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Robust $L_2 - L_\infty$ Filter Design for Uncertain 2-D Continuous Nonlinear Delayed Systems With Saturation

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ABSTRACT This paper discusses the $L_2 - L_\infty$ filter design problem for non-linear two-dimensional (2-D) uncertain continuous systems with state delays and saturation. The non-linear function under consideration is assumed to satisfy the Lipschitz condition while the saturation term is being dealt by using a memory-less sector region methodology. A suitable Lyapunov-Krasovskii functional is considered, and the Wirtinger-based integral inequality method is used to derive some sufficient conditions which ensure that the resultant filtering error system is robustly asymptotically stable along-with the specified $L_2 - L_\infty$ disturbance attenuation level γ. A suitable example explains the derived results' usefulness.

INDEX TERMS Roesser model, 2-D systems, non-linear systems, saturation, uncertainties, $L_2 - L_\infty$ filter.

I. INTRODUCTION

There exist a variety of practical systems in the area of signal and image processing, electrical transmission systems and thermal processes modeling [1], [2], whose dynamics are entirely different from conventional one-dimensional (1-D) systems due to the dependence of systems' states upon two independent variables. The dynamics of such systems can be modeled by the theory of 2-D systems and can be expressed in several standard 2-D state-space forms such as Roesser and FM models [3], [4] etc. Similar to 1-D systems, researchers have made efforts to exploit the Lyapunov approach for the stability analysis and controller synthesis of 2-D systems [5]–[7]. Some other relevant and useful works can be found in [8] and [9].

The excessive applications of state estimation in control has made this topic a focus of research among the researchers during the recent decades. Among the filtering techniques, the standard Kalman filtering [10]–[13] is most popular but rather conservative technique due to its reliance upon the assumptions of the accurate system model and the known noise, which would consequently bring limitation to its application in practice. This motivated the practitioners to focus on $L_2 - L_\infty$ filter and H_∞ filter [14] design methodologies which make no assumption on the noise other than the bounded energy one. Among 2-D systems, the work in [15], mainly discussed the design procedure for H_∞ filter by making use of 1-D system results. Wu *et al.* [16] established a method to design H_∞ and $l_2 - l_\infty$ filters ensuring the asymptotical stability of the FE dynamics with guaranteed desired performance. Considering the uncertain systems, researchers in [17]–[19] explored H_∞ filtering problem for both discrete and continuous 2-D systems.

In fact, a dominant role is played by the non-linearities and time delays which naturally exists in practical situations and may cause degradation in the system's performance or, in some situations, the stability of system is even compromised. For 2-D non-linear systems, H_∞ filter design strategy was proposed in [20], with the non-linearity satisfying the assumption of sector boundedness. Discussion about similar problems can be seen in [21] and references therein. The saturation problem in actuators, sensors, and states occurs most frequently in real systems with a typical non-linear nature and many 2-D systems studies were devoted to this problem [22]–[25]. It should be highlighted that most of the 2-D delayed and non-delayed systems' literature have mostly solved H_∞ controller synthesis problem for the systems with non-linearities and actuator saturation [26]–[28]. Recently, a few studies have also been devoted to the H_∞ filtering problem of discrete delayed systems with sensor failures [29]. Other relevant works can be found in [30]–[33].

Moreover, the utilization of Jensen inequality is very common in integral inequality method [34] for time-delayed systems but it introduces conservatism, which lead Seuret *et al.* [35] to propose a less conservative class of inequality known as Wirtinger inequality which prompted its applications to many 1-D systems works [36], [37]. It must be indicated here that most of the 2-D continuous delayed systems' literature at hand so far have employed Jensen inequality during the analysis and synthesis rather than the Wirtinger inequality.

It should be pointed out that the aforementioned 2-D system studies have considered the practical factors such as uncertainties, time-delays, non-linearities, external disturbance and saturation either individually or in combination. However, it is a well-established fact that almost all practical systems have non-linear nature, and there might exist time-delays in system states due to several reasons. Moreover, the saturation problem in sensor occurs frequently in practice, whenever they are pushed beyond their design boundaries. Therefore, the simultaneous consideration of the aforementioned practical factors with an emphasis on utilizing the WBI inequality technique would make the analysis and synthesis problem more complex and challenging especially in the framework of 2-D systems.

In light of above-motivating factors, we propose to solve robust $L_2 - L_\infty$ filter design problem for 2-D continuous non-linear delayed systems with saturation that has not been studied as yet according to best of our knowledge. The non-linearities and uncertainties satisfy the Lipschitz and norm-bounded conditions, respectively. By considering a suitable LKF and utilizing WBI inequality, a criterion for $L_2 - L_\infty$ filter is devised to ensure that FE system is asymptotically stable with desired $L_2 - L_\infty$ performance index. An example explains the advantage of the proposed strategy.

This paper adopts the following organization. Section 2 formulates the problem and presents the necessary preliminaries. The discussion about main results is given in Section 3. Then, Section 4 and Section 5 present numerical example and concluding remarks.

Notations: In this paper, $M^T \in R^{b\times a}$ simply means the transposition of $M \in R^{a\times b}$; $A > 0$ and $A \geq 0$ show the positive definite and semi-definite matrices, respectively; $diag\{.\}$ is notation reserved for the block diagonal matrix. The symmetric terms in matrices are shown by "$*$". The zero and the identity matrices with suitable dimensions are shown by the notations I and 0, respectively; $0_{n,n}$ means $n \times n$ zeros matrix. If the matrix dimensions are not explicitly specified then they should be assumed compatible for algebric operations. The L_2-norm of a 2-D signal $w(t_1, t_2)$ is given by:

$$\|w(t_1,t_2)\|_2^2 = \int_0^\infty \int_0^\infty w^T(t_1,t_2)w(t_1,t_2)dt_1dt_2.$$

II. PROBLEM FORMULATION AND PRELIMINARIES

Let us consider the following 2-D continuous non-linear Roesser model with uncertainties, time-delays and saturation that is frequently used to represent the dynamics of many practical systems such as chemical reactors, pipe furnaces, heat exchangers, transmission lines [1] and also has large applications in image deblurring, image enhancement, and signal processing [2], [4]:

$$\begin{aligned}
\begin{bmatrix} \dfrac{\partial h(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial v(t_1,t_2)}{\partial t_2} \end{bmatrix} &= \bar{A}_1 x(t_1,t_2) + \bar{A}_2 x(t_1 - h(t_1), t_2 - v(t_2)) \\
&\quad + \bar{B} w(t_1,t_2) + \psi(t,x), \\
y(t_1,t_2) &= sat\left[\bar{D}_1 x(t_1,t_2)\right] \\
&\quad + sat\left[\bar{D}_2 x(t_1 - h(t_1), t_2 - v(t_2))\right] \\
&\quad + \bar{E} w(t_1,t_2), \\
z(t_1,t_2) &= F x(t_1,t_2), \qquad (1)
\end{aligned}$$

with

$$x(t_1,t_2) = \begin{bmatrix} h(t_1,t_2) \\ v(t_1,t_2) \end{bmatrix},$$
$$x(t_1 - h(t_1), t_2 - v(t_2)) = \begin{bmatrix} h(t_1 - h(t_1), t_2) \\ v(t_1, t_2 - v(t_2)) \end{bmatrix},$$

$\psi(t,x) = \psi(t_1, t_2, x(t_1,t_2), x(t_1 - h(t_1), t_2 - v(t_2)))$; where, the horizontal state vector and the vertical one are denoted by $h(t_1,t_2) \in R^{n_h}$ and $v(t_1,t_2) \in R^{n_v}$, respectively. $x(t_1,t_2)$ is the whole state in R^n with $n = n_h + n_v$. $y(t_1,t_2) \in R^r$ is the measured output; $w(t_1,t_2) \in R^p$ is the exogenous input that belongs to $L_2\{[0,\infty),[0,\infty)\}$; $z(t_1,t_2) \in R^q$ is the signal to be estimated; $\bar{A}_1 = A_1 + \Delta A_1$, $\bar{A}_2 = A_2 + \Delta A_2$, $\bar{B} = B + \Delta B$, $\bar{D}_1 = D_1 + \Delta D_1$, $\bar{D}_2 = D_2 + \Delta D_2$, $\bar{E} = E + \Delta E$, with A_1, A_2, B, D_1, D_2 and E are real, known appropriately dimensioned matrices while ΔA_1, ΔA_2, ΔB, ΔD_1, ΔD_2 and ΔE are uncertain matrices that are denoted as:

$$\begin{bmatrix} \Delta A_1 & \Delta A_2 & \Delta B \\ \Delta D_1 & \Delta D_2 & \Delta E \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \Lambda \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix}, \qquad (2)$$

with M_1, M_2, G_1, G_2 and G_3 being real-valued appropriately dimensioned known matrices and Λ represents an unknown time-varying matrix that satisfies $\Lambda^T\Lambda \leq I$. The delays $h(t_1)$ and $v(t_2)$ denote the functions that are time-varying in the horizontal and vertical direction, respectively and satisfy:

$$\begin{aligned}
&h(t_1) \in [h_m,\ h_M], \quad \dot{h}(t_1) \in [d_{hm}, d_{hM}], \\
&v(t_2) \in [v_m,\ v_M], \quad \dot{v}(t_2) \in [d_{vm}, d_{vM}], \qquad (3)
\end{aligned}$$

where $0 \leq h_m \leq h_M$, $0 \leq v_m \leq v_M$, $d_{hm} \leq d_{hM} < 1$, and $d_{vm} \leq d_{vM} < 1$. Assume that $\tau(t) = diag\{h(t_1)I_h, v(t_2)I_v\}$, $\dot{\tau}(t) = diag\{\dot{h}(t_1)I_h, \dot{v}(t_2)I_v\}$, $m = diag\{h_m I_h, v_m I_v\}$, $M = diag\{h_M I_h, v_M I_v\}$, $\dot{m} = diag\{d_{hm}I_h, d_{vm}I_v\}$, $\dot{M} = diag\{d_{hM}I_h, d_{vM}I_v\}$, then (3) becomes:

$$\tau(t) \in [m,\ M], \quad \dot{\tau}(t) \in [\dot{m},\ \dot{M}], \qquad (4)$$

with $0 \le m \le M$ and $\dot m \le \dot M < I$. We express the boundary conditions as follows:

$$\begin{aligned} &h(g_1,t_2) = \zeta_{g_1}(t_2), \quad \forall -h_M \le g_1 \le 0,\ 0 \le t_2 \le T_2,\\ &h(g_1,t_2) = 0, \quad \forall -h_M \le g_1 \le 0,\ t_2 > T_2,\\ &v(t_1,g_2) = \zeta_{g_2}(t_1), \quad \forall -v_M \le g_2 \le 0,\ 0 \le t_1 \le T_1,\\ &v(t_1,g_2) = 0, \quad \forall -v_M \le g_2 \le 0,\ t_1 > T_1, \end{aligned} \tag{5}$$

where $\zeta_{g_1}(t_2)$, $\zeta_{g_2}(t_1)$ are the given vectors while $T_1 < \infty$ and $T_2 < \infty$ are the positive constants.

