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The Uncertainty Analysis of Vague Sets in Rough Approximation Spaces

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ABSTRACT Vague sets, as well as intuitionistic fuzzy sets, are extensions of fuzzy sets. Based on fuzzy sets, vague sets generalize the membership degree from a single value to an interval value. Vague sets have a more powerful ability to process fuzzy information than fuzzy sets to some degree. In addition as we all know, human cognition is usually a gradual process. As a result, in a given multi-granularity space, how to characterize a vague concept and further measure its uncertainty, become a hot issue worth studying. However, the uncertainty of vague sets in rough approximation spaces is still lacking relative studies. Therefore, in order to more effectively excavate knowledge from vague sets, this paper focuses on the uncertainty of vague sets and reveals its hidden rules. First, change rules of the average fuzziness of the vague value with changing its truth membership degree and false membership degree are discussed. Second, in rough approximation spaces, the uncertainty of vague sets, i.e., the uncertainty of average-step-vague sets are analyzed. Then, its change rules with changing granularity are analyzed and discussed. Next, to better approximate a vague concept, change rules of uncertainty for approximation sets of vague sets with changing knowledge granularity are discussed. Finally, several illustration examples are listed to verify the obtained conclusions. These rules are in accordance with human cognitive mechanisms in multi-granularity knowledge spaces.

INDEX TERMS Vague sets, intuitionistic fuzzy sets, uncertainty, rough approximation, multi-granularity.

I. INTRODUCTION

In artificial intelligence research, granular computing (GrC) is a new methodology for simulating human thinking and solving complicated problems [34], [35], [46], [62]–[64]. GrC is regarded as an umbrella covering the theories, methodologies and techniques on granularity [69] and a powerful tool for solving complex problems in different fields, such as data mining, fuzzy information processing, large-scale computing, cloud computing, etc. [8], [16], [17], [48]. There are three main GrC theoretical models: fuzzy sets, rough sets and quotient space. As an important tool for dealing with uncertain and imprecise problems, fuzzy sets [65] were proposed by Zadeh in 1965. Since then, plenty of researchers have paid much attention to this theory and applied it to many different fields, such as fuzzy system [51], [53]–[55], [75], [78], fuzzy clustering [18], [19], [49], fuzzy control [6], [37], [52], [74], fuzzy reasoning [7], [62], [66], fuzzy decision [24], [33], [59], etc. As the generalization models of fuzzy sets, intuitionistic fuzzy sets [2] were proposed by Atanassov and vague sets [14] were introduced by Gau and Buehrer. Subsequently, Bustince and Burillo pointed out that vague sets and intuitionistic fuzzy sets are essentially the same [5]. They investigated the similarities and differences between vague sets and intuitionistic fuzzy sets in the literature [26] and concluded that vague sets are more universally applicable than intuitionistic fuzzy sets to some degree. As a result, intuitionistic fuzzy sets and vague sets are collectively referred to as vague sets in this paper.

Vague sets, as a typical soft computing method, have attracted much attention and have been applied to various fields [1], [56], [57], [59], [71], [76], [77]. Applied in decision-making, pattern recognition, knowledge discovery, etc., research on uncertainty of vague sets has also become a hot issue [11], [21], [25], [27], [30], [68], [72], [73]. Vagueness, as well as fuzziness and roughness, has been introduced to characterize the uncertainty of vague concepts [22], [29], [39]. On the basis of fuzzy sets, the fuzzy entropy was proposed to measure the uncertainty of vague sets [23], [24] and intuitionistic fuzzy sets [21]. Based on vague-rough sets, Feng *et al.* [12], [13] used the uncertainty knowledge acquisition to measure uncertainty. Based on the similarity among sets, Zhang *et al.* [72] proposed the similarity measure of vague sets. Based on intuitionistic fuzzy soft sets, Muthukumar and Krishnan [28] introduced the weighted similarity measure. Moreover, a similarity measure based on implication functions was proposed by Zeng *et al.* [67]. Summing up and analyzing existing methods, Zhang *et al.* [71] further characterized the uncertainty of vague sets with integral calculus method and produced a series of concepts, i.e., fuzziness interval of vague sets, average fuzziness of vague value and a new measuring method. However, in real applications, decision-making is usually a step-by-step process. Thus, with the development of GrC, multi-granularity knowledge discovery has become an important direction in the research of artificial intelligence. The uncertainty of the decision information systems is an important parameter for obtaining a good decision-making [9], [20], [38], [40], [50], [61]. In a great deal of literatures [15], [22], [29], [31], [32], [39], many researchers came into conclusion that how to describe a vague concept with an approximation set in rough knowledge space is the priority. Plenty of well-known researchers, such as Skowron, Szczuka, Dutta, Nguyen, Bazan and Polkowski, have conducted many research studies in this field [3], [10], [36], [41]–[45], [47]. Based on these aforementioned works, putting the vague sets in rough approximation spaces, Zhang proposed the approximation sets of a vague concept if there is no additional information and focused on constructing the approximation set of a vague set, such as the 0.5-crisp set, approximation set, step-vague set and average-step-vague set [70]. Furthermore, the change rules of the similarity degree between vague sets and its approximation sets with different knowledge granularity are summed up. On this foundation, several researchers studied on vague sets in rough approximation spaces including uncertainty. Bonikowski and Wybraniec-Skardowska [4] proposed a new formal approach to vagueness, and many important conditions concerning the membership relation for vague sets were established as well. John and Amirtharaj [20] proposed a novel similarity measure based on statistical confidence intervals and discussed its rules in multi-granularity spaces.

However, there are still several shortcomings in current research as follows:

(1) In data mining, it is important to understand the rules and internal structure of a model. However, there is still few theoretical analysis on the change rules of uncertainty of vague sets when changing its intrinsic parameters;

(2) In real applications, decision-making is usually a step-by-step progress and data are also accumulating. Thus, it is necessary to excavate the change rules of uncertainty from the perspective of changing knowledge granularity;

(3) In decision-making, how to characterize a vague concept is a hot issue of great interest. There is little research on the change rules of uncertainty for characterizing a vague concept, as well as the uncertainty of the approximation set in rough approximation spaces.

Thus, in order to solve these aforementioned problems, in this paper, the main contribution are as follows:

(1) We would further study fuzziness of the vague sets and excavate its hidden rules. It could provide a good theoretical foundation for judging system stability in uncertain information processing;

(2) We focus on the vague sets in rough approximation spaces and discuss the change rules of uncertainty of vague sets with changing knowledge granularity. The rules of uncertainty with changing granularity which we excavate provide an important basis for granularity selection and decision making.

Many relevant preliminary concepts are reviewed briefly and presented in Section 2. In Section 3, change rules of average fuzziness of the vague value with changing its truth membership degree and false membership degree are discussed and proved. Examples are cited to verify these rules. In Section 4, the uncertainty of vague sets in rough approximation spaces is established and discussed. Furthermore, with changing knowledge granularity, change rules of uncertainty of average-step-vague sets and change rules of uncertainty for approximation sets of vague sets are found and proved.

II. PRELIMINARIES

In order to better present the context of this paper, many preliminary concepts, definitions and results related to vague sets and uncertainty measure are reviewed as follows.

Definition 1 (Fuzzy Set [60], [65], [66]): Given a mapping in a universe of discourse U,

$$\mu_A : U \to [0, 1], \quad x \mapsto \mu_A(x),$$

where $U = \{x_1, x_2, \ldots, x_n\}$, $A = \{\langle x, \mu_A(x)\rangle \mid x \in U\}$ is called a fuzzy set in U, and $\mu_A(x)$ is called a membership function of A. The membership degree $\mu_A(x)$ $(0 \leq \mu_A(x) \leq 1)$ denotes the degree of the element belonging to the fuzzy set A.

Definition 2 (Vague Set [14]): A vague set V in a universe of discourse U is characterized by a truth membership function $t_V(x)$ and a false membership function $f_V(x)$. $t_V(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_V(x)$ is a lower bound on the negation of x derived from the evidence against x. Both $t_V(x)$ and $f_V(x)$ associate a real number in the interval [0, 1] with each point in x, where $t_V(x) + f_V(x) \leq 1$. That is, $t_V(x) : U \to [0, 1]$ and $f_V(x) : U \to [0, 1]$. When U is continuous, a vague set V can be represented by

$$V = \int_U [t_V(x), 1 - f_V(x)]/x dx.$$

When U is discrete, a vague set V can be represented by

$$V=\sum_{i=1}^{n}\left[t_V\left(x_i\right), 1-f_V\left(x_i\right)\right] / x_i .$$

Here, $\left[t_V(x), 1-f_V(x)\right]$ denotes a vague value of x, where $t_V(x) \leq \mu_V(x) \leq 1-f_V(x)$. Actually, the fuzzy set is a special vague set (that is, $t_V(x)=1-f_V(x)$), i.e., if a vague value interval $\left[t_V(x), 1-f_V(x)\right]$ becomes a single point set, the vague set will degenerate into a fuzzy set.

Definition 3 (Intuitionistic Fuzzy Set [2]): An intuitionistic fuzzy set $I=\{\langle x, \mu_I(x), v_I(x)\rangle \mid x \in U\}$ in a universe of discourse U is characterized by a membership function, μ_I, and a non-membership function, v_I, as follows, $\mu_I: U \rightarrow[0,1], v_I: U \rightarrow[0,1]$ and $0 \leq \mu_I+v_I \leq 1$.

