Received July 30, 2018, accepted October 3, 2018, date of publication November 1, 2018, date of current version March 8, 2019.

Digital Object Identifier 10.1109/ACCESS.2018.2877141

Synchronization of Interconnected Discontinuous Neural Networks With Nonlinear Coupling Functions

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This work was supported in part by the National Natural Science Foundation of China under Grant 11601143 and Grant 1573003, in part by a Key Project Supported by the Scientific Research Fund of the Hunan Provincial Education Department under Grant 15k026 and Grant 15A038, in part by the China Postdoctoral Science Foundation under Grant 2018M632207, and in part by the China Scholarship Council and NSERC Canada.

ABSTRACT Some criteria on exponential synchronization of multiple identical discontinuous neural networks with time delay are given in this paper. Different from the available results, the nodes of complex networks are coupled via nonlinear discontinuous functions. First, we study the existence of Filippov solution of the nonlinear coupled discontinuous neural networks (NCDNNs) based on differential inclusion theory. In addition, we derive some exponential synchronization criteria of the NCDNNs. Finally, we design a pinning impulsive controller to realize the exponential synchronization of NCDNNs. Numerical examples are presented to illustrate the obtained results.

INDEX TERMS Exponential synchronization, nonlinear coupled neural networks, discontinuous activations, pinning impulsive controller.

I. INTRODUCTION

As one of the most important features of collective behavior, synchronization has received great attention due to its wide applications in many fields [1]–[3] since Pecora and Carroll [4] proposed a driving response law for synchronization of two identical chaotic systems. Many results on synchronization of complex networks with different coupling structures have been obtained such as constant coupling, time varying coupling, impulsive coupling and time delay coupling [5]–[8]. Moreover, the identical nodes are interconnected via a linear diffusive function in these papers. Under this case, the state of the identical nodes must be detectable. However, in various cases, the states of the neuron are unknown. Instead, we can only observe an nonlinear function of the state of the uncoupled neuron, which represent the coupling functions of the coupled complex networks. Therefore, we need to study the synchronization properties of the interconnected neuron via these nonlinear coupling functions. Recently, synchronization on nonlinear coupled complex networks has become a popular topic of research [9]–[11]. Liu and Chen [9] studied synchronization for complex networks with nonlinear coupling functions, which need to satisfy three strict properties. Yang *et al.* [10] and Guo *et al.* [11] studied the synchronization of multiple memristive neural networks via nonlinear coupling function, which is composed by a discontinuous "sign" term and a liner diffusive term. However, the neuron functions of the complex networks are required to be continuous in these papers. According to what we know, almost no results on synchronization of NCDNNs have existed in the literature.

Discontinuous neural networks have received much attention since Mauro Forti first proposed the stability criteria under the framework of the differential inclusion [12]. However, most of these results are concerned with the stability of equilibria and periodic solutions [13]–[20]. Recently, the analysis on synchronization of discontinuous drive-response systems has become a hot research issue [21]–[23]. As well as we know, the robust state estimation problem of uncertain discontinuous neural networks were studied in [21] according to differential inclusion theory. Later, qusi-synchronization of discontinuous complex networks were investigated. In addition, the linear feedback controller were out of work to realize the complete synchronization because of the discontinuities of the neuron functions [22]. Subsequently, the exponential periodic synchronization for discontinuous drive-response systems was studied in [23].

Recently, synchronization on multiple discontinuous neural networks have published in many journals [24]–[30]. Different kinds of control strategies, such as sate feedback control strategy, adaptive control strategy and impulsive control strategy, were used to guarantee the synchronization of discontinuous interconnected networks in [24], [25], and [29], respectively. Later, general pinning controllers were designed to ensure the complete synchronization of discontinuous complex networks with linear coupling functions. Soon afterwards, finite-time synchronization of discontinuous complex networks were studied via state feedback control strategy and impulsive control strategy in [27] and [28]. Liu *et al.* [30] studied the synchronization of discontinuous complex networks in terms of LMIs, which is difficult to verify since the integral inequalities contains the information of output solution. Moreover, the author studied the finite-time synchronization of uncertain discontinuous neural networks with time delay in [31]. In conclusion, all of these results exhibit the following shortcomings: (1) there exists a ''sign'' term in the designed controller to eliminate the effects of discontinuous activations, which may lead to chattering problem if the control gain is not well designed;(2) though several papers studied the existence of the Filippov solution for discontinuous neural networks, there's almost no results on the existence of Filippov solution of discontinuous complex networks; (3) in the aforementioned papers, the coupling functions of the complex networks are linearly coupled with undirected topology or directed topology. In this paper, we use a new method to overcome the effects brought by the discontinuous neuron function, which is designing the discontinuous coupling function with a ''sign'' term.

In the past decade, many control strategies were presented to realize the synchronization of complex networks [9], [22], [25]–[27], [29], [32]–[34], such as intermittent control [9], state feedback control [22], [27], [32], adaptive control [25], impulsive control [29], event-triggered control [33], and pinning control [26], [34]. Among these methods, pinning control method is an efficient way to control large networks since the pinning controllers act on fraction of the coupled neurons. Moreover, Impulsive control method is a low-cost way because the action only applied at finitely discrete time steps. Combined the advantages of these two methods, pinning-impulsive control strategy is proposed to study the synchronization of complex networks in [35] and [36]. Pinning impulsive controllers, which only depends on the neighboring state information, was first proposed in [35] to investigate the synchronization. In [36], impulsive controllers are designed to applied partial of nodes to realize the synchronization of multiple memristive neural networks. Moreover, the author proposed a method to select the pinned nodes of the complex networks to ensure synchronization via pinning impulsive control method. Furthermore, in [38] present some results on complex networks with hybrid switching and impulsive effects. In this paper, we design a pinning impulsive controller by using the partial local state information to realize the exponential synchronization of NCDNNs.

In the following, we will focus on the following four problems on discontinuous complex networks based on differential inclusion theory.

1) (Existence of solution for discontinuous complex network): Some papers have studied the solution existence in the Fillipov sense of discontinuous neural networks [24], [28], [29]. However, almost no papers has studied the Filippiv solution existence of discontinuous complex networks. So the solution existence of discontinuous complex networks need further studied under the framework of Filippov.
2) (Synchronization condition): A small quantity of papers concerned on complex networks with nonlinear coupling functions, where the coupling functions are continuous. In this paper, both the neuron functions and the coupling functions are discontinuous. It is obvious that both of their discontinuity can lead to chattering problem. Can we find sufficient conditions such that the oscillations caused by these two discontinuous function can be offset against each other?
3) (Conditions of synchronization) Can we find sufficient conditions for discontinuous complex networks with nonlinear coupling function to realized exponential synchronization?
4) (Pinning impulsive control) As we know, the synchronization condition always contains a balance condition on the parameters on the identical node and the coupling matrix. Therefore, sometimes it is not easy to satisfy these conditions. Pinning impulsive control method is an efficient and low-cost way to realize the synchronization. Can we used the pinning impulsive control strategy to realize synchronization for NCDNNs?

The main contribution of this paper are aimed to solve the above proposed problems under the framework of Filippov. In section II, we introduce the nonlinear coupled complex networks with discontinuous neural networks and some preliminaries. Furthermore, we study the existence of Filippov solution of NCDNNs under the framework of Filippov. Section III proposes conditions to ensure the exponential synchronization of NCDNNs. In section IV, a pinning impulsive control law is developed and some sufficient conditions for ascertaining globally exponential synchronization are derived. In section V, two numerical simulations are given to show the correctness of the theoretical results and the availability of the designed controller. Finally, we make some concluding remarks in Section VI.