The saturation function $sat(\cdot) : R^r \to [-1, 1]$ is defined as:

$$sat(\chi) = \left[sat(\chi_1)\ sat(\chi_2)\ \cdots\ sat(\chi_r) \right]^T, \tag{6}$$

where $sat(\chi_j) = sign(\chi_j)\, min(1, |\chi_j|), j = 1, 2, \ldots, r.$

The non-linear time-varying function $\psi(t,x) \in R^n$ satisfies $\psi(0,0) = 0$, and

$$\begin{aligned} \psi^T(t,x)\,\psi(t,x) &\le \alpha x^T(t_1,t_2)\,x(t_1,t_2)\\ &+ \beta x^T(t_1-h(t_1), t_2-v(t_2))\,x(t_1-h(t_1), t_2-v(t_2)). \end{aligned} \tag{7}$$

This work aims to estimate $z(t_1,t_2)$, with the help of following 2-D continuous filter:

$$\begin{aligned} \begin{bmatrix} \dfrac{\partial \hat h(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial \hat v(t_1,t_2)}{\partial t_2} \end{bmatrix} &= A_{1f} \begin{bmatrix} \hat h(t_1,t_2) \\ \hat v(t_1,t_2) \end{bmatrix} + Y_f y(t_1,t_2)\\ \hat z(t_1,t_2) &= F_f \begin{bmatrix} \hat h(t_1,t_2) \\ \hat v(t_1,t_2) \end{bmatrix}, \end{aligned} \tag{8}$$

where, the horizontal state vector and the vertical one of the filter are denoted by $\hat h(t_1,t_2) \in R^{n_h}$ and $\hat v(t_1,t_2) \in R^{n_v}$, respectively; $\hat z(t_1,t_2)$ is the estimate of $z(t_1,t_2)$; A_{1f}, Y_f and F_f are appropriately dimensioned filter parameter matrices to be designed.

The main results, mentioned later are established on the basis of some lemmas, which are revisited below:

Lemma 1 [38]: Consider the saturation function $sat(\chi)$ defined in (6), then the following relationship stands true:

$$\rho^T(\chi)\,\rho(\chi) \le \chi^T \chi,$$

where $\rho(\chi) = sat(\chi) - \chi$.

Lemma 2 [39]: Let W_1, W_2, W_3 and Ω be the real appropriately dimensioned matrices, then for an arbitrary scalar $\lambda > 0$ and the matrix $\Omega > 0$ satisfying $\lambda I - W_2 W_2^T > 0$, the following inequality holds:

$$\begin{aligned} &[W_1 + W_2 W_3]^T \Omega [W_1 + W_2 W_3]\\ &\qquad \le W_1^T \left[\Omega^{-1} - \lambda^{-1} W_2 W_2^T\right]^{-1} W_1 + \lambda W_3^T W_3. \end{aligned}$$

Lemma 3 [35]: For a given matrix $V > 0$ and a continuously differentiable function ω in $[\alpha_1, \alpha_2] \to R^n$, the following inequality holds:

$$-\int_{\alpha_1}^{\alpha_2} \dot\omega^T(s) V \dot\omega(s)\, ds \le \frac{1}{\alpha_2 - \alpha_1} \varpi^T \theta \varpi,$$

where $\varpi = \left[\omega^T(\alpha_2)\ \omega^T(\alpha_1)\ \frac{1}{\alpha_2-\alpha_1}\int_{\alpha_1}^{\alpha_2} \omega^T(s)\, ds \right]^T$, $\theta = \begin{bmatrix} -4V & -2V & 6V \\ * & -4V & 6V \\ * & * & -12V \end{bmatrix}$.

Lemma 4 [40]: For given integers $m_1 > 0$, $m_2 > 0$ and a scalar β in the interval (0,1), a given $m_1 \times m_1$-matrix $R > 0$, two matrices $Z_1, Z_2 \in R^{m_1 \times m_2}$; define, for all vectors $\upsilon \in R^{m_2}$ the function $\Phi(\beta, R)$ is given by:

$$\Phi(\beta, R) = \frac{1}{\beta} \upsilon^T Z_1^T R Z_1 \upsilon + \frac{1}{1-\beta} \upsilon^T Z_2^T R Z_2 \upsilon,$$

then, the following inequality holds if there exists a matrix $X \in R^{m_1 \times m_1}$, such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$:

$$\min_{\beta \in (0,1)} \Phi(\beta, R) \ge \begin{bmatrix} Z_1 \upsilon \\ Z_2 \upsilon \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} Z_1 \upsilon \\ Z_2 \upsilon \end{bmatrix}.$$

Now, by using Lemma 1, concatenation of system (1) and filter (8) results in:

$$\begin{aligned} \begin{bmatrix} \dfrac{\partial \widehat h(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial \widehat v(t_1,t_2)}{\partial t_2} \end{bmatrix} &= \widehat A_1 \widehat x(t_1,t_2) + \widehat A_2 \widehat x(t_1 - h(t_1), t_2 - v(t_2))\\ &\quad + \widehat B w(t_1,t_2) + \widehat A_3 \widehat A_3^T \widehat \psi(t,x) + \widehat Y_1 \tilde\rho\left(\widehat D_1 \widehat x(t_1,t_2)\right)\\ &\quad + \widehat Y_2 \tilde\rho\left(\widehat D_2 \widehat x(t_1 - h(t_1), t_2 - v(t_2))\right),\\ e(t_1,t_2) &= \widehat F \widehat x(t_1,t_2). \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\widehat A_1 = \Xi \tilde A_1 \Xi^T, \quad \widehat A_2 = \Xi \tilde A_2 \Xi^T, \quad \widehat B = \Xi \tilde B,\\ &\widehat F = \tilde F \Xi^T, \quad \widehat Y_1 = \Xi \tilde Y_1, \quad \widehat D_1 = \tilde D_1 \Xi^T,\\ &\widehat D_2 = \tilde D_2 \Xi^T, \quad \widehat Y_2 = \Xi \tilde Y_2,\\ &\tilde A_1 = \begin{bmatrix} \bar A_1 & 0 \\ Y_f \bar D_1 & A_{1f} \end{bmatrix},\\ &\tilde A_2 = \begin{bmatrix} \bar A_2 & 0 \\ Y_f \bar D_2 & 0 \end{bmatrix},\\ &\tilde B = \begin{bmatrix} \bar B \\ Y_f \bar E \end{bmatrix}, \quad \widehat A_3 = \Xi \begin{bmatrix} I \\ 0 \end{bmatrix},\\ &\tilde D_1 = \left[\bar D_1\ 0 \right], \quad \tilde D_2 = \left[\bar D_2\ 0 \right],\\ &\tilde F = \left[F\ -F_f \right], \quad \tilde Y_1 = \tilde Y_2 = \begin{bmatrix} 0 \\ Y_f \end{bmatrix},\\ &\Xi = \begin{bmatrix} I_{n_h} & 0 & 0 & 0 \\ 0 & 0 & I_{n_h} & 0 \\ 0 & I_{n_v} & 0 & 0 \\ 0 & 0 & 0 & I_{n_v} \end{bmatrix},\\ &\widehat h(t_1,t_2) = \begin{bmatrix} h(t_1,t_2) \\ \hat h(t_1,t_2) \end{bmatrix},\\ &\widehat v(t_1,t_2) = \begin{bmatrix} v(t_1,t_2) \\ \hat v(t_1,t_2) \end{bmatrix},\\ &\widehat x(t_1,t_2) = \begin{bmatrix} \widehat h(t_1,t_2) \\ \widehat v(t_1,t_2) \end{bmatrix}, \end{aligned}$$

$$\widehat{h}(t_1 - h(t_1), t_2) = \begin{bmatrix} h(t_1 - h(t_1), t_2) \\ \hat{h}(t_1 - h(t_1), t_2) \end{bmatrix},$$
$$\widehat{v}(t_1, t_2 - v(t_2)) = \begin{bmatrix} v(t_1, t_2 - v(t_2)) \\ \hat{v}(t_1, t_2 - v(t_2)) \end{bmatrix},$$
$$\widehat{x}(t_1 - h(t_1), t_2 - v(t_2)) = \begin{bmatrix} \widehat{h}(t_1 - h(t_1), t_2) \\ \widehat{v}(t_1, t_2 - v(t_2)) \end{bmatrix},$$
$$e(t_1, t_2) = z(t_1, t_2) - \hat{z}(t_1, t_2)$$

and

$$\widehat{\psi}(t, x) = \begin{bmatrix} \psi^T(t, x) & 0 & 0 & \psi^T(t, x) \end{bmatrix}^T.$$

Definition 1 [13]: A filtering error system, represented by (9), is asymptotically stable with a specified $L_2 - L_\infty$ disturbance attenuation level γ, if under zero boundary conditions the following holds true:

$$\|e(t_1, t_2)\|_\infty^2 < \gamma^2 \|w(t_1, t_2)\|_2^2,$$

with $\|e(t_1, t_2)\|_\infty^2 = \sup_{t_1, t_2} e(t_1, t_2)^T e(t_1, t_2)$.