Definition 4 [58]: Given an information system $S=(U, A)$, where $U=\{x_1, x_2, \ldots, x_n\}$, A is an attribute set. Let R be a subset of $A(R \subseteq A)$. For any vague set $V=\{[t_V(x), 1-f_V(x)] \mid x \in U\}$ in a universe of discourse U, and a pair of parameters (α, β) $(0 \leq \beta<\alpha \leq 1)$, the upper approximation set and lower approximation set of the vague set are defined as follows:

$$\underline{R}^{(\alpha, \beta)}(V)=\{x \in U \mid t_V(x) \geq \alpha\},$$
$$\overline{R}^{(\alpha, \beta)}(V)=\{x \in U \mid 1-f_V(x)>\beta\}.$$

The discourse U is divided into three disjoint regions as follows:

$$POS_R^{(\alpha, \beta)}(V)=\{x \in U \mid t_V(x) \geq \alpha\},$$
$$BND_R^{(\alpha, \beta)}(V)=\{x \in U \mid t_V(x)<\alpha \wedge 1-f_V(x)>\beta\},$$
$$NEG_R^{(\alpha, \beta)}(V)=\{x \in U \mid 1-f_V(x) \leq \beta\}.$$

Definition 5 (Step-Vague Set [70]): Let V be a vague set on U; R be an equivalence relation on U and $U / R=\{X_1, X_2, \ldots, X_m\}$. If for any $x \in X_1$, $\left[t_V(x), 1-f_V(x)\right]=\left[t_1, 1-f_1\right]$ is always satisfied, and for any $x \in X_2$, $\left[t_V(x), 1-f_V(x)\right]=\left[t_2, 1-f_2\right]$ always is satisfied, ..., and for any $x \in X_m$, $\left[t_V(x), 1-f_V(x)\right]=\left[t_m, 1-f_m\right]$ is held also, then the vague set V is called a step-vague set on U / R, and denoted as V_J, where $0 \leq t_i \leq 1$, $0 \leq f_i \leq 1$ and $t_i+f_i \leq 1\,(i=1,2, \ldots, m)$.

Definition 6 (Average-Step-Vague Set [70]): Let V be a vague set on U, R be an equivalence relation on U and $U / R=\{[x]_R \mid x \in U\}$. For any $x(x \in U)$.

$$\overline{V}_J(x)=\left[\frac{\sum_{y \in[x]_R} t_V(y)}{\left|[x]_R\right|}, 1-\frac{\sum_{y \in[x]_R} f_V(y)}{\left|[x]_R\right|}\right].$$

Then the vague set $\overline{V}_J$ is called average-step-vague set in the approximation space (U, R).

Definition 7 (Average Fuzziness of Vague Value [71]): Let $\left[t_V(x), 1-f_V(x)\right]$ be a vague value of x $(x \in U)$, the average fuzziness of x is defined as follows,

$$\overline{H}_V(x)=\frac{4}{1-f_V(x)-t_V(x)} \int_{t_V(x)}^{1-f_V(x)} \mu_V(x)\left[1-\mu_V(x)\right] d \mu_V(x).$$

If the discourse U is discrete, the average fuzziness of vague set V is defined as follows:

$$H_V(U)=\frac{1}{|U|} \sum_{i=1}^{|U|} \overline{H}_V\left(x_i\right).$$

If the discourse U is continuous in an interval $[a, b]$, the average fuzziness of the vague set V is defined as follows:

$$H_V(U)=\frac{1}{b-a} \int_a^b \overline{H}_V(x) d x.$$

Definition 8 (Containment [26]): A vague set V_1 is contained in another vague set V_2, i.e., $V_1 \subseteq V_2$, if and only if $t_{V_1}(x) \leq t_{V_2}(x)$ and $1-f_{V_1}(x) \leq 1-f_{V_2}(x)$ for any point x in U.

Definition 9 (Intersection [26]): The union of two vague sets V_1 and V_2 is a vague set V_3, written as $V_3=V_1 \cup V_2$, whose truth membership and false membership functions are related to those of V_1 and V_2, that is, $t_{V_3}(x)=\max\left\{t_{V_1}(x), t_{V_2}(x)\right\}$ and $f_{V_3}(x)=\min\left\{f_{V_1}(x), f_{V_2}(x)\right\}$.

Definition 10 (Union [26]): The intersection of two vague sets V_1 and V_2 is a vague set V_3, written as $V_3=V_1 \cap V_2$, whose truth membership and false membership functions are related to those of V_1 and V_2, that is, $t_{V_3}(x)=\min\left\{t_{V_1}(x), t_{V_2}(x)\right\}$ and $f_{V_3}(x)=\max\left\{f_{V_1}(x), f_{V_2}(x)\right\}$.

Definition 11 (Complement [26]): The complement of a vague set V is denoted by $\sim V$ and $\sim V$ is defined as follow: for any point x in a universe of discourse U, $t_{\sim V}(x)=f_V(x)$ and $f_{\sim V}(x)=t_V(x)$.

III. CHANGE RULES OF UNCERTAINTY OF A VAGUE VALUE

Uncertainty is an important characteristic of vague sets, and many researchers have focused on how to measure uncertainty of vague sets [11], [24], [25], [27], [29], [68]. Comparing with existing methods, Zhang *et al.* [71] proposed a new method for measuring fuzziness of vague sets, which was named the average fuzziness of a vague value. Furthermore, exploring properties of average fuzziness is also an essential study. In this section, change rules of an average fuzziness of the vague value with changing truth membership degree and false membership degree are discussed respectively.

Let V be a vague set in a universe of discourse U, $\left[t_V\left(x_1\right), 1-f_V\left(x_1\right)\right]$ be a vague value of $x_1\left(x_1 \in U\right)$ and $\left[t_V\left(x_2\right), 1-f_V\left(x_2\right)\right]$ be a vague value of $x_2\left(x_2 \in U\right)$. According to **Definition 7**, for any $x(x \in V)$, the average fuzziness of x is shown as follows:

$$\begin{aligned}
&\overline{H}_V(x)\\
&=\frac{4}{1-f_V(x)-t_V(x)} \int_{t_V(x)}^{1-f_V(x)} \mu_V(x)\left[1-\mu_V(x)\right] d \mu_V(x)\\
&=2\left(1-f_V(x)\right)+2 t_V(x)-\frac{4}{3}\left(1-f_V(x)\right)^2-\frac{4}{3} t_V^2(x)\\
&\quad-\frac{4}{3} t_V(x)\left(1-f_V(x)\right). \qquad (1)
\end{aligned}$$

Next, the size of $\overline{H}_V\left(x_1\right)-\overline{H}_V\left(x_2\right)$ would be discussed in the following situations:

Case 1: $t_V(x_1) = t_V(x_2) = t$ and $1-f_V(x_1) \geq 1-f_V(x_2)$. Thus, we could obtain

$$\begin{aligned} &\overline{H}_V(x_1) - \overline{H}_V(x_2) \\ &= 2(1-f_V(x_1)) - \frac{4}{3}(1-f_V(x_1))^2 - \frac{4}{3}t(1-f_V(x_1)) \\ &\quad - \left[2(1-f_V(x_2)) - \frac{4}{3}(1-f_V(x_2))^2 - \frac{4}{3}t(1-f_V(x_2))\right]. \end{aligned} \tag{2}$$

Suppose $1 - f_V(x)$ is an independent variable and $Y[1-f_V(x)] = -\frac{4}{3}(1-f_V(x))^2 + \left(2-\frac{4}{3}t\right)(1-f_V(x))$ is a dependent variable. Then $\overline{H}_V(x_1) - \overline{H}_V(x_2) = Y[1-f_V(x_1)] - Y[1-f_V(x_2)]$.

Therefore, we just need to discuss the variation trend of $Y[1-f_V(x)]$ when $1-f_V(x)$ increases. Additionally, $1-f_V(x_0) = -\frac{-\left(2-\frac{4}{3}t\right)}{2\times\left(-\frac{4}{3}\right)} = -\frac{1}{2}t + \frac{3}{4}$. In other words, if $1-f_V(x) \geq -\frac{1}{2}t + \frac{3}{4}$, $Y[1-f_V(x)]$ would decrease with increasing $1-f_V(x)$, and $1-f_V(x) < -\frac{1}{2}t+\frac{3}{4}$, $Y[1-f_V(x)]$ would increase with increasing $1-f_V(x)$.

