies that $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $t=\xi+1$. By induction, $\sum_{i\in B}Row_i(M^t)=\mathbf{1}_{2^n}$ holds for $\forall\ t\ \geq\ \tau$, $t\in\mathbb{N}$.

From (7) and (8),

$$\begin{aligned}\mathbb{P}\{x(t)\in A \mid x(0)=\delta_{2^n}^{\beta}\}&=\sum_{i\in B}(M^t)_{i,\beta}\\&=Col_{\beta}\Big(\sum_{i\in B}Row_i(M^t)\Big)=Col_{\beta}(\mathbf{1}_{2^n})=1\end{aligned}$$

holds for $\forall\ t\ \geq\ \tau$, $t\ \in\ \mathbb{N}$, and $\forall\ \beta\ \in\ \{1,2,\cdots,2^n\}$, which together with Definition 2 shows that system (1) with probabilistic function perturbation is stable at the set A with probability one. □

Remark 1: When (7) holds for some $p_k\ \geq\ 0$ with $\sum_{k=1}^{2^n}p_k=1$, it will hold for any p_k satisfying $p_k\geq 0$ and $\sum_{k=1}^{2^n}p_k=1$.

Similarly, we can obtain the following necessary and sufficient condition for the set stability with positive probability of system (1) with probabilistic function perturbation.

Theorem 2: System (1) with probabilistic function perturbation is stable at the set A with positive probability, iff $\exists\ \tau\leq 2^n$, $\tau\in\mathbb{Z}_+$ such that

$$\sum_{i\in B}Row_i(M^{\tau})>0. \tag{10}$$

Remark 2: As a special kind of one-column function perturbations, one-bit function perturbation [36] of system (1) occurs if some logical function f_i, $i=1,\cdots,n$ has an alteration by modifying the value on the j-th ($j\in\{1,\cdots,n\}$) entry of its truth table, that is, the j-th column of M_i changes its value to M_i'. In this case, $\widetilde{L}$ only has two possible choices, that is, $\mathbb{P}\{\widetilde{L}=L\}=p$ and $\mathbb{P}\{\widetilde{L}=M_1*\cdots*M_i'*\cdots*M_n\}=1-p$, where $0<p<1$. By letting the probability of other 2^n-2 possible choices of L be 0, one can study the set stability of system (1) with one-bit function perturbation based on Theorem 1.

B. MARKOV JUMP FUNCTION PERTURBATION

This part studies the set stability problem of system (1) with Markov jump function perturbation.

With the Markov jump function perturbation, we can convert system (1) into equivalent Markov jump switched system as follows:

$$x(t+1)=L_{\sigma(t)}x(t), \tag{11}$$

where $\sigma:\mathbb{N}\mapsto\{1,2,\cdots,2^n\}$ is a switching signal satisfying $\mathbb{P}\{\sigma(t+1)=s \mid \sigma(t)=l\}=p_{sl}$, $\sum_{s=1}^{2^n}p_{sl}=1$, $l=1,\cdots,2^n$, and L_k, $k=1,2,\cdots,2^n$ is given in (3).

Identifying $\sigma(t)=l$ as the vector form $\sigma(t)=\delta_{2^n}^{l}$ and setting

$$\hat{L}=[L_1\ \ L_2\ \ \cdots\ \ L_{2^n}]\in\mathcal{L}_{2^n\times 2^{2n}}, \tag{12}$$

we have the equivalent form of system (11) as follows:

$$x(t+1)=\hat{L}\sigma(t)x(t), \tag{13}$$

where $\{\sigma(t) : t \in \mathbb{N}\} \subseteq \Delta_{2^n}$ denotes the discrete-time homogeneous Markov chain and the transition probability matrix $P = (p_{sl})_{2^n \times 2^n}$, where

$$\mathbb{P}\{\sigma(t+1) = \delta_{2^n}^s \mid \sigma(t) = \delta_{2^n}^l\} = p_{sl}. \tag{14}$$

Given a probability distribution $\sigma(t) \in \Upsilon_{2^n \times 1}$, we can easily obtain that

$$\sigma(t+1) = P\sigma(t). \tag{15}$$

The above analysis can lead to the following result.