III. MAIN RESULTS

The aim of this section is to design the parameters of the filter proposed in (8) such that the FE system (9) is asymptotically stable with a specified $L_2 - L_\infty$ performance index γ.

Theorem 1: Given the values of $\tau(t)$, $\dot{\tau}(t)$, satisfying (4) and the parameters $\alpha > 0$, $\beta > 0$, and $\gamma > 0$. Assume that there exist matrices $P = diag\{P_h, P_v\} > 0$, $Q = diag\{Q_h, Q_v\} > 0$, $R = diag\{R_h, R_v\} > 0$, $S = diag\{S_h, S_v\} > 0$, $X_{11} = diag\{X_{h11}, X_{v11}\}$, $X_{12} = diag\{X_{h12}, X_{v12}\}$, $X_{21} = diag\{X_{h21}, X_{v21}\}$, $X_{22} = diag\{X_{h22}, X_{v22}\}$, with suitable dimensions and scalars $\lambda_{w_1} > 0$, $w_1 = 1, 2, ..., 6$, such the below mentioned inequalities hold:

$$\tilde{\Omega} = \begin{bmatrix} \bar{\Omega} & \Psi_a & 0 \\ * & \Psi_b & \Psi_c \\ * & * & -M\lambda_4 I \end{bmatrix} < 0, \quad (10a)$$
$$\begin{bmatrix} R & 0 & X_{11} & X_{12} \\ * & 3R & X_{21} & X_{22} \\ * & * & R & 0 \\ * & * & * & 3R \end{bmatrix} > 0, \quad (10b)$$
$$\begin{bmatrix} P & \widehat{F}^T \\ * & \gamma^2 I \end{bmatrix} > 0, \quad (10c)$$

then, the FE system, represented by (9), is asymptotically stable with a specified $L_2 - L_\infty$ disturbance attenuation level γ; where

$$\bar{\Omega} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & P\widehat{B} \\ * & \bar{\Omega}_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & 0 \\ * & * & \Omega_{33} & 0 & \Omega_{35} & 0 \\ * & * & * & \Omega_{44} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & -I \end{bmatrix},$$

$$\bar{\Omega}_{11} = \widehat{A}_1^T P + P\widehat{A}_1 + \lambda_1 \alpha \widehat{A}_3 \widehat{A}_3^T + Q + S + M\lambda_4 \alpha \widehat{A}_3 \widehat{A}_3^T - \bar{M}4R, \quad \bar{\Omega}_{12} = P\widehat{A}_2 - \bar{M} \times (2R + X_{11} + X_{21} + X_{12} + X_{22}),$$
$$\Omega_{13} = -\bar{M}(-X_{11} - X_{21} + X_{12} + X_{22}),$$
$$\Omega_{14} = \tau(t) P + \bar{M}(6R),$$
$$\Omega_{15} = -\bar{M}(-2X_{12} - 2X_{22}), \quad \bar{\Omega}_{22} = \lambda_1 \beta \widehat{A}_3 \widehat{A}_3^T - (I - \dot{\tau}(t)) Q + M\lambda_4 \beta \widehat{A}_3 \widehat{A}_3^T - \bar{M}\big(8R - X_{11}^T + X_{21}^T - X_{12}^T + X_{22}^T - X_{11} + X_{21} - X_{12} + X_{22}\big),$$
$$\Omega_{23} = -\bar{M}(X_{11} - X_{21} - X_{12} + X_{22} + 2R),$$
$$\Omega_{24} = -\tau(t)(I - \dot{\tau}(t)) P + \bar{M}6R,$$
$$\Omega_{25} = (M - \tau(t))(I - \dot{\tau}(t)) P - \bar{M} \times (2X_{12} - 2X_{22} - 6R), \quad \Omega_{33} = -S - \bar{M}(4R - 2X_{22}),$$
$$\Omega_{35} = -(M - \tau(t)) P + \bar{M}(6R - 4X_{22}),$$
$$\Omega_{44} = \Omega_{55} = -\bar{M}(12R),$$
$$\Psi_a = \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_3 & \Psi_4 & \Psi_5 & \Psi_6 \end{bmatrix},$$
$$\Psi_1 = \begin{bmatrix} \widehat{A}_3^T P & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Psi_2 = \begin{bmatrix} \widehat{Y}_1^T P & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Psi_3 = \begin{bmatrix} \widehat{Y}_2^T P & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Psi_4 = \begin{bmatrix} \left(\widehat{D}_1 (\lambda_2 I + M\lambda_5 I)\right)^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Psi_5 = \begin{bmatrix} 0 & \left(\widehat{D}_2 (\lambda_3 I + M\lambda_6 I)\right)^T & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$
$$\Psi_6 = \begin{bmatrix} \left(M\widehat{A}_1^T R\right)^T & \left(M\widehat{A}_2^T R\right)^T & 0 & 0 & 0 & \left(M\widehat{B}^T R\right)^T \end{bmatrix}^T,$$
$$\Psi_b = diag\{-\lambda_1 I, -\lambda_2 I, -\lambda_3 I, -(\lambda_2 + M\lambda_5) I, -(\lambda_3 + M\lambda_6) I, -MR\},$$
$$\Psi_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & R^T M^T \end{bmatrix}^T.$$

Proof: The proof of Theorem 1 is given in the Appendix A.

Remark 1: Please note that Lemma 2 facilitates in avoiding the multiplication of saturation and non-linearity terms during the calculation of unidirectional derivative of the terms $\int_{t_1 - h_M}^{t_1} \int_{\theta_2}^{t_1} \frac{\partial \widehat{h}^T(\theta_1, t_2)}{\partial \theta_1} R_h \frac{\partial \widehat{h}(\theta_1, t_2)}{\partial \theta_1} d\theta_1 d\theta_2$ and $\int_{t_2 - v_M}^{t_2} \int_{\theta_2}^{t_2} \frac{\partial \widehat{v}^T(t_1, \theta_1)}{\partial \theta_1} R_v \frac{\partial \widehat{v}(t_1, \theta_1)}{\partial \theta_1} d\theta_1 d\theta_2$. Thus, a non-linear problem is transformed into a linear one.

Remark 2: It can be noticed that the results presented in Theorem 1 cannot be directly used to find the filter parameters. Therefore, in order to find the filter parameters we present Theorem 2 below.

Theorem 2: For some known paramteres $\alpha > 0$, $\beta > 0$ and $\gamma > 0$; provided that there exist some matrices $P_1 = diag\{P_{h1}, P_{v1}\} > 0$, $P_2 = diag\{P_{h2}, P_{v2}\} > 0$, $R_1 = diag\{R_{h1}, R_{v1}\} > 0$, $Q_1 = diag\{Q_{h1}, Q_{v1}\} > 0$, $S_1 = diag\{S_{h1}, S_{v1}\} > 0$, X_{11}, X_{12}, X_{21}, X_{22}, ι_{Yf}, ι_{A1f}, ι_f of appropriate dimensions and scalars $\lambda_{w_1} > 0$, $\varepsilon_{w_2} > 0$, $w_1 = 1, 2, ..., 6$, $w_2 = 1, 2, ..., 5$, such that (4) is satisfied and following inequalities are true:

$$\widehat{\phi} = \begin{bmatrix} \widehat{\phi}_a & \widehat{\phi}_b \\ * & \widehat{\phi}_d \end{bmatrix} < 0, \quad (11a)$$

$$\begin{bmatrix} \hat{R}_1 & 0 & \hat{X}_{11} & \hat{X}_{12} \\ * & 3\hat{R}_1 & \hat{X}_{21} & \hat{X}_{22} \\ * & * & \hat{R}_1 & 0 \\ * & * & * & 3\hat{R}_1 \end{bmatrix} > 0, \tag{11b}$$

$$\begin{bmatrix} P_1 & P_2 & F^T \\ * & \delta P_2 & -\iota_f^T \\ * & * & \gamma^2 I \end{bmatrix} > 0, \tag{11c}$$

then, a filter having structure as proposed in (8) exists such that the FE system (9) is not only asymptotically stable but also satisfies a specified $L_2 - L_\infty$ disturbance attenuation level γ. Consequently, the desired filter matrices can be found by:

$$Y_f = P_2^{-1}\iota_{Yf}, \quad A_{1f} = P_2^{-1}\iota_{A1f}, \; \iota_f = F_f. \tag{12}$$

where

$$\widehat{\phi}_a = \begin{bmatrix} \widehat{\phi}_{11} & \widehat{\phi}_{12} & \widehat{\phi}_{13} & \widehat{\phi}_{14} & \widehat{\phi}_{15} & \widehat{\phi}_{16} & \widehat{\phi}_{17} & \widehat{\phi}_{18} & \widehat{\phi}_{z} & \widehat{\phi}_{a1} \\ * & \widehat{\phi}_{22} & \widehat{\phi}_{23} & \widehat{\phi}_{24} & \widehat{\phi}_{25} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{29} & \widehat{\phi}_{a2} \\ * & * & \widehat{\phi}_{33} & \widehat{\phi}_{z} & \widehat{\phi}_{35} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ * & * & * & \widehat{\phi}_{44} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ * & * & * & * & \widehat{\phi}_{55} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ * & * & * & * & * & \widehat{\phi}_{66} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{a3} \\ * & * & * & * & * & * & \widehat{\phi}_{77} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ * & * & * & * & * & * & * & \widehat{\phi}_{88} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ * & * & * & * & * & * & * & * & \widehat{\phi}_{99} & \widehat{\phi}_{z} \\ * & * & * & * & * & * & * & * & * & \widehat{\phi}_{a4} \end{bmatrix},$$