1) When $t \geq \frac{1}{2}$, $1-f_V(x_0) = -\frac{1}{2}t + \frac{3}{4} \leq \frac{1}{2}$. Thus, we can obtain $1-f_V(x_1) \geq 1-f_V(x_2) \geq t \geq \frac{1}{2} \geq 1-f_V(x_0)$. Then $\overline{H}_V(x_1) \leq \overline{H}_V(x_2)$ is obtained. Every case is illustrated in Fig 1.
   In order to verify the above conclusion, **Example 1** is shown as follows:
   Example 1: Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, and V be a vague set in a universe of discourse U:

$$\begin{aligned} V &= \frac{[0.1, 0.6]}{x_1} + \frac{[0.1, 0.7]}{x_2} + \frac{[0.2, 0.4]}{x_3} + \frac{[0.2, 0.6]}{x_4} \\ &+ \frac{[0.3, 0.4]}{x_5} + \frac{[0.3, 0.45]}{x_6} + \frac{[0.3, 0.6]}{x_7} + \frac{[0.3, 0.7]}{x_8} \\ &+ \frac{[0.3, 0.9]}{x_9} + \frac{[0.4, 0.6]}{x_{10}} + \frac{[0.4, 0.8]}{x_{11}} + \frac{[0.4, 0.9]}{x_{12}} \\ &+ \frac{[0.7, 0.8]}{x_{13}} + \frac{[0.7, 0.9]}{x_{14}} + \frac{[0.8, 0.9]}{x_{15}}. \end{aligned}$$

   For $\frac{[0.7,0.8]}{x_{13}}$ and $\frac{[0.7,0.9]}{x_{14}}$, according to **Definition 7**, we have

$$\overline{H}_V(x_{13}) \approx 0.7467, \quad \overline{H}_V(x_{14}) \approx 0.6267.$$

   A fact is drawn that $t_V(x_{13}) = t_V(x_{14}) = 0.7 > \frac{1}{2}$, $1-f_V(x_{14}) < 1-f_V(x_{13})$ and $\overline{H}_V(x_{14}) < \overline{H}_V(x_{13})$. This fact validates the conclusion.
2) When $t < \frac{1}{2}$, $1-f_V(x_0) = -\frac{1}{2}t + \frac{3}{4} > \frac{1}{2}$. Thus, if $\frac{1}{2} \geq 1-f_V(x_1) \geq 1-f_V(x_2)$ and $1-f_V(x_0) \geq 1-f_V(x_1) \geq 1-f_V(x_2)$ are constant, then $\overline{H}_V(x_1) \geq \overline{H}_V(x_2)$ is obtained.
   (continued) **Example 1.** For $\frac{[0.3,0.4]}{x_5}$ and $\frac{[0.3,0.45]}{x_6}$, we have

$$\overline{H}_V(x_5) \approx 0.9067, \quad \overline{H}_V(x_6) = 0.93.$$

   A fact is drawn that $t_V(x_5) = t_V(x_6) = 0.3 < \frac{1}{2}$, $\frac{1}{2} > 1-f_V(x_6) > 1-f_V(x_5)$ and $\overline{H}_V(x_6) > \overline{H}_V(x_5)$. This fact validates the conclusion.
3) When $t < \frac{1}{2}$, because $t \geq 0$ is constant, $1-f_V(x_0) = -\frac{1}{2}t + \frac{3}{4} \leq \frac{3}{4}$. Thus, if $1-f_V(x_1) \geq 1-f_V(x_2) \geq \frac{3}{4}$ and $1-f_V(x_1) \geq 1-f_V(x_2) \geq 1-f_V(x_0)$ are constant, then $\overline{H}_V(x_1) \leq \overline{H}_V(x_2)$ is obtained.
   (continued) **Example 1.** For $\frac{[0.4,0.8]}{x_{11}}$ and $\frac{[0.4,0.9]}{x_{12}}$, we have

$$\overline{H}_V(x_{11}) \approx 0.9067, \quad \overline{H}_V(x_{12}) \approx 0.8267.$$

   A fact is drawn that $t_V(x_{11}) = t_V(x_{12}) = 0.4 < \frac{1}{2}$, $1-f_V(x_{12}) > 1-f_V(x_{11}) > \frac{3}{4}$ and $\overline{H}_V(x_{12}) < \overline{H}_V(x_{11})$. This fact validates the conclusion.
4) According to above three conditions, when $t < \frac{1}{2}$ and $\frac{3}{4} > 1-f_V(x_1) \geq 1-f_V(x_2) > \frac{1}{2}$, the size relationship between $\overline{H}_V(x_1)$ and $\overline{H}_V(x_2)$ is uncertain.
   (continued) **Example 1.**
   - For $\frac{[0.1,0.6]}{x_1}$ and $\frac{[0.1,0.7]}{x_2}$, we have

$$\overline{H}_V(x_1) \approx 0.8267, \quad \overline{H}_V(x_2) = 0.84.$$

     A fact is drawn that $t_V(x_1) = t_V(x_2) = 0.1 < \frac{1}{2}$, $\frac{1}{2} < 1-f_V(x_1) < 1-f_V(x_2) < \frac{3}{4}$ and $\overline{H}_V(x_1) < \overline{H}_V(x_2)$.
   - For $\frac{[0.3,0.6]}{x_7}$ and $\frac{[0.3,0.7]}{x_8}$, we have

$$\overline{H}_V(x_7) = 0.96, \quad \overline{H}_V(x_8) \approx 0.9467.$$

     A fact is drawn that $t_V(x_7) = t_V(x_8) < \frac{1}{2}$, $\frac{1}{2} < 1-f_V(x_7) < 1-f_V(x_8) < \frac{3}{4}$ and $\overline{H}_V(x_7) > \overline{H}_V(x_8)$.

   In conclusion, when $t_V(y) = t_V(z) < \frac{1}{2}$ and $\frac{1}{2} < 1-f_V(y) \leq 1-f_V(z) < \frac{3}{4}$ $(y, z \in U)$, the size relationship between $\overline{H}_V(y)$ and $\overline{H}_V(z)$ is uncertain.

Case 2: $1-f_V(x_1) = 1-f_V(x_2) = f$ and $t_V(x_1) \geq t_V(x_2)$.
Similarly, suppose $t_V(x)$ is an independent variable and $Y[t_V(x)] = -\frac{4}{3}t_V^2(x) + \left(2-\frac{4}{3}f\right)t_V(x)$ is a dependent variable. Then $\overline{H}_V(x_1) - \overline{H}_V(x_2) = Y[t_V(x_1)] - Y[t_V(x_2)]$. And $t_V(x_0) = -\frac{-\left(2-\frac{4}{3}f\right)}{2\times\left(-\frac{4}{3}\right)} = -\frac{1}{2}f + \frac{3}{4}$. In other words, if $t_V(x) < -\frac{1}{2}f + \frac{3}{4}$, $Y[t_V(x)]$ would increase with increasing $t_V(x)$, and if $t_V(x) \geq -\frac{1}{2}f + \frac{3}{4}$, $Y[t_V(x)]$ would decrease with increasing $t_V(x)$.

1) When $f < \frac{1}{2}$, $t_V(x_0) = -\frac{1}{2}f + \frac{3}{4} > \frac{1}{2}$. Thus, we can obtain $t_V(x_0) > \frac{1}{2} > f \geq t_V(x_1) \geq t_V(x_2)$. Then $\overline{H}_V(x_1) \geq \overline{H}_V(x_2)$ is obtained.
   (continued) **Example 1.** For $\frac{[0.2,0.4]}{x_3}$ and $\frac{[0.3,0.4]}{x_5}$, we have

$$\overline{H}_V(x_3) \approx 0.8267, \quad \overline{H}_V(x_5) \approx 0.9067.$$

   A fact is drawn that $1-f_V(x_3) = 1-f_V(x_5) < \frac{1}{2}$, $t_V(x_5) > t_V(x_3)$ and $\overline{H}_V(x_5) > \overline{H}_V(x_3)$. This fact validates the conclusion.
2) When $f \geq \frac{1}{2}$, $t_V(x_0) = -\frac{1}{2}f + \frac{3}{4} \leq \frac{1}{2}$. Thus if $t_V(x_1) \geq t_V(x_2) \geq \frac{1}{2}$ and $t_V(x_1) \geq t_V(x_2) \geq t_V(x_0)$ are constant, then $\overline{H}_V(x_1) \leq \overline{H}_V(x_2)$ is obtained.

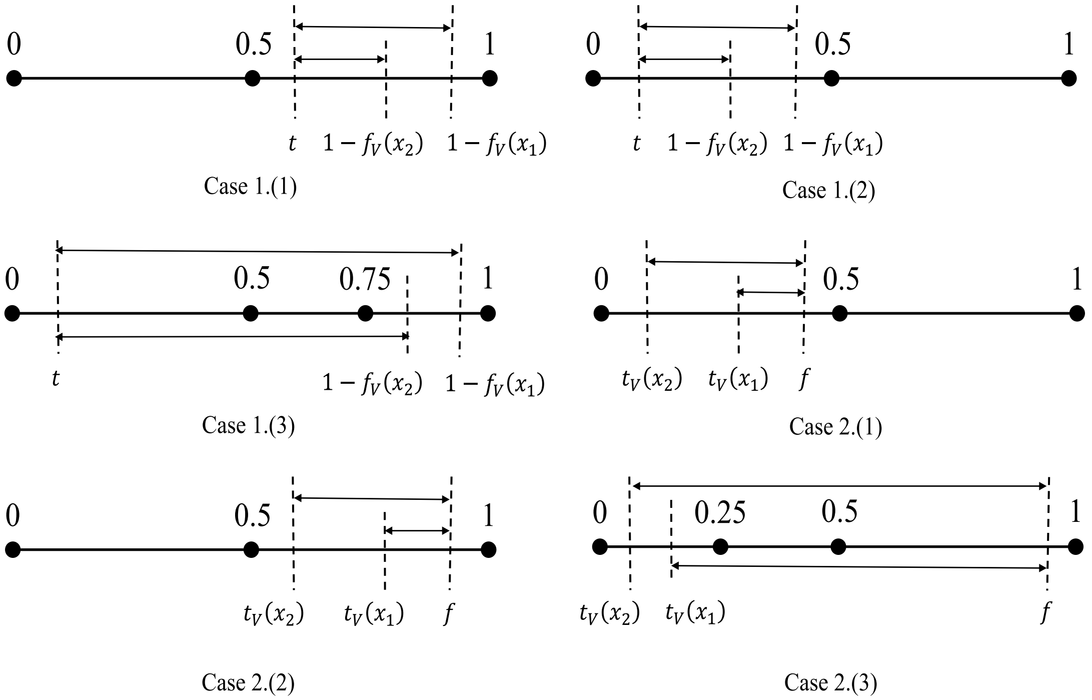

FIGURE 1. Various circumstances with changing $[t_V(x), 1 - f_V(x)]$.