II. MODEL DESCRIPTION AND SOME PRELIMINARIES

The nonlinear coupled discontinuous neural networks are described as follows:

$$\dot{x}_i(t) = -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I + \sum_{j=1}^{N} g_{ij}\Gamma\phi(x_j(t), x_i(t)), \quad i = 1, 2, \cdots, N, \quad (1)$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^{\mathrm{T}} \in R^n$ denotes the ith neuron state, $D = \operatorname{diag}\{d_1, d_2, \cdots, d_n\}$ is a constant matrix with $d_i > 0$; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are connection weight matrix; $I = (I_1, I_2, \cdots, I_n)^{\mathrm{T}}$ is the neuron input vector and $\tau \geq 0$ means the time delay; $f(x_i) = (f_1(x_{i1}), f_2(x_{i2}), \cdots, f_n(x_{in}))^{\mathrm{T}}$ denotes the discontinuous neuron function satisfying the following two assumptions, which are proposed in [12] and [36], respectively.

($\mathscr{A}1$) The neuron function f_i is discontinuous at a number of isolate points ρ_k^i, where both the right limit $f_i^+(\rho_k^i)$ and left limit $f_i^-(\rho_k^i)$ exist. Moreover, on any compact subinterval of R, f_i has at most finite discontinuous points.

($\mathscr{A}2$) For $\forall u, v \in R$, there are two nonnegative constant K_i and N_i satisfying $\sup|\xi_i - \eta_i| \leq K_i|u - v| + N_i$, where $\xi_i \in K[f_i(u)]$, $\eta_i \in K[f_i(v)]$, and $K[f_i(x)] = [\min\{f_i^-(x), f_i^+(x)\}, \max\{f_i^-(x), f_i^+(x)\}]$.

Furthermore, $G = [g_{ij}]_{N\times N}$ means the outer-coupling matrix and $\Gamma = \operatorname{diag}\{r_1, r_2, \cdots, r_n\}$ with $r_i > 0$ denotes the inner coupling matrix. The nonlinear coupling function $\phi : R^n \mapsto R^n$ is defined as $\phi(x_j, x_i) = x_j - x_i + \operatorname{sign}(x_j - x_i)$, where $\operatorname{sign}(x_j, x_i) = (\operatorname{sign}(x_{j1} - x_{i1}), \operatorname{sign}(x_{j2} - x_{i2}), \cdots, \operatorname{sign}(x_{jn} - x_{in}))^{\mathrm{T}}$.

Remark 1: In many papers [9], [24], [26], [27], [30], [34], the outer-coupling matrix is assumed to be symmetric, which means that the coupling topology is an undirected graph. In this paper, G is not required to be symmetric. That is to say the coupling topology can be either a directed graph or an undirected graph.

Remark 2: Different from many synchronization results on linearly coupled discontinuous neural networks [24]–[29], we study the synchronization of nonlinear coupled networks with discontinuous activation functions. The nonlinear coupling function ϕ is composed by a diffusive linear term and a discontinuous "sign" term proposed in [10] and [11], which is used to ensure the synchronization of coupled memristive neural networks.

Remark 3: Based on the results shown in [9], there are three main improvements as follows: (1) The neuron function of the neural networks can be discontinuous, which is more general in real applications. (2) We drop the assumption of local Lipschitz continuity of the nonlinear coupling function. (3) We remove the following condition of nonlinear coupling function: there is a constant $\alpha > 0$ satisfying $(x-y)\phi(y, x) \leq -\alpha(x - y)^2$ for $x \neq y$. Therefore, the nonlinear coupling function can be more general than that in [9].

Define Laplacian matrix $L \in R^{N\times N}$ corresponding to G by $l_{ij} = -g_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{k=1,k\neq i}^{N} g_{ik}$ for $i, j \in \{1, \cdots, N\}$. Given a vector $x \in R^N$, it is easy to get $(Lx)_i = \sum_{j=1}^{N} g_{ij}(x_i - x_j)$. For convenience, we define $X = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \cdots, x_N^{\mathrm{T}}))^{\mathrm{T}}, f(X) = (f(x_1), f(x_2), \cdots, f(x_N)^{\mathrm{T}}$, $\mathbf{D} = I_N \otimes D, \mathbf{A} = I_N \otimes A, \mathbf{B} = I_N \otimes B, \mathbf{I} = (I^{\mathrm{T}}, I^{\mathrm{T}}, \cdots, I^{\mathrm{T}})^{\mathrm{T}}$, $\mathbf{L} = L \otimes \Gamma$. Then the nonlinear coupled discontinuous neural networks (1) is expressed in a matrix form as follows:

$$\dot{X} = -\mathbf{D}X + \mathbf{A}f(X) + \mathbf{B}f(X^\tau) + \mathbf{I} - \mathbf{L}X + \Psi, \qquad (2)$$

where $X^\tau = X(t - \tau)$, $\Psi = (\Psi_1^{\mathrm{T}}, \Psi_2^{\mathrm{T}}, \cdots, \Psi_N^{\mathrm{T}})^{\mathrm{T}}$ with $\Psi_k = \sum_{j=1}^{N} g_{1j}\Gamma\operatorname{sign}(x_j - x_k)$. Let $\mathcal{C}([-\tau, 0], R^n)$ is a Banach space form $[-\tau, 0]$ to R^n, we can give the initial condition by $X = \varphi : [-\tau, 0] \mapsto R^n)$, where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)^{\mathrm{T}}$ and $\varphi_i \in \mathcal{C}([-\tau, 0], R^n)$.

Since the activation function and the coupling function in (2) are discontinuous, the existence of the conventional solutions might be lost. For investigating the synchronization of the multiple interconnected works (1), we construct the corresponding differential inclusion via Filippov regularization [39], and define the Fillippov solution for system (2). Now, we give some basic definitions and lemmas, firstly.

Definition 1 [15], [23]: For a discontinuous function $f : \mathcal{C}([-\tau, 0]) \mapsto R^n$, we define the corresponding Filippov set-valued map as follows:

$$F(x_t) = \bigcap_{\delta>0} \bigcap_{\mu(\mathbb{N})=0} K[f(B(x_t, \delta)\backslash\mathbb{N})].$$

Here, $f(x_t)$ is measurable and essentially locally bounded, $B(x_t, \delta) = \{z_t^* \in C| \|x_t^* - x_t\| \leq \delta\}$, $K(\cdot)$ is the closure of the convex hull of set $(\cdot)$ and $\mu(\cdot)$ means Lebesgue measure of $(\cdot)$.

If a function x is absolutely continuous on every subset of $[0, T)$ and satisfies $\dot{x} \in F(x_t)$ for $t \in [0, T)$, then x is defined as a Filippov solution of $\dot{x} = f(x_t)$. We can select a measurable function $\zeta \in F(x_t)$ satisfying $\dot{x} = \zeta$ for $t \in [0, T)$ based on the measurable selection theorem in [40]. From the above analysis and Definition 1, we can give the solution of system (2) in sense of Filippov.

Definition 2: A continuous function $X : [-\tau, T) \to R^{Nn}$ is called a solution of system (2) in sense of Filippov if

(i) X is absolutely continuous on any compact subset of $[0, T)$.

(ii) $X : [-\tau, T) \mapsto R^{Nn}$ satisfies

$$\begin{aligned}\dot{X} &\in -\mathbf{D}X + \mathbf{A}F(X) + \mathbf{B}F(X^\tau) + \mathbf{I} - \mathbf{L}X + \bar{\Psi} \\ &\triangleq F(X, X^\tau), \qquad t \in [0, T),\end{aligned} \qquad (3)$$

where $F(X) = [F^{\mathrm{T}}(x_1), \cdots, F^{\mathrm{T}}(x_N)]^{\mathrm{T}}$, $F(x_k) = [F(x_{k1}), \cdots, F(x_{kn})] \in 2^{R^n}$ and $\bar{\Psi} = [\bar{\Psi}_1^{\mathrm{T}}, \cdots, \bar{\Psi}_N^{\mathrm{T}}]^{\mathrm{T}}$, $\bar{\Psi}_k = \sum_{j=1}^{N} g_{kj}\Gamma F(\operatorname{sign}(x_j - x_k)) \in 2^{R^n}$.

