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Discrete-Time Adaptive Control for Systems With Input Time-Delay and Non-Sector Bounded Nonlinear Functions

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ABSTRACT This paper presents a discrete-time adaptive control approach for nonlinear systems with input delay. The nonlinearity is assumed to be non-sector bounded, resulting in the key technical lemma being inapplicable. The main aim of this paper is to present a general implementation inspired from Kanellakopoulos and Fu, *et al*. for uncertain scalar and multivariable input delay systems with uncertain parameters as well as uncertain input gain. While it has been shown by Kanellakopoulos and Fu, *et al*. that it is possible to design adaptive control laws that compensate for the growth of the nonlinearity for single parameter scalar systems, a rigorous analysis of multiple parameter systems is not shown. In this paper, it is shown that an adaptive controller design that compensates for the growth of the nonlinearity is possible for both multiple parameter scalar and multivariable systems with input delay. Rigorous stability proofs and simulations are presented to verify the validity of the approach.

INDEX TERMS Adaptive control, discrete-time systems, nonlinear control, time-delay systems.

I. INTRODUCTION

Stabilization of systems with actuator delays has always been a challenge in controller design. The celebrated Smith Predictor [3], proved to be the first practical solution to dealing with actuator delays although it was limited by the requirement of exact model parameters as well as the time-delay. Later on, adaptive control designs for uncertain linear time invariant systems with known time-delays were presented by Ortega and Lozano [4]. This was expanded further in [5]–[12], for various cases including input delays, state delays, distributed delays, time-varying delays, etc. In addition, various practical implementations have been presented in [13]–[15]. The survey paper [19] provides a comprehensive list of papers published prior to 2003 that discuss the stabilization of time delay systems. Also, the book [20] presents predictive feedback in delay systems with extensions to nonlinear systems, delay-adaptive control and actuator dynamics modeled by PDEs. More recently, compensation approaches for input delays using truncated predictor feedback are shown in [16]–[25].

Successful studies on the adaptive control of linear, discrete-time uncertain systems with time-delay can be found in [21]–[25]. For nonlinear discrete-time adaptive control, implementations have always been limited by the requirement that the system nonlinearities are sector bounded. This is a strict requirement of the Key Technical Lemma [26] (page 181) that guarantees asymptotic stability of the system. In order to eliminate this limitation a new approach was proposed in [1]. This approach allowed for the relaxation of the bound conditions on the nonlinearity while still guaranteeing asymptotic stability. The approach was developed for a scalar system (with a single uncertain parameter) without an uncertain input gain or input time-delay and it was highlighted that extension to more general cases is difficult. In [2], the same problem is addressed without assuming a growth condition on the nonlinearity, in the presence of bounded disturbances. The results are proven for a system similar to that in [1] and the algorithm for multivariable systems is given without any rigorous analysis or stability proofs.

In this paper, a more general implementation inspired by [1] and [2] is presented for uncertain scalar input delay systems with multiple uncertain parameters as well as uncertain input gain. The approach is further extended to multivariable input delay systems. For the scalar case, the approach is based on the prediction of future signals through successive substitution of the system model as is shown in [27]. Following the approach in [1], a coefficient is introduced into the adaptive law that guarantees asymptotic convergence in the presence of non sector bounded nonlinearities. The approach is further extended to multivariable systems and it is shown that this extension is not trivial and needs to be investigated rigorously. Stability proofs are given with simulation results for a scalar and a multivariable system to verify the proposed approach.

The organization of this paper is as follows: In Section II, the main result and a discussion of scalar systems are presented. In Section III, an extension to multivariable systems is provided. In Section IV, simulation examples are presented and concluding remarks are given in Section V.

Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm and $O(\cdot)$ denotes order of '$\cdot$'. For notational convenience, the mathematical expression "f_k" represents the value of the signal f at the k'th sampling instant.

II. MAIN RESULT

In this section, the controller design is presented starting with a simple scalar first-order system.

A. CONTROL OF A SCALAR INPUT-DELAY SYSTEM IN DISCRETE-TIME

Consider the following discrete-time system with input delay

$$x_{k+1} = \boldsymbol{\phi}^\top \boldsymbol{\xi}(x_k) + b u_{k-p} + \delta_k \tag{1}$$

where $x_k \in \Re$ is the system output, the parameters $\boldsymbol{\phi} \in \Re^{q^*}$, the function $\boldsymbol{\xi}(x_k) \in \Re^{q^*}$ is a known polynomial function of x_k, $q^* \in Z^+$ is the number of parameters, $b \in \Re$ is assumed to be known, p is the delay in number of steps and $|\delta_k| \in O(1)$ is an uncertain smooth time-varying disturbance. For the system (1), the following assumptions are made:

Assumption 1: The delay p is known *a priori*.

Assumption 2: The function $\boldsymbol{\xi}(x_k)$ is bounded for a bounded x_k. Furthermore, $\|\boldsymbol{\xi}(x_k)\| \leq c_0 + c_1|x_k|^g$ for some positive constant c_0, c_1 and $g \in Z^+$ is the order of the polynomial function $\boldsymbol{\xi}(x_k)$.

Assumption 3: From the structure of the system (1), there exist constants κ_0 and κ_1 such that the control input is bounded as $|u_{k-p}| \leq \kappa_0 + \kappa_1 \max_{i\in[0,k+1]} |x_i|^g$.

The goal is to force the system (1) to track the reference model

$$x_{\mathrm{m},k+1} = a_{\mathrm{m}} x_{\mathrm{m},k-p} + b_{\mathrm{m}} r_{k-p} \tag{2}$$

where $a_{\mathrm{m}} \in \Re$ is in the unit-disk. Extending the work in [3] and [28] a controller is chosen as

$$u_k = b^{-1}\left(a_{\mathrm{m}} x_k - \boldsymbol{\phi}^\top \boldsymbol{\xi}\left(x_{k+p}\right) + b_{\mathrm{m}} r_k - \hat{\delta}_{k+p}\right) \tag{3}$$

where $\hat{\delta}_k$ is an estimate of the disturbance. Substitution of the controller (3) into (1) leads to error dynamics of the form

$$e_{k+1} = a_{\mathrm{m}} e_{k-p} + \tilde{\delta}_k \tag{4}$$

where $e_k = x_{\mathrm{m},k} - x_k$ and $\tilde{\delta}_k$ is the disturbance estimation error. Since $|a_{\mathrm{m}}| < 1$ and if the term $|\tilde{\delta}_k|$ is bounded such $|\tilde{\delta}_k| \leq \Delta$ for some constant Δ, then (4) is stable. Note that the controller (3) is a function of x_{k+p} and $\hat{\delta}_{k+p}$. Therefore, x_{k+p} and $\hat{\delta}_{k+p}$ are needed for the computation of the control input u_k. Rather than deriving seperate estimations for x_{k+p} and $\hat{\delta}_{k+p}$, the system (1) will be rewritten into a form that attenuates the influence of the disturbance δ_k and that form will be used to derive delay free system dynamics. From the delay free system dynamics, a causal control law is derived.

Consider the system (1), according to the assumptions on the disturbance δ_k and the results in [29], it follows that $\delta_k - 2\delta_{k-1} + \delta_{k-2} \in O(T^2)$ where $T < 1$ is the sampling interval. Using this result, the system can be written in a disturbance compensated form as

$$x_{k+1} - 2x_k + x_{k-1} = \boldsymbol{\phi}^\top\left(\boldsymbol{\xi}_k - 2\boldsymbol{\xi}_{k-1} + \boldsymbol{\xi}_{k-2}\right) + b\left(u_{k-p} - 2u_{k-p-1} + u_{k-p-2}\right) + \upsilon_k, \tag{5}$$

$$x_{k+1} = \boldsymbol{\phi}^\top\left(\boldsymbol{\xi}_k - 2\boldsymbol{\xi}_{k-1} + \boldsymbol{\xi}_{k-2}\right) + 2x_k - x_{k-1} + b\left(u_{k-p} - 2u_{k-p-1} + u_{k-p-2}\right) + \upsilon_k \tag{6}$$

where $\boldsymbol{\xi}_k \equiv \boldsymbol{\xi}(x_k)$ and $\upsilon_k = \delta_k - 2\delta_{k-1} + \delta_{k-2} \in O(T^2)$. Using successive substitutions a delay-free system is obtained as

$$x_{k+p+1} = \boldsymbol{\theta}^\top \boldsymbol{\zeta}(x_k, u_{k-1}, \ldots, u_{k-p+1}) + b u_k + \boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1} + \bar{\upsilon}_{k+p} \tag{7}$$

where $\boldsymbol{\theta}$ is the augmented parameter vector, $\boldsymbol{\zeta}(\cdot)$ is the augmented nonlinear function that is a function of the state x_k and control history $u_{k-1}, \ldots, u_{k-p+1}$, and $\boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1}$, $\bar{\upsilon}_{k+p}$ are the augmented disturbance terms due to the successive substitutions. Note that as a result of the successive substitutions $\boldsymbol{\rho}_{k-1}$ will be a function of ϕ, b and υ_k and that $\|\boldsymbol{\rho}_{k-1}\| \in O(\lambda T^2)$ for some constant λ.