$$\widehat{\phi}_b = \begin{bmatrix} \widehat{\phi}_{z} & \widehat{\phi}_{b2} & \widehat{\phi}_{b2} & \widehat{\phi}_{b2} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{8a} & \widehat{\phi}_{z} \\ \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{z} & \widehat{\phi}_{9a} \\ \widehat{\phi}_{b1} & \widehat{\phi}_{b3} & \widehat{\phi}_{b3} & \widehat{\phi}_{b3} & \widehat{\phi}_{z} & \widehat{\phi}_{z} \end{bmatrix},$$

$$\widehat{\phi}_d = diag\{-M\lambda_4 I, -\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I\},$$

$$\begin{aligned}
\widehat{\phi}_{11} &= \begin{bmatrix} \widehat{\phi}_{11a} & \widehat{\phi}_{11b} \\ * & \widehat{\phi}_{11d} \end{bmatrix} + \hat{Q} + \hat{S} - \bar{M}4\hat{R} \\
&\quad + (\varepsilon_1 I + \varepsilon_4 I)\tilde{G}_1^T\tilde{G}_1 + \widehat{A}_3(\lambda_1\alpha I + M\lambda_4\alpha I)\widehat{A}_3^T, \\
\widehat{\phi}_{11a} &= P_1A_1 + \iota_{Yf}D_1 + (P_1A_1 + \iota_{Yf}D_1)^T, \\
\widehat{\phi}_{11b} &= \iota_{A1f} + (P_2A_1 + \delta\iota_{Yf}D_1)^T, \\
\widehat{\phi}_{11d} &= \delta\iota_{A1f} + (\delta\iota_{A1f})^T, \\
\widehat{\phi}_{12} &= \begin{bmatrix} P_1A_2 + \iota_{Yf}D_2 & 0 \\ P_2A_2 + \delta\iota_{Yf}D_2 & 0 \end{bmatrix} \\
&\quad - \bar{M}\left(2\hat{R} + \hat{X}_{11} + \hat{X}_{21} + \hat{X}_{12} + \hat{X}_{22}\right), \\
\widehat{\phi}_{13} &= -\bar{M}\left(-\hat{X}_{11} - \hat{X}_{21} + \hat{X}_{12} + \hat{X}_{22}\right), \\
\widehat{\phi}_{14} &= \tau(t)\hat{P} + 6\bar{M}\hat{R}, \; \widehat{\phi}_{15} = -\bar{M}\left(-2\hat{X}_{12} - 2\hat{X}_{22}\right), \\
\widehat{\phi}_{15} &= -\bar{M}\left(-2\hat{X}_{12} - 2\hat{X}_{22}\right), \\
\widehat{\phi}_{16} &= \begin{bmatrix} P_1B + \iota_{Yf}E & P_1 \\ P_2B + \delta\iota_{Yf}E & P_2 \end{bmatrix}, \quad \widehat{\phi}_{17} = \begin{bmatrix} \iota_{Yf} & \iota_{Yf} \\ \delta\iota_{Yf} & \delta\iota_{Yf} \end{bmatrix}, \\
\widehat{\phi}_{18} &= \begin{bmatrix} (\lambda_2 I + M\lambda_5 I)D_1^T & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{\phi}_z = 0_{2,2}, \\
\widehat{\phi}_{a1} &= M\begin{bmatrix} A_1^TR_1 + \delta_1 D_1^T\iota_{Yf}^T & \delta_1 A_1^TP_2 + \delta_2 D_1^T\iota_{Yf}^T \\ \delta_1\iota_{A1f}^T & \delta_2\iota_{A1f}^T \end{bmatrix}, \\
\widehat{\phi}_{b2} &= \begin{bmatrix} P_1M_1 + \iota_{Yf}M_2 & 0 \\ P_2M_1 + \delta\iota_{Yf}M_2 & 0 \end{bmatrix}, \\
\widehat{\phi}_{22} &= \widehat{A}_3(\lambda_1\beta I + M\lambda_4\beta I)\widehat{A}_3^T \\
&\quad + (\varepsilon_2 I + \varepsilon_5 I)\tilde{G}_2^T\tilde{G}_2 - (I - \dot{\tau}(t))\hat{Q} \\
&\quad - \bar{M}\Big(8\hat{R} - \hat{X}_{11}^T + \hat{X}_{21}^T - \hat{X}_{12}^T \\
&\quad + \hat{X}_{22}^T - \hat{X}_{11} + \hat{X}_{21} - \hat{X}_{12} + \hat{X}_{22}\Big), \\
\widehat{\phi}_{23} &= -\bar{M}\left(\hat{X}_{11} - \hat{X}_{21} - \hat{X}_{12} + \hat{X}_{22} + 2\hat{R}\right), \\
\widehat{\phi}_{24} &= -\tau(t)(I - \dot{\tau}(t))\hat{P} + \bar{M}6\hat{R}, \\
\widehat{\phi}_{25} &= (M - \tau(t))(I - \dot{\tau}(t))\hat{P} - \bar{M} \\
&\quad \times \left(2\hat{X}_{12} - 2\hat{X}_{22} - 6\hat{R}\right), \\
\widehat{\phi}_{29} &= \begin{bmatrix} (\lambda_3 I + MI\lambda_6)D_2^T & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{\phi}_{a2} = M \\
&\quad \times \begin{bmatrix} A_2^TR_1 + \delta_1 D_2^T\iota_{Yf}^T & \delta_1 A_2^TP_2 + \delta_2 D_2^T\iota_{Yf}^T \\ 0 & 0 \end{bmatrix}, \\
\widehat{\phi}_{33} &= -\hat{S} - \bar{M}\left(4\hat{R} - 2\hat{X}_{22}\right), \\
\widehat{\phi}_{35} &= -(M - \tau(t))\hat{P} + \bar{M}\left(6\hat{R} - 4\hat{X}_{22}\right), \\
\widehat{\phi}_{44} &= \widehat{\phi}_{55} = -\bar{M}\left(12\hat{R}\right), \\
\widehat{\phi}_{66} &= diag\left\{-I + \varepsilon_3 IG_3^TG_3, -\lambda_1 I\right\}, \\
\widehat{\phi}_{a3} &= M \\
&\quad \times \begin{bmatrix} B^TR_1 + \delta_1 E^T\iota_{Yf}^TP_2 & \delta_1 B^TP_2 + \delta_2 E^T\iota_{Yf}^TP_2 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\widehat{\phi}_{77} = diag\{-\lambda_2 I, -\lambda_3 I\},$$
$$\widehat{\phi}_{88} = -(\lambda_2 I + M\lambda_5 I),\ \widehat{\phi}_{8a} = (\lambda_2 I + M\lambda_5 I)M_2,$$
$$\widehat{\phi}_{99} = -(\lambda_3 I + M\lambda_6 I),\ \widehat{\phi}_{9a} = (\lambda_3 I + M\lambda_6 I)M_2,$$
$$\widehat{\phi}_{a4} = \begin{bmatrix} -MR_1 & -\delta_1 MP_2 \\ * & -\delta_2 MP_2 \end{bmatrix},$$
$$\widehat{\phi}_{b1} = \begin{bmatrix} MR_1 & \delta_1 MP_2 \\ * & \delta_2 MP_2 \end{bmatrix},$$
$$\widehat{\phi}_{b3} = \begin{bmatrix} MR_1M_1 + \delta_1 M\iota_{Yf}M_2 & 0 \\ \delta_1 MP_2M_1 + \delta_2 M\iota_{Yf} & 0 \end{bmatrix}.$$

Proof: The inequality (10a) can be decomposed into two parts as $\tilde{\Omega} = \phi + \Delta\phi$. Replacing $\widehat{A}_1, \widehat{A}_2, \widehat{B}, \widehat{D}_1$ and $\widehat{D}_2$ in (10a) by $\hat{A}_1, \hat{A}_2, \hat{B}, \hat{D}_1$ and $\hat{D}_2$, respectively results in ϕ; where $\hat{A}_1 = \Xi \begin{bmatrix} A_1 & 0 \\ Y_fD_1 & A_{1f} \end{bmatrix} \Xi^T, \hat{A}_2 = \Xi \begin{bmatrix} A_2 & 0 \\ Y_fD_2 & 0 \end{bmatrix} \Xi^T, \hat{B} = \Xi \begin{bmatrix} B \\ Y_fE \end{bmatrix}$, $\hat{D}_1 = \begin{bmatrix} D_1 & 0 \end{bmatrix} \Xi^T, \hat{D}_2 = \begin{bmatrix} D_2 & 0 \end{bmatrix} \Xi^T$. While, $\Delta\phi$ is given by:

$$\Delta\phi = \begin{bmatrix} \Delta\phi_\alpha & \Delta\phi_\beta \\ * & \Delta\phi_\gamma \end{bmatrix}, \tag{13}$$

with

$$\Delta\phi_\alpha = \begin{bmatrix} \Delta\hat{\phi}_{11} & P\Delta\hat{A}_2 & 0 & 0 & 0 & P\Delta\hat{B} \\ * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\Delta\phi_\beta = \begin{bmatrix} 0 & 0 & 0 & \Delta\phi_{a1} & 0 & M\Delta\hat{A}_1^TR & 0 \\ 0 & 0 & 0 & 0 & \Delta\phi_{b1} & M\Delta\hat{A}_2^TR & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M\Delta\hat{B}^TR & 0 \end{bmatrix},$$

