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On the Metric Dimension of Generalized Petersen Multigraphs

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ABSTRACT In this paper, we study the metric dimension of barycentric subdivision of Möbius ladders and the metric dimension of generalized Petersen multigraphs. We prove that the generalized Petersen multigraphs denoted by $P(2n, n)$ have metric dimension 3 when n is even and 4 otherwise. We also study the exchange property for resolving sets of barycentric subdivisions of Möbius ladders and generalized Petersen multigraphs and prove that the exchange property of the bases in a vector space does not hold for minimal resolving sets of these graphs.

INDEX TERMS Metric dimension, basis, resolving set, barycentric subdivision, Möbius ladders, generalized Petersen graphs.

I. INTRODUCTION AND PRELIMINARY RESULTS

In a connected graph G, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. For an ordered set $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ and a vertex $v \in V(G)$, the *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), d(v, w_3), \dots, d(v, w_k))$. If every vertex of G is uniquely identified by its distances from the vertices of W, or equivalently, if every two distinct vertices of G have distinct representations with respect to the ordered set W, then W is called a *resolving set* [5] or *locating set* [20]. A resolving set of minimum cardinality is called a *basis* for G and this cardinality is referred as the *metric dimension* or *location number* of G, denoted by $\beta(G)$ [3]. The concept of resolving sets and metric dimension have appeared previously in the literature. For more details, see [9], [10], [12]–[14], [16], [19], [23].

For a given ordered set of vertices $W = \{w_1, w_2, \dots, w_k\}$ of a graph G, the i-th component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G)\backslash W$.

Tomescu and Imran [22] determined a useful property to find $\beta(G)$ in the following lemma:

Lemma 1: Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u, w) = d(v, w)$ for all vertices $w \in V(G) \setminus \{u, v\}$, then $\{u, v\} \cap W \neq \emptyset$.

Let $\mathcal{F}$ be an infinite family of connected graphs. If each graph in the family has bounded metric dimension, then we say that $\mathcal{F}$ has bounded metric dimension; otherwise $\mathcal{F}$ has unbounded metric dimension.

If all graphs in $\mathcal{F}$ have the same metric dimension then $\mathcal{F}$ is called a family with constant metric dimension [15]. A connected graph G has $\beta(G) = 1$ if and only if G is a *path* [5]; *cycles* C_n have metric dimension 2 for every $n \geq 3$. Also generalized Petersen graphs $P(n, 2)$, antiprisms A_n and circulant graphs C_n^2 are families of graphs with constant metric dimension [15].

In [3], if W_n denotes a *wheel* with n spokes and J_{2n} the graph deduced from the wheel W_{2n} by alternately deleting n spokes, then $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for every $n \geq 7$ and in [24] if $n \geq 4$ then $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

An example of a family which has bounded metric dimension is the family of *prisms*. In [4], for any $n \geq 3$ it was proved that

$$\beta(P_m \times C_n) = \begin{cases} 2, & \text{if n is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

Prisms D_n are the cubic plane graphs obtained by the cartesian product of a path P_2 with a cycle C_n, so prisms constitute a family of *cubic graphs* with bounded metric dimension. The generalized Petersen graphs $P(n, 3)$ have bounded metric dimension [11].

Let $u, v \in V(G)$ and $e = uv$ be an edge of G. *Subdividing the edge e* means that a new vertex w is added to $V(G)$, and that edge $e = uv$ is replaced in $E(G)$ by an edge $e' = uw$ and an edge $e'' = wv$.

By an inverse operation we mean the replacement of two edges that meet at a vertex of degree two by a single edge that join their end points and this inverse operation is called *smoothing away a vertex*. *Subdividing a graph G* means performing a sequence of edge-subdivision operations. The resulting graph is called a *subdivision of the graph G*. The operation of subdivision can be used to convert a general graph into a simple graph. The *barycentric subdivision* of a graph G is the subdivision in which one new vertex is inserted in the interior of each edge.

The barycentric subdivision of a graph G is bipartite and loopless, and if G itself is loopless, then its barycentric subdivision is simple (see [7]).

A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivisions of graphs play a very important role in the characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be *homeomorphic* if they are subdivisions of same graph G. Gross and Yellen [7] give a characterization of planar graphs and proved the following theorem.