Consider the term $\boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1}$, based on the structure of $\boldsymbol{\xi}(x_k)$, the augmented nonlinear function $\boldsymbol{\zeta}(\cdot)$ and $\boldsymbol{\varphi}_k$ will contain cross terms of the states $x_{1,k}$, $x_{2,k}, \ldots$, the control inputs $u_{1,k}, u_{2,k}, \ldots$ and υ_k. However, $\boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1}$ can still be written in parametric form. Using **Assumption 3**, it can be shown that $|\boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1}| \leq \kappa_2 + \kappa_3 \max_{i\in[0,k]} |x_i|^{(g^{p+1}-g)}$ where $\kappa_2, \kappa_3 \in O(\lambda T^2)$ are some positive constants. Furthermore, it can be shown that since $\bar{\upsilon}_k$ is a function of υ_k history and the uncertain parameter vector $\boldsymbol{\phi}$, a bound can be found such that $|\bar{\upsilon}_k| \leq \bar{\Delta} \in O(T^2)$.

Proceeding with the control law design, subtracting (7) from a $k+p$ steps ahead form of (2) results in an error dynamics of the form

$$e_{k+p+1} = a_{\mathrm{m}} e_k + a_{\mathrm{m}} x_k - \boldsymbol{\theta}^\top \boldsymbol{\zeta}(x_k, u_{k-1}, \ldots, u_{k-p+1}) - b u_k + b_{\mathrm{m}} r_k - \boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1} - \bar{\upsilon}_{k+p}. \tag{8}$$

From (8) a control law is selected as

$$u_k = b^{-1}\left(a_{\mathrm{m}} x_k - \boldsymbol{\theta}^\top \boldsymbol{\zeta}(x_k, u_{k-1}, \ldots, u_{k-p+1}) + b_{\mathrm{m}} r_k\right) \tag{9}$$

such that an error dynamics is achieved as

$$e_{k+1} = a_{\mathrm{m}} e_{k-p} + \boldsymbol{\rho}_{k-p-1}^\top \boldsymbol{\varphi}_{k-p-1} + \bar{\upsilon}_k. \tag{10}$$

Assume that $\kappa_2 \approx \kappa_2 + \bar{\Delta}$, then (10) is further written in the form

$$\begin{aligned}|e_{k+1}| &\le \left(|a_{\rm m}| + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}\right)|e_{k-p}| + \kappa_2 + |\bar{\upsilon}_k| \\ &\quad + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}|r_{k-p}| \\ &\le \left(|a_{\rm m}| + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}\right)|e_{k-p}| + \kappa_2 \\ &\quad + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}|r_{k-p}| \end{aligned} \tag{11}$$

which is asymptotically stable if and only if the state x_k lies in a neighborhood that satisfies the condition $\left(|a_{\rm m}| + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}\right) < 1$.

Remark 1: Upon careful inspection of the result (11), it can be seen that the term $\left(|a_{\rm m}| + \kappa_3|x_{k-p}|^{(g^{p+1}-g-1)}\right)$ is a function of the delay p. Therefore, the stability of (11) is guaranteed if and only if

$$|x_k| < \left(\frac{1-|a_{\rm m}|}{\kappa_3}\right)^{\frac{1}{g^{p+1}-g-1}}. \tag{12}$$

The condition (12) gives the neighborhood for the stability of (11). It is possible to select the sampling-interval T such that $\kappa_3 \in O(\lambda T^2)$ is small enough resulting in a large enough neighborhood for stability.

B. ADAPTIVE CONTROL OF AN INPUT-DELAY SYSTEM

Consider now that the parameters $\boldsymbol{\phi}$ and b in system (1) are uncertain constants. This will result in the parameter vector $\boldsymbol{\theta}$ being uncertain and the control law (9) is revised as

$$u_k = \hat{b}_k^{-1}\left(a_{\rm m}x_k - \hat{\boldsymbol{\theta}}_k^\top \boldsymbol{\zeta}(x_k, u_{k-1}, \ldots, u_{k-p+1}) + b_{\rm m}r_k\right) \tag{13}$$

where $\hat{\boldsymbol{\theta}}_k$ and $\hat{b}_k$ are the estimates of $\boldsymbol{\theta}$ and b respectively. The parameter estimates must be computed such that the system (1) tracks the reference model (2). Now that the goal of the adaptive law is defined, it is possible to proceed with the derivation. In order to derive the adaptive law, substituting the control law (13) in (7) it is obtained that

$$x_{k+p+1} = a_{\rm m}x_k + \tilde{\boldsymbol{\theta}}_k^\top \boldsymbol{\zeta}_k + b_{\rm m}r_k + \tilde{b}_k u_k + \boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1} + \bar{\upsilon}_k \tag{14}$$

where $\boldsymbol{\zeta}_k \equiv \boldsymbol{\zeta}\left(x_k, u_{k-1}, \ldots, u_{k-p+1}\right)$ and $\tilde{\boldsymbol{\theta}}_k, \tilde{b}_k$ are the parameter estimation errors respectively. Substracting (2) from a p steps delayed (14), it is obtained that

$$\begin{aligned} e_{k+1} &= a_{\rm m}e_{k-p} + \tilde{\boldsymbol{\theta}}_{k-p}^\top \boldsymbol{\zeta}_{k-p} + \tilde{b}_{k-p}u_{k-p} + \boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1} + \bar{\upsilon}_k \\ &= a_{\rm m}e_{k-p} + \tilde{\boldsymbol{\psi}}_{k-p}^\top \bar{\boldsymbol{\zeta}}_{k-p} + \boldsymbol{\rho}_{k-1}^\top \boldsymbol{\varphi}_{k-1} + \bar{\upsilon}_k \end{aligned} \tag{15}$$

where $\tilde{\boldsymbol{\psi}}_k^\top = \left[\tilde{\boldsymbol{\theta}}_k^\top \,\vdots\, \tilde{b}_k\right] \in \Re^{q+1}$ is the lumped parameter estimation error and $\bar{\boldsymbol{\zeta}}_k^\top = \left[\boldsymbol{\zeta}_k^\top \,\vdots\, u_k\right] \in \Re^{q+1}$. Using (15), it is possible to formulate the adaptive law as follows

$$\hat{\boldsymbol{\psi}}_{k+1} = \begin{cases} \hat{\boldsymbol{\psi}}_{k-p} + \alpha_{k+1}\beta_k\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1} & \forall k \in [k_0, \infty) \\ \hat{\boldsymbol{\psi}}_{k_0} & \forall k \in [0, k_0) \end{cases} \tag{16}$$

$$P_{k+1} = \begin{cases} P_{k-p} - \alpha_{k+1} \\ \times \dfrac{\beta_k\gamma_k P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}}{1+\alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}} & \forall k \in [k_0, \infty) \\ P_{k_0} > 0 & \forall k \in [0, k_0) \end{cases} \tag{17}$$

where $\bar{e}_{k+1} = e_{k+1} - a_{\rm m}e_{k-p}$, $\beta_k > 0$ is a scalar coefficient used to prevent a singular $\hat{b}_k$, k_0 is some initial time step and the coefficient α_k is positive and will be defined later. The matrix $P_k \in \Re^{q+1\times q+1}$ is the symmetric positive-definite covariance matrix. The coefficient γ_k is given as

$$\gamma_k = \begin{cases} 1 - \dfrac{(1+\alpha_{\max}d_0)\,\omega_k^2}{\bar{e}_{k+1}^2}, & \text{if } |\bar{e}_{k+1}| \ge (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k \\ 0, & \text{if } |\bar{e}_{k+1}| < (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k \end{cases} \tag{18}$$

where $\omega_k = \left(\kappa_2 + \kappa_3|x_{k-p}|^{(g^{p+1}-g)}\right)$. The constants $\alpha_{\max}$ and d_0 are positive and will be defined in **Lemma 2** and **Lemma 3**. Finally, with respect to the coefficient β_k in (16) and (17), consider the control law (9). The term $\hat{b}_k$ is an adaptive term and there is a risk of division by zero if $\hat{b}_k$ is singular. In order to guarantee that $\hat{b}_k$ in (13) is not singular, consider the adaptive law from (16), namely

$$\hat{\boldsymbol{\psi}}_{k+1} = \hat{\boldsymbol{\psi}}_{k-p} + \alpha_{k+1}\beta_k\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1} \tag{19}$$

and let $\mathbf{s} = [0\cdots 0\ 1]^\top$ such that $\hat{b}_k = \mathbf{s}^\top\hat{\boldsymbol{\psi}}_k$. Then premultiplying both sides of (19) with $\mathbf{s}^\top$ it is obtained that

$$\begin{aligned} \hat{b}_{k+1} &= \mathbf{s}^\top\hat{\boldsymbol{\psi}}_{k+1} = \hat{b}_{k-p} + \alpha_{k+1}\beta_k\gamma_k\mathbf{s}^\top P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1} \\ &= \hat{b}_{k-p}\beta_k\left(\frac{1}{\beta_k} + \hat{b}_{k-p}^{-1}\mathbf{s}^\top\alpha_{k+1}\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1}\right) \end{aligned} \tag{20}$$

and if the initial choice of $\hat{b}_{k-p}$ is nonsingular and $\beta_k^{-1} \ne -\hat{b}_{k-p}^{-1}\mathbf{s}^\top\alpha_{k+1}\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1}$ then $\hat{b}_{k+1}$ will be nonsingular. The value of β_k can be selected from a predefined set as long as it satisfies $\beta_k^{-1} \ne -\hat{b}_{k-p}^{-1}\mathbf{s}^\top\alpha_{k+1}\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1}$.

Before proceeding with the stability analysis it is necessary to define the following **Lemmas**:

Lemma 1: For the system (15) and the adaptive laws (16) and (17), the following conditions are true:

(a) $\lim_{k\to\infty} \frac{\alpha_{k+1}\beta_k\gamma_k^2}{1+\alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}}\bar{e}_{k+1}^2 = 0$

(b) $\|\bar{\boldsymbol{\zeta}}_k\| \le c_0\|\boldsymbol{\zeta}_k\|$, for some constants c_0.