(continued) **Example 1.** For $\frac{[0.7,0.9]}{x_{14}}$ and $\frac{[0.8,0.9]}{x_{15}}$, we have

$$\overline{H}_V(x_{14}) \approx 0.6267, \quad \overline{H}_V(x_{15}) \approx 0.5067.$$

A fact is drawn that $1 - f_V(x_{14}) = 1 - f_V(x_{15}) > \frac{1}{2}$, $t_V(x_{15}) > t_V(x_{14}) > \frac{1}{2}$ and $\overline{H}_V(x_{15}) < \overline{H}_V(x_{14})$. This fact validates the conclusion.

3) When $f \geq \frac{1}{2}$, because $f \leq 1$ is constant, $t_V(x_0) = -\frac{1}{2}f + \frac{3}{4} \geq \frac{1}{4}$. Thus if $\frac{1}{4} \geq t_V(x_1) \geq t_V(x_2)$ and $t_V(x_0) \geq t_V(x_1) \geq t_V(x_2)$ are constant, then $\overline{H}_V(x_1) \geq \overline{H}_V(x_2)$ is obtained.
(continued) **Example 1.** For $\frac{[0.1,0.6]}{x_1}$ and $\frac{[0.2,0.6]}{x_4}$, we have

$$\overline{H}_V(x_1) \approx 0.8267, \quad \overline{H}_V(x_4) \approx 0.8867.$$

A fact is drawn that $1 - f_V(x_1) = 1 - f_V(x_4) > \frac{1}{2}$, $\frac{1}{2} > t_V(x_4) > t_V(x_1)$ and $\overline{H}_V(x_4) > \overline{H}_V(x_1)$. This fact validates the conclusion.

4) According to above three conditions, when $f \geq \frac{1}{2}$ and $\frac{1}{2} > t_V(x_1) \geq t_V(x_2) > \frac{1}{4}$, the size relationship between $\overline{H}_V(x_1)$ and $\overline{H}_V(x_2)$ is uncertain.
(continued) **Example 1.**

- For $\frac{[0.3,0.6]}{x_7}$ and $\frac{[0.4,0.6]}{x_{10}}$, we have

$$\overline{H}_V(x_3) = 0.96, \quad \overline{H}_V(x_6) \approx 0.9867.$$

A fact is drawn that $1 - f_V(x_7) = 1 - f_V(x_{10}) > \frac{1}{2}$, $\frac{1}{4} < t_V(x_7) < t_V(x_{10}) < \frac{1}{2}$ and $\overline{H}_V(x_7) < \overline{H}_V(x_{10})$.

- For $\frac{[0.3,0.9]}{x_9}$ and $\frac{[0.4,0.9]}{x_{12}}$, we have

$$\overline{H}_V(x_9) = 0.84, \quad \overline{H}_V(x_{12}) \approx 0.8267.$$

A fact is drawn that $1 - f_V(x_9) = 1 - f_V(x_{12}) > \frac{1}{2}$, $\frac{1}{4} < t_V(x_{12}) < t_V(x_{15}) < \frac{1}{2}$ and $\overline{H}_V(x_9) > \overline{H}_V(x_{12})$.

In conclusion, when $1 - f_V(y) = 1 - f_V(z) \geq \frac{1}{2}$ and $\frac{1}{4} < t_V(y) \leq t_V(z) < \frac{1}{2}\,(y, z \in U)$, the size relationship between $\overline{H}_V(y)$ and $\overline{H}_V(z)$ is uncertain.

Note: In this section, change rules of average fuzziness of the vague value with changing truth membership degree and false membership degree are discussed respectively. These conclusions provide a basis for understanding the uncertainty structure of a vague value. These examples and diagrams show that these aforementioned conclusions are consistent with human cognition. Therefore, these conclusions lay the foundation for further data mining based on vague sets.

IV. UNCERTAINTY OF VAGUE SETS IN ROUGH APPROXIMATION SPACES

Based on the view of granular computing, Zhang *et al.* [70] put vague sets in rough approximation spaces and proposed average-step-vague sets. In this section, with changing knowledge granularity, change rules of uncertainty for average-step-vague sets and change rules of uncertainty for approximation sets of vague sets are discussed.

A. UNCERTAINTY OF AVERAGE-STEP-VAGUE SETS IN ROUGH APPROXIMATION SPACES

In rough approximation spaces, given an equivalence relation, vague sets would become corresponding average-step-vague sets. Then the uncertainty of the average-step-vague sets could be obtained as follows:

Example 2: Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}$, V be a vague set in a universe of discourse U and R_1 be an equivalence relation on U.

$$
\begin{aligned}
V = & \frac{[0.1, 0.6]}{x_1} + \frac{[0.2, 0.4]}{x_2} + \frac{[0.2, 0.5]}{x_3} + \frac{[0.2, 0.6]}{x_4} \\
& + \frac{[0.2, 0.7]}{x_5} + \frac{[0.3, 0.8]}{x_6} + \frac{[0.4, 0.7]}{x_7} + \frac{[0.5, 0.6]}{x_8} \\
& + \frac{[0.5, 0.7]}{x_9} + \frac{[0.5, 0.8]}{x_{10}} + \frac{[0.6, 0.7]}{x_{11}} + \frac{[0.6, 0.8]}{x_{12}} \\
& + \frac{[0.6, 0.9]}{x_{13}} + \frac{[0.7, 0.8]}{x_{14}} + \frac{[0.8, 1]}{x_{15}}.
\end{aligned}
$$

$$
\begin{aligned}
U/R_1 = & \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6, x_7, x_8\}, \\
& \{x_9, x_{10}, x_{11}, x_{12}\}, \{x_{13}, x_{14}, x_{15}\}\}.
\end{aligned}
$$

According to the **Definition 6**, we have

$$
\begin{aligned}
\overline{V}_J^{R_1}(x_1) = \overline{V}_J^{R_1}(x_2) & \\
= & \left[\frac{0.1 + 0.2}{2}, \frac{0.6 + 0.4}{2}\right] = [0.15, 0.5].
\end{aligned}
$$

Similarly, the average-step-vague set of the vague set based on R_1 is obtained:

$$
\begin{aligned}
\overline{V}_J^{R_1} = & \frac{[0.15, 0.5]}{x_1} + \frac{[0.15, 0.5]}{x_2} + \frac{[0.2, 0.6]}{x_3} + \frac{[0.2, 0.6]}{x_4} \\
& + \frac{[0.2, 0.6]}{x_5} + \frac{[0.4, 0.7]}{x_6} + \frac{[0.4, 0.7]}{x_7} + \frac{[0.4, 0.7]}{x_8} \\
& + \frac{[0.55, 0.75]}{x_9} + \frac{[0.55, 0.75]}{x_{10}} + \frac{[0.55, 0.75]}{x_{11}} \\
& + \frac{[0.55, 0.75]}{x_{12}} + \frac{[0.7, 0.9]}{x_{13}} + \frac{[0.7, 0.9]}{x_{14}} + \frac{[0.7, 0.9]}{x_{15}}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\overline{H}_{\overline{V}_J^{R_1}}(x_1) = & \overline{H}_{\overline{V}_J^{R_1}}(x_2) \\
= & \frac{4}{0.5 - 0.15} \int_{0.15}^{0.5} \mu_V(x)\left[1 - \mu_V(x)\right] d\mu_V(x) \\
\approx & 0.8367.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& \overline{H}_{\overline{V}_J^{R_1}}(x_3) = \overline{H}_{\overline{V}_J^{R_1}}(x_4) = \overline{H}_{\overline{V}_J^{R_1}}(x_5) \approx 0.9067, \\
& \overline{H}_{\overline{V}_J^{R_1}}(x_6) = \overline{H}_{\overline{V}_J^{R_1}}(x_7) = \overline{H}_{\overline{V}_J^{R_1}}(x_8) = 0.96, \\
& \overline{H}_{\overline{V}_J^{R_1}}(x_9) = \overline{H}_{\overline{V}_J^{R_1}}(x_{10}) = \overline{H}_{\overline{V}_J^{R_1}}(x_{11}) = \overline{H}_{\overline{V}_J^{R_1}}(x_{12}) \\
& \qquad \approx 0.8967, \\
& \overline{H}_{\overline{V}_J^{R_1}}(x_{13}) = \overline{H}_{\overline{V}_J^{R_1}}(x_{14}) = \overline{H}_{\overline{V}_J^{R_1}}(x_{15}) \approx 0.6267.
\end{aligned}
$$