Lemma 1: For $\forall\, t \in \mathbb{Z}_+$, $\forall\, x(0) \in \Delta_{2^n}$ and $\forall\, \sigma(0) \in \Delta_{2^n}$, the probability distribution of $x(t)$ for system (13) satisfies the following formula:

$$x(t) = Q_t\sigma(0)x(0), \tag{16}$$

where

$$Q_t = \begin{cases} \hat{L}, & t = 1, \\ \hat{L}P^{t-1}(I_{2^n} \otimes Q_{t-1})M_{r,2^n}, & t \geq 2. \end{cases}$$

Proof: When $t = 1$, from (13), one can easily see that $x(1) = \hat{L}\sigma(0)x(0) = Q_1\sigma(0)x(0)$ holds for $\forall\, x(0) \in \Delta_{2^n}$ and $\forall\, \sigma(0) \in \Delta_{2^n}$, that is, $Q_1 = \hat{L}$.

For $t = 2$, by (13) and (15), we have

$$\begin{aligned} x(2) &= \hat{L}\sigma(1)x(1) \\ &= \hat{L}P\sigma(0)x(1) \\ &= \hat{L}P\sigma(0)Q_1\sigma(0)x(0) \\ &= \hat{L}P(I_{2^n} \otimes Q_1)M_{r,2^n}\sigma(0)x(0) \\ &= Q_2\sigma(0)x(0), \end{aligned}$$

where $Q_2 = \hat{L}P(I_{2^n} \otimes Q_1)M_{r,2^n} \in \Upsilon_{2^n \times 2^{2n}}$.

Similarly, for any integer $t \geq 3$,

$$\begin{aligned} x(t) &= \hat{L}\sigma(t-1)x(t-1) \\ &= \hat{L}P\sigma(t-2)x(t-1) \\ &= \hat{L}P^{t-1}\sigma(0)Q_{t-1}\sigma(0)x(0) \\ &= \hat{L}P^{t-1}(I_{2^n} \otimes Q_{t-1})M_{r,2^n}\sigma(0)x(0) \\ &= Q_t\sigma(0)x(0), \end{aligned}$$

where $Q_t = \hat{L}P^{t-1}(I_{2^n} \otimes Q_{t-1})M_{r,2^n} \in \Upsilon_{2^n \times 2^{2n}}$.

Hence, (16) holds. This completes the proof. □

If system (1) with Markov jump function perturbation is stable at the set A with probability one, then A should be an invariant set for any L_i, $i = 1, 2, \cdots, 2^n$. Thus, we give the following assumption.

Assumption 1: The set A satisfies $L_iA \subseteq A$, $\forall\, i = 1, 2, \cdots, 2^n$, where $L_iA := \{L_ix : x \in A\}$.

Based on the above analysis, we present the following result.

Theorem 3: Suppose that Assumption 1 holds. System (1) with Markov jump function perturbation is stable at the set A with probability one, if and only if there exists a positive integer $\tau \leq 2^{2n}$ such that

$$\sum_{i \in B} Row_i(Q_\tau) = \mathbf{1}_{2^{2n}}. \tag{17}$$

Proof (Necessity): Presume that system (1) with Markov jump function perturbation is stable at the set A with probability one. Then, from Definition 2 and Lemma 1, there exists a $\tau \in \mathbb{Z}_+$ such that

$$\mathbb{P}\{x(t) \in A \mid \sigma(0), x(0)\} = \sum_{i \in B} Row_i(Q_t)\sigma(0)x(0) = 1 \tag{18}$$

holds for $\forall\, t \geq \tau$, $t \in \mathbb{N}$, $\forall\, x(0) \in \Delta_{2^n}$ and $\forall\, \sigma(0) \in \Delta_{2^n}$. Hence, $\sum_{i \in B} Row_i(Q_t) = \mathbf{1}_{2^{2n}}$, $\forall\, t \geq \tau$.