Proof: Consider the positive function

$$V_k = \sum_{i=0}^{p}\tilde{\boldsymbol{\psi}}_{k-i}^\top P_{k-i}^{-1}\tilde{\boldsymbol{\psi}}_{k-i}. \tag{21}$$

The forward difference $V_{k+1} - V_k$ can be found as, [22],

$$\Delta V_k = V_{k+1} - V_k = \tilde{\boldsymbol{\psi}}_{k+1}^\top P_{k+1}^{-1}\tilde{\boldsymbol{\psi}}_{k+1} - \tilde{\boldsymbol{\psi}}_{k-p}^\top P_{k-p}^{-1}\tilde{\boldsymbol{\psi}}_{k-p}. \tag{22}$$

Substitution of (16) in (22) and following the approach in [27], it is obtained that

$$\Delta V_k = \tilde{\boldsymbol{\psi}}_{k-p}^{\top}\left(P_{k+1}^{-1} - P_{k-p}^{-1}\right)\tilde{\boldsymbol{\psi}}_{k-p} - 2\alpha_{k+1}\beta_k\tilde{\boldsymbol{\psi}}_{k-p}^{\top}\bar{\boldsymbol{\zeta}}_{k-p}\bar{e}_{k+1} + \bar{\boldsymbol{\zeta}}_{k-p}^{\top}P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\alpha_{k+1}^2\beta_k^2\gamma_k^2\bar{e}_{k+1}^2. \tag{23}$$

To proceed further, consider (17). According to [22], the covariance matrix P_k satisfies $P_{k+1}^{-1} = P_{k-p}^{-1} + \alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}^{\top}$. Using this condition and the fact that α_{k+1}, β_k, γ_k are positive coefficients, then (23) can be simplified further to obtain

$$\Delta V_k \leq -\frac{\alpha_{k+1}\beta_k\gamma_k^2}{1+\alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^{\top}P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}}\bar{e}_{k+1}^2 \tag{24}$$

which implies that $\tilde{\boldsymbol{\psi}}_k^{\top} \equiv \left[\tilde{\boldsymbol{\theta}}_k^{\top} \,\vdots\, \tilde{b}_k\right]$ is bounded and, therefore, $\hat{\boldsymbol{\psi}}_k^{\top} \equiv \left[\hat{\boldsymbol{\theta}}_k^{\top} \,\vdots\, \hat{b}_k\right]$ is also bounded, [22]. Note that for any $k \in [k_0, \infty)$ the following is true

$$V_{k+1} = V_{k_0} + \sum_{i=0}^{k-k_0}\Delta V_{k_0+i} \tag{25}$$

Substituting (24) in (25), it is obtained that

$$\lim_{k\to\infty}V_{k+1} < V_{k_0} - \lim_{k\to\infty}\sum_{i=0}^{k-k_0} \times \frac{\alpha_{k_0+i+1}\beta_{k_0+i}\gamma_{k_0+i}^2\bar{e}_{k_0+i+1}^2}{1+\alpha_{k_0+i+1}\beta_{k_0+i}\gamma_{k_0+i}\bar{\boldsymbol{\zeta}}_{k_0+i-p}^{\top}P_{k_0+i-p}\bar{\boldsymbol{\zeta}}_{k_0+i-p}}. \tag{26}$$

Since by definition, V_{k+1} is non-negative and V_{k_0} is finite, then according to the convergence theorem of the sum of series condition (a) of **Lemma 1** is established as

$$\lim_{k\to\infty}\frac{\alpha_{k+1}\beta_k\gamma_k^2}{1+\alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^{\top}P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}}\bar{e}_{k+1}^2 = 0. \tag{27}$$

Finally to verify part (b), consider the definition of $\bar{\boldsymbol{\zeta}}_k$ and the control law (13). It is obtained that

$$\begin{aligned}\bar{\boldsymbol{\zeta}}_k^{\top} &= \left[\boldsymbol{\zeta}_k^{\top} \,\vdots\, u_k\right] \\ &= \left[\boldsymbol{\zeta}_k^{\top} \,\vdots\, \hat{b}_k^{-1}\left(a_{\mathrm{m}}x_k - \hat{\boldsymbol{\theta}}_k^{\top}\boldsymbol{\zeta}_k + b_{\mathrm{m}}r_k\right)\right].\end{aligned} \tag{28}$$

Consider (28), then from condition (a) it follows that the adaptive parameters $\hat{b}_k$ and $\hat{\boldsymbol{\theta}}_k$ are bounded. Furthermore, the reference signal r_k is bounded and $\boldsymbol{\zeta}_k$ is not sector bounded w.r.t x_k. Then it is obtained that

$$\|\bar{\boldsymbol{\zeta}}_k\| \leq c_0\|\boldsymbol{\zeta}_k\|. \tag{29}$$

■

Lemma 2: For the matrix P_k in (17), the term $\bar{\boldsymbol{\zeta}}_{k-p}^{\top}P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}$ is bounded as

$$\bar{\boldsymbol{\zeta}}_{k-p}^{\top}P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p} \leq d_0 \tag{30}$$

if α_k is selected such that

$$\alpha_k \geq \frac{f_k - d_1\, g_k}{d_1\, h_k - l_k} \tag{31}$$

where f_k, g_k, h_k and l_k are functions of the elements of $\bar{\boldsymbol{\zeta}}_k$ and α_k history while d_0, d_1 are some positive constants.

Proof: The inverse of the covariance matrix satisfies the condition $P_{k+1}^{-1} = P_{k-p}^{-1} + \alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}^{\top}$. Therefore, the solution $P_k^{-1}\forall k \in [k_0+p, \infty)$ can be computed as

$$P_k^{-1} = P_{k_0}^{-1} + \sum_{i=0}^{\lfloor\frac{k-k_0-p}{2}\rfloor}\alpha_{k-ip}\beta_{k-ip-1}\gamma_{k-ip-1}\bar{\boldsymbol{\zeta}}_{k-(i+1)p-1}\bar{\boldsymbol{\zeta}}_{k-(i+1)p-1}^{\top} \tag{32}$$

where k_0 is an initial time step and $\lfloor\cdot\rfloor$ is the floor function. Rewriting (32) as

$$P_k^{-1} = P_0^{-1} + \alpha_k\beta_{k-1}\gamma_{k-1}\bar{\boldsymbol{\zeta}}_{k-p-1}\bar{\boldsymbol{\zeta}}_{k-p-1}^{\top} + \sum_{i=1}^{\lfloor\frac{k-p}{2}\rfloor}\Big(\alpha_{k-ip} \times \beta_{k-ip-1}\gamma_{k-ip-1}\bar{\boldsymbol{\zeta}}_{k-(i+1)p-1}\bar{\boldsymbol{\zeta}}_{k-(i+1)-1}^{\top}\Big) \tag{33}$$

where $k_0 = 0$ for the sake of simplicity and considering that the initial value of P_0 is selected such that $P_0^{-1} = diag(p_1, p_2, \ldots, p_{q+1})$, then the matrix P_k can be evaluated by computing the inverse of P_k^{-1}. Therefore, the expression of P_k is obtained as

$$P_k = \frac{1}{\alpha_k h_k + g_k}\Big(\alpha_k M_{1,k} + M_{2,k}\Big) \tag{34}$$

where $\det\left(P_k^{-1}\right) = \alpha_k h_k + g_k$ and $\mathrm{adj}\left(P_k^{-1}\right) = \alpha_k M_{1,k} + M_{2,k}$. Premultiplying (34) with $\bar{\boldsymbol{\zeta}}_k^{\top}$ and postmultiplying with $\bar{\boldsymbol{\zeta}}_k$, it is obtained that

$$\bar{\boldsymbol{\zeta}}_k^{\top}P_k\bar{\boldsymbol{\zeta}}_k = \frac{1}{\alpha_k h_k + g_k}\Big(\alpha_k\bar{\boldsymbol{\zeta}}_k^{\top}M_{1,k}\bar{\boldsymbol{\zeta}}_k + \bar{\boldsymbol{\zeta}}_k^{\top}M_{2,k}\bar{\boldsymbol{\zeta}}_k\Big). \tag{35}$$

Furthermore, using matrix and vector norms on the right-hand-side of (35), the upperbound on $\bar{\boldsymbol{\zeta}}_k^{\top}P_k\bar{\boldsymbol{\zeta}}_k$ is obtained as

$$\begin{aligned}\bar{\boldsymbol{\zeta}}_k^{\top}P_k\bar{\boldsymbol{\zeta}}_k &\leq \frac{1}{\alpha_k h_k + g_k}\Big(\alpha_k\left\|M_{1,k}\right\| + \left\|M_{2,k}\right\|\Big)\left\|\bar{\boldsymbol{\zeta}}_k\right\|^2 \\ &\leq \frac{c_0}{\alpha_k h_k + g_k}\Big(\alpha_k\left\|M_{1,k}\right\| + \left\|M_{2,k}\right\|\Big)\left\|\boldsymbol{\zeta}_k\right\|^2.\end{aligned} \tag{36}$$

Now consider (36). If $\bar{\boldsymbol{\zeta}}_k^{\top}P_k\bar{\boldsymbol{\zeta}}_k \leq d_0$ then

$$\frac{c_0}{\alpha_k h_k + g_k}\Big(\alpha_k\left\|M_{1,k}\right\| + \left\|M_{2,k}\right\|\Big)\left\|\boldsymbol{\zeta}_k\right\|^2 \leq d_0 \tag{37}$$

and solving for α_k results in a condition on α_k that is given as

$$\alpha_k \geq \frac{f_k - d_1\, g_k}{d_1\, h_k - l_k} \tag{38}$$

where $f_k = \left\|M_{2,k}\right\|\left\|\boldsymbol{\zeta}_k\right\|^2$, $l_k = \left\|M_{1,k}\right\|\left\|\boldsymbol{\zeta}_k\right\|^2$ and $d_1 = \frac{d_0}{c_0}$. ■

Remark 2: Note that the constant d_1 can be adjusted to avoid division by zero. Furthermore, a lower bound on α_k can be imposed to ensure that α_k is always positive.