$$\Delta\phi_\gamma = 0_{7n,7n}, \quad \Delta\hat{\phi}_{11} = P\Delta\hat{A}_1 + \Delta\hat{A}_1^TP,$$
$$\Delta\phi_{a1} = (\lambda_2 I + M\lambda_5 I)\Delta\hat{D}_1^T,$$
$$\Delta\phi_{b1} = (\lambda_3 I + M\lambda_6 I)\Delta\hat{D}_2^T,$$
$$\Delta\hat{A}_1 = \Xi \begin{bmatrix} M_1\Lambda G_1 & 0 \\ Y_fM_2\Lambda G_1 & 0 \end{bmatrix} \Xi^T,$$
$$\Delta\hat{A}_2 = \Xi \begin{bmatrix} M_1\Lambda G_2 & 0 \\ Y_fM_2\Lambda G_2 & 0 \end{bmatrix} \Xi^T,$$
$$\Delta\hat{B} = \Xi \begin{bmatrix} M_1\Lambda G_3 \\ Y_fM_2\Lambda G_3 \end{bmatrix}, \quad \Delta\hat{D}_1 = \begin{bmatrix} M_2\Lambda G_1 & 0 \end{bmatrix} \Xi^T$$

and

$$\Delta\hat{D}_2 = \begin{bmatrix} M_2\Lambda G_2 & 0 \end{bmatrix} \Xi^T.$$

By (2), $\Delta\phi$ can be dealt easily, which would consequently transform the non-linear inequality (10a) into a linear one. To elaborate the procedure, we only consider the terms $\Delta\hat{\phi}_{11}$, $M\Delta\hat{A}_1^TR$ and $MR\Delta\hat{A}_1$ from (13), as follows:

$$\begin{bmatrix} \Delta\hat{\phi}_{11} & \Delta\hat{\phi}_{12} \\ * & 0_{12n,12n} \end{bmatrix} \le \varepsilon_1^{-1}\eta_a\eta_a^T + \varepsilon_1\eta_b\eta_b^T,$$

where

$$\Delta\hat{\phi}_{12} = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,M\Delta\hat{A}_1^TR\,0 \end{bmatrix},$$
$$\eta_a = \begin{bmatrix} \left(P\tilde{M}_1\right)^T & 0\,0\,0\,0 & 0\,0\,0\,0\,0\,0 & \left(MR\tilde{M}_1\right)^T & 0 \end{bmatrix}^T$$

and

$$\eta_b = \begin{bmatrix} \tilde{G}_1 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}^T.$$

Rest of the uncertain terms in $\Delta\phi$ can be dealt in the similar manner and thus, by Schur complement following can be obtained:

$$\Delta\phi \le \begin{bmatrix} \eta_{11} & \eta_{12} \\ * & \eta_{22} \end{bmatrix},$$

where

$$\eta_{12} = \begin{bmatrix} P\tilde{M}_1 & P\tilde{M}_1 & P\tilde{M}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{12a} & 0 \\ 0 & 0 & 0 & 0 & \eta_{12b} \\ MR\tilde{M}_1 & MR\tilde{M}_1 & MR\tilde{M}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\eta_{12a} = (\lambda_2 I + M\lambda_5 I)M_2,$$
$$\eta_{12b} = (\lambda_3 I + M\lambda_6 I)M_2,$$
$$\eta_{11} = diag\Big\{(\varepsilon_1 I + \varepsilon_4 I)\tilde{G}_1^T\tilde{G}_1, (\varepsilon_2 I + \varepsilon_5 I)\tilde{G}_2^T\tilde{G}_2,$$
$$0, 0, 0, \varepsilon_3 I\tilde{G}_3^T\tilde{G}_3\Big\},$$
$$\eta_{22} = diag\{-\varepsilon_1 I, -\varepsilon_2 I, -\varepsilon_3 I, -\varepsilon_4 I, -\varepsilon_5 I\},$$
$$\tilde{M}_1 = \Xi \begin{bmatrix} M_1 \\ Y_fM_2 \end{bmatrix},$$
$$\tilde{G}_1 = \begin{bmatrix} G_1 & 0 \end{bmatrix} \Xi^T,$$

and

$$\tilde{G}_2 = \begin{bmatrix} G_2 & 0 \end{bmatrix} \Xi^T.$$

Then, $\tilde{\Omega}$ can be equivalently expressed as:

$$\begin{bmatrix} \phi + \eta_{11} & \eta_{12} \\ * & \eta_{22} \end{bmatrix} < 0. \tag{14}$$

The inequalities (11a)-(11c) can be readily obtained from (10a)-(10c), respectively, if we assume that matrices $P, Q, R, S, X_{11}, X_{12}, X_{21}$, and X_{22} have the following form with the assumption that matrix P_2 is non-singular:

$$\Xi P \Xi^T = \Xi diag\left\{\begin{bmatrix} P_{h1} & P_{h2} \\ P_{h2} & \delta P_{h2} \end{bmatrix}, \begin{bmatrix} P_{v1} & P_{v2} \\ P_{v2} & \delta P_{v2} \end{bmatrix}\right\} \Xi^T$$
$$= \begin{bmatrix} P_1 & P_2 \\ P_2 & \delta P_2 \end{bmatrix} = \hat{P}, \quad \Xi R \Xi^T = \Xi$$
$$\times\, diag\left\{\begin{bmatrix} R_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix}, \begin{bmatrix} R_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix}\right\} \Xi^T$$

$$= \begin{bmatrix} R_1 & \delta_1 P_2 \\ \delta_1 P_2 & \delta_2 P_2 \end{bmatrix} = \hat{R}, \quad \Xi Q \Xi^T = \Xi \times diag\left\{ \begin{bmatrix} Q_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix}, \begin{bmatrix} Q_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix} \right\} \Xi^T = \begin{bmatrix} Q_1 & \delta_1 P_2 \\ \delta_1 P_2 & \delta_2 P_2 \end{bmatrix} = \hat{Q}, \quad \Xi S \Xi^T = \Xi \times diag\left\{ \begin{bmatrix} S_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix}, \begin{bmatrix} S_{h1} & \delta_1 P_{h2} \\ \delta_1 P_{h2} & \delta_2 P_{h2} \end{bmatrix} \right\} \Xi^T = \begin{bmatrix} S_1 & \delta_1 P_2 \\ \delta_1 P_2 & \delta_2 P_2 \end{bmatrix} = \hat{S},$$

$$\Xi X_{11} \Xi^T = \begin{bmatrix} X_{11a} & X_{11b} \\ X_{11c} & X_{11d} \end{bmatrix} = \hat{X}_{11},$$

$$\Xi X_{12} \Xi^T = \begin{bmatrix} X_{12a} & X_{12b} \\ X_{12c} & X_{12d} \end{bmatrix} = \hat{X}_{12},$$

$$\Xi X_{21} \Xi^T = \begin{bmatrix} X_{21a} & X_{21b} \\ X_{21c} & X_{21d} \end{bmatrix} = \hat{X}_{21},$$

$$\Xi X_{22} \Xi^T = \begin{bmatrix} X_{22a} & X_{22b} \\ X_{22c} & X_{22d} \end{bmatrix} = \hat{X}_{22}.$$

This concludes our proof.

Remark 3: It is important to highlight that the results established here are more general and advantageous due to the utilization of WBI inequality, and the simultaneous consideration of practical factors such as uncertainties, time-delays, non-linearities, saturation and disturbance which make the analysis and synthesis problem more challenging and exciting. To discuss a few similar relevant 2-D system studies, H_∞ filtering problem for discrete linear systems has been presented in [18] and [19] for the systems with discrete dynamics. Contrarily, the similar results related to 2-D non-linear (continuous and discrete) systems have been presented in [20], [22], [23], [26], [28], [30]–[33], and [35]. It can be noticed that the authors in above cited relevant references have either considered these practical factors individually or in combination but not simultaneously, which explains the importance of the problem being addressed in our paper.

Remark 4: In the following, a procedure is presented that can be used to find the filter matrices by solving the inequalities in Theorem 2:

Step 1: Input matrices $A_1, A_2, B, D_1, D_2, F, E, G_1, G_2, G_3$, M_1 and M_2.

Step 2: Input the parameters m, M, $\dot{m}$, $\dot{M}$, α, β, δ, δ_1, δ_2 and γ.

Step 3: Feasible solution of inequalities (11a)-(11c), results in matrices P_1, P_2, R_1, Q_1, S_1, X_{11}, X_{12}, X_{21}, X_{22}, ι_{Yf}, ι_{A1f}, ι_f and the parameters $\lambda_{w_1} > 0$, $\varepsilon_{w_2} > 0$, $w_1 = 1, 2, \ldots, 6$, $w_2 = 1, 2, \ldots, 5$, then filter matrices can obtained by (12).

IV. ILLUSTRATIVE EXAMPLE

Let us consider the following mathematical model that represents the dynamics of some process in gas absorption, water stream heating and air drying [1].

$$\frac{\partial^2 \theta(s,\tau)}{\partial s \partial \tau} = a_2 \frac{\partial \theta(s,\tau)}{\partial s} + a_1 \frac{\partial \theta(s,\tau)}{\partial \tau} + a_0 \theta(s,\tau) + a_d s(s, t - d(\tau)) + b\psi(s,\tau) + ew(s,\tau) \quad (15)$$

where $\theta(s,\tau)$ represents some unknown function at $s \in [0, s_f]$(space) and $\tau \in [0, \infty)$ (time). The parameters a_2, a_0, a_1, e, a_d and b are some real known coefficients; $d(\tau)$, $\psi(s,\tau)$ (satisfying (7)) and $w(s,\tau)$ (L_2 norm-bounded) denote the time-varying delay, non-linearity, and the disturbance, respectively.