Let τ be the smallest integer such that

$$\sum_{i \in B} Row_i(Q_\tau) = \mathbf{1}_{2^{2n}}.$$

If $\tau > 2^{2n}$, then there exist two positive integers $p \leq 2^n$, $p \notin B$ and $q \leq 2^{2n}$ such that $(Q_{2^{2n}})_{p,q} > 0$. Therefore, for $\sigma(0)x(0) = \delta_{2^{2n}}^q$, there exists a path $\{\sigma(0), \cdots, \sigma(2^{2n}-1)\}$ such that

$$\begin{aligned} &\mathbb{P}\{x(2^{2n}) = \delta_{2^n}^p \mid x(0), \sigma(0)\} \\ &\quad \geq \mathbb{P}\{\sigma(2^{2n}-1) \mid \sigma(2^{2n}-2)\}\mathbb{P}\{\sigma(2^{2n}-2) \mid \\ &\qquad \sigma(2^{2n}-3)\} \cdots \mathbb{P}\{\sigma(1) \mid \sigma(0)\} > 0. \end{aligned}$$

Since $\{\sigma(t)x(t) : t \in \{0, 1, \cdots, 2^{2n}\}\}$ has 2^{2n} different elements, there exist $t_1, t_2 \leq 2^{2n}$ ($t_1 < t_2, t_1, t_2 \in \mathbb{N}$) such that

$$\sigma(t_1)x(t_1) = \sigma(t_2)x(t_2) := \delta_{2^n}^\theta \ltimes \delta_{2^n}^\gamma$$

and

$$\mathbb{P}\{x(t_2 - t_1) = \delta_{2^n}^\gamma \mid \delta_{2^n}^\gamma, \delta_{2^n}^\theta\} > 0.$$

Then, for $\sigma(0)x(0) = \delta_{2^{2n}}^q$, we can obtain that

$$\begin{aligned} &\mathbb{P}\{x(2^{2n} + s(t_2 - t_1)) = \delta_{2^n}^p \mid x(0), \sigma(0)\} \\ &\quad \geq \left(\mathbb{P}\{x(t_2 - t_1) = \delta_{2^n}^\gamma \mid \delta_{2^n}^\gamma, \delta_{2^n}^\theta\}\right)^s \\ &\qquad \times \mathbb{P}\{x(2^{2n}) = \delta_{2^n}^p \mid x(0), \sigma(0)\} > 0 \end{aligned}$$

holds for $\forall\, s > 0, s \in \mathbb{Z}_+$. Therefore, there exists $s \in \mathbb{Z}_+$ satisfying $2^{2n} + s(t_2 - t_1) > \tau$, which contradicts the minimality of τ. Hence, $\tau \leq 2^{2n}$.

(Sufficiency) Presume that (17) holds. By induction, we prove that $\sum_{i \in B} Row_i(Q_t) = \mathbf{1}_{2^{2n}}$ holds for $\forall\, t \geq \tau, t \in \mathbb{N}$ at first. Obviously, $\sum_{i \in B} Row_i(Q_t) = \mathbf{1}_{2^{2n}}$ holds for $t = \tau$. Provided that $\sum_{i \in B} Row_i(Q_t) = \mathbf{1}_{2^{2n}}$ holds for $t = \xi > \tau$. By (18), we obtain that $\mathbb{P}\{x(\xi) \in A \mid \sigma(0), x(0)\} = 1$ holds for $\forall\, x(0) \in \Delta_{2^n}$ and $\forall\, \sigma(0) \in \Delta_{2^n}$. Then, from Assumption 1, one can see that

$$\begin{aligned} &\mathbb{P}\{x(\xi+1) \in A \mid \sigma(0), x(0)\} \\ &\quad = \sum_{i \in B} \mathbb{P}\{x(\xi+1) = \delta_{2^n}^i \mid \sigma(0), x(0)\} \\ &\quad = \sum_{i \in B} \sum_{i_\xi \in B} \mathbb{P}\{x(\xi+1) = \delta_{2^n}^i \mid \sigma(\xi) = P\sigma(\xi-1), \end{aligned}$$

$$x(\xi) = \delta_{2^n}^{i_\xi}\} \times \Big[\sum_{\delta_{2^n}^{i_1}, \cdots, \delta_{2^n}^{i_{\xi-1}} \in \Delta_{2^n}} \mathbb{P}\{x(1) = \delta_{2^n}^{i_1} \mid \sigma(0), x(0)\} \times \cdots \times \mathbb{P}\{x(\xi) = \delta_{2^n}^{i_\xi} \mid \sigma(\xi) = P\sigma(\xi - 1), x(\xi - 1) = \delta_{2^n}^{i_{\xi-1}}\}\Big] = 1$$

holds for $\forall\, x(0) \in \Delta_{2^n}$ and $\forall\, \sigma(0) \in \Delta_{2^n}$, which implies that $\sum_{i \in B} Row_i(Q_{\xi+1}) = \mathbf{1}_{2^{2n}}$. By induction, $\sum_{i \in B} Row_i(Q_t) = \mathbf{1}_{2^{2n}}$ holds for $\forall\, t \geq \tau, t \in \mathbb{N}$.