Remark 3: It is not possible to generalize the expression for α_k for multiple uncertain parameters, however, the procedure will be presented for a system with two uncertain parameters in order to illustrate the implementation of the adaptive law. The procedure is similar for any number of uncertain parameters.

Example 1: Consider the system given by (15) and assume that $\tilde{\boldsymbol{\psi}}_k^\top = [\,\tilde{\psi}_{1,k} \,\vdots\, \tilde{\psi}_{2,k}\,] \in \Re^2$ and $\bar{\boldsymbol{\zeta}}_k^\top = [\,\bar{\zeta}_{1,k} \,\vdots\, \bar{\zeta}_{2,k}\,] \in \Re^2$. Also let $P_0^{-1} = diag(p_1, p_2)$. Then the matrices $M_{1,k}$ and $M_{2,k}$ are obtained as

$$M_{1,k} = \bar{\beta}_{k-1}\begin{bmatrix} \bar{\zeta}^2_{2,k-p-1} & -\bar{\zeta}_{1,k-p-1}\bar{\zeta}_{2,k-p-1} \\ -\bar{\zeta}_{1,k-p-1}\bar{\zeta}_{2,k-p-1} & \bar{\zeta}^2_{1,k-p-1}\end{bmatrix} \quad (39)$$

and

$$M_{2,k} = \sum_{i=1}^{\lfloor \frac{k-p}{2} \rfloor} \alpha_{k-ip}\bar{\beta}_{k-ip-1}\begin{bmatrix} \bar{\zeta}^2_{2,j(k,i)} & -\bar{\zeta}_{1,j(k,i)}\bar{\zeta}_{2,j(k,i)} \\ -\bar{\zeta}_{1,j(k,i)}\bar{\zeta}_{2,j(k,i)} & \bar{\zeta}^2_{1,j(k,i)}\end{bmatrix} + \begin{bmatrix} p_2 & 0 \\ 0 & p_1 \end{bmatrix} \quad (40)$$

where $\bar{\beta}_{k-1} = \beta_{k-1}\gamma_{k-1}$ and $j(k,i) = k-(i+1)p-1$. Furthermore, the functions h_k and g_k are obtained as

$$\begin{aligned} h_k &= \bar{\beta}_{k-1}p_2\zeta^2_{1,k-p-1} + \bar{\beta}_{k-1}p_1\zeta^2_{2,k-p-1} + \bar{\beta}_{k-1} \\ &\quad \times \sum_{i=1}^{\lfloor \frac{k-p}{2} \rfloor} \alpha_{k-ip}\bar{\beta}_{k-ip-1} \\ &\quad \times \left(\zeta_{2,k-p-1}\bar{\zeta}_{1,j(k,i)} - \zeta_{1,k-p-1}\bar{\zeta}_{2,j(k,i)}\right)^2 \end{aligned} \quad (41)$$

and

$$g_k = p_1\, p_2 + \sum_{i=1}^{\lfloor \frac{k-p}{2} \rfloor} \alpha_{k-ip}\bar{\beta}_{k-ip-1}\left(p_2\bar{\zeta}^2_{1,j(k,i)} + p_1\bar{\zeta}^2_{2,j(k,i)}\right). \quad (42)$$

Finally, the results (39), (40), (41) and (42) can be susbstituted in (38) for the computation of α_k.

As can be seen from **Example 1**, the procedure for calculating α_k is straightforward and it is possible to extend it to a higher number of uncertain parameters.

Lemma 3: If α_k is computed from the lower bound in (38) such that

$$\alpha_k = \frac{f_k - d_1\, g_k}{d_1\, h_k - l_k} \quad (43)$$

and that $\bar{\boldsymbol{\zeta}}_k, \boldsymbol{\zeta}_k$ are bounded, then there exists an upper-bound $\alpha_{\max}$ such that $\max_{0\leq k<\infty} \alpha_k \leq \alpha_{\max}$.

Proof: Consider the expression (43), using the results (39), (40), (41) and (42) from **Example 1**, then it is obtained that

$$\begin{aligned} \alpha_k &= \frac{f_k - d_1\, g_k}{d_1\, h_k - l_k} \\ &= \frac{\nu_k}{d_1\, h_k - l_k} + \frac{\mu_{1,k}}{d_1\, h_k - l_k}\alpha_{k-p} + \frac{\mu_{2,k}}{d_1\, h_k - l_k}\alpha_{k-2p} + \cdots \end{aligned} \quad (44)$$

where $\nu_k, \mu_{1,k}, \mu_{2,k}, \ldots$ are functions of $p_1, p_2, d_1, \beta_k, \gamma_k$, $\bar{\boldsymbol{\zeta}}_k$ and $\boldsymbol{\zeta}_k$. Furthermore, the system (44) is augmented to the form

$$\begin{aligned} \alpha_k &= \frac{\nu_k}{d_1\, h_k - l_k} + \frac{\mu_{1,k}}{d_1\, h_k - l_k}\alpha_{k-p} + \frac{\mu_{2,k}}{d_1\, h_k - l_k}\alpha_{k-2p} + \cdots \\ \alpha_{k-p} &= \alpha_{k-p} \\ \alpha_{k-2p} &= \alpha_{k-2p} \\ \vdots &= \vdots \end{aligned} \quad (45)$$

which can be written in a vector form as

$$\begin{aligned} \bar{\boldsymbol{\alpha}}_k &= \begin{bmatrix} \frac{\mu_{1,k}}{d_1 h_k - l_k} & \frac{\mu_{2,k}}{d_1 h_k - l_k} & \cdots & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \cdots \end{bmatrix}\bar{\boldsymbol{\alpha}}_{k-p} + \begin{bmatrix} \frac{\nu_k}{d_1 h_k - l_k} \\ 0 \\ \vdots \end{bmatrix} \\ &= H_k\bar{\boldsymbol{\alpha}}_{k-p} + \boldsymbol{\vartheta}_k \end{aligned} \quad (46)$$

where $\bar{\boldsymbol{\alpha}}_k^\top = \left[\,\alpha_k \,\vdots\, \alpha_{k-p} \,\vdots\, \cdots\,\right]$. Consider that all terms other than $\bar{\boldsymbol{\alpha}}_k$ are bounded. From (41), the function h_k is a function of $\bar{\boldsymbol{\alpha}}_{k-p}$ and appears in the denominator of the first row elements of H_k given in (46). If α_k grows large enough with all other terms bounded then the first row elements of H_k and $\boldsymbol{\vartheta}_k$ will shrink such that there exists $\alpha_k = \alpha_{\max}$ which results in the norm $\|H_k\| < 1$. Thus, it is obtained that

$$\|\bar{\boldsymbol{\alpha}}_k\| \leq \|H_k\| \left\|\bar{\boldsymbol{\alpha}}_{k-p}\right\| + \|\boldsymbol{\vartheta}_k\| \quad (47)$$

which is stable due to the fact that $\|H_k\| < 1$ and that $\|\boldsymbol{\vartheta}_k\|$ is bounded. Then it is obtained that $\max_{0\leq k<\infty} \alpha_k \leq \alpha_{\max}$. ■

Remark 4: Even though the results from **Example 1** are used, an expression similar to (44) can be obtained for a system with any number of uncertain parameters.

Remark 5: From **Lemma 3** it is seen that a constant $\alpha_{\max}$ exists that will satisfy (38). Thus, $\alpha_k = \alpha_{\max}$ can be tuned rather than using (43) to compute α_k.

C. STABILITY ANALYSIS

Stability is given by the following **Theorem**:

Theorem 1: The closed-loop system, consisting of the system (1) with uncertain parameters $\boldsymbol{\phi}$ and b, the controller (13) with adaptive laws (16) and (17), is stable if and only if

$$\left(|a_{\mathrm{m}}| + \kappa_3\,(1+\alpha_{\max}d_0)^{\frac{1}{2}}\,|x_{k-p}|^{(g^{p+1}-g-1)}\right) < 1.$$

Furthermore, the tracking error, $e_k = x_k - x_{\mathrm{m},k}$, converges asymptotically to a bound ϵ.

Proof: The first part of **Theorem 1** discusses the boundedness of the signals in the closed loop system while the second part discusses the asymptotic convergence of the tracking error. However, note that the boundedness of α_k can only be considered after the convergence of the tracking error is established.