Finally, the uncertainty of this average-step-vague set could be calculated as follows:

$$
\begin{aligned}
H_{\overline{V}_J^{R_1}}(U) = & \frac{1}{|U|} \sum_{i=1}^{|U|} \overline{H}_{\overline{V}_J^{R_1}}(x_i) \\
= & \frac{1}{15}(0.8367 \times 2 + 0.9067 \times 3 + 0.96 \times 3 \\
& + 0.8967 \times 4 + 0.6267 \times 3) \\
\approx & 0.8494.
\end{aligned}
$$

Furthermore, the uncertainty of the corresponding average-step-vague set $H_{\overline{V}_J^{R}}(U)$ satisfies the following conditions or properties:

1) If a vague set V degenerates into a crisp set, $H_{\overline{V}_J^{R}}(U)$ reaches its minimal value 0. In other words, the uncertainty of a crisp set in all rough approximation spaces is equal to 0.
2) If a vague set V degenerates into a fuzzy set, $H_{\overline{V}_J^{R}}(U)$ becomes an ordinary fuzziness equation.
3) In all rough approximation spaces, $H_{\overline{V}_J^{R}}(U) = H_{\sim\overline{V}_J^{R}}(U)$ is always held, where $\sim \overline{V}_J^{R}$ is the complement set of $\overline{V}_J^{R}$.

Proof:

1) If a vague set V degenerates into a crisp set, that is, for any point x in U, $t_V(x) = 1, f_V(x) = 0$ or $t_V(x) = 0, f_V(x) = 1$. In rough approximation spaces, $\overline{V}_J^{R}(x) = [1, 1]$ or $\overline{V}_J^{R}(x) = [0, 0]$. According to Section 3, $\overline{H}_{\overline{V}_J^{R}}(x) = 0$. Then $H_{\overline{V}_J^{R}}(U) = 0$ is always held.
2) If a vague set V degenerates into a fuzzy set, that is to say, for any point x in U, $t_V(x)$ is equal to $1 - f_V(x)$. The $\overline{V}_J^{R}$ is also a fuzzy set. Then according to literature [26], 2) is easy to prove.
3) $\overline{V}_J^{R}$ is a vague set V in a universe of discourse U. Thus, according to literature [26], 3) is easy to prove. ■

With changing knowledge granularity, vague sets corresponding average-step-vague sets would change. As a result, the uncertainty of vague sets is different in different rough approximation spaces. Therefore, change rules of the uncertainty of vague sets with changing knowledge granularity would be discussed as follows:

Theorem 1: Let V be a vague set in a universe of discourse U, R_1 and R_2 be two equivalence relations on U. If $R_1 \subseteq R_2$, then $H_{\overline{V}_J^{R_1}}(U) \geq H_{\overline{V}_J^{R_2}}(U)$.

Proof: Suppose $U/R_1 = \{X_1, X_2, \ldots, X_m\}$ and $U/R_2 = \{Y_1, Y_2, \ldots, Y_k\}$. Because $R_1 \subseteq R_2$, $U/R_1 \preceq U/R_2$. Thus, for any $Y_i \in U/R_2$, $\exists X_j \in U/R_1$, $Y_i \subseteq X_j$. For simplicity, we suppose there only exists one granule X_1 $(X_1 \in U/R_1)$ which is subdivided into two finer sub-granules (the more complicated cases can be transformed into this case; therefore, we do not repeat them here). Suppose $X_1 = Y_1 \cup Y_2$, $X_2 = Y_3$, $X_3 = Y_4$, ..., $X_m = Y_k$ $(k = m + 1)$. Thus, we can obtain

$$H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U) = \frac{1}{|U|}\sum_{x_i\in U}\overline{H}_{\overline{V}_J^{R_1}}(x_i) - \frac{1}{|U|}\sum_{x_i\in U}\overline{H}_{\overline{V}_J^{R_2}}(x_i). \quad (3)$$

Therefore,

$$\begin{aligned}|U|\left[H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U)\right] &= \sum_{x_i\in U}\overline{H}_{\overline{V}_J^{R_1}}(x_i) - \sum_{x_i\in U}\overline{H}_{\overline{V}_J^{R_2}}(x_i)\\ &= \sum_{x_i\in X_1}\overline{H}_{\overline{V}_J^{R_1}}(x_i) + \sum_{x_i\in U-X_1}\overline{H}_{\overline{V}_J^{R_1}}(x_i) - \sum_{x_i\in Y_1}\overline{H}_{\overline{V}_J^{R_2}}(x_i)\\ &\quad - \sum_{x_i\in Y_2}\overline{H}_{\overline{V}_J^{R_2}}(x_i) - \sum_{x_i\in U-Y_1-Y_2}\overline{H}_{\overline{V}_J^{R_2}}(x_i)\\ &= \sum_{x_i\in X_1}\overline{H}_{\overline{V}_J^{R_1}}(x_i) - \sum_{x_i\in Y_1}\overline{H}_{\overline{V}_J^{R_2}}(x_i) - \sum_{x_i\in Y_2}\overline{H}_{\overline{V}_J^{R_2}}(x_i). \quad (4)\end{aligned}$$

Because $\overline{V}_J$ is an average-step vague set, $\overline{H}_{\overline{V}_J^{R}}(x) = H_{\overline{V}_J^{R}}([x]_R)$.

$$|U|\left[H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U)\right] = |X_1|\,H_{\overline{V}_J^{R_1}}(X_1) - |Y_1|\,H_{\overline{V}_J^{R_2}}(Y_1) - |Y_2|\,H_{\overline{V}_J^{R_2}}(Y_2). \quad (5)$$

For convenience, let $X_1 = \{x_1, x_2, ..., x_{a+b}\}$, $Y_1 = \{x_1, x_2, \ldots, x_a\}$ and $Y_2 = \{x_{a+1}, x_{a+2}, ..., x_{a+b}\}$. Thus, $t_V(X_1) = \frac{1}{a+b}\sum_{i=1}^{a+b} t_V(x_i)$, $1 - f_V(X_1) = \frac{1}{a+b}\sum_{i=1}^{a+b}(1-f_V(x_i))$, $t_V(Y_1) = \frac{1}{a}\sum_{i=1}^{a} t_V(x_i)$, $1 - f_V(Y_1) = \frac{1}{a}\sum_{i=1}^{a}(1-f_V(x_i))$, $t_V(Y_2) = \frac{1}{b}\sum_{i=a+1}^{a+b} t_V(x_i)$ and $1 - f_V(Y_2) = \frac{1}{b}\sum_{i=a+1}^{a+b}(1-f_V(x_i))$. According to Section 3,

$$\overline{H}_V(x) = 2(1-f_V(x)) + 2t_V(x) - \frac{4}{3}(1-f_V(x))^2 - \frac{4}{3}t_V^2(x) - \frac{4}{3}t_V(x)(1-f_V(x)). \quad (6)$$

Thus,

$$\overline{H}_{\overline{V}_J^{R_1}}(X_1) = 2(1-f_V(X_1)) + 2t_V(X_1) - \frac{4}{3}(1-f_V(X_1))^2 - \frac{4}{3}t_V^2(X_1) - \frac{4}{3}t_V(X_1)(1-f_V(X_1)), \quad (7)$$

$$\overline{H}_{\overline{V}_J^{R_2}}(Y_1) = 2(1-f_V(Y_1)) + 2t_V(Y_1) - \frac{4}{3}(1-f_V(Y_1))^2 - \frac{4}{3}t_V^2(Y_1) - \frac{4}{3}t_V(Y_1)(1-f_V(Y_1)), \quad (8)$$

$$\overline{H}_{\overline{V}_J^{R_2}}(Y_2) = 2(1-f_V(Y_2)) + 2t_V(Y_2) - \frac{4}{3}(1-f_V(Y_2))^2 - \frac{4}{3}t_V^2(Y_2) - \frac{4}{3}t_V(Y_2)(1-f_V(Y_2)). \quad (9)$$

Therefore, $|U|\left[H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U)\right] = (\mathrm{a}+\mathrm{b})\,\overline{H}_{\overline{V}_J^{R_1}}(X_1) - a\overline{H}_{\overline{V}_J^{R_2}}(Y_1) - b\overline{H}_{\overline{V}_J^{R_2}}(Y_2)$.