Following the procedure as adopted in [27], the dynamics of the system (15) can be transformed into system (1). For an illustrative purpose, let us consider system (1) with the parameters given in Table 1.

TABLE 1. System parameters.

$A_1 = \begin{bmatrix} -0.1 & 0.11 \\ 0 & 0.01 \end{bmatrix}$	$A_2 = \begin{bmatrix} -0.1 & -0.05 \\ 0 & -0.1 \end{bmatrix}$
$B = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}$	$D_1 = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix}$
$D_2 = \begin{bmatrix} 0.05 & 0 \end{bmatrix}$	$F = \begin{bmatrix} -0.02 & 0.81 \end{bmatrix}$
$G_1 = \begin{bmatrix} 0.01 & 0 \end{bmatrix}$	$G_2 = \begin{bmatrix} -0.02 & -0.03 \end{bmatrix}$
$G_3 = 1.55$	$E = 1.55$
$M_1 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}$	$M_2 = -0.34$
$\Lambda = 0.5 \sin(t_1 + t_2)$	$h(t_1) = 0.5 + 0.5 \sin\left(\frac{\pi t_1}{2}\right)$
$v(t_2) = 1 + \sin\left(\frac{\pi t_2}{2}\right)$	$\alpha = \beta = 0.001$
$\delta = 0.38$	$\delta_1 = 0.19$
$\delta_2 = 0.53$	$\gamma = 1$

TABLE 2. Filter parameters.

$A_{1f} = \begin{bmatrix} -0.6121 & -0.0451 \\ -0.0910 & -0.3059 \end{bmatrix}$	$Y_f = \begin{bmatrix} -0.0203 \\ -0.0442 \end{bmatrix}$
$F_f = \begin{bmatrix} 0.0067 & -0.2791 \end{bmatrix}$	

Then, upon solving the inequalities (11)-(12) established in Theorem 2, the filter (8) with the specifications mentioned in Table 2 can be obtained. Moreover, the maximum upper bound of time-delay has also been estimated in Table 3 by checking the feasibility of inequalities established in Theorem 2.

TABLE 3. Maximum upper bound of delay $h_M = v_M$ and the feasibility of inequalities in Theorem 2 with $h_m = v_m = 0$ and the parameters given in Table 1.

$h_M = v_M$	0.5	1	1.5	2	2.1	2.2
Feasibility	✓	✓	✓	✓	×	×

The simulation is carried out to depict the responses $\widehat{h}(t_1,t_2)$, $\widehat{v}(t_1,t_2)$ and $e(t_1,t_2)$ of the filtering error system (9) in the Figures 1-3, respectively. We consider the noise as $w(t_1,t_2) = e^{-0.5(t_1+t_2)}$.

The boundary conditions defined in (5) are chosen as follows:

$$h(g_1,t_2) = 0.5, \quad \forall -1 \le g_1 \le 0, \ 0 \le t_2 \le 3,$$
$$h(g_1,t_2) = 0, \quad \forall -1 \le g_1 \le 0, \ t_2 > 3,$$
$$v(t_1,g_2) = 0.5, \quad \forall -2 \le g_2 \le 0, \ 0 \le t_1 \le 3,$$
$$v(t_1,g_2) = 0, \quad \forall -2 \le g_2 \le 0, \ t_1 > 3.$$

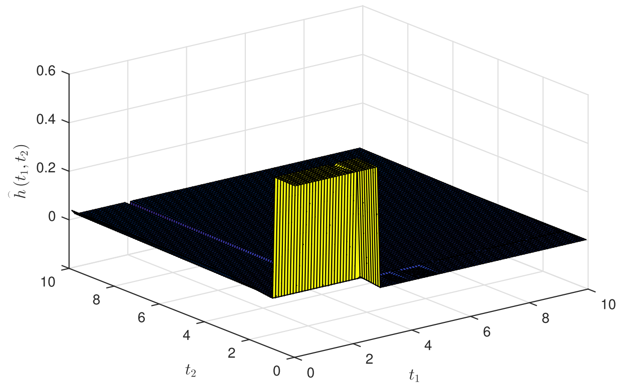

FIGURE 1. The response of $\widehat{h}(t_1, t_2)$.

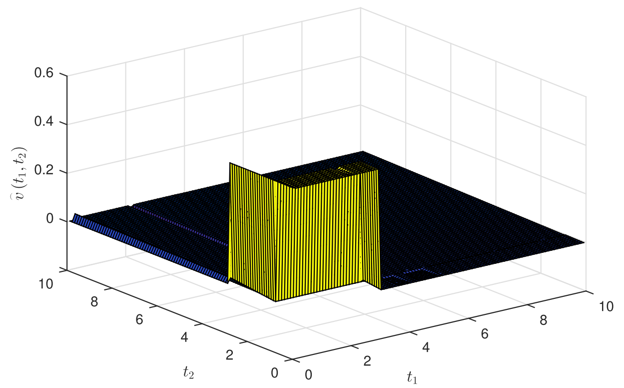

FIGURE 2. The response of $\widehat{v}(t_1, t_2)$.

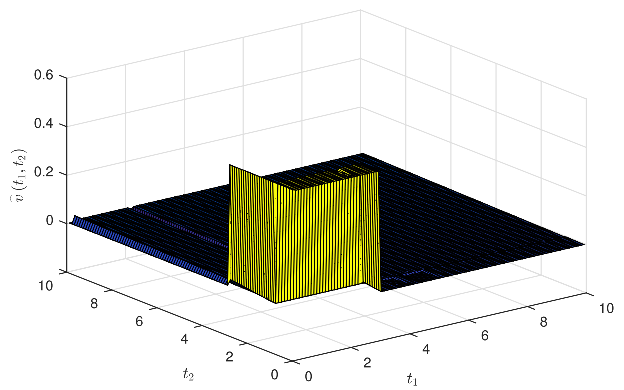

FIGURE 3. The filtering error response $e(t_1, t_2)$.

It is evident in Figures 1-3 that the FE system represented by (9) converges ultimately to zero and hence is asymptotically stable. Therefore, based upon the above responses it can be deduced that our proposed $L_2 - L_\infty$ filter meets the desired requirements.

V. CONCLUSION

The solution of $L_2 - L_\infty$ filter design problem for uncertain 2-D continuous non-linear system with time-delays has been obtained in this paper. During the computation of derivative of LKF, we dealt with some terms by the WBI inequality technique which is considered to reduce conservatism of the Jensen inequality method. Lemma 1 and Lipschitz conditions were used to linearize the saturation and non-linearity, respectively. Thus, the asymptotical stability conditions for the FE system (9) having a desired $L_2 - L_\infty$ performance were formulated in terms of inequalities. The usefulness of the proposed results has been demonstrated through a suitable example. In future, we aim to extend these results for 2-D systems with discrete dynamics.

APPENDIX A

Proof: Let us take into account the functional of the following form:

$$V\left(\widehat{x}(t_1,t_2)\right) = V_h\left(\widehat{h}(t_1,t_2)\right) + V_v\left(\widehat{v}(t_1,t_2)\right), \quad (16)$$

with

$$\begin{aligned}
&V_h\left(\widehat{h}(t_1,t_2)\right) \\
&\quad = \xi_h^T(t_1,t_2) P_h \xi_h(t_1,t_2) \\
&\qquad + \int_{t_1-h(t_1)}^{t_1} \widehat{h}^T(\theta_1,t_2) Q_h \widehat{h}(\theta_1,t_2)\, d\theta_1 \\
&\qquad + \int_{t_1-h_M}^{t_1} \widehat{h}^T(\theta_1,t_2) S_h \widehat{h}(\theta_1,t_2)\, d\theta_1 \\
&\qquad + \int_{t_1-h_M}^{t_1} \int_{\theta_2}^{t_1} \frac{\partial \widehat{h}^T(\theta_1,t_2)}{\partial\theta_1} R_h \frac{\partial \widehat{h}(\theta_1,t_2)}{\partial\theta_1} d\theta_1 d\theta_2, \\
&V_v\left(\widehat{v}(t_1,t_2)\right) \\
&\quad = \xi_v^T(t_1,t_2) P_v \xi_v(t_1,t_2) \\
&\qquad + \int_{t_2-v(t_2)}^{t_2} \widehat{v}^T(t_1,\theta_2) Q_v \widehat{v}(t_1,\theta_2)\, d\theta_2 \\
&\qquad + \int_{t_2-v_M}^{t_2} \widehat{v}^T(t_1,\theta_2) S_v \widehat{v}(t_1,\theta_2)\, d\theta_2 \\
&\qquad + \int_{t_2-v_M}^{t_2} \int_{\theta_2}^{t_2} \frac{\partial \widehat{v}^T(t_1,\theta_1)}{\partial\theta_1} R_v \frac{\partial \widehat{v}(t_1,\theta_1)}{\partial\theta_1} d\theta_1 d\theta_2,
\end{aligned}$$

having $\xi_h(t_1,t_2) = \Big[\widehat{h}^T(t_1,t_2),\ \int_{t_1-h(t_1)}^{t_1} \widehat{h}^T(\theta_1,t_2)d\theta_1,$ $\int_{t_1-h_M}^{t_1-h(t_1)} \widehat{h}^T(\theta_1,t_2)d\theta_1\Big]^T$ and $\xi_v(t_1,t_2) = \Big[\widehat{v}^T(t_1,t_2),$ $\int_{t_2-v(t_2)}^{t_2} \widehat{v}^T(t_1,\theta_2)\,d\theta_2, \int_{t_2-v_M}^{t_2-v(t_2)} \widehat{v}^T(t_1,\theta_2)\,d\theta_2\Big]^T.$