It was shown in **Lemma 1** and **Lemma 2** that the adaptive paramaters $\hat{\boldsymbol{\phi}}_k$ and $\hat{b}_k$ are bounded. Now consider the condition (a) of **Lemma 1** given as

$$\lim_{k\to\infty} \frac{\alpha_{k+1}\beta_k\gamma_k^2}{1+\alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}}\bar{e}_{k+1}^2 = 0 \tag{48}$$

which is true for $|\bar{e}_{k+1}| \geq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$. To guarantee that $\lim_{k\to 0}|\bar{e}_{k+1}| \leq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$ it must be guaranteed that $\|\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}\|$ is bounded. From **Lemma 2** it is shown that this is indeed the case. Therefore, since, α_{k+1}, β_k and γ_k are positive in addition to $\|\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}\| \leq d_0$, where d_0 is some positive constant, then (48) implies that

$$\lim_{k\to\infty} \frac{\alpha_{k+1}\beta_k\gamma_k^2}{1+\alpha_{k+1}\beta_k\gamma_k d_0}\bar{e}_{k+1}^2 = 0 \tag{49}$$

and ultimately, $\lim_{k\to\infty}|\bar{e}_{k+1}| \leq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$. Consider now the error dynamics given by

$$e_{k+1} = a_{\mathrm{m}}e_{k-p} + \bar{e}_{k+1} \tag{50}$$

and

$$\begin{aligned}|e_{k+1}| \leq &\left(|a_{\mathrm{m}}| + \kappa_3\,(1+\alpha_{\max}d_0)^{\frac{1}{2}}\,|x_{k-p}|^{(g^{p+1}-g-1)}\right)|e_{k-p}| \\ &+\kappa_3\,(1+\alpha_{\max}d_0)^{\frac{1}{2}}\,|x_{k-p}|^{(g^{p+1}-g-1)}|r_{k-p}| + \kappa_2.\end{aligned} \tag{51}$$

Consider that κ_3 is small enough and x_k lies in a neighborhood such that $\left(|a_{\mathrm{m}}| + \kappa_3\,(1+\alpha_{\max}d_0)^{\frac{1}{2}}\,|x_{k-p}|^{(g^{p+1}-g-1)}\right) < \bar{a}_{\mathrm{m}} < 1$ and $\kappa_3\,(1+\alpha_{\max}d_0)^{\frac{1}{2}}\,|x_{k-p}|^{(g^{p+1}-g-1)}|r_{k-p}| + \kappa_2 < \Delta_{\max}$ for some positive constants $\bar{a}_{\mathrm{m}}$ and $\Delta_{\max}$, then (51) has a solution that satisfies

$$|e_k| \leq \bar{a}_{\mathrm{m}}^{\lfloor\frac{k-p}{2}\rfloor}|e_0| + \sum_{i=1}^{\lfloor\frac{k-p}{2}\rfloor}\bar{a}_{\mathrm{m}}^i\Delta_{\max}. \tag{52}$$

Also, since $\bar{a}_{\mathrm{m}}$ is in the unit disk, it follows that $\lim_{k\to\infty}\bar{a}_{\mathrm{m}}^{\lfloor\frac{k-p}{2}\rfloor}|e_0| = 0$ and $\lim_{k\to\infty}\sum_{i=1}^{\lfloor\frac{k-p}{2}\rfloor}\bar{a}_{\mathrm{m}}^i\Delta_{\max} = \epsilon$ for some positive constant ϵ. Therefore,

$$\lim_{k\to\infty}|e_k| \leq \epsilon \tag{53}$$

which establishes the boundedness of $|e_k|$.

Finally, consider the coefficient α_k given by (38). Since it has been established that $|e_k|$ is bounded, then $\bar{\boldsymbol{\zeta}}_k$ and $\boldsymbol{\zeta}_k$ are also bounded. Therefore, using **Lemma 3** it is concluded that α_k is bounded. ■

III. EXTENSION TO MULTIVARIABLE SYSTEMS

In this section the proposed discrete-time adaptive controller is extended to multivariable nonlinear systems with time-delay.

Consider the n^{th} order feedback linearizable nonlinear system of the form

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi^\top\boldsymbol{\xi}(\mathbf{x}_k) + \Gamma\mathbf{u}_{k-p} + \Gamma\boldsymbol{\delta}_k \\ \mathbf{y}_k &= C^\top\mathbf{x}_k\end{aligned} \tag{54}$$

where $\mathbf{x}_k \in \Re^n$ is the state vector, $\mathbf{u}_k \in \Re^m$ is the control input vector, $\mathbf{y}_k \in \Re^m$ is the output vector, $\Phi \in \Re^{q^*\times n}$ is a matrix of uncertain parameters, $\Gamma \in \Re^{n\times m}$ is the uncertain input gain, $C \in \Re^{m\times n}$ is the output matrix and $\boldsymbol{\delta}_k \in \Re^m$ is a smooth time-varying disturbance vector such that $\|\boldsymbol{\delta}_k\| \in O(1)$. The function $\boldsymbol{\xi}(\mathbf{x}_k) \in \Re^{q^*}$ is a vector of known polynomial functions $\mathbf{x}_k$. For the system (54), the following assumptions are made:

Assumption 4: The delay p is known *a priori*.

Assumption 5: $C^\top\Gamma$ is non-singular.

Assumption 6: The norm of the function vector $\|\boldsymbol{\xi}\,(\mathbf{x}_k)\,\|$ is bounded for a bounded $\|\mathbf{x}_k\|$. Furthermore, $\|\boldsymbol{\xi}(\mathbf{x}_k)\| \leq c_0 + c_1\|\mathbf{x}_k\|^g$ for some positive contants c_0, c_1 and $g \in Z^+$.

Assumption 7: There exists a $\Theta_{\mathrm{x}} \in \Re^{q\times m}$ and a positive definite $\Omega \in \Re^{m\times m}$ such that $\Theta = \Theta_{\mathrm{x}}\Gamma_{\mathrm{n}}^\top$ and $\Gamma = \Gamma_{\mathrm{n}}\Omega$ where Θ is an augmented parameter matrix and Γ_{n} is a known nominal input gain matrix.

Consider now the sampled-data reference model

$$\begin{aligned}\mathbf{x}_{\mathrm{m},k+1} &= \Phi_{\mathrm{m}}\mathbf{x}_{\mathrm{m},k-p} + \Gamma_{\mathrm{m}}\mathbf{r}_{k-p} \\ \mathbf{y}_{\mathrm{m},k} &= C^\top\mathbf{x}_{\mathrm{m},k}\end{aligned} \tag{55}$$

where $\mathbf{x}_{\mathrm{m},k} \in \Re^n$ is the reference model state vector, $\mathbf{r}_k \in \Re^m$ is the reference vector, $\mathbf{y}_{\mathrm{m},k} \in \Re^m$ is the reference model output vector, $\Phi_{\mathrm{m}} \in \Re^{n\times n}$ is a known Hurwitz matrix and $\Gamma_{\mathrm{m}} \in \Re^{n\times m}$ is a known matrix. The control objective is to force the system (54) to follow the reference model (55).

Before proceeding with the controller design, consder the system (54). Using **Assumption 5** and the fact that $\|\boldsymbol{\delta}_k - 2\boldsymbol{\delta}_{k-1} + \boldsymbol{\delta}_{k-2}\| \in O(T^2)$, it is obtained that

$$\begin{aligned}\mathbf{x}_{k+1} = &\Phi^\top\boldsymbol{\xi}_k + \Gamma(C^\top\Gamma)^{-1}C^\top\Big(2\mathbf{x}_k - \mathbf{x}_{k-1} - \Phi^\top(2\boldsymbol{\xi}_{k-1} \\ &-\boldsymbol{\xi}_{k-2})\Big) + \Gamma\left(\mathbf{u}_{k-p} - 2\mathbf{u}_{k-p-1} + \mathbf{u}_{k-p-2}\right) + \Gamma\tilde{\boldsymbol{\delta}}_k\end{aligned} \tag{56}$$

where $\boldsymbol{\xi}_k \equiv \boldsymbol{\xi}(\mathbf{x}_k)$ and $\tilde{\boldsymbol{\delta}}_k = \boldsymbol{\delta}_k - 2\boldsymbol{\delta}_{k-1} + \boldsymbol{\delta}_{k-2}$. Consider the system (56), then using successive substitutions it is obtained that

$$\mathbf{x}_{k+p+1} = \Theta^\top\boldsymbol{\zeta}_k + \Gamma\mathbf{u}_k + \Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \Gamma\boldsymbol{\upsilon}_{k+p} \tag{57}$$

where Θ is the augmented parameter vector, $\boldsymbol{\zeta}_k$ is the augmented nonlinear function that is a function of the state $\mathbf{x}_k$ and control history $\mathbf{u}_{k-1}^\top, \cdots, \mathbf{u}_{k-p+1}^\top$. The terms $\Upsilon_k^\top\boldsymbol{\varphi}_k$ and $\boldsymbol{\upsilon}_k$ are the augmented disturbances due to successive substitution. The system (54) is now in a disturbance compensating form, (57), and this will allow the design of a controller that performs well in the presence of external disturbances.