Definition 3 (VIP): For a continuous function $\varphi(s)$ and two measurable selections $\bar{\varphi}(s)$, $\bar{\bar{\varphi}}(s)$ satisfying $\bar{\varphi}(s) \in F(\varphi(s))$, $\bar{\bar{\varphi}}(s) \in ((\sum_{j=1}^{N} g_{1j}\Gamma F(\operatorname{sign}(\varphi_j(s) - \varphi_1(s))))^{\mathrm{T}}, \cdots, (\sum_{j=1}^{N} g_{1j}\Gamma F(\operatorname{sign}(\varphi_j(s) - \varphi_N(s))))^{\mathrm{T}})^{\mathrm{T}}$ for almost every $s \in [-\tau, 0]$, the initial value problem related to (2) with $\varphi, \bar{\varphi}, \bar{\bar{\varphi}}$ can be described as follows: find coupled functions $[X, \zeta, \upsilon] : [0, T) \mapsto R^{Nn} \times R^{Nn} \times R^{Nn}$ such that X

is a Filippov solution of (3) on $[-\tau, T)$ and

$$\begin{cases} \dot{X} = -\mathbf{D}X + \mathbf{A}\zeta + \mathbf{B}\zeta^{\tau} + \mathbf{I} - \mathbf{L}X + \upsilon, \\ \qquad\qquad \text{for a.a.} t \in [0, T) \\ \zeta \in F(X), \quad \upsilon \in \bar{\Psi}, \ \text{for a.a.} t \in [0, T) \\ x(s) = \varphi(s), \quad s \in [-\tau, 0] \\ \zeta(s) = \bar{\varphi}(s), \quad \upsilon(s) = \bar{\bar{\varphi}}(s), \ \text{for a.a.} s \in [-\tau, 0] \end{cases} \tag{4}$$

i.e.

$$\dot{x}_i = -Dx_i + A\zeta_i + B\zeta_i^{\tau} + I + \sum_{j=1}^{N} g_{ij}\Gamma x_j + \upsilon_i, \tag{5}$$

where $\upsilon_i = \sum_{j=1}^{N} g_{ij}\Gamma\upsilon_{ji}$, $\upsilon_{ji} = (\upsilon_{ji,1}, \upsilon_{ji,2}, \cdots, \upsilon_{ji,n})^{\mathrm{T}}$, $\upsilon_{ji,p} \in F(\text{sign}(x_{jp} - x_{ip}))$ and $\zeta_i^{\tau} = \zeta_i(t-\tau)$.

Under the assumption of ($\mathscr{A}1$), one obtains that the set-valued map $F(X, X^{\tau})$ is upper semi-continuous with nonempty bounded convex closed values. From ($\mathscr{A}2$), the generalized linear growth condition is satisfied. Then we can get the following results for system (2) by using the same method in [15] and [23].

Theorem 1: Suppose ($\mathscr{A}1$) and ($\mathscr{A}2$) hold, the Filippov solution X of system (2) with any give initial condition exist on $[0, +\infty)$.

Remark 4: There are many results on the control of synchronization of discontinuous complex networks [21]–[23], [25]–[27], [30], however, few of them discussed about the existence of the solution. Though the authors studied the existence of discontinuous neural networks in [24], [28], and [29], they cannot ensure the solution existence of the discontinuous complex networks. In this paper, we studied the Filippov solution existence for NCDNNs based on differential inclusion theory.

Since the Filippov solution of system (2) exists on $[0, +\infty)$, i.e. system (1) has solution on $[0, +\infty)$ for $i \in \{1, 2, \cdots, N\}$. Next, we will give the definition of exponential synchronization for system (1) and present some lemmas, which will be used later.

Definition 4 [11]: The nonlinear coupled delayed neural networks (1) is exponentially synchronizable if there are positive constants $M > 0$ and $\alpha > 0$ such that

$$\|x_i - x_j\| \leq M\|\varphi_i - \varphi_j\|e^{-\alpha t}, \quad t \geq 0$$

for any initial condition $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_N)^{\mathrm{T}}$ and $i, j \in \{1, 2, \cdots, N\}$.

Definition 5 [2]: Denote $\mathscr{M}_1^N$ is a subset of matrix with N columns, if for any matrix $M = M_{ij}$, there exist i_1 and i_2 such that $M_{ii_1} = M_{ii_2} = 1$ for every i, then we said $M \in \mathscr{M}_1^N$. Moreover, if for any indexes i and j, there exist columns with indexes $i_1, i_2, \cdots, i_l$ satisfying $i_1 = i$ and $i_l = j$, and rows with indexes $p_1, \cdots, p_{l-1}$ in M such that the entries $M_{p_q,i_q} \neq 0$ and $M_{p_q,i_{q+1}} \neq 0$ for $q = 1, \ldots \cdots, l-1$, then we said $M \in \mathscr{M}_2^N$

Definition 6 [2]: Let $\mathcal{T}_1^N(0) \subset R^{N\times N}$ is a subset of matrix, for any symmetric matrix T with nonnegative off-diagonal elements, if the sum of the elements in each row is zero, then $T \in \mathcal{T}_1^N(0)$.

Lemma 1 [41]: If the matrix $A \in \mathcal{T}_1^N(0)$, then A is irreducible when and only when that there is an $p \times N$ matrix $M \in \mathscr{M}_2^N$ satisfying $A = -M^{\mathrm{T}}M$.

Lemma 2 [13]: Suppose $V(x)$ is a C-regular function, and x is absolutely continuous on any subintervals of $[0, +\infty)$, then $V(x) : [0, +\infty) \to \mathbb{R}$ is differentiable for almost every $t \in [0, +\infty)$. Furthermore,

$$\frac{d}{dt}V(x) = \zeta^{\mathrm{T}}\dot{x}, \quad \forall\zeta \in \partial_c V(x),$$

where $\partial_c V(x)$ is the Clark generalized gradient of V at x and ζ is a measurable function.

Lemma 3 [42]: Suppose a nonnegative continuous function $W(t)$ satisfies

$$D^{+}W(t) \leq -pW(t) + q \max_{t-\tau \leq s \leq t} W(s), \quad t \geq 0, \tag{6}$$

for almost every $t \in (-\tau, +\infty)$, where $p > q$ are positive constants. Then

$$W(t) \leq \max_{-\tau \leq s \leq 0} W(s)e^{-\rho t}, t \geq 0,$$

where $\rho > 0$ is the unique positive solution of $\rho - p + qe^{\rho\tau} = 0$.

III. EXPONENTIAL SYNCHRONIZATION CONDITION

Some sufficient conditions to ensure the exponential synchronization of NCDNNs will be given in this section. Denote $\nu_{ij} = g_{ij} + g_{ji} - \sum_{k=1,k\neq i,j}^{N}(g_{ik} + g_{jk})$. In addition, $\nu_{ij} = \nu_{ji}$. Moreover, for any given matrix $M \in \mathscr{M}_2^N$, define $\check{\nu}(M) = \min\{\nu_{i_1 i_2} : (i_1, i_2) \text{satisfies} M_{ii_1} \neq 0, M_{ii_2} \neq 0\}$. Furthermore, let $\{A\}^s = (A + A^{\mathrm{T}})/2$ and $\check{r} = \min_{1\leq i\leq n}\{r_i\}$.