Because $X_1 = Y_1 \cup Y_2$, suppose $\sum_{i=1}^{a}(1-f_V(x_i)) = A$, $\sum_{i=a+1}^{a+b}(1-f_V(x_i)) = B$, $\sum_{i=1}^{a} t_V(x_i) = G$, $\sum_{i=a+1}^{a+b} t_V(x_i) = H$. Thus,

$$\begin{aligned}&\frac{3}{4}|U|\left[H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U)\right]\\ &= \frac{A^2+G^2+AG}{a} + \frac{B^2+H^2+BH}{b} - \frac{1}{a+b}\Big[(A+B)^2\\ &\quad + (G+H)^2 + (A+B)(G+H)\Big]\\ &= \frac{1}{ab(a+b)}\Big[b^2A^2 + b^2G^2 + b^2AG + a^2B^2 + a^2H^2 + a^2\\ &\quad \times BH - 2abAB - 2abGH\Big]. \quad (10)\end{aligned}$$

Because $ab(a+b) > 0$, we can obtain,

$$\begin{aligned}&\frac{3}{4}ab(a+b)|U|\left[H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U)\right]\\ &= b^2A^2 + b^2G^2 + b^2AG + a^2B^2 + a^2H^2 + a^2BH - 2ab\\ &\quad \times AB - 2abGH\\ &= (bA-aB)^2 + (bG-aH)^2 + b^2AG + a^2BH \geq 0. \quad (11)\end{aligned}$$

As such, $H_{\overline{V}_J^{R_1}}(U) - H_{\overline{V}_J^{R_2}}(U) \geq 0$. In other words, $H_{\overline{V}_J^{R_1}}(U) \geq H_{\overline{V}_J^{R_2}}(U)$. Hence, **Theorem 1** is proven successfully. ■

In order to verify **Theorem 1**, **Example 2** is shown as follows:

(continued) **Example 2.** Let R_2 be an equivalence relation on U and $R_1 \subseteq R_2$. So $U/R_1 \preceq U/R_2$.

$$U/R_2 = \{\{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6, x_7, x_8\}, \{x_9, x_{10}\}, \{x_{11}, x_{12}\}, \{x_{13}, x_{14}, x_{15}\}\}.$$

Then, the average-step-vague set of the vague set based on R_2 is obtained:

$$\begin{aligned}\overline{V}_J^{R_2} &= \frac{[0.15, 0.5]}{x_1} + \frac{[0.15, 0.5]}{x_2} + \frac{[0.2, 0.5]}{x_3} + \frac{[0.2, 0.65]}{x_4}\\ &\quad + \frac{[0.2, 0.65]}{x_5} + \frac{[0.4, 0.7]}{x_6} + \frac{[0.4, 0.7]}{x_7} + \frac{[0.4, 0.7]}{x_8}\\ &\quad + \frac{[0.5, 0.75]}{x_9} + \frac{[0.5, 0.75]}{x_{10}} + \frac{[0.6, 0.75]}{x_{11}} + \frac{[0.6, 0.75]}{x_{12}}\\ &\quad + \frac{[0.7, 0.9]}{x_{13}} + \frac{[0.7, 0.9]}{x_{14}} + \frac{[0.7, 0.9]}{x_{15}}.\end{aligned}$$

Next,

$$\overline{H}_{\overline{V}_J^{R_2}}(x_1) = \overline{H}_{\overline{V}_J^{R_2}}(x_2) \approx 0.8367, \overline{H}_{\overline{V}_J^{R_2}}(x_3) = 0.88,$$

$$\overline{H}_{\overline{V}_J^{R_2}}(x_4) = \overline{H}_{\overline{V}_J^{R_2}}(x_5) = 0.91,$$

$$\overline{H}_{\overline{V}_J^{R_2}}(x_6) = \overline{H}_{\overline{V}_J^{R_2}}(x_7) = \overline{H}_{\overline{V}_J^{R_2}}(x_8) = 0.96,$$

$$\overline{H}_{\overline{V}_J^{R_2}}(x_9) = \overline{H}_{\overline{V}_J^{R_2}}(x_{10}) \approx 0.9167,$$

$$\overline{H}_{\overline{V}_J^{R_2}}(x_{11}) = \overline{H}_{\overline{V}_J^{R_2}}(x_{12}) = 0.87,$$

$$\overline{H}_{\overline{V}_J^{R_2}}(x_{13}) = \overline{H}_{\overline{V}_J^{R_2}}(x_{14}) = \overline{H}_{\overline{V}_J^{R_2}}(x_{15}) \approx 0.6267.$$

Thus, we can obtain:

$$\begin{aligned} H_{\overline{V}_J^{R_2}}(U) &= \frac{1}{|U|}\sum_{i=1}^{|U|}\overline{H}_{\overline{V}_J^{R_2}}(x_i) \\ &= \frac{1}{15}(0.8367 \times 2 + 0.88 + 0.91 \times 2 + 0.96 \\ &\quad \times 3 + 0.9167 \times 2 + 0.87 \times 2 + 0.6267 \times 3) \\ &\approx 0.8471. \end{aligned}$$

According to **Example 2**, when $R_1 \subseteq R_2$, $H_{\overline{V}_J^{R_1}}(U) > H_{\overline{V}_J^{R_2}}(U)$ holds. The example validates **Theorem 1**.

In this paper, equivalence relations corresponding to the finest rough approximation spaces and the coarsest rough approximation spaces in an information system are denoted by R_O and R_U, respectively. Then, we have the following corollaries,

Corollary 1: Let V be a vague set in a universe of discourse U and R be an equivalence relation on U. Then $H_{\overline{V}_J^{R}}(U) \geq H_{\overline{V}_J^{R_O}}(U)$ holds.

Proof: From **Theorem 1**, **Corollary 2** is easy to prove. ■

Corollary 2: Let V be a vague set in a universe of discourse U, R be an equivalence relation on U. Then $H_{\overline{V}_J^{R}}(U) \geq H_{\overline{V}_J^{R_U}}(U)$ holds.

Proof: From **Theorem 1**, **Corollary 2** is easy to prove. ■

In order to verify **Corollary 1** and **Corollary 2**, **Example 2** is shown as follows:

(continued) **Example 2.** Let R_O be an equivalence relation corresponding the finest rough approximation spaces on U and let R_U be an equivalence relation corresponding the coarsest rough approximation spaces on U. Thus,

$$\begin{aligned} &U/R_U \preceq U/R_1 \preceq U/R_2 \preceq U/R_O. \\ &U/R_U = \{\{x_1\},\{x_2\},\{x_3\},\{x_4\},\{x_5\},\{x_6\},\{x_7\},\{x_8\}, \\ &\qquad \{x_9\},\{x_{10}\},\{x_{11}\},\{x_{12}\},\{x_{13}\},\{x_{14}\},\{x_{15}\}\}, \\ &U/R_U = \{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12}, \\ &\qquad x_{13},x_{14},x_{15}\}\} \end{aligned}$$

Then, we have $\overline{V}_J^{R_O} = V$.

$$\begin{aligned} &\overline{H}_{\overline{V}_J^{R_O}}(x_1) \approx 0.8267, \overline{H}_{\overline{V}_J^{R_O}}(x_2) \approx 0.8267, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_3) = 0.88, \overline{H}_{\overline{V}_J^{R_O}}(x_4) \approx 0.8867, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_5) \approx 0.9067, \overline{H}_{\overline{V}_J^{R_O}}(x_6) \approx 0.9067, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_7) = 0.96, \overline{H}_{\overline{V}_J^{R_O}}(x_8) \approx 0.9867, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_9) \approx 0.9467, \overline{H}_{\overline{V}_J^{R_O}}(x_{10}) = 0.88, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_{11}) \approx 0.9067, \overline{H}_{\overline{V}_J^{R_O}}(x_{12}) \approx 0.8267, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_{13}) \approx 0.72, \overline{H}_{\overline{V}_J^{R_O}}(x_{14}) \approx 0.63, \\ &\overline{H}_{\overline{V}_J^{R_O}}(x_{15}) \approx 0.3533. \end{aligned}$$

$$H_{\overline{V}_J^{R_O}}(U) = \frac{1}{|U|}\sum_{i=1}^{|U|}\overline{H}_{\overline{V}_J^{R_o}}(x_i) \approx 0.8296.$$

Additionally,

$$\begin{aligned} \overline{V}_J^{R_U} &= \frac{[0.42, 0.7067]}{x_1} + \frac{[0.42, 0.7067]}{x_2} + \frac{[0.42, 0.7067]}{x_3} \\ &+ \frac{[0.42, 0.7067]}{x_4} + \frac{[0.42, 0.7067]}{x_5} + \frac{[0.42, 0.7067]}{x_6} \\ &+ \frac{[0.42, 0.7067]}{x_7} + \frac{[0.42, 0.7067]}{x_8} + \frac{[0.42, 0.7067]}{x_9} \\ &+ \frac{[0.42, 0.7067]}{x_{10}} + \frac{[0.42, 0.7067]}{x_{11}} + \frac{[0.42, 0.7067]}{x_{12}} \\ &+ \frac{[0.42, 0.7067]}{x_{13}} + \frac{[0.42, 0.7067]}{x_{14}} + \frac{[0.42, 0.7067]}{x_{15}}. \end{aligned}$$

$$\begin{aligned} &H_{\overline{V}_J^{R_U}}(U) \\ &= \frac{1}{|U|}\sum_{i=1}^{|U|}\overline{H}_{\overline{V}_J^{R_o}}(x_i) \approx 9565. \end{aligned}$$

According to **Example 2**, $H_{\overline{V}_J^{R_U}}(U) > H_{\overline{V}_J^{R_1}}(U) > H_{\overline{V}_J^{R_2}}(U) > H_{\overline{V}_J^{R_o}}(U)$. The example validates **Corollary 1** and **Corollary 2**.