Proceeding with the controller design, a p-steps ahead reference model (55) is subtracted from (57) and using **Assumption 7** results in the error dynamics of the form

$$\begin{aligned}\mathbf{e}_{k+p+1} = &\Phi_{\mathrm{m}}\mathbf{e}_k + \Gamma_{\mathrm{n}}\Theta_{\mathrm{x}}^\top\boldsymbol{\zeta}_k - \Gamma_{\mathrm{n}}\Theta_{\mathrm{m}}\mathbf{x}_k - \Gamma_{\mathrm{n}}\Theta_{\mathrm{r}}\mathbf{r}_k + \Gamma_{\mathrm{n}}\Omega\mathbf{u}_k \\ &+\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \boldsymbol{\upsilon}_{k+p}\end{aligned} \tag{58}$$

where $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}_{\mathrm{m},k}$ and $\Gamma_{\mathrm{m}} = \Gamma_{\mathrm{n}}\Theta_{\mathrm{r}}$ with $\Theta_{\mathrm{r}} \in \Re^{m\times m}$ a known constant matrix. Define $C_\gamma^\top = \left(C^\top\Gamma_{\mathrm{n}}\right)^{-1}C^\top$ and the state $\mathbf{z}_k \in \Re^m$ such that

$$\mathbf{z}_{k+p+1} = C_\gamma^\top(\mathbf{e}_{k+p+1} - \Phi_\mathrm{m} e_k). \quad (59)$$

Note that since it is assumed that $C^\top\Gamma$ and Ω are non-singular then $C^\top\Gamma_\mathrm{n}$ is non-singular. Substitution of (58) in (59), gives

$$\begin{aligned}\mathbf{z}_{k+p+1} &= \Theta_\mathrm{x}^\top\boldsymbol{\zeta}_k - \Theta_\mathrm{m}\mathbf{x}_k - \Theta_\mathrm{r}\mathbf{r}_k + \Omega\mathbf{u}_k \\ &\quad + C_\gamma^\top\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \bar{\boldsymbol{\upsilon}}_{k+p} \end{aligned} \quad (60)$$

where $\Gamma\boldsymbol{\upsilon}_k = \Gamma_\mathrm{n}\bar{\boldsymbol{\upsilon}}_k$. Similar to the scalar case, it can be shown that $\|\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1}\| \leq \kappa_0 + \kappa_1\max_{i\in[0,k]}\|\mathbf{x}_i\|^{(g^{p+1}-g)}$ where κ_0 and κ_1 are positive constants. To achieve stability, the controller is formulated as

$$\mathbf{u}_k = -\Omega^{-1}\left(\Theta_\mathrm{x}^\top\boldsymbol{\zeta}_k - \Theta_\mathrm{m}\mathbf{x}_k - \Theta_\mathrm{r}\mathbf{r}_k\right). \quad (61)$$

However, since the parameters Θ and Ω are assumed to be uncertain the controller is modified to the form

$$\mathbf{u}_k = -\hat{\Omega}_k^{-1}\left(\hat{\Theta}_{\mathrm{x},k}^\top\boldsymbol{\zeta}_k - \Theta_\mathrm{m}\mathbf{x}_k - \Theta_\mathrm{r}\mathbf{r}_k\right). \quad (62)$$

Substitution of (62) in (60) results in

$$\mathbf{z}_{k+p+1} = \tilde{\Theta}_{\mathrm{x},k}^\top\boldsymbol{\zeta}_k + \tilde{\Omega}_k\mathbf{u}_k + C_\gamma^\top\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \bar{\boldsymbol{\upsilon}}_{k+p} \quad (63)$$

where $\tilde{\Theta}_{\mathrm{x},k} = \Theta_\mathrm{x} - \hat{\Theta}_{\mathrm{x},k}$ and $\tilde{\Omega}_k = \Omega - \hat{\Omega}_k$. Rewriting (63) and delaying by p-time steps it is obtained that

$$\begin{aligned}\mathbf{z}_{k+1} &= \tilde{\Theta}_{\mathrm{x},k-p}^\top\boldsymbol{\zeta}_{k-p} + \tilde{\Omega}_k\mathbf{u}_{k-p} + C_\gamma^\top\Upsilon_{k-p-1}^\top\boldsymbol{\varphi}_{k-p-1} + \bar{\boldsymbol{\upsilon}}_k \\ &= \tilde{\Psi}_{k-p}^\top\bar{\boldsymbol{\zeta}}_{k-p} + C_\gamma^\top\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \bar{\boldsymbol{\upsilon}}_k \end{aligned} \quad (64)$$

where $\tilde{\Psi}_k^\top = \left[\tilde{\Theta}_{\mathrm{x},k}^\top \,\vdots\, \tilde{\Omega}_k\right] \in \Re^{m\times(q+m)}$ is the augmented parameter estimate error vector and $\bar{\boldsymbol{\zeta}}_k^\top = \left[\boldsymbol{\zeta}_k^\top \,\vdots\, \mathbf{u}_k^\top\right] \in \Re^{q+m}$ is the augmented vector of known functions. Using (64), it is possible to formulate the adaptation law as follows

$$\hat{\Psi}_{k+1} = \begin{cases}\hat{\Psi}_{k-p} + \alpha_{k+1}\beta_k\gamma_k P_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\mathbf{z}_{k+1}^\top & \forall k \in [k_0, \infty) \\ \hat{\Psi}_{k_0} & \forall k \in [0, k_0)\end{cases} \quad (65)$$

$$P_{k+1} = \begin{cases}P_{k-p} - \dfrac{\alpha_{k+1}\beta_k\gamma_k P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}}{1 + \alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}} & \forall k \in [k_0, \infty) \\ P_{k_0} > 0 & \forall k \in [0, k_0)\end{cases} \quad (66)$$

where $P_k \in \Re^{(q+m)\times(q+m)}$ is a symmetric positive-definite covariance matrix, α_k is a positive coefficient and $\beta_k > 0$ is a scalar coefficient used to prevent a singular $\hat{\Omega}_k$. The coefficient γ_k is defined similar to (18) and is given as

$$\gamma_k = \begin{cases}1 - \dfrac{(1+\alpha_{\max}d_0)\,\omega_k^2}{\|\mathbf{z}_{k+1}\|^2}, & \text{if } \|\mathbf{z}_{k+1}\| \geq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k \\ 0, & \text{if } \|\mathbf{z}_{k+1}\| < (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k\end{cases} \quad (67)$$

where $\omega_k = \|C_\gamma^\top\|\left(\kappa_0 + \kappa_1\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g)}\right)$, $d_0 \geq \bar{\boldsymbol{\zeta}}_k^\top P_k\bar{\boldsymbol{\zeta}}_k$ and $\alpha_{\max} \geq \alpha_k$.

Remark 6: Similar to the scalar case, if β_k is selected such that β_k^{-1} is not an eigenvalue of $-\alpha_{k+1}\gamma_k\hat{\Omega}_{k-p}^{-1}SP_{k+1}\bar{\boldsymbol{\zeta}}_{k-p}\mathbf{z}_{k+1}^\top$, where $S = [0\cdots 0\ C_\gamma^\top]$, then it is guaranteed that $\hat{\Omega}_{k+1}$ will never be singular.

Stability of (64) is summarized in the following theorem:

Theorem 2: The closed loop system (64) with adaptive laws (65) and (66), is stable if and only if

$$\|\Phi_\mathrm{m}\| + \kappa_1\left(1 + 2\|C_\gamma^\top\|\|\Gamma_\mathrm{n}\|\right) \times (1+\alpha_{\max}d_0)^{\frac{1}{2}}\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g-1)} < 1.$$

Furthermore, the tracking error, $\|\mathbf{e}_k\| = \|\mathbf{x}_k - \mathbf{x}_{\mathrm{m},k}\|$, converges asymptotically to a bound ϵ.

Proof: Consider the positive function

$$V_k = \sum_{j=1}^{m}\sum_{i=0}^{p}\left(\tilde{\boldsymbol{\psi}}_{j,k-i}^\top P_{k-i}^{-1}\tilde{\boldsymbol{\psi}}_{j,k-i}\right) \quad (68)$$

where $\tilde{\Psi}_k^\top = \left[\tilde{\boldsymbol{\psi}}_{1,k}^\top \,\vdots\, \tilde{\boldsymbol{\psi}}_{2,k}^\top \,\vdots\, \cdots \,\vdots\, \tilde{\boldsymbol{\psi}}_{m,k}^\top\right]$. The difference between two time steps is

$$\Delta V_k = \sum_{j=1}^{m}\left(\tilde{\boldsymbol{\psi}}_{j,k+1}^\top P_{k+1}^{-1}\tilde{\boldsymbol{\psi}}_{j,k+1} - \tilde{\boldsymbol{\psi}}_{j,k-p}^\top P_{k-p}^{-1}\tilde{\boldsymbol{\psi}}_{j,k-p}\right). \quad (69)$$

Substitution of (65) and (66) into (69) and following the same procedures as in **Lemma 1** it is obtained that

$$\Delta V_k = -\frac{\alpha_{k+1}\beta_k\gamma_k^2}{1 + \alpha_{k+1}\beta_k\gamma_k\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}}\mathbf{z}_{k+1}^\top\mathbf{z}_{k+1} \leq 0 \quad (70)$$

which is true for $\|\mathbf{z}_{k+1}\| \geq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$. To guarantee that $\lim_{k\to 0}\|\mathbf{z}_{k+1}\| \leq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$ it must be guaranteed that $\|\bar{\boldsymbol{\zeta}}_{k-p}^\top P_{k-p}\bar{\boldsymbol{\zeta}}_{k-p}\|$ is bounded. Using the results in **Lemma 2**, **Lemma 3** and **Theorem 1** it can be concluded that $\lim_{k\to\infty}\|\mathbf{z}_{k+1}\| \leq (1+\alpha_{\max}d_0)^{\frac{1}{2}}\,\omega_k$.