Theorem 2: Besides ($\mathscr{A}1$) and ($\mathscr{A}2$) hold, if there exist an $m \times N$ matrix $M \in \mathscr{M}_2^N$ and a diagonal matrix $\Theta = \text{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$ such that

$$(\||A|\|_1 + \||B|\|_1)\|N\|_1 - \check{r}\check{\nu}(M) \leqslant 0, \tag{7}$$

$$\{M^{\mathrm{T}}M(r_i L - \theta_i I_N)\}^s \geq 0, \tag{8}$$

$$\lambda_{\min}(2(D + \Theta) - 2|A|K - |B|K^2|B|^{\mathrm{T}}) \geq 1, \tag{9}$$

hold, then the NCDNNs (1) is globally exponentially synchronizable, where $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $K = \text{diag}(K_1, K_2, \cdots, K_n)$, $N = (N_1, N_2, \cdots, N_n)^{\mathrm{T}}$.

Proof: Let $\mathbf{M} = M \otimes I_n$ and $Y = \mathbf{M}X = (y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_m^{\mathrm{T}})^{\mathrm{T}}$, where $y_i = (y_{i1}, y_{i2}, \cdots, y_{in})^{\mathrm{T}}$. According to Definition 5, one obtains $y_i = x_{i_1} - x_{i_2}$. It follows from Definition 4 that exponential synchronization of system (1) when and only when Y is exponentially convergent to 0. Construct the following Lyapunov function:

$$W(X) = X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X. \tag{10}$$

Obviously, $W(X) = 0$ when and only when $\mathbf{M}X = 0$. Obviously, $W(X) > 0$ when $\mathbf{M}X \neq 0$. By Lemma 3 (Chain Rule), calculate and estimate the derivative of V along the solution of (4), one obtains

$$\begin{aligned}\dot{W} &= 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}[-\mathbf{D}X + \mathbf{A}\zeta + \mathbf{B}\zeta^{\tau} + \mathbf{I} \\ &\quad - \mathbf{L}X + \upsilon - \Theta X + \Theta X] \\ &= -2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}(\mathbf{D} + \Theta)X + 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MA}\zeta \\ &\quad + 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MB}\zeta^{\tau} - 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}(L \otimes \Gamma \\ &\quad - \Theta)X + 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MI} + 2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\upsilon, \qquad (11)\end{aligned}$$

where $\Theta = I_N \otimes \Theta$ is a free diagonal matrix. Since $\mathbf{M} = M \otimes I_n$, one obtains $2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MI} = 0$. Note that

$$\begin{aligned}&-2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}(\mathbf{D} + \Theta)X \\ &\quad = -2\sum_{i=1}^{m}(x_{i_1} - x_{i_2})^{\mathrm{T}}((D + \Theta)x_{i_1} - (D + \Theta)x_{i_2}) \\ &\quad = -2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}(D + \Theta)|x_{i_1} - x_{i_2}|, \qquad (12)\end{aligned}$$

where $|x_{i_1} - x_{i_2}| = (|x_{i_1 1} - x_{i_2 1}|, \cdots, |x_{i_1 n} - x_{i_2 n}|)^{\mathrm{T}}$.

$$\begin{aligned}&2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MA}\zeta \\ &= 2\sum_{i=1}^{m}(x_{i_1} - x_{i_2})^{\mathrm{T}}(A\zeta_{i_1} - A\zeta_{i_2}) \\ &= 2\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{n}(x_{i_1 p} - x_{i_2 p})a_{pq}(\zeta_{i_1 q} - \zeta_{i_2 q}) \\ &\le 2\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{n}|x_{i_1 p} - x_{i_2 p}||a_{pq}||\zeta_{i_1 q} - \zeta_{i_2 q}| \\ &\le 2\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{n}|x_{i_1 p} - x_{i_2 p}||a_{pq}|(K_q|x_{i_1 q} - x_{i_2 q}| + N_q) \\ &= 2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|A|K|x_{i_1} - x_{i_2}| \\ &\quad + 2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|A|N \\ &\le 2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|A|K|x_{i_1} - x_{i_2}| + 2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|A|N \\ &\le 2\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|A|K|x_{i_1} - x_{i_2}| \\ &\quad + 2\sum_{i=1}^{m}\|x_{i_1} - x_{i_2}\|_1\||A|\|_1\|N\|_1. \qquad (13)\end{aligned}$$

With the similar method, one obtains

$$\begin{aligned}&2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{MB}\zeta^{\tau} = \sum_{i=1}^{m}(x_{i_1} - x_{i_2})^{\mathrm{T}}(B\zeta_{i_1}(t-\tau) - B\zeta_{i_2}^{\tau}) \\ &\quad \le 2\sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{q=1}^{n}|x_{i_1 p} - x_{i_2 p}||b_{pq}|(K_q|x_{i_1 q}^{\tau} - x_{i_2 q}^{\tau}| + N_q) \\ &\quad \le \sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|B|K^2|B|^{\mathrm{T}}|x_{i_1} - x_{i_2}| \\ &\qquad + \sum_{i=1}^{m}|x_{i_1}^{\tau} - x_{i_2}^{\tau}|^{\mathrm{T}}|x_{i_1}^{\tau} - x_{i_2}^{\tau}| \\ &\qquad + 2\sum_{i=1}^{m}\|x_{i_1} - x_{i_2}\|_1\||B|\|_1\|N\|_1, \qquad (14)\end{aligned}$$

and

$$\begin{aligned}&2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\upsilon \\ &\quad = 2\sum_{i=1}^{m}(x_{i_1} - x_{i_2})^{\mathrm{T}}(\upsilon_{i_1} - \upsilon_{i_2}) \\ &\quad = 2\sum_{i=1}^{m}(x_{i_1} - x_{i_2})^{\mathrm{T}}[\sum_{j=1}^{N}g_{i_1 j}\Gamma\upsilon_{ji_1} - \sum_{j=1}^{N}g_{i_2 j}\Gamma\upsilon_{ji_2}] \\ &\quad \le -2\sum_{i=1}^{m}\nu_{i_1 i_2}\|\Gamma(x_{i_1} - x_{i_2})\|_1 \\ &\quad \le -2\sum_{i=1}^{m}\check{\nu}(M)\check{r}\|x_{i_1} - x_{i_2}\|_1. \qquad (15)\end{aligned}$$

Denote $\bar{x}_j = (x_{1j}, x_{2j}, \cdots, x_{Nj})^{\mathrm{T}}$. Then according to condition (8), one obtains

$$\begin{aligned}&2X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}(L \otimes \Gamma - \Theta)X \\ &\qquad = \sum_{i=1}^{n}\bar{x}_i^{\mathrm{T}}M^{\mathrm{T}}M(r_iL - \theta_iI_N)^s\bar{x}_i \ge 0. \qquad (16)\end{aligned}$$

According to formula (12)-(16) and (7), we obtain

$$\begin{aligned}\dot{W} &\le -\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}(2(D + \Theta) - 2|A|K \\ &\quad - |B|K^2|B|^{\mathrm{T}})|x_{i_1} - x_{i_2}| + \sum_{i=1}^{m}|x_{i_1}^{\tau} - x_{i_2}^{\tau}|^{\mathrm{T}}|x_{i_1}^{\tau} \\ &\quad - x_{i_2}^{\tau}| + 2\sum_{i=1}^{m}((\||A|\|_1 + \||B|\|_1)\|N\|_1 \\ &\quad - \check{\nu}(M)\check{r})\|x_{i_1} - x_{i_2}\|_1 \\ &\le -\eta\sum_{i=1}^{m}|x_{i_1} - x_{i_2}|^{\mathrm{T}}|x_{i_1} - x_{i_2}| \\ &\quad + \sum_{i=1}^{m}|x_{i_1}^{\tau} - x_{i_2}^{\tau}|^{\mathrm{T}}|x_{i_1}^{\tau} - x_{i_2}^{\tau}| \\ &= -\eta X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X + (X^{\tau})^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X^{\tau} \\ &= -\eta W + W(t - \tau), \qquad (17)\end{aligned}$$

where $\eta = \lambda_{\min}(2(D + \Theta) - 2|A|K - |B|K^2|B|^{\mathrm{T}})$.