As the matrices P_h, Q_h, S_h, R_h, P_v, Q_v, S_v and R_v are positive definite. Consequently, the functional $V\left(\widehat{x}(t_1,t_2)\right) > 0$. Now, computing the unidirectional derivative of (16) along system (9) trajectories yields:

$$\dot{V}_u\left(\widehat{x}(t_1,t_2)\right)$$

$$\leq 2\begin{bmatrix} \widehat{h}(t) \\ h(t_1)\dfrac{1}{h(t_1)} \int_{t_1-h(t_1)}^{t_1} \widehat{h}(\theta_1)d\theta_1 \\ (h_M - h(t_1))\dfrac{1}{(h_M - h(t_1))} \int_{t_1-h_M}^{t_1-h(t_1)} \widehat{h}(\theta_1)d\theta_1 \end{bmatrix}^T P_h$$

$$\times\begin{bmatrix} \widehat{h}(\dot{t}) \\ \widehat{h}(t)-(1-\dot{h}(t_1))\,\widehat{h}_{d(t_1)} \\ (1-\dot{h}(t_1))\left(\widehat{h}_{d(t_1)}-\widehat{h}_{dM}\right)\end{bmatrix}+\widehat{h}^T(t)\,Q_h\widehat{h}(t)$$
$$-(1-\dot{h}(t_1))\,\widehat{h}^T_{d(t_1)}Q_h\widehat{h}_{d(t_1)}+\widehat{h}^T(t)\,S_h\widehat{h}(t)$$
$$-\widehat{h}^T_{dM}S_h\widehat{h}_{dM}+h_M\widehat{h}^T(\dot{t})\,R_h\widehat{h}(\dot{t})$$
$$-\int_{t_1-h_M}^{t_1}\widehat{h}^T(\dot{\theta}_1)R^h\widehat{h}(\dot{\theta}_1)\,d\theta_1$$
$$+2\begin{bmatrix} \widehat{v}(t) \\ v(t_2)\frac{1}{v(t_2)}\int_{t_2-v(t_2)}^{t_2}\widehat{v}(\theta_2)d\theta_2 \\ (v_M-v(t_2))\frac{1}{(v_M-v(t_2))}\int_{t_2-v_M}^{t_2-v(t_2)}\widehat{v}(\theta_2)d\theta_2\end{bmatrix}^T P_v$$
$$\times\begin{bmatrix} \widehat{v}(\dot{t}) \\ \widehat{v}(t)-(1-\dot{v}(t_2))\,\widehat{v}_{d(t_2)} \\ (1-\dot{v}(t_2))\left(\widehat{v}_{d(t_2)}-\widehat{v}_{dM}\right)\end{bmatrix}+\widehat{v}^T(t)\,Q_v\widehat{v}(t)$$
$$-(1-\dot{v}(t_2))\,\widehat{v}^T_{d(t_2)}Q_v\widehat{v}_{d(t_2)}+\widehat{v}^T(t)\,S_v\widehat{v}(t)$$
$$-\widehat{v}^T_{dM}S_v\widehat{v}_{dM}+v_M\widehat{v}^T(\dot{t})\,R_v\widehat{v}(\dot{t})$$
$$-\int_{t_2-v_M}^{t_2}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2. \quad (17)$$

where $\widehat{h}(t_1,t_2)=\widehat{h}(t)$, $\frac{\partial\widehat{h}(t_1,t_2)}{\partial t_1}=\widehat{h}(\dot{t})$, $\frac{\partial\widehat{h}(\theta_1,t_2)}{\partial\theta_1}=\widehat{h}(\dot{\theta}_1)$, $\frac{\partial\widehat{v}(t_1,t_2)}{\partial t_2}=\widehat{v}(\dot{t})$, $\widehat{h}(t_1-h(t_1),t_2)=\widehat{h}_{d(t_1)}$, $\widehat{h}(t_1-h_M,t_2)=\widehat{h}_{dM}$, $\widehat{v}(t_1,t_2)=\widehat{v}(t)$, $\frac{\partial\widehat{v}(t_1,\theta_2)}{\partial\theta_2}=\widehat{v}(\dot{\theta}_2)$, $\widehat{v}(t_1,t_2-v(t_2))=\widehat{v}_{d(t_2)}$, $\widehat{v}(t_1,t_2-v_M)=\widehat{v}_{dM}$, $w(t_1,t_2)=w$, $\widehat{h}(\theta_1,t_2)=\widehat{h}(\theta_1)$, $\widehat{v}(t_1,\theta_2)=\widehat{v}(\theta_2)$.

Next, we split $-\int_{t_1-h_M}^{t_1}\widehat{h}^T(\dot{\theta}_1)R^h\widehat{h}(\dot{\theta}_1)\,d\theta_1$ and $-\int_{t_2-v_M}^{t_2}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2$ as follows:

$$-\int_{t_1-h_M}^{t_1}\widehat{h}^T(\dot{\theta}_1)R_h\widehat{h}(\dot{\theta}_1)\,d\theta_1 = -\int_{t_1-h_M}^{t_1-h(t_1)}\widehat{h}^T(\dot{\theta}_1)R_h\widehat{h}(\dot{\theta}_1)\,d\theta_1 - \int_{t_1-h(t_1)}^{t_1}\widehat{h}^T(\dot{\theta}_1)R_h\widehat{h}(\dot{\theta}_1)\,d\theta_1, \quad (18)$$

$$-\int_{t_2-v_M}^{t_2}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2 = -\int_{t_2-v_M}^{t_2-v(t_2)}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2 - \int_{t_2-v(t_2)}^{t_2}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2. \quad (19)$$

After utilization of Lemma 3, (18) and (19) can be exhibited as follows:

$$-\int_{t_1-h_M}^{t_1}\widehat{h}^T(\dot{\theta}_1)R_h\widehat{h}(\dot{\theta}_1)\,d\theta_1 \le -\zeta_h^T(t_1,t_2)\left(\frac{1}{h(t_1)}L_{h12}^T\widehat{R}_hL_{h12} + \frac{1}{h_M-h(t_1)}L_{h34}^T\widehat{R}_hL_{h34}\right)\zeta_h(t_1,t_2), \quad (20)$$

$$-\int_{t_2-v_M}^{t_2}\widehat{v}^T(\dot{\theta}_2)R_v\widehat{v}(\dot{\theta}_2)\,d\theta_2 \le -\zeta_v^T(t_1,t_2)\left(\frac{1}{v(t_2)}L_{v12}^T\widehat{R}_vL_{v12} + \frac{1}{v_M-v(t_2)}L_{v34}^T\widehat{R}_vL_{34}\right)\zeta_v(t_1,t_2), \quad (21)$$

where

$$\widehat{R}_h = diag\{R_h, 3R_h\},\quad L_{h12}=\begin{bmatrix}L_{h1}\\L_{h2}\end{bmatrix},\quad L_{h34}=\begin{bmatrix}L_{h3}\\L_{h4}\end{bmatrix},$$
$$\widehat{R}_v = diag\{R_v, 3R_v\},\quad L_{h1}=\begin{bmatrix}I_h & -I_h & 0 & 0 & 0 & 0\end{bmatrix},\quad L_{v12}=\begin{bmatrix}L_{v1}\\L_{v2}\end{bmatrix},$$
$$L_{h2}=\begin{bmatrix}I_h & I_h & 0 & -2I_h & 0 & 0\end{bmatrix},\quad L_{v34}=\begin{bmatrix}L_{v3}\\L_{v4}\end{bmatrix},\quad L_{h3}=\begin{bmatrix}0 & I_h & -I_h & 0 & 0 & 0\end{bmatrix},$$
$$\Theta_h=\begin{bmatrix}\widehat{R}_h & X_h\\ * & \widehat{R}_h\end{bmatrix},\quad L_{h4}=\begin{bmatrix}0 & I_h & I_h & 0 & -2I_h & 0\end{bmatrix},\quad \Theta_v=\begin{bmatrix}\widehat{R}_v & X_v\\ * & \widehat{R}_v\end{bmatrix},$$
$$L_{v1}=\begin{bmatrix}I_v & -I_v & 0 & 0 & 0 & 0\end{bmatrix},\quad L_h=\begin{bmatrix}L_{h12}^T & L_{h34}^T\end{bmatrix}^T,\quad L_{v2}=\begin{bmatrix}I_v & I_v & 0 & -2I_v & 0 & 0\end{bmatrix},$$
$$L_v=\begin{bmatrix}L_{v12}^T & L_{v34}^T\end{bmatrix}^T,\quad L_{v3}=\begin{bmatrix}0 & I_v & -I_v & 0 & 0 & 0\end{bmatrix},\quad X_h=\begin{bmatrix}X_{h11} & X_{h12}\\ X_{h21} & X_{h22}\end{bmatrix},$$
$$L_{v4}=\begin{bmatrix}0 & I_v & I_v & 0 & -2I_v & 0\end{bmatrix},\quad X_v=\begin{bmatrix}X_{v11} & X_{v12}\\ X_{v21} & X_{v22}\end{bmatrix},$$
$$\zeta_h(t_1,t_2)=\begin{bmatrix}\zeta_{h1}^T(t_1,t_2) & \zeta_{h2}^T(t_1,t_2)\end{bmatrix}^T,$$
$$\zeta_{h1}^T(t_1,t_2)=\begin{bmatrix}\widehat{h}^T(t) & \widehat{h}^T_{d(t_1)} & \widehat{h}^T_{dM}\end{bmatrix},$$

$$\zeta_v(t_1,t_2) = \left[\zeta_{v1}^T(t_1,t_2)\ \zeta_{v2}^T(t_1,t_2)\right]^T,$$
$$\zeta_{h2}^T(t_1,t_2) = \left[\frac{1}{h(t_1)}\int_{t_1-h(t_1)}^{t_1}\hat{h}^T(\theta_1)d\theta_1 \times \frac{1}{(h_M-h(t_1))}\int_{t_1-h_M}^{t_1-h(t_1)}\hat{h}^T(\theta_1)d\theta_1 w^T\right]^T,$$
$$\zeta_{v2}^T(t_1,t_2) = \left[\frac{1}{v(t_2)}\int_{t_2-v(t_2)}^{t_2}\hat{v}^T(\theta_2)d\theta_2 \times \frac{1}{(v_M-v(t_2))}\int_{t_2-v_M}^{t_2-v(t_2)}\hat{v}^T(\theta_2)d\theta_2\ w^T\right]^T.$$