Theorem 1: A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

Note that the problem of determining whether $\beta(G) < k$ is an *NP*-complete problem [6].

In this paper, we study the metric dimension of the subdivision of Möbius ladders and use this construction to study the metric dimension of generalized Petersen multigraphs $P(2n, n)$.

We prove that the generalized Petersen graphs $P(2n, n)$ which are multigraphs have metric dimension equal to 3 when n is even and equal to 4 otherwise. We also study the exchange property for resolving sets of subdivisions of Möbius ladders. We prove that the exchange property of the bases in a vector space does not hold for minimal resolving sets of barycentric subdivisions of Möbius ladders and also does not hold for minimal resolving sets of generalized Petersen multigraphs $P(2n, n)$ when n is even.

II. APPLICATIONS OF METRIC DIMENSION

- **Application in robot navigation**
  The idea of metric dimension comes initially from the systems networking. Persuaded by the issue of deciding particularly the area of an interloper in a given system, the idea of metric dimension was presented by Slater [20] and independent by Harary and Melter in [8]. Uses of this invariant to the route of robots in systems are examined in [17] while applications to issue of example acknowledgment and picture handling, some of which include the utilization of progressive information structures are given in [18].
  Route can be considered in a diagram organized structure in which the route operator (which we will accept to be a point robot) moves from hub to hub of a *graph space*. The robot can find itself by the nearness of unmistakably named *landmark* hubs in the graph space. For a robot exploring in Euclidean space, visual recognition of a particular milestone gives data about the course to the milestone, and enables the robot to decide its situation by triangulation. On a graph, in any case, there is neither the idea of heading nor that of perceivability. Rather, we will accept that a robot exploring on a diagram can detect the separations to an arrangement of tourist spots.
- **Application in chemistry**
  A fundamental issue in chemistry is to give numerical portrayals to an arrangement of chemical mixes in a way that gives unmistakable portrayals to particular mixes. As portrayed in [5], the structure of a substance compound can be spoken to by a marked diagram whose vertex and edge names determine the iota and bond composes, individually. Therefore, a graph theoretic understanding of this issue is to give portrayals to the vertices of a diagram so that unmistakable vertices have particular portrayals. This is the subject of papers [3], [18], [20], and [21].

III. METRIC DIMENSION OF BARYCENTRIC SUBDIVISION OF MöBIUS LADDERS

The *barycentric subdivision of Möbius ladders* denoted by SM_n is obtained by subdividing the edges of Möbius ladders by putting a vertex of degree two on each edge. It has $\frac{5n}{2}$ vertices and $3n$ edges. There are n vertices of degree 3 and $\frac{3n}{2}$ vertices are of degree 2. Two different views of SM_{12} are shown in Figure 1. Also, view SM_n as a barycentric subdivision of the prism D_n (the cartesian product of the path on two vertices with a cycle C_n) with one cross edge.

We denote vertices $\{v_0, v_1, \ldots, v_{2n-1}\} \subset V(SM_n)$ that numbered clockwise, induce the cycle of length $2n$. Moreover $\{v_{2i+1} : 0 \leq i \leq n-1\}$ and $\{v_{2i} : 0 \leq i \leq n-1\}$ are the vertices of degree 2 and 3 respectively. The set of vertices $\{u_i : 0 \leq i \leq \frac{n}{2} - 1\} \subset V(SM_n)$ are internal vertices of $v_{2i} - v_{2i+n}$ paths, where $0 \leq i \leq \frac{n}{2} - 1$ and each u_i is a vertex of degree 2. So we have $V(SM_n) = \{v_i : 0 \leq i \leq 2n-1\} \cup \{u_i : 0 \leq i \leq \frac{n}{2} - 1\}$. Also, the subscripts of vertices v_i are taken modulo $2n$ and the subscripts of vertices u_i are taken modulo $\frac{n}{2}$.

The metric dimension of Möbius ladders has been studied in [1] where it was proved that Möbius ladders constitute a family of cubic graphs with constant metric dimension 3 except when $n \equiv 2 \pmod 8$. In the next theorem, we extend this study to the metric dimension of barycentric subdivisions of Möbius ladders denoted by SM_n. Note that the choice of appropriate basis vertices is the core of the problem.