Based on Lemma 4 and formula (9), it follows that Y converges to 0 exponentially. Then the NCDNNs (1) is globally exponentially synchronizable.

Remark 5: It is easy to see that the discontinuous 'sign' term in the nonlinear coupling function can eliminate the influence of the discontinuous neuron function if the coupling matrix G satisfies condition (8). Compared with the existing results [24]–[30], the complete synchronization can be ensured in this paper without any designed controllers.

Remark 6: The crucial part of the proof is to find a proper matrix $M \in \mathscr{M}_2^N$ to construct the Lyapunov function. Obviously, if G is an undirected graph, then one obtains $-L \in \mathcal{T}_1^N(0)$. Furthermore, if the graph is connected, i.e. $-L$ is also irreducible. On the basis of Lemma2, we can find an $m \times N$ matrix $M \in \mathscr{M}_2^N$ satisfying $L = M^{\mathrm{T}}M$. Therefore, condition (8) can be satisfied for $\Theta = 0$, if G is symmetric and irreducible.

Corollary 1: Suppose $(\mathscr{A}1)$ and $(\mathscr{A}2)$ hold and G is symmetric and irreducible. The nonlinear coupled complex networks (1) is globally exponentially synchronizable if

$$(\||A|\|_1 + \||B|\|_1)\|N\|_1 - \check{r}\check{\nu}(M) \leqslant 0 \tag{18}$$

and

$$\lambda_{\min}(2(D+\Theta) - 2|A|K - |B|K^2|B|^{\mathrm{T}}) \geq 1 \tag{19}$$

where M can be obtained from L based on Lemma 2 and $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $K = \mathrm{diag}(K_1, K_2, \cdots, K_n)$ and $N = (N_1, N_2, \cdots, N_n)^{\mathrm{T}}$.

Remark 7: The free diagonal matrix Θ makes Theorem 2 become more applicable because it can make a balance between parameter D and the coupling matrix L. It can be seen that a larger Θ will be favorable to the solvability of inequality (9). However, it is difficult to satisfy condition (8) for larger Θ. Therefore, it is difficult to guarantee the existence of Θ such that both (8) and (9) are satisfied. In the following part, we design pinning impulsive controller to guarantee complete synchronization of NCDNNs (1).

IV. PINNING IMPULSIVE CONTROL

Pinning impulsive controllers are designed to ensure exponential synchronization of NCDNNs (1) in this section. Rearranging the order of the nodes, system (1) can be rewritten as follows:

$$\begin{cases} \dot{x}_i(t) = -Dx_i + Af(x_i(t)) + Bf(x_i(t-\tau)) + I \\ \qquad + \sum_{j=1}^N g_{ij}\Gamma\phi(x_j(t), x_i(t)) + u_i(t), \\ \qquad i = 1, 2, \cdots, l, \\ \dot{x}_i(t) = -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau)) + I \\ \qquad + \sum_{j=1}^N g_{ij}\Gamma\phi(x_j(t), x_i(t)), \\ \qquad i = l+1, l+2, \cdots, N, \end{cases} \tag{20}$$

where $u_i(t)$ represents the designed controller, which relies on part of neighborhood neuron states of ith system. Namely, $u_i = u(x_j : j \in \tilde{\mathfrak{N}}_i)$, where $\tilde{\mathfrak{N}}_i \subset \mathfrak{N}_i = \{j \in \mathfrak{L} | g_{ij} > 0\}$.

We design the pinning impulsive controller to be

$$u_i = \sum_{k=1}^{+\infty} D_k(\sum_{j\in\tilde{\mathfrak{N}}_i} \beta_{ij}(x_j(t) - x_i(t)))\delta(t - t_k), \qquad i = 1, 2, \cdots, l, \tag{21}$$

where $k \in \mathbb{N}_+$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ are impulsive instants satisfying $\lim_{k\to\infty} t_k = \infty$. $\delta(\cdot)$ and $D_k \in R^{n\times n}$ represent Dirac impulsive function and control gains, respectively. Moreover, β_{ij} is the influence weight of the $j-$th neuron state on $i-$th neuron, which is a positive number. As studied in [11] and [35], if we let $\beta_{ij} = 0$ where $i = 1, 2, \cdots, l$, and $j \in \mathfrak{L} \setminus \tilde{\mathfrak{N}}_i$, or $i = l+1, \cdots, N$, and $j \in \mathfrak{L}$, then the matrix $\beta = (\beta_{ij})_{N\times N}$ is defined as control topology, which represents a weighted directed graph. With the same method, we can define Laplacian matrix Π as follows: $\pi_{ij} = -\beta_{ij}$ for $i \neq j$, and $\pi_{ii} = \sum_{j=1}^N \beta_{ij}$.

Remark 8: Different form the traditional impulsive controller, the designed impulsive controller (21) only use the local neuron state information. Moreover, besides only portion of neurons are controlled, the "pinning" here also means that a part of local information of the pinned neurons are used.

Under the designed pinning impulsive controller (21), NCDNNs (1) becomes:

$$\begin{cases} \dot{x}_i = -Dx_i + Af(x_i) + Bf(x_i^\tau) + I \\ \qquad + \sum_{j=1}^N g_{ij}\Gamma\phi(x_j, x_i), \quad t \neq t_k \\ \triangle x_i(t_k) = -D_k \sum_{j=1}^N \beta_{ij}x_j(t_k), \quad t = t_k, \end{cases} \tag{22}$$

where $\triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k)$ and $x_i(t_k^+) = \lim_{t\to t_k^+} x_i$. Next, we assume $x(t_k) = x(t_k^-)$.

With the similar analysis in Section 2, we can rewrite system (22) in a matrix form in sense of Filippov as follows:

$$\begin{cases} \dot{X} = -\mathbf{D}X + \mathbf{A}\zeta + \mathbf{B}\zeta^\tau + \mathbf{I} - \mathbf{L}X + \upsilon, \\ \qquad \text{for a.a. } t \in (t_{k-1}, t_k) \\ X(t_k^+) = (I_{Nn} - \mathbf{D}_k\Pi)X(t_k), \quad t = t_k, \end{cases} \tag{23}$$

where $\mathbf{D}_k = I_N \otimes D_k$ and $\Pi = \Pi \otimes I_n$.

Theorem 3: Suppose $(\mathscr{A}1)$ and $(\mathscr{A}2)$ hold and $\tau \leq t_k - t_{k-1}$ for all $k \in N_+$. The nonlinear coupled complex networks (1) is global exponentially synchronizable if there exist an $m \times N$ matrix $M \in \mathscr{M}_2^N$, a diagonal matrix $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$ and a positive number α such that

$$(\||A|\|_1 + \||B|\|_1)\|N\|_1 - \check{r}\check{\nu}(M) \leqslant 0, \tag{24}$$

$$\{M^{\mathrm{T}}M(r_iL - \theta_iI_N)\}^s \geq 0, \tag{25}$$

$$\ln(\delta_k + \tau) \leq -(\alpha + c)(t_{k+1} - t_k), \quad k \in N_+, \tag{26}$$

where $c = -\lambda_{\min}(2(D+\Theta) - 2|A|K - |B|K^2|B|^{\mathrm{T}}) + 1 \geq 0$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $K = \mathrm{diag}(K_1, K_2, \cdots, K_n)$, $N = (N_1, N_2, \cdots, N_n)^{\mathrm{T}}$, $\delta_k = \lambda_{\max}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})^{\mathrm{T}}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})$ and $\bar{\mathbf{D}}_k$, $\bar{\Pi}$ satisfying $\mathbf{MD}_k = \bar{\mathbf{D}}_k\mathbf{M}$, $\mathbf{M}\Pi = \bar{\Pi}\mathbf{M}$.