If there exist matrices X^h and X^v, such that $\Theta_h > 0$, and $\Theta_v > 0$, then Lemma 4 yields:

$$-\int_{t_1-h_M}^{t_1}\hat{h}^T(\dot{\theta}_1)R_h\hat{h}(\dot{\theta}_1)\,d\theta_1 \le -\frac{1}{h_M}\zeta_h^T(t_1,t_2)L_h^T\Theta_h L_h\zeta_h(t_1,t_2), \tag{22}$$
$$-\int_{t_2-v_M}^{t_2}\hat{v}^T(\dot{\theta}_2)R_v\hat{v}(\dot{\theta}_2)\,d\theta_2 \le -\frac{1}{v_M}\zeta_v^T(t_1,t_2)L_v^T\Theta_v L_v\zeta_v(t_1,t_2). \tag{23}$$

Using lemma 2 on the terms $h_M\hat{h}^T(\dot{t})R_h\hat{h}(\dot{t})$, $v_M\hat{v}^T(\dot{t})R_v\hat{v}(\dot{t})$ in (17) and taking $\tilde{R}_h = \left(R_h^{-1} - \lambda_{4h}^{-1}I_h\right)^{-1}$, $\tilde{R}_v = \left(R_v^{-1} - \lambda_{4v}^{-1}I_v\right)^{-1}$ and $\tilde{R} = diag\left\{\tilde{R}_h, \tilde{R}_v\right\}$, the following can be obtained:

$$\begin{aligned}
M&\begin{bmatrix}\frac{\partial\hat{h}(t_1,t_2)}{\partial t_1}\\ \frac{\partial\hat{v}(t_1,t_2)}{\partial t_2}\end{bmatrix}^T R\begin{bmatrix}\frac{\partial\hat{h}(t_1,t_2)}{\partial t_1}\\ \frac{\partial\hat{v}(t_1,t_2)}{\partial t_2}\end{bmatrix}\\
&\le M\left[\hat{x}^T(t_1,t_2)\hat{A}_1^T + \hat{x}^T(t_1-h(t_1),t_2-v(t_2))\hat{A}_2^T + w^T(t_1,t_2)\hat{B}^T\right]\tilde{R}\left[\hat{A}_1\hat{x}(t_1,t_2)\right.\\
&\quad\left.+\hat{A}_2\hat{x}(t_1-h(t_1),t_2-v(t_2)) + \hat{B}w(t_1,t_2)\right]\\
&\quad+M\lambda_4\left[\alpha\hat{x}^T(t_1,t_2)\hat{A}_3\hat{A}_3^T\hat{x}(t_1,t_2)\right.\\
&\quad+\beta\hat{x}^T(t_1-h(t_1),t_2-v(t_2))\hat{A}_3\hat{A}_3^T\\
&\quad\left.\times\hat{x}(t_1-h(t_1),t_2-v(t_2))\right]\\
&\quad+M\lambda_6\hat{x}^T(t_1-h(t_1),t_2-v(t_2))\hat{D}_2^T\hat{D}_2\\
&\quad\times\hat{x}(t_1-h(t_1),t_2-v(t_2))\\
&\quad+M\lambda_5\hat{x}^T(t_1,t_2)\hat{D}_1^T\hat{D}_1\hat{x}(t_1,t_2).
\end{aligned} \tag{24}$$

Denoting:

$$\hat{x} = \left[\hat{h}^T(t)\quad \hat{v}^T(t)\right]^T,\quad \hat{x}_{d(t)} = \left[\hat{h}_{d(t_1)}^T\quad \hat{v}_{d(t_2)}^T\right]^T,$$
$$\hat{x}_{dM} = \left[\hat{h}_{dM}^T\quad \hat{v}_{dM}^T\right]^T,$$
$$\hat{x}_{i1} = \left[\frac{1}{h(t_1)}\int_{t_1-h(t_1)}^{t_1}\hat{h}^T(\theta_1)d\theta_1,\ \frac{1}{v(t_2)}\int_{t_2-v(t_2)}^{t_2}\hat{v}^T(\theta_2)d\theta_2\right]^T,$$
$$\hat{x}_{i2} = \left[\frac{1}{(h_M-h(t_1))}\int_{t_1-h_M}^{t_1-h(t_1)}\hat{h}^T(\theta_1)\,d\theta_1,\ \frac{1}{(v_M-v(t_2))}\int_{t_2-v_M}^{t_2-v(t_2)}\hat{v}^T(\theta_2)d\theta_2\right]^T,$$
$$L = \left[L_h^T\quad L_v^T\right]^T,$$
$$\zeta(t_1,t_2) = \left[\hat{x}^T\quad \hat{x}_{d(t)}^T\quad \hat{x}_{dM}^T\quad \hat{x}_{i1}^T\quad \hat{x}_{i2}^T\quad w^T\right]^T.$$

Now, by considering the inequalities (20)-(24), the following is easy to obtain from (17):

$$\dot{V}_u\left(\hat{x}(t_1,t_2)\right) \le \zeta^T(t_1,t_2)\,\Omega\zeta(t_1,t_2) + w^T(t_1,t_2)\,w(t_1,t_2), \tag{25}$$

where

$$\Omega = \begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16}\\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} & \Omega_{26}\\ * & * & \Omega_{33} & 0 & \Omega_{35} & 0\\ * & * & * & \Omega_{44} & 0 & 0\\ * & * & * & * & \Omega_{55} & 0\\ * & * & * & * & * & \Omega_{66}\end{bmatrix},$$

$$\begin{aligned}
\Omega_{11} &= \bar{\Omega}_{11} + \lambda_2^{-1}P\hat{Y}_1\hat{Y}_1^TP + \lambda_2\hat{D}_1^T\hat{D}_1 + \lambda_3^{-1}P\hat{Y}_2\hat{Y}_2^TP\\
&\quad+\lambda_1^{-1}P\hat{A}_3\hat{A}_3^TP + M\hat{A}_1^T\tilde{R}\hat{A}_1 + \lambda_5M\hat{D}_1^T\hat{D}_1,\\
\Omega_{22} &= \bar{\Omega}_{22} + \lambda_3\hat{D}_2^T\hat{D}_2 + M\hat{A}_2^T\tilde{R}\hat{A}_2 + M\lambda_6\hat{D}_2^T\hat{D}_2,\\
\Omega_{12} &= \bar{\Omega}_{12} + M\hat{A}_1^T\tilde{R}\hat{A}_2,\quad \Omega_{16} = P\hat{B} + M\hat{A}_1^T\tilde{R}\hat{B}\\
\Omega_{26} &= M\hat{A}_2^T\tilde{R}\hat{B},\quad \Omega_{66} = M\hat{B}^T\tilde{R}\hat{B}.
\end{aligned}$$

Thus, from the inequalities (25) and (10), it follows:

$$\dot{V}_u\left(\hat{x}(t_1,t_2)\right) < w^T(t_1,t_2)\,w(t_1,t_2). \tag{26}$$

Consequently, $\dot{V}_u\left(\hat{x}(t_1,t_2)\right) < 0$, when $w(t_1,t_2) = 0$, in accordance with Lemma 1 derived in [41], which implies that the filtering error system (9) is asymptotically stable. The following is easy to obtain from (26):

$$\int_0^{t_2}\int_0^{t_1}\dot{V}_u\left(\hat{x}(t_1,t_2)\right)dt_1dt_2 < \int_0^{t_2}\int_0^{t_1}w^T(t_1,t_2)\,w(t_1,t_2)\,dt_1dt_2. \tag{27}$$

By Schur complement, the inequality (10C) implies $\hat{F}\hat{F}^T < \gamma^2P$. Moreover, under the zero boundary conditions the following can be obtained from (27):

$$\begin{aligned} & e^T(t_1,t_2)\,e(t_1,t_2) \\ & = \widehat{x}^T(t_1,t_2)\,\widehat{F}^T\widehat{F}\widehat{x}(t_1,t_2) \\ & < \widehat{x}^T(t_1,t_2)\,P\widehat{x}(t_1,t_2) \\ & < \gamma^2 \int_0^{t_2}\int_0^{t_1} w^T(t_1,t_2)\,w(t_1,t_2)\,dt_1dt_2 \\ & \leq \gamma^2 \int_0^{\infty}\int_0^{\infty} w^T(t_1,t_2)\,w(t_1,t_2)\,dt_1dt_2. \end{aligned}$$

That is

$$\|e(t_1,t_2)\|_\infty^2 < \gamma^2\,\|w(t_1,t_2)\|_2^2,$$

for any nonzero arbitrary $w(t_1,t_2) \in L_2\{[0,\infty),[0,\infty)\}$. This concludes our proof.

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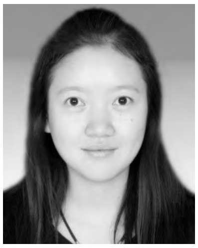

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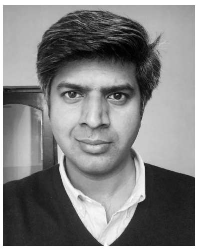

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