Consider the system (58), then substitution of the control law (62) gives

$$\begin{aligned}\mathbf{e}_{k+p+1} &= \Phi_\mathrm{m}\mathbf{e}_k + \Gamma_\mathrm{n}\Theta_\mathrm{x}^\top\boldsymbol{\zeta}_k - \Gamma_\mathrm{n}\Theta_\mathrm{m}\mathbf{x}_k - \Gamma_\mathrm{n}\Theta_\mathrm{r}r_k + \Gamma_\mathrm{n}\Omega\mathbf{u}_k \\ &\quad + \Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} + \Gamma_\mathrm{n}\bar{\boldsymbol{\upsilon}}_{k+p} \\ &= \Phi_\mathrm{m}\mathbf{e}_k + \Gamma_\mathrm{n}\mathbf{z}_{k+p+1} + \left(I - \Gamma_\mathrm{n}C_\gamma^\top\right)\Upsilon_{k-1}^\top\boldsymbol{\varphi}_{k-1} \end{aligned} \quad (71)$$

and a p time-steps delayed (71) satisfies

$$\begin{aligned}\|\mathbf{e}_{k+1}\| &\leq \left(\|\Phi_\mathrm{m}\| + \kappa_1\left(1 + 2\|C_\gamma^\top\|\|\Gamma_\mathrm{n}\|\right)(1+\alpha_{\max}d_0)^{\frac{1}{2}} \right.\\ &\quad \left.\times\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g-1)}\right)\|\mathbf{e}_{k-p}\| + \kappa_1\left(1+2\|C_\gamma^\top\|\|\Gamma_\mathrm{n}\|\right) \\ &\quad \times(1+\alpha_{\max}d_0)^{\frac{1}{2}}\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g-1)}\|\mathbf{r}_{k-p}\| \\ &\quad + \kappa_0\left(1 + 2\|C_\gamma^\top\|\|\Gamma_\mathrm{n}\|\right). \end{aligned} \quad (72)$$

Consider that κ_1 is small enough with $\mathbf{x}_k$ lying in a neighborhood such that $\|\Phi_\mathrm{m}\| + \kappa_1\left(1 + 2\|C_\gamma^\top\|\|\Gamma_\mathrm{n}\|\right) \times (1+\alpha_{\max}d_0)^{\frac{1}{2}}\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g-1)} < \|\bar{\Phi}_\mathrm{m}\| < 1$ and that $\kappa_1\left(1+2\|C_\gamma^\top\|\|\Gamma_{\rm n}\|\right)(1+\alpha_{\max}d_0)^{\frac{1}{2}}\|\mathbf{x}_{k-p}\|^{(g^{p+1}-g-1)}$ $\|\mathbf{r}_{k-p}\|+\kappa_0\left(1+2\|C_\gamma^\top\|\|\Gamma_{\rm n}\|\right)<\Delta_{\max}$ for some hurwitz $\|\bar{\Phi}_{\rm m}\|$ and a positive $\Delta_{\max}$, then the expression (72) satisfies

$$\|\mathbf{e}_k\|\leq\|\bar{\Phi}_{\rm m}\|^{\lfloor\frac{k-p}{2}\rfloor}\|\mathbf{e}_0\|+\sum_{i=1}^{\lfloor\frac{k-p}{2}\rfloor}\|\bar{\Phi}_{\rm m}\|^i\Delta_{\max}. \tag{73}$$

Since $\bar{\Phi}_{\rm m}$ is Hurwitz then $\lim_{k\to\infty}\|\bar{\Phi}_{\rm m}\|^{\lfloor\frac{k-p}{2}\rfloor}\|\mathbf{e}_0\| = 0$ and $\lim_{k\to\infty}\sum_{i=1}^{\lfloor\frac{k-p}{2}\rfloor}\|\bar{\Phi}_{\rm m}\|^i\Delta_{\max} = \epsilon$ for some positive constant ϵ. Therefore,

$$\lim_{k\to\infty}\|\mathbf{e}_k\|\leq\epsilon \tag{74}$$

which establishes the boundedness of $\|\mathbf{e}_k\|$. ■

IV. SIMULATION EXAMPLE

In this section, a scalar system and a multivariable system will be used to demonstrate the performance of the controller. The scalar system example will compare the proposed approach to that in [2].

A. SCALAR SYSTEM WITHOUT TIME-DELAY

Consider the nonlinear discrete-time system presented in [2]

$$x_{k+1}=-3\,x_k^2+u_k+\sin\left(\frac{k}{50}\pi\right) \tag{75}$$

with $x_0=0$. To attenuate the influence of the disturbance the system is written in the form

$$\begin{aligned}x_{k+1}&=2\,x_k-x_{k-1}-3\left(x_k^2-2\,x_{k-1}^2+x_{k-2}^2\right)\\&\quad+u_k-2\,u_{k-1}+u_{k-2}+\upsilon_k.\end{aligned} \tag{76}$$

The design objective is to track the reference model

$$x_{{\rm m},k+1}=0.9\,x_{{\rm m},k}+0.25\,r_k \tag{77}$$

where $r_k=1$. Using (76) and (77), the control law is derived as

$$\begin{aligned}u_k&=2\,u_{k-1}-u_{k-2}+0.9\,x_k+0.25\,r_k-2\,x_k+x_{k-1}\\&\quad-\hat{\phi}_k\left(x_k^2-2\,x_{k-1}^2+x_{k-2}^2\right).\end{aligned} \tag{78}$$

As in [2], the parameter uncertainty is assumed to be 90% and the initial value of the adaptive parameter is selected as $\hat{\phi}_0=-0.3$. After a number of trials, the remaining parameters are selected as $P_0=100$, $\alpha_k=1$ and $d_0\alpha_{\max}=0.1$. The value $P_0=100$ is the same as in [2]. The system is simulated using both control approaches and the tracking performance is shown in Fig.1. From the results, it can be seen that both approaches result in stable performance, however, the approach proposed in this work can attenuate the effects of external disturbances leading to better tracking performance. In Fig.2 the parameter convergence for both approaches is shown. It can be seen that in both approaches the adaptive parameter converges to the true value which is to be expected for the case of a single uncertain parameter.

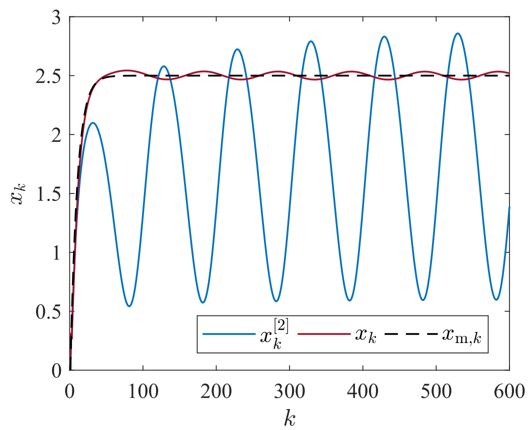

FIGURE 1. Tracking performance of the proposed controller and the approach in [2].

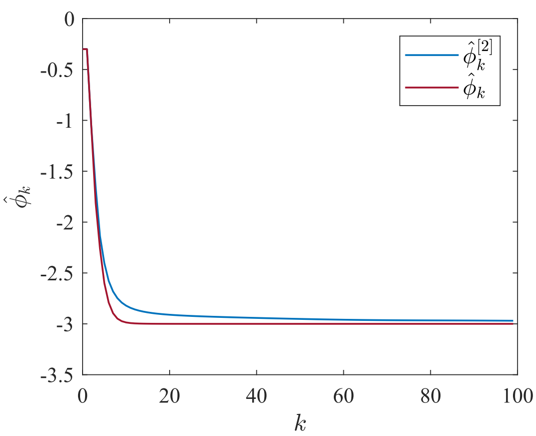

FIGURE 2. Parameter convergence of the proposed controller and the approach in [2].

B. SCALAR SYSTEM WITH TIME-DELAY

Consider the system (75) with a control input time-delay of $p=1$ given as

$$x_{k+1}=-3\,x_k^2+u_{k-1}+\sin\left(\frac{k}{50}\pi\right). \tag{79}$$

Using successive substitutions, it is obtained that

$$\begin{aligned}x_{k+1}&=-27\,x_{k-1}^4-3\,u_{k-2}^2+18\,x_{k-1}^2u_{k-2}+u_{k-1}\\&\quad+18\delta_{k-1}x_{k-1}^2-6\delta_{k-1}u_{k-2}-3\delta_{k-1}^2+\delta_k\\&=\boldsymbol{\phi}^\top\boldsymbol{\xi}_{k-1}+u_{k-1}+\boldsymbol{\rho}_{k-1}^\top\boldsymbol{\varphi}_{k-1}+\upsilon_k\end{aligned} \tag{80}$$

where $\boldsymbol{\phi}^\top=[-27|-3|18]$, $\boldsymbol{\xi}_k^\top=\left[x_k^4|u_{k-1}^2|x_k^2u_{k-1}\right]$ and $\delta_k=\sin\left(\frac{k}{50}\pi\right)$. The terms $\boldsymbol{\rho}_{k-1}^\top\boldsymbol{\varphi}_{k-1}$ and υ_k are the augmented disturbance terms as a result of successive substitutions. The system (80) is now written in the disturbance attenuating form as

$$\begin{aligned}x_{k+1}&=2\,x_k-x_{k-1}+\boldsymbol{\phi}^\top\left(\boldsymbol{\xi}_{k-1}-2\boldsymbol{\xi}_{k-2}+\boldsymbol{\xi}_{k-3}\right)+u_{k-1}\\&\quad-2u_{k-2}+u_{k-3}+\bar{\boldsymbol{\rho}}_{k-1}^\top\bar{\boldsymbol{\varphi}}_{k-1}+\bar{\upsilon}_k.\end{aligned} \tag{81}$$

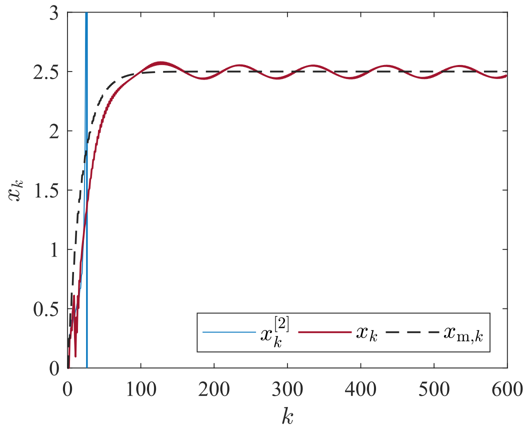

FIGURE 3. Tracking performance of the proposed controller and the approach in [2].