Proof: We consider the Lyapunov-Krasovskii function

$$W(X) = W_1(X) + W_2(X), \tag{27}$$

where $W_1(X) = X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X$ and $W_2(X) = \int_{t-\tau}^t X^{\mathrm{T}}(s)\mathbf{M}^{\mathrm{T}}\mathbf{M}X(s)\mathrm{d}s$.

For any $t \in (t_{k-1}, t_k]$, we get the upper Dini derivatives of W_1 along with the solution of (23) based on Lemma 3. According to condition (24) and (25), with the same analysis in Theorem 2, one obtains:

$$\begin{aligned} \dot{W}_1 &\leq -\lambda_{\min}(2(D+\Theta) - 2|A|L - |B|L^2|B|^{\mathrm{T}}) \\ &\quad \times X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X + (X^\tau)^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X^\tau, \\ \dot{W}_2 &= X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X - (X^\tau)^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X^\tau. \end{aligned} \tag{28}$$

It follows that

$$\begin{aligned}\dot{W} &\leq (-\lambda_{\min}(2(D+\Theta) - 2|A|L - |B|L^2|B|^{\mathrm{T}}) \\ &\quad + 1)X^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}X \\ &= cW_1 \leq cW. \end{aligned} \tag{29}$$

Next, we investigate how the impulses affect the Lyapunov-Krasovskii function. when $t = t_k$, one has

$$\begin{aligned} W_1(t_k^+) &= X^{\mathrm{T}}(t_k)(I_{Nn} - \mathbf{D}_k\Pi)^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}(I_{Nn} - \mathbf{D}_k\Pi)X(t_k) \\ &= X^{\mathrm{T}}(t_k)\mathbf{M}^{\mathrm{T}}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})^{\mathrm{T}}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})\mathbf{M}X(t_k) \\ &\leq \delta_k X^{\mathrm{T}}(t_k)\mathbf{M}^{\mathrm{T}}\mathbf{M}X(t_k) \\ &= \delta_k W_1(t_k). \end{aligned} \tag{30}$$

For W_2, we get

$$W_2(t_k^+) = W_2(t_k), \quad k \in \mathbb{N}_+. \tag{31}$$

In the following, we use mathematical induction to get that

$$W \leq Me^{-(\alpha+c)(t_k-t_0)}e^{c(t-t_0)} \tag{32}$$

for all $k \in \mathbb{N}_+$, where $M = V(t_0)e^{(\alpha+c)(t_1-t_0)}$. For $t \in [t_0, t_1]$, from (29), we can conclude that

$$\begin{aligned} W &\leq W(t_0)e^{c(t-t_0)} \\ &= W(t_0)e^{(\alpha+c)(t_1-t_0)}e^{-(\alpha+c)(t_1-t_0)}e^{c(t-t_0)} \\ &= Me^{-(\alpha+c)(t_1-t_0)}e^{c(t-t_0)}. \end{aligned} \tag{33}$$

Then, (30) and (33) imply that

$$W_1(t_1^+) \leq \delta_1 W_1(t_1) \leq \delta_1 W(t_1) \leq \delta_1 Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)}. \tag{34}$$

Based on (31), we can obtain that

$$\begin{aligned} W_2(t_1^+) = W_2(t_1) &= \int_{t-\tau}^{t} X^{\mathrm{T}}(s)\mathbf{M}^{\mathrm{T}}\mathbf{M}X(s)\mathrm{d}s \\ &= \int_{t-\tau}^{t} V_1(s)\mathrm{d}s \\ &\leq \tau \sup_{s\in[t_1-\tau,t_1]} \{V_1(s)\} \\ &\leq \tau Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)}. \end{aligned} \tag{35}$$

It follows from (26), (34) and (35), one obtains that

$$\begin{aligned} W(t_1^+) &= W_1(t_1^+) + W_2(t_1^+) \\ &\leq \delta_1 Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)} \\ &\quad + \tau Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)} \\ &= (\delta_1+\tau)Me^{-(\alpha+c)(t_1-t_0)}e^{c(t_1-t_0)} \\ &\leq Me^{-(\alpha+c)(t_2-t_0)}e^{c(t_1-t_0)}. \end{aligned} \tag{36}$$

Then, for $t \in (t_1, t_2]$, one has

$$\begin{aligned} W &\leq W(t_1^+)e^{c(t-t_1)} \\ &\leq Me^{-(\alpha+c)(t_2-t_0)}e^{c(t_1-t_0)}e^{c(t-t_1)} \\ &= Me^{-(\alpha+c)(t_2-t_0)}e^{c(t-t_0)}. \end{aligned} \tag{37}$$

i.e. (32) holds for $k = 2$.

Suppose (32) works for $k = j$ ($j > 2$), that is

$$W \leq Me^{-(\alpha+c)(t_j-t_0)}e^{c(t-t_0)}, \quad t \in (t_{j-1}, t_j]. \tag{38}$$

We will show that (32) holds for $k = j+1$. As discussed in (34) and (35), we can get

$$\begin{aligned} W(t_j^+) &= W_1(t_j^+) + W_2(t_j^+) \\ &\leq (\delta_j + \tau)Me^{-(\alpha+c)(t_j-t_0)}e^{c(t_j-t_0)} \\ &\leq Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_j-t_0)}. \end{aligned} \tag{39}$$

Then for $t \in (t_j, t_{j+1}]$, one obtains

$$\begin{aligned} W &\leq W(t_j^+)e^{c(t-t_j)} \\ &\leq Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t_j-t_0)}e^{c(t-t_j)} \\ &= Me^{-(\alpha+c)(t_{j+1}-t_0)}e^{c(t-t_0)}. \end{aligned} \tag{40}$$

i.e. (32) holds for $k \in \mathbb{N}_+$.

From (32), for $t \in (t_{k-1}, t_k]$, we obtain

$$\begin{aligned} W &\leq Me^{-(\alpha+c)(t_k-t_0)}e^{c(t-t_0)} \\ &\leq Me^{-\alpha(t_k-t_0)} \\ &\leq Me^{-\alpha(t-t_0)} \\ &= V(t_0)e^{(\alpha+c)(t_1-t_0)}e^{-\alpha(t-t_0)} \\ &\leq \sup_{s\in[t_0-\tau,t_0]} \|\mathbf{M}\varphi(s)\|_2^2 e^{(\alpha+c)(t_1-t_0)}e^{-\alpha(t-t_0)}. \end{aligned} \tag{41}$$

i.e.

$$\begin{aligned} \|Y\|_2 &= \sqrt{V_1} \leq \sqrt{V} \\ &\leq \bar{M} \sup_{s\in[t_0-\tau,t_0]} \|\mathbf{M}\varphi(s)\|_2 e^{-\frac{\alpha}{2}(t-t_0)}, \end{aligned} \tag{42}$$

for $t \geq t_0$, where $\bar{M} = \sqrt{e^{(\alpha+c)(t_1-t_0)}}$. That is to say Y exponential converge to 0. On the basis of Definition 4, the NCDNNs (1) is global exponentially synchronizable.