Theorem 2: If $n \geq 8$ and SM_n denote the barycentric subdivisions of a Möbius ladder, then $\beta(SM_n) = 3$.

Proof: The proof of this theorem follows directly from the proofs of Lemmas 2 to 6. □

In the proofs of Lemmas 2 (as well as 3 - 6) each entry in the code-tables is the distance between the vertices of column 1

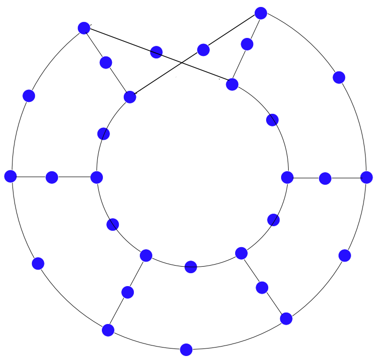
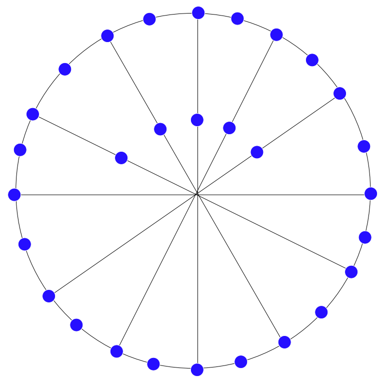

FIGURE 1. Two different views of SM_{12}

TABLE 1. (a) Codes for the outer vertices of SM_n. (b) Codes for the outer vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1}: 0 \leq j \leq 4k-3$	$j+1$	$j+4$
$v_{i+4k+j-1}: 0 \leq j \leq 2$	$4k+j-1$	$4k-j$
$v_{i+n-j-2}: 0 \leq j \leq 4k-4$	$j+4$	$j+1$
$v_{i+n+j}: 0 \leq j \leq 4k-2$	$j+2$	$j+1$
$v_{i+2n-j-2}: 0 \leq j \leq 4k-2$	$j+2$	$j+3$

(a)

$d(.,.)$	u_{i+2k-1}
$v_{i+j+1}: 0 \leq j \leq 4k-3$	$4k-j-2$
$v_{i+4k+j-1}: 0 \leq j \leq 2$	$j+2$
$v_{i+n-j-2}: 0 \leq j \leq 4k-4$	$4k-j+1$
$v_{i+n+j}: 0 \leq j \leq 4k-2$	$4k-j-1$
$v_{i+2n-j-2}: 0 \leq j \leq 4k-2$	$4k-j+1$

(b)

TABLE 2. Codes for the inner vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+2k-1}
$u_{i+j}: 0 \leq j \leq 2k-2$	$2j+1$	$2j+2$	$4k-2j$
$u_{i+2k+j}: 0 \leq j \leq 2k-1$	$4k-2j+1$	$4k-2j$	$2j+4$

and the vertices of row 1. Each row represents the code of a vertex, with respect to row 1, lying in column 1 of that row.

Lemma 2: If $n \equiv 0 \pmod 8$ and SM_n is the barycentric subdivision of a Möbius ladder, then $\beta(SM_n) \leq 3$.

Proof: We can write $n = 8k$ where $k \geq 1$. We will prove that for a chosen index i such that $0 \leq i \leq 2n-1$, the set $W = \{v_i, v_{i+n-1}, u_{i+2k-1}\}$ is a resolving set for SM_n. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are the following: $r(v_{i+n+4k-1}|W) = (4k+1, 4k, 2)$, $r(v_{i+2n-1}|W) = (1, 4, 4k)$ and in tables 1(a), 1(b) and table 2.

Since all the vertices lying in the first column of above two tables have distinct codes with respect to W, it implies that $\beta(SM_n) \leq 3$ when $n \equiv 0 \pmod 8$. □

Lemma 3: If $n \equiv 2 \pmod 8$ and SM_n is the barycentric subdivision of a Möbius ladder, then $\beta(SM_n) \leq 3$.