In this section, a conclusion is excavated that the finer the granularity, the smaller uncertainty. On the contrary, the coarser the granularity, the larger uncertainty. In other words, the more information we obtain, the more certainty the information system would be.

B. UNCERTAINTY OF APPROXIMATION SET IN ROUGH APPROXIMATION SPACES

In rough approximation spaces, given a universe, it is usually necessary to make a decision through approximating a target concept. As a result, change rules of the uncertainty for upper approximation sets of vague sets with changing knowledge granularity are discussed as follows.

Suppose $U/R_1 = \{X_1, X_2, ..., X_m\}$, $U/R_2 = \{Y_1, Y_2, ..., Y_k\}$. Because $R_1 \subseteq R_2$, $U/R_1 \preceq U/R_2$ holds. Thus, for any $Y_i \in U/R_2$, $\exists X_j \in U/R_1$, $Y_i \subseteq X_j$ holds. For simplicity, we suppose there only exists one granule X_1 $(X_1 \in U/R_1)$ which is subdivided into two finer sub-granules (the more complicated cases can be transformed into this case. Therefore, we do not repeat them here). Suppose $X_1 = Y_1 \cup Y_2$, $X_2 = Y_3$, $X_3 = Y_4$, ..., $X_m = Y_k$ $(k = m + 1)$.

There is

$$\begin{aligned} H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) &= H\left(POS_{R_1}^{(\alpha,\beta)}(V)\right) \\ &= \frac{1}{\left|POS_{R_1}^{(\alpha,\beta)}(V)\right|}\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x). \end{aligned} \quad (12)$$

For convenience, let $X_1 = \{x_1, x_2, ..., x_{a+b}\}$, $Y_1 = \{x_1, x_2, ..., x_a\}$ and $Y_2 = \{x_{a+1}, x_{a+2}, ..., x_{a+b}\}$. Thus, $t_V(X_1) = \frac{1}{a+b}\sum_{i=1}^{a+b} t_V(x_i)$, $1 - f_V(X_1) = \frac{1}{a+b}\sum_{i=1}^{a+b}(1 - f_V(x_i))$, $t_V(Y_1) = \frac{1}{a}\sum_{i=1}^{a} t_V(x_i)$, $1 - f_V(Y_1) = \frac{1}{a}\sum_{i=1}^{a}(1 - f_V(x_i))$, $t_V(Y_2) = \frac{1}{b}\sum_{i=a+1}^{a+b} t_V(x_i)$ and $1 - f_V(Y_2) = \frac{1}{b}\sum_{i=a+1}^{a+b}(1 - f_V(x_i))$. Next we discuss as follows:

Case 1: If $t_V(X_1) = 1 - f_V(X_1) = 1$, then $t_V(Y_1) = 1 - f_V(Y_1) = 1$ and $t_V(Y_2) = 1 - f_V(Y_2) = 1$. That is, $X_1 \subseteq POS_{R_1}(V) \subseteq POS_{R_1}^{(\alpha,\beta)}(V)$, $Y_1 \subseteq POS_{R_1}(V) \subseteq POS_{R_1}^{(\alpha,\beta)}(V)$ and $Y_2 \subseteq POS_{R_1}(V) \subseteq POS_{R_1}^{(\alpha,\beta)}(V)$. We easily have $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) = H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$.

Case 2: If $t_V(X_1) = 1 - f_V(X_1) = 0$, that is, $X_1 \subseteq NEG_{R_1}(V)$. All of the finer sub-granules are in $NEG_{R_2}(V)$. It does not have effect on $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right)$.

Case 3: If $\alpha > t_V(X_1) > 0$, that is, $X_1 \subseteq BND_{R_1}^{(\alpha,\beta)}(V)$ or $X_1 \subseteq NEG_{R_1}^{(\alpha,\beta)}(V)$. Suppose $\left|\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right| = c$, $t_V(Y_2) \geq \alpha$ and $\left|\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right| = c + |Y_2|$. Then

$$H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) = \frac{1}{\left|POS_{R_1}^{(\alpha,\beta)}(V)\right|}\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x) = \frac{1}{c}\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x) \tag{13}$$

and

$$\begin{aligned} &H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right) \\ &= \frac{1}{\left|POS_{R_2}^{(\alpha,\beta)}(V)\right|}\Big[\sum_{x\in POS_{R_2}^{(\alpha,\beta)}(V)-Y_2} H(x) + \sum_{x\in Y_2} H(x)\Big] \\ &= \frac{1}{c+b}\Big[\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x) + \frac{4b}{1-f_V(Y_2)-t_V(Y_2)} \\ &\quad\times \int_{t_V(Y_2)}^{1-f_V(Y_2)} \mu_V(x)(1-\mu_V(x))\,d\mu_V(x)\Big]. \end{aligned} \tag{14}$$

Because $POS_{R_2}^{(\alpha,\beta)}(V) = POS_{R_1}^{(\alpha,\beta)}(V) + Y_2$,

$$\begin{aligned} &c(c+b)\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\ &= b\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x) - bc\,[2t_V(Y_2) + 2(1-f_V(Y_2)) \\ &\quad - \frac{4}{3}t_V^2(Y_2) - \frac{4}{3}(1-f_V(Y_2))^2 - \frac{4}{3}t_V(Y_2)(1-f_V(Y_2))\Big]. \end{aligned} \tag{15}$$

Then,

$$\begin{aligned} &\frac{(c+b)}{b}\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\ &= \frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x)}{c} - 2t_V(Y_2) - 2(1-f_V(Y_2)) + \frac{4}{3} \\ &\quad\times t_V^2(Y_2) + \frac{4}{3}(1-f_V(Y_2))^2 + \frac{4}{3}t_V(Y_2)(1-f_V(Y_2)). \end{aligned} \tag{16}$$

Suppose $G(y) = \frac{3}{4}y^2 - 2y$, it is easy to know that when $y \geq \frac{3}{4}$, $G(y)$ would increase with the increase of y. Thus, when $1 - f_V(Y_2) \geq t_V(Y_2) \geq \alpha \geq \frac{3}{4}$,

$$\begin{aligned} &\frac{(c+b)}{b}\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\ &\geq \frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)} H(x)}{c} - 2\alpha - 2\alpha + \frac{4}{3}\alpha^2 + \frac{4}{3}\alpha^2 \\ &\quad + \frac{4}{3}t_V(Y_2)(1-f_V(Y_2)) \\ &\geq \alpha - 2\alpha - 2\alpha + \frac{4}{3}\alpha^2 + \frac{4}{3}\alpha^2 + \frac{4}{3}\alpha^2 \\ &= 4\alpha^2 - 3\alpha. \end{aligned} \tag{17}$$

When $\alpha \geq \frac{3}{4}$, $4\alpha^2 - 3\alpha \geq 0$. Thus, when $\alpha \geq \frac{3}{4}$, then $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \geq H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$.

Case 4: If $1 > t_V(X_1) \geq \alpha$, that is, $X_1 \subseteq POS_{R_1}^{(\alpha,\beta)}(V)$. Suppose $\left|\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right| = c + |X_1|$, $t_V(Y_2) \geq \alpha$ and $\left|\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right| = c + |Y_2|$. Then

$$\begin{aligned} &H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \\ &= \frac{1}{\left|POS_{R_1}^{(\alpha,\beta)}(V)\right|}\Big[\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-X_1} H(x) + \sum_{x\in X_1} H(x)\Big] \\ &= \frac{1}{c+a+b} \times \sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-X_1} H(x) \\ &\quad + \frac{1}{c+a+b}\Big[\frac{4(a+b)}{1-f_V(X_1)-t_V(X_1)} \\ &\quad\times \int_{t_V(X_1)}^{1-f_V(X_1)} \mu_V(x)(1-\mu_V(x))\,d\mu_V(x)\Big]. \end{aligned} \tag{18}$$

$$\begin{aligned} &H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right) \\ &= \frac{1}{\left|POS_{R_2}^{(\alpha,\beta)}(V)\right|}\Big[\sum_{x\in POS_{R_2}^{(\alpha,\beta)}(V)-Y_2} H(x) + \sum_{x\in Y_2} H(x)\Big] \\ &= \frac{1}{c+b}\Big[\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x) + \frac{4b}{1-f_V(Y_2)-t_V(Y_2)} \\ &\quad\times \int_{t_V(Y_2)}^{1-f_V(Y_2)} \mu_V(x)(1-\mu_V(x))\,d\mu_V(x)\Big]. \end{aligned} \tag{19}$$

Because $X_1 = Y_1 \cup Y_2$,

$$\begin{aligned}
&(c+a+b)(c+b)\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\
&\quad = ac[2t_V(Y_2) + 2(1-f_V(Y_2)) - \frac{4}{3}t_V^2(Y_2) \\
&\qquad - \frac{4}{3}(1-f_V(Y_2))^2 - \frac{4}{3}t_V(Y_2)(1-f_V(Y_2)) \\
&\qquad - \frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x)}{c}] - (c+b)(a+b)\frac{4}{3} \\
&\qquad \times \left[t_V^2(Y_1) + (1-f_V(Y_1))^2 + 2t_V(Y_1)t_V(Y_2)\right. \\
&\qquad + 2(1-f_V(Y_1))(1-f_V(Y_2)) + t_V(Y_1)(1-f_V(Y_1)) \\
&\qquad + t_V(Y_1)(1-f_V(Y_2)) + t_V(Y_2)(1-f_V(Y_1))]. \qquad (20)
\end{aligned}$$