Therefore, for $\forall\ t \geq \tau$, $t \in \mathbb{N}$, $\forall\ x(0) \in \Delta_{2^n}$ and $\forall\ \sigma(0) \in \Delta_{2^n}$, we have $\mathbb{P}\{x(t) \in A \mid \sigma(0), x(0)\} = \sum_{i \in B} Row_i(Q_t)\sigma(0)x(0) = 1$, which together with Definition 2 shows that system (1) with Markov jump function perturbation is stable at the set A with probability one. □

Similar to Theorem 3, we have the following result on the set stability with positive probability of system (1) with Markov jump function perturbation.

Theorem 4: Suppose that Assumption 1 holds. System (1) with Markov jump function perturbation is stable at the set A with positive probability, if and only if there exists a positive integer $\tau \leq 2^{2n}$ such that

$$\sum_{i \in B} Row_i(Q_\tau) > 0. \tag{19}$$

Remark 3: When $|A| = 1$, one can use Theorems 1 and 3 to study the stability to an equilibrium of BNs with stochastic function perturbation.

IV. ILLUSTRATIVE EXAMPLES

Example 1: The D. melanogaster segmentation polarity gene network is given as follows:

$$\begin{cases} x_1(t+1) = x_1(t) \wedge \neg x_2(t) \wedge \neg x_4(t), \\ x_2(t+1) = \neg x_1(t) \wedge x_2(t) \wedge \neg x_3(t), \\ x_3(t+1) = x_1(t) \vee x_3(t), \\ x_4(t+1) = x_2(t) \vee x_4(t), \\ x_5(t+1) = \{\neg x_2(t) \wedge \neg x_4(t)\} \vee \{x_5(t) \\ \qquad \wedge \neg x_1(t) \wedge \neg x_3(t)\}, \\ x_6(t+1) = \{\neg x_1(t) \wedge \neg x_3(t)\} \vee \{x_6(t) \\ \qquad \wedge \neg x_2(t) \wedge \neg x_4(t)\}, \end{cases} \tag{20}$$

where $x_1 = wg_1, x_2 = wg_2, x_3 = wg_3, x_4 = wg_4, x_5 = PTC_1$ and $x_6 = PTC_2$, wg and PTC denote the secreted proteins wingless and the transmembrane receptor proteins patched, respectively [1].

Setting $x(t) = \ltimes_{i=1}^{6} x_i(t)$, by the STP method, system (20) has the algebraic representation as follows:

$$x(t+1) = Lx(t),$$

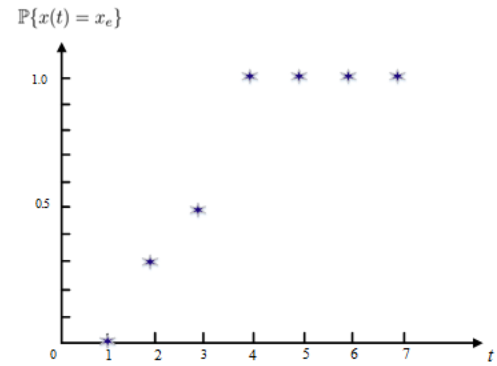

FIGURE 1. The probability trajectory of equilibrium $x_e = \delta_4^3$ under the Markov jump function perturbation $\sigma(t)$, where $\sigma(0) = \delta_4^2$ and $x(0) = \delta_4^4$.

where

$$\begin{aligned} L = \delta_{64}[\ & 52\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 52\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 52\ 52\ 52\ 52\ 21\ 22\ 21\ 22 \\ & 52\ 52\ 52\ 52\ 21\ 22\ 21\ 22 \\ & 52\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 41\ 41\ 43\ 43\ 41\ 41\ 43\ 43 \\ & 52\ 52\ 52\ 52\ 53\ 54\ 53\ 54 \\ & 57\ 57\ 59\ 59\ 61\ 61\ 61\ 61]. \end{aligned}$$

From the algebraic form, we can easily obtain the total 10 fixed points of system (20) as $\delta_{64}^{21}, \delta_{64}^{22}, \delta_{64}^{41}, \delta_{64}^{43}, \delta_{64}^{52}, \delta_{64}^{53}, \delta_{64}^{54}, \delta_{64}^{57}, \delta_{64}^{59}, \delta_{64}^{61}$.