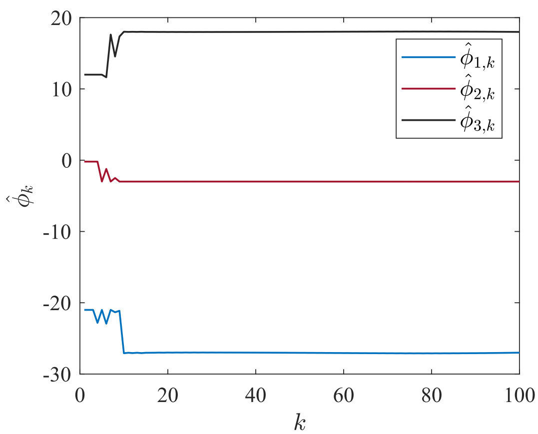

FIGURE 4. Parameter convergence of the proposed controller.

The goal is for the system (81) to track reference model given as

$$x_{m,k+1} = 0.9\, x_{m,k-1} + 0.25\, r_{k-1} \tag{82}$$

resulting in the control law of the form

$$u_k = 2\, u_{k-1} - u_{k-2} + 0.9\, x_k + 0.25\, r_k - 2\, x_k + x_{k-1} - \hat{\boldsymbol{\phi}}_k^\top \left(\boldsymbol{\xi}_k - 2\boldsymbol{\xi}_{k-1} + \boldsymbol{\xi}_{k-2}\right). \tag{83}$$

After a number of trials, the controller parameters are selected as $\hat{\boldsymbol{\phi}}_0^\top = [-21|0|12]$, $P_0 = 200\, I$, $\alpha_k = 70$ and $d_0\alpha_{\max} = 0.2$. For the approach in [2], the parameters are similarly selected as $\hat{\boldsymbol{\phi}}_0^\top = [-21|0|12]$ and $P_0 = 200\, I$. The system is simulated using both control approaches and the tracking performance is shown in Fig.3. The results show that the proposed approach can produce stable performance while minimizing the effects of the external disturbance. On the otherhand, the approach in [2] is unable to handle the effects of the time-delay coupled with the external disturbance and results in unstable performance. In Fig.4 the convergence of the adaptive parameters is shown while Fig.5 shows the non-linear growth of $\boldsymbol{\xi}_k$ with respect to $|x_k|$. As it can be seen from the results, the proposed approach guarantees convergence

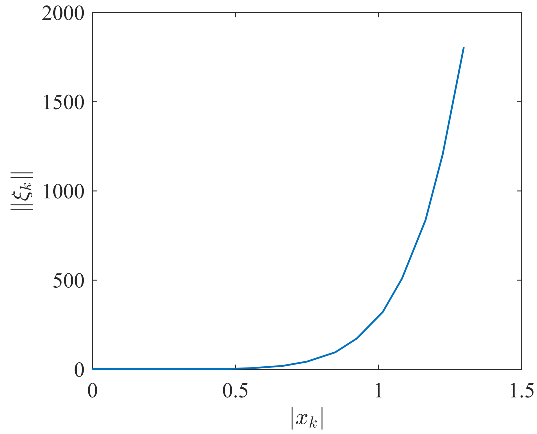

FIGURE 5. Growth of $\|\boldsymbol{\xi}_k\|$ w.r.t $|x_k|$.

even though the nonlinear function does not satisfy the sector bound condition that is required for the classical discrete-time adaptive control approach.

C. MULTIVARIABLE SYSTEMS

Consider a nonlinear discrete-time system with matched disturbance of the form

$$\begin{aligned}\mathbf{x}_{k+1} &= \begin{bmatrix} 1.5 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \begin{bmatrix} x_{2,k}^2 \\ x_{3,k} \\ x_{1,k} \end{bmatrix} + \begin{bmatrix} 1.2 & 0 \\ 0 & 1.3 \\ 0 & 0 \end{bmatrix} \mathbf{u}_{k-p} \\ &\quad + \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix} \sin\left(\frac{k}{50}\pi\right) \\ \mathbf{y}_{k+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}_k \end{aligned} \tag{84}$$

with delay $p = 4$. The reference model is selected as

$$\begin{aligned}\mathbf{x}_{m,k+1} &= \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}_{m,k-p} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{r}_{k-p} \\ \mathbf{y}_{m,k+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}_{m,k}. \end{aligned} \tag{85}$$

The reference is selected as $\mathbf{r}_k = [0.15| - 0.15]^\top$. The nominal gain matrix and controller parameters are set as

$$\Gamma_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Theta_m = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \\ 0 & 0 \end{bmatrix}^\top, \quad \hat{\Omega}_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$d_0\alpha_{\max} = 0.2$, $\alpha_k = 10$ and $P_0 = 5 \times 10^4 I_{16\times16}$. The matrix $\hat{\Theta}_{x,0} \in \Re^{14\times2}$ is computed using a nominal $\Phi = I_{3\times3}$. The system is simulated and the results can be seen in Fig.6. It can be seen that the system output converges to the desired trajectory. To demonstrate the ability to handle unknown control directions, the matrix $\hat{\Omega}_0$ is set as $\hat{\Omega}_0 = -0.5 I_{2\times2}$ while some of the controller parameters are retuned as $\alpha_k = 20$ and $P_0 = 1 \times 10^2 I_{16\times16}$. It can be seen from Fig.7 that the system output converges to the desired trajectory. In Fig.8 the

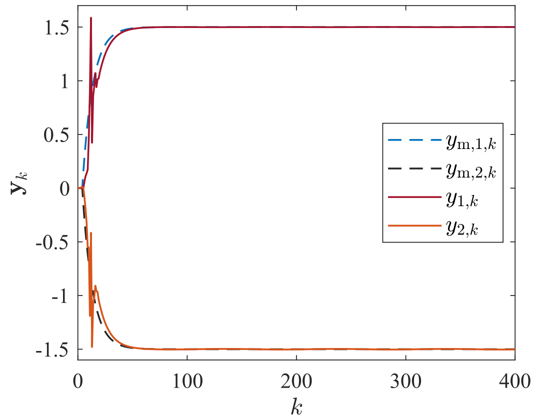

FIGURE 6. Tracking performance of the controller for a multivariable system.

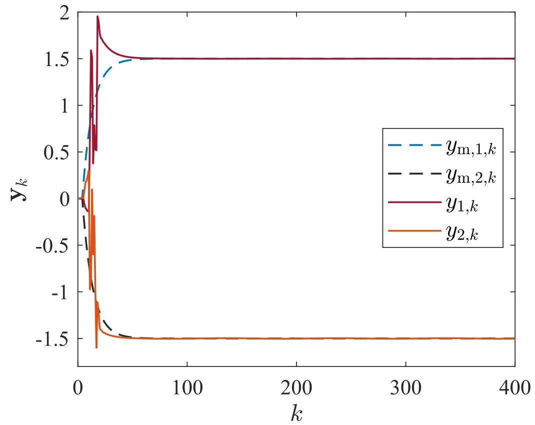

FIGURE 7. Tracking performance of the controller for a multivariable system.

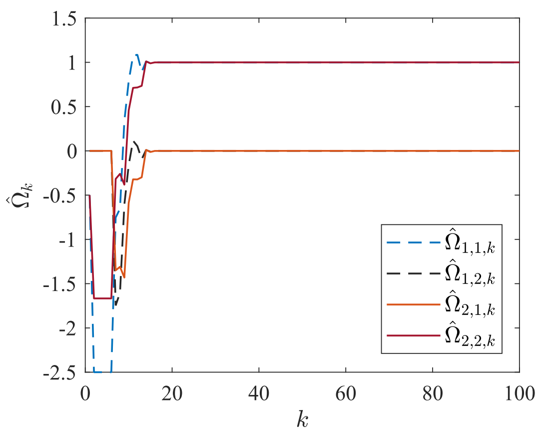

FIGURE 8. Elements of the matrix $\hat{\Omega}_k$.

convergence of the elements of $\hat{\Omega}_k$ is shown. It can be seen in the results that the adaptive law is capable of correcting $\hat{\Omega}_k$ to match the actual system control direction.

Finally, the system is simulated with different values of the input delay p using the controller parameters that led to the results in Fig.7. The average of the norm of the tracking error

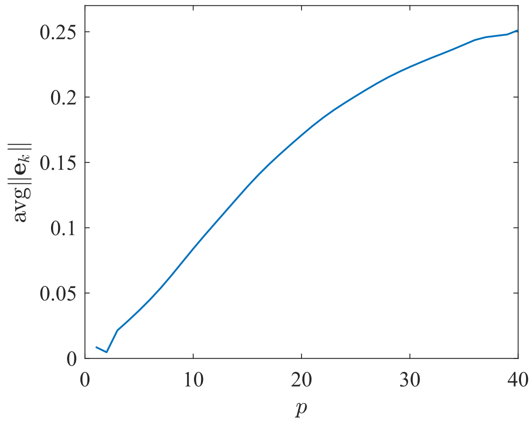

FIGURE 9. avg$\|\mathbf{e}_k\|$ for different values of the input delay p.

avg$\|\mathbf{e}_k\|$ is computed over an interval of 1000 steps and plotted in Fig.9. The results show that the tracking performance may degrade with the increase in input delay p. This is due to the fact that the delay p influences the transient performance of the system (see **Remark 1**).

V. CONCLUSION

In this paper, a discrete-time adaptive controller for nonlinear systems with non-sector bounded nonlinearities is proposed. Although numerous approaches have been proposed in other works that can handle non-sector bounded nonlinearities, stability proofs are shown only for single parameter adaptive laws. This paper presents stability proofs for systems with multiple parameters while at the same time demonstrating the difficulty of addressing systems with non-sector bounded nonlinearities. Simulation results are given to demonstrate the effectiveness of the controller for a nonlinear system.

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