Remark 9: From the proof of Theorem 3, we can get that the exponential synchronization rate equals to $\alpha/2$, which depends on the parameters c, time delay τ, controller parameters δ_k and impulsive interval lengths $t_{k+1} - t_k$. condition (26) gives a balancing condition on δ_k and $t_{k+1} - t_k$ to design a appropriate pinning impulsive controller. However, Theorem 3 is invalid for network (1) with large time-delay since $\tau \leq t_k - t_{k-1}$.

Theorem 4: Besides ($\mathscr{A}1$) and ($\mathscr{A}2$) hold, if there exist an $m \times N$ matrix $M \in \mathscr{M}_2^N$, a diagonal matrix $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$ and a positive number α satisfying

$$(\||A|\|_1 + \||B|\|_1)\|N\|_1 - \check{r}\check{\nu}(M) \leqslant 0, \tag{43}$$

$$\{M^{\mathrm{T}}M(r_iL - \theta_i I_N)\}^s \geq 0, \tag{44}$$

$$\ln(\delta_k + \tau e^{\alpha\tau}) \leq -(\alpha+c)(t_{k+1} - t_k), \quad k \in N_+, \tag{45}$$

then the NCDNNs (1) is global exponentially synchronizable, where $c = -\lambda_{\min}(2(D+\Theta) - 2|A|K - |B|K^2|B|^{\mathrm{T}}) + 1 \geq 0$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$, $K = \mathrm{diag}(K_1, K_2, \cdots, K_n)$, $N = (N_1, N_2, \cdots, N_n)^{\mathrm{T}}$, $\delta_k = \lambda_{\max}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})^{\mathrm{T}}(I_{mn} - \bar{\mathbf{D}}_k\bar{\Pi})$ and $\bar{\mathbf{D}}_k$, $\bar{\Pi}$ satisfying $\mathbf{M}\mathbf{D}_k = \bar{\mathbf{D}}_k\mathbf{M}$, $\mathbf{M}\Pi = \bar{\Pi}\mathbf{M}$.

Proof: Construct the same Lyapunov-Krasovskii functional given in (30). We have shown that (29), (30) and (31) are valid according to the proof of Theorem 3. Since $\lim_{k\to\infty} t_k = \infty$, one has an integer $i \geq 1$ satisfying $t_i - \tau \geq t_0$, and for $t \in [t_0, t_i]$, we have

$$W(X) = W(X)e^{\alpha(t-t_0)}e^{-\alpha(t-t_0)} \leq Me^{-\alpha(t-t_0)}, \quad (46)$$

where $M = \sup_{t\in[t_0-\tau,t_i]} \{V(x)\}e^{\alpha(t_i-t_0)}$.

Next, we shall show that

$$W(X) \leq Me^{-(\alpha+c)(t_{k+1}-t_0)}e^{c(t-t_0)} \quad (47)$$

for $t \in (t_k, t_{k+1}]$ and $k \geq i$. When $k = i$, we obtain from (46) that $t_0 \leq t_i + s \leq t_i$ for $s \in [-\tau, 0]$, and

$$\begin{aligned}\sup_{s\in[t_i-\tau,t_i]} &\{\|\mathbf{M}X(s)\|_2^2\} \\ &= \sup_{s\in[t_i-\tau,t_i]} \{V_1(s)\} \leq \sup_{s\in[t_i-\tau,t_i]} \{V(s)\} \\ &= Me^{-\alpha(t_i-t_0)} \leq Me^{-\alpha(t_i-\tau-t_0)} \\ &= Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)}e^{\alpha\tau}. \end{aligned} \quad (48)$$

With the similar analysis on $W_1(t_i^+)$ and $W_2(t_i^+)$ in (34) and (35), we obtain

$$\begin{aligned} W(t_i^+) &= W_1(t_i^+) + W_2(t_i^+) \\ &\leq \delta_i W_1(t_i) + W_2(t_i) \\ &\leq \delta_i Me^{-\alpha(t_i-t_0)} + \tau e^{\alpha\tau}Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)} \\ &= (\delta_i + \tau e^{\alpha\tau})Me^{-(\alpha+c)(t_i-t_0)}e^{c(t_i-t_0)} \\ &\leq Me^{-(\alpha+c)(t_{i+1}-t_0)}e^{c(t_i-t_0)}. \end{aligned} \quad (49)$$

Along with (29), one has

$$W(X) \leq Me^{-(\alpha+c)(t_{i+1}-t_0)}e^{c(t-t_0)} \quad (50)$$

for $t \in (t_i, t_{i+1}]$.

With the same discussion in the proof in Theorem 3, we conclude that (47) holds for all $k \geq i$ by mathematical induction. Then, for $t \in (t_k, t_{k+1}]$, one obtains

$$\begin{aligned} W(X) &\leq Me^{-(\alpha+c)(t_{k+1}-t_0)}e^{c(t-t_0)} \\ &\leq Me^{-\alpha(t_{k+1}-t_0)} \\ &\leq Me^{-\alpha(t-t_0)}. \end{aligned} \quad (51)$$

Along with (46) and (51), we can see that

$$W(X) \leq Me^{-\alpha(t-t_0)}$$

for $t \geq t_0$. With the same analysis in Theorem 3, it is easy to check Y exponentially converges to 0. According to Definition 4, the nonlinear coupled complex networks (1) is globally exponentially synchronizable.

Remark 10: The Lyapunov-Krasovskii functional candidates is composed by a function part and a functional part, which has been widely used to investigate the synchronization of impulsive system [37], [38]. The impulse can change the value of the function dramatically, but it cannot alter the valued of functional part instantaneously. However, this truth brings difficulties to theoretic analysis of our results. Thus we use the generalized derivative proposed in [13], we calculate the derivative of the Lyapunov-Krasovskii functionals.

V. NUMERICAL SIMULATIONS

In this part, we give an example to show the correctness of Theorem 2. Furthermore, we provide another example to demonstrate the effectiveness of the designed controllers in Theorem 3.

The dynamics model of single neuron can be given by the following discontinuous delayed neural networks:

$$\dot{x} = -Dx + Af(x) + Bf(x^\tau) + I, \quad (52)$$

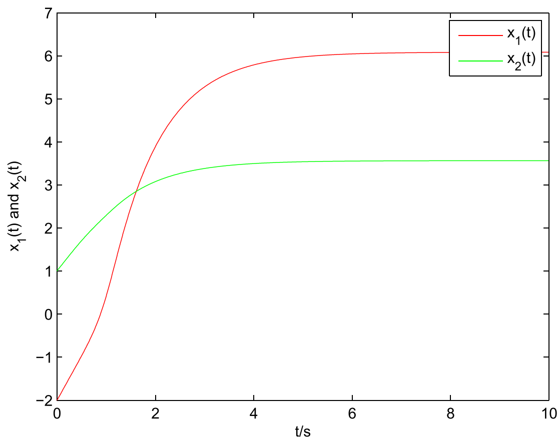
where $x = (x_1, x_2^T)$, $\tau = 0.02$, $I = (0, 0)^T$, and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 4 \\ 0.1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.7 & -0.9 \\ 0.1 & 1.2 \end{bmatrix}.$$

The neuron function $f(x) = (f_1(x_1), f_2(x_2))^T$ is given as follows:

$$f_i(x_i) = \begin{cases} \tanh(x_i) + 0.05, & x_i \geq 0, \\ \tanh(x_i) - 0.03, & x_i < 0, \end{cases}$$

which is the same as that in [26]. Obviously, the discontinuous function satisfies ($\mathscr{A}1$) and ($\mathscr{A}2$) with $K_i = 1$ and $N_i = 0.08$, $i = 1, 2$. The state trajectories of system (52) with initial condition $\phi(s) = (-2, 1)^T$ $s \in [-0.02, 0]$ is shown in Fig.1.