Assume that after probabilistic function perturbation, $\mathbb{P}\{Col_1(L) = \delta_{64}^{20}\} = 0.6$ and $\mathbb{P}\{Col_1(L) = \delta_{64}^{52}\} = 0.4$. In the following, we verify whether system (20) is stable at $A = \{\delta_{64}^{21}, \delta_{64}^{22}, \delta_{64}^{41}, \delta_{64}^{43}, \delta_{64}^{52}, \delta_{64}^{53}, \delta_{64}^{54}, \delta_{64}^{57}, \delta_{64}^{59}, \delta_{64}^{61}\}$ with probability one or not.

Setting $M = L_1 \times 0.6 + L_2 \times 0.4$, where

$$\begin{aligned} L_1 = \delta_{64}[\ & 20\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 52\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 52\ 52\ 52\ 52\ 21\ 22\ 21\ 22 \\ & 52\ 52\ 52\ 52\ 21\ 22\ 21\ 22 \\ & 52\ 52\ 52\ 52\ 52\ 52\ 52\ 52 \\ & 41\ 41\ 43\ 43\ 41\ 41\ 43\ 43 \\ & 52\ 52\ 52\ 52\ 53\ 54\ 53\ 54 \\ & 57\ 57\ 59\ 59\ 61\ 61\ 61\ 61] \end{aligned}$$

and $L_2 = L$. A simple calculation shows that

$$\sum_{i \in \{21,22,41,43,52,53,54,57,59,61\}} Row_i(M^2) = \mathbf{1}_{64}.$$

Therefore, by Theorem 1, system (20) with probabilistic function perturbation is stable at the set A with probability one.

Example 2: The considered BN is:

$$\begin{cases} x_1(t+1) = \neg x_2(t), \\ x_2(t+1) = x_1(t) \vee x_2(t). \end{cases} \tag{21}$$

Setting $x(t) = x_1(t) \ltimes x_2(t)$, system (21) has the algebraic representation as follows:

$$x(t+1) = Lx(t), \tag{22}$$

where $L = \delta_4[3\ 1\ 3\ 2]$. Obviously, system (21) is stable at $x_e = \delta_4^3$.

Suppose that the 4-th column of L is perturbed by a Markov jump function perturbation with

$$P = \begin{bmatrix} 0.3 & 0.1 & 0.2 & 0 \\ 0.2 & 0.5 & 0.3 & 0 \\ 0.1 & 0.2 & 0.4 & 1 \\ 0.4 & 0.2 & 0.1 & 0 \end{bmatrix}.$$

According to (12), $\hat{L} = [L_1\ \ L_2\ \ L_3\ \ L_4]$, where $L_i = \delta_4[3\ 1\ 3\ i]$, $i = 1, 2, 3, 4$. One can easily see that $L_i x_e = x_e$, $i = 1, 2, 3, 4$. Thus, Assumption 1 holds.

By Lemma 1, a simple calculation shows that

$$Q_3 = \delta_4[3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3\ 3].$$

Therefore, $Row_3(Q_3) = \mathbf{1}_{16}$. By Theorem 3, system (21) with the Markov jump function perturbation is stable at δ_4^3 with probability one.

V. CONCLUSION

We have considered the set stability of BNs with two kinds of stochastic function perturbations, that is, probabilistic function perturbation and Markov jump function perturbation. By constructing a state transition matrix, we have proposed a new criterion for the set stability of BNs with probabilistic function perturbation. In addition, by calculating the state probability distribution, we have studied the set stability of BNs with Markov jump function perturbation.

It is noted that we only consider stochastic one-column function perturbations in this paper. Future works will study the case of stochastic multi-column function perturbations, and the set stabilization of BNs with stochastic function perturbations.

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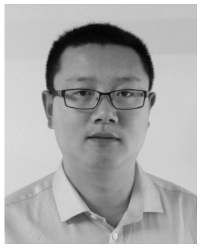

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