Suppose $G(y) = -\frac{3}{4}y^2 + 2y$, it is easy to know that when $y \geq \frac{3}{4}$, $G(y)$ would decrease with the increase of y. Suppose

$$\begin{aligned}
&(c+a+b)(c+b)\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\
&\quad = ac[G(t_V(Y_2)) + G(1-f_V(Y_2)) - \frac{4}{3}t_V(Y_2) \\
&\qquad \times (1-f_V(Y_2)) - \frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x)}{c}] - A. \qquad (21)
\end{aligned}$$

Thus, when $\frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x)}{c} \geq \alpha$, and $1 - f_V(Y_2) \geq t_V(Y_2) \geq \alpha \geq \frac{3}{4}$,

$$\begin{aligned}
&(c+a+b)(c+b)\left[H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) - H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)\right] \\
&\quad \leq ac[G(t_V(Y_2)) + G(t_V(Y_2)) - \frac{4}{3}t_V(Y_2)(1-f_V(Y_2)) \\
&\qquad - \frac{\sum_{x\in POS_{R_1}^{(\alpha,\beta)}(V)-Y_2} H(x)}{c}] - A \\
&\quad \leq ac[G(\alpha) + G(\alpha) - \frac{4}{3}\alpha^2 - \alpha] - A \\
&\quad = ac\left(-4\alpha^2 + 3\alpha\right) - A \\
&\quad \leq ac\left(-4\alpha^2 + 3\alpha\right). \qquad (22)
\end{aligned}$$

When $\alpha \geq \frac{3}{4}$, $-4\alpha^2 + 3\alpha \leq 0$ holds. Therefore, $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \leq H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$.

As a result, according to the analysis and demonstration of **Case 3** and **Case 4**, **Theorem 2** and **Theorem 3** are obviously obtained:

Theorem 2: Let V be a vague set in a universe of discourse U, R_1 and R_2 be two equivalence relations on U. Suppose $R_1 \subseteq R_2$. If $\underline{R}_{R_1}^{(\alpha,\beta)}(V) \subseteq \underline{R}_{R_2}^{(\alpha,\beta)}(V)$ and $\alpha \geq \frac{3}{4}$, then $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \geq H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$.

Theorem 3: Let V be a vague set in a universe of discourse U, R_1 and R_2 be two equivalence relations on U. Suppose $R_1 \subseteq R_2$. If $\underline{R}_{R_2}^{(\alpha,\beta)}(V) \subseteq \underline{R}_{R_1}^{(\alpha,\beta)}(V)$ and $\alpha \geq \frac{3}{4}$, then $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \leq H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$.

In order to verify **Theorem 2** and **Theorem 3**, **Example 3** is shown as follows:

Example 3: Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, V be a vague set in a universe of discourse U and R_1, R_2 and R_3 be two equivalence relations on U.

$$\begin{aligned}
V = {} & \frac{[0.1, 0.2]}{x_1} + \frac{[0.2, 0.4]}{x_2} + \frac{[0.2, 0.6]}{x_3} + \frac{[0.3, 0.5]}{x_4} \\
& + \frac{[0.3, 0.6]}{x_5} + \frac{[0.5, 0.7]}{x_6} \\
& + \frac{[0.6, 0.75]}{x_7} + \frac{[0.75, 0.9]}{x_8} + \frac{[0.8, 0.9]}{x_9} + \frac{[0.9, 1]}{x_{10}}.
\end{aligned}$$

$$\begin{aligned}
U/R_1 &= \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6, x_9\}, \{x_7, x_8, x_{10}\}\}, \\
U/R_2 &= \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6, x_9\}, \{x_7\}, \{x_8, x_{10}\}\}, \\
U/R_3 &= \{\{x_1, x_2\}, \{x_3, x_4, x_5\}, \{x_6\}, \{x_9\}, \{x_7, x_8, x_{10}\}\}.
\end{aligned}$$

It is obvious that $R_1 \subseteq R_2$ and $R_1 \subseteq R_3$. Thus, suppose $\alpha = 0.75$, we can obtain:

$$\begin{aligned}
\underline{R}_{R_1}^{(\alpha,\beta)}(V) &= \frac{[0.75, 0.8833]}{x_7} + \frac{[0.75, 0.8833]}{x_8} \\
&\quad + \frac{[0.75, 0.8833]}{x_{10}}, \\
\underline{R}_{R_2}^{(\alpha,\beta)}(V) &= \frac{[0.75, 0.875]}{x_8} + \frac{[0.75, 0.875]}{x_{10}}, \\
\underline{R}_{R_3}^{(\alpha,\beta)}(V) &= \frac{[0.8, 0.9]}{x_9} + \frac{[0.75, 0.8833]}{x_7} + \frac{[0.75, 0.8833]}{x_8} \\
&\quad + \frac{[0.75, 0.8833]}{x_{10}}.
\end{aligned}$$

Thus,

$$H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) = 0.5930,$$

$$H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right) = 0.6042,$$

$$H\left(\underline{R}_{R_3}^{(\alpha,\beta)}(V)\right) = 0.5714.$$

When $R_1 \subseteq R_3$ and $\underline{R}_{R_1}^{(\alpha,\beta)}(V) \subseteq \underline{R}_{R_3}^{(\alpha,\beta)}(V)$, $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) > H\left(\underline{R}_{R_3}^{(\alpha,\beta)}(V)\right)$. When $R_1 \subseteq R_2$ and $\underline{R}_{R_2}^{(\alpha,\beta)}(V) \subseteq \underline{R}_{R_1}^{(\alpha,\beta)}(V)$, $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) > H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$. Therefore, the example validates **Theorem 2** and **Theorem 3**.

According to **Theorem 2** and **Theorem 3**, two practical facts could be summarized:

1) In the process of adding attributes, when the positive region(lower approximation sets)$\underline{R}^{(\alpha,\beta)}(V)$ increases, its uncertainty would decrease. By contrast, when the positive region(lower approximation sets)$\underline{R}^{(\alpha,\beta)}(V)$ decreases, its uncertainty would increase. This conclusion is consistent with real life. For example, when people choose items, they usually choose items that have reached the standard first. For items that cannot be judged, they usually do not make decisions until you know more about these attributes or they would be refined. In the next round of decision-making, only

the items that reach the standard will be added to the measured items. Items that have been selected would not be re-measured again. In other words, the "positive region" will only increase instead of decreasing in decision making. In the process of decision-making step by step, the information system will gradually reach to stability. Its uncertainty would be smaller and smaller.

2) Only if $\alpha \geq \frac{3}{4}$ and $\underline{R}_{R_1}^{(\alpha,\beta)}(V) \subseteq \underline{R}_{R_2}^{(\alpha,\beta)}(V)$, then $H\left(\underline{R}_{R_1}^{(\alpha,\beta)}(V)\right) \geq H\left(\underline{R}_{R_2}^{(\alpha,\beta)}(V)\right)$. As an extension model of the Pawlak rough set model, the probabilistic rough set model introduces a pair of thresholds (α, β) in order to increase tolerance for error. However, infinitely increased misclassification probability would only make the cost of the decision become larger and larger. And only if control the appropriate degree of error tolerance, the uncertainty of decision results would become smaller and smaller.

V. CONCLUSIONS

Vague sets are the further generalization of fuzzy sets, and there are many interesting features for handling vague data. In many application fields, how to measure the uncertainty of vague sets is a really important issue. The uncertainty of vague sets roots in two aspects, one is the fuzziness and the other is hesitation degree. Zhang *et al.* [71] proposed a new method for measuring fuzziness of vague sets, named average fuzziness of vague sets. What'more, multi-granularity knowledge discovery has attracted a great deal of researchers attention in this field. Putting vague sets in rough approximation spaces, Zhang *et al.* [70] proposed average-step-vague sets and discussed its hidden properties. Based on these aforementioned studies, in this paper, we focus on the uncertainty rules of vague sets in rough approximation spaces. Firstly, change rules of average fuzziness of the vague value with changing its truth membership degree and false membership degree are discussed and proved. Secondly, The uncertainty of vague sets in rough approximation spaces is established and discussed. And then, with changing knowledge granularity the change rules of uncertainty of vague sets are dicussed and proved. Next, change rules of uncertainty for the approximation set of vague sets are revealed and proved. Finally, Some examples are presented to verify these rules. These conclusions may provide a theoretical basis for decision-making in multi-granularity spaces. Furthermore, we hope this study would contribute to the development of research on vague sets, uncertainty measure and GrC. Our future work will focus on how to further excavate uncertainty of vague sets and how to acquire more fuzzy knowledge from vague sets in rough approximation spaces.

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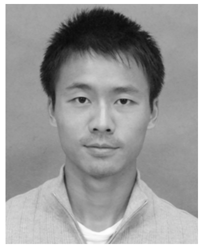

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