(a)

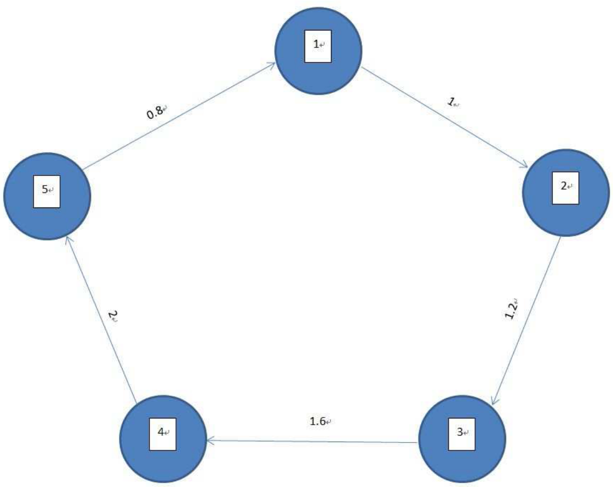
FIGURE 1. The state trajectories for system (52).

Since Theorem 2 and Theorem 3 are available for NCDNNs with directed topology and undirected topology, we present two examples with directed and undirected topology in Example 1 and Example 2, respectively.

Example 1: Suppose the identical neuron state can be described in (52). Five identical neurons are coupled in a directed ring structure (see Fig.2), which is the same as that in [11]. The adjacent matrix of the directed ring graph is

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.8 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 1.6 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \quad (53)$$

and $\Gamma = \text{diag}\{10, 10\}$.



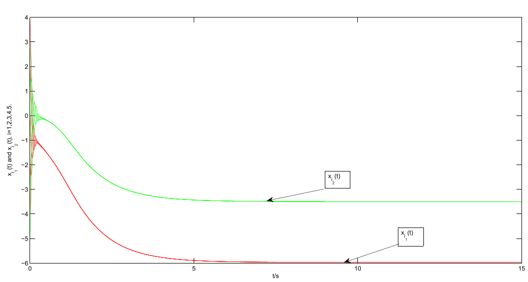
(a)

FIGURE 2. **The coupling configuration in Example 1.**

Let

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \tag{54}$$

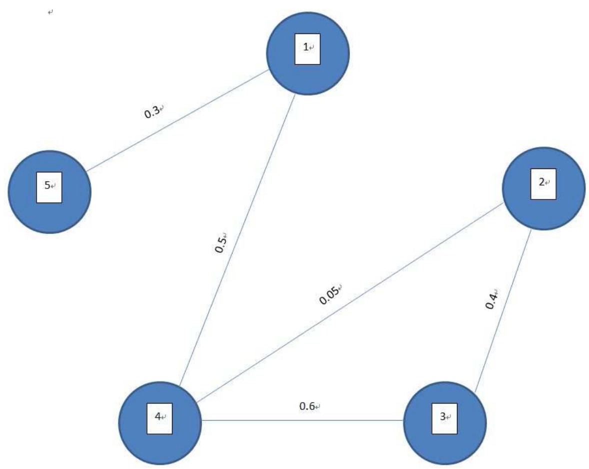
it is easy to get $\check{\nu}(M) = 0.2$ and $\check{r} = 10$ through simple computation. Since $\||A|\|_1 = 6$, $\||B|\|_1 = 1.6$ and $\|K\| = 1$, $\|N\|_1 = 0.08$, which satisfies condition (7). Choose $\Theta = \text{diag}\{4, 4\}$, it is easy to verify $\{M^{\text{T}}M(r_iL - \theta_i I_N)\}^s \geq 0$, i.e. condition (8) holds. The left-hand side of condition (9) can be calculated as $\lambda_{\min}(2(D+\Theta) - 2|A|K - |B|K^2|B|^{\text{T}}) = 1.3803$, which is larger than 1. According to Theorem 2, the NCDNNs is exponentially synchronizable. Choose the initial conditions for each discontinuous neural networks as $x_i(0) = (-6+2i, 5-2i)^{\text{T}}$, the numerical simulation results are shown in Fig.3.



(a)

FIGURE 3. **The trajectories of NCDNNs in Example 1.**

Example 2: Consider five discontinuous neural networks (52) coupled in an undirected ring structure (see Fig.4).



(a)

FIGURE 4. **The undirected coupling configuration in Example 2.**

The outer coupling matrix G is

$$G = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0.3 \\ 0 & 0 & 0.4 & 0.05 & 0 \\ 0 & 0.4 & 0 & 0.6 & 0 \\ 0.5 & 0.05 & 0.6 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

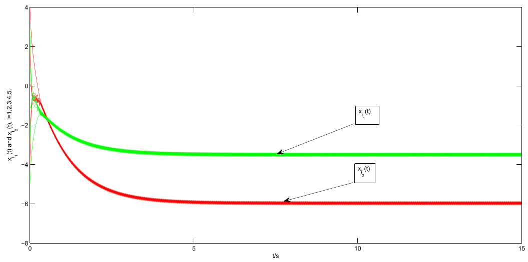
and $\Gamma = \text{diag}\{14, 14\}$.

Since the graph is connected and undirected, i.e. G is symmetric and irreducible. From analysis in Remark 6, it is obvious that condition (25) is satisfied for $\Theta = 0$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}, \tag{56}$$

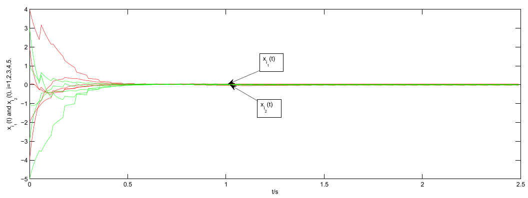
one obtains $\check{\nu}(M) = 0.05$ and $\check{r} = 14$ though simple computation. Since $\||A|\|_1 = 6$, $\||B|\|_1 = 1.6$ and $\|N\|_1 = 0.08$, it is easy to testify condition (24). Since $\lambda_{\min}(2(D+\Theta) - 2|A|K - |B|K^2|B|^{\text{T}}) = -6.8904 < 1$, Theorem 3 cannot ensure the realization of synchronization.

Here we design a pinning impulsive controller to realize the synchronization. Consider the pinning impulsive controller with $t_{k+1} - t_k = 0.06$ and D_k $(k = 1, 2, \cdots)$ are all set to be identity matrices with dimension 2. Choose the controlled edges as $\{e_{14}, e_{23}, e_{34}, e_{51}\}$, and the influence weight are set to be $\beta_{14} = \beta_{23} = \beta_{34} = 0.5$, $\beta_{51} = 0.8$. Though computation, we can get $\delta_k = 0.5208$. Therefore, if we choose $\alpha = 2 > 0$, it is easy to verify that condition (26) holds. Hence, all conditions in Theorem 3 hold and consequently the discontinuous complex networks are synchronized under the designed pinning impulsive controller. The simulation results are shown in Fig.5 (without impulsive controller) and Fig.6 (under impulsive controller), where the initial conditions of each discontinuous neural networks are selected as $x_i(0) = (-6+2i, 5-2i)^{\text{T}}$ for $i = 1, 2, \cdots, 5$.



(a)

FIGURE 5. **The trajectories of NCDNNs in Example 2 without controller.**



(a)

FIGURE 6. **The trajectories of NCDNNs in Example 2 under pinning impulsive controllers.**

VI. CONCLUSION

In this paper, we investigated the exponential synchronization for NCDNNs based on differential inclusion theory. We obtained the conditions for the existence of Filippov solution and the exponential synchronization for NCDNNs. Compared with the existing results, we used the nonlinear coupling function to eliminate the effects of the discontinuous activation functions. In pinning control strategy, how to select the control nodes and what is the minimum number of control nodes to realize the complete synchronization of the nonlinear coupled discontinuous complex networks is our future work.

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