Proof: We can write $n = 8k+2$ where $k \geq 1$. We will prove that for a chosen index i such that $0 \leq i \leq 2n-1$,

TABLE 3. (a) Codes for the outer vertices of SM_n. (b) Codes for the outer vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1}: 0 \leq j \leq 4k-2$	$j+1$	$j+4$
$v_{i+4k+j}: 0 \leq j \leq 2$	$4k+j$	$4k-j+1$
$v_{i+n-j-2}: 0 \leq j \leq 4k-3$	$j+4$	$j+1$
$v_{i+n+j+1}: 0 \leq j \leq 4k-1$	$j+3$	$j+2$
$v_{i+2n-j-2}: 0 \leq j \leq 4k-2$	$j+2$	$j+3$

(a)

$d(.,.)$	u_{i+2k+1}
$v_{i+j+1}: 0 \leq j \leq 4k-2$	$4k-j+2$
$v_{i+4k+j}: 0 \leq j \leq 2$	$3-j$
$v_{i+n-j-2}: 0 \leq j \leq 4k-3$	$4k-j-1$
$v_{i+n+j+1}: 0 \leq j \leq 4k-1$	$4k-j+2$
$v_{i+2n-j-2}: 0 \leq j \leq 4k-2$	$4k-j-1$

(b)

TABLE 4. (a) Codes for the inner vertices of SM_n. (b) Codes for the inner vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$u_{i+j+1}: 0 \leq j \leq 2k-1$	$2j+3$	$2j+4$
$u_{i+2k+j+2}: 0 \leq j \leq 2k-2$	$4k-2j-1$	$4k-2j-2$

(a)

$d(.,.)$	u_{i+2k+1}
$u_{i+j+1}: 0 \leq j \leq 2k-1$	$4k-2j+2$
$u_{i+2k+j+2}: 0 \leq j \leq 2k-2$	$2j+4$

(b)

$W = \{v_i, v_{i+n-1}, u_{i+2k+1}\}$ is a resolving set for $V(SM_n)$ where $k = \frac{n-2}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $r(v_{i+n}|W) = (2, 1, 4k+1)$, $r(v_{i+n+4k+1}|W) = (4k+1, 4k+2, 2)$, $r(v_{i+2n-1}|W) = (1, 4, 4k)$, $r(u_i|W) = (1, 2, 4k+2)$ and in tables 3(a), 3(b) and tables 4(a), 4(b).

It can be seen that all the vertices lying in the first column of tables 3(a), 3(b) and tables 4(a), 4(b) have distinct codes with respect to W implying that $\beta(SM_n) \leq 3$ when $n \equiv 2 \pmod 8$. □

Lemma 4: If $n \equiv 4 \pmod 8$ and SM_n is the barycentric subdivision of a Möbius ladder, then $\beta(SM_n) \leq 3$.

TABLE 5. (a) Codes for the outer vertices of SM_n. (b) Codes for the outer vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1}: 0 \le j \le 4k-1$	$j+1$	$j+4$
$v_{i+4k+j+1}: 0 \le j \le 2$	$4k+j+1$	$4k-j+2$
$v_{i+n-j-2}: 0 \le j \le 4k-2$	$j+4$	$j+1$
$v_{i+n+j}: 0 \le j \le 4k$	$j+2$	$j+1$
$v_{i+2n-j-2}: 0 \le j \le 4k$	$j+2$	$j+3$

(a)

$d(.,.)$	u_{i+2k}
$v_{i+j+1}: 0 \le j \le 4k-1$	$4k-j$
$v_{i+4k+j+1}: 0 \le j \le 2$	$j+2$
$v_{i+n-j-2}: 0 \le j \le 4k-2$	$4k-j+3$
$v_{i+n+j}: 0 \le j \le 4k$	$4k-j+1$
$v_{i+2n-j-2}: 0 \le j \le 4k$	$4k-j+3$

(b)

TABLE 6. (a) Codes for the inner vertices of SM_n. (b) Codes for the inner vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$u_{i+j}: 0 \le j \le 2k-1$	$2j+1$	$2j+2$
$u_{i+2k+j+1}: 0 \le j \le 2k$	$4k-2j+3$	$4k-2j+2$

(a)

$d(.,.)$	u_{i+2k}
$u_{i+j}: 0 \le j \le 2k-1$	$4k-2j+2$
$u_{i+2k+j+1}: 0 \le j \le 2k$	$2j+4$

(b)

Proof: We can write $n = 8k+4$ where $k \ge 1$. We will show that for a chosen index i such that $0 \le i \le 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+2k}\}$ is a resolving set for SM_n, where $k = \frac{n-4}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $r(v_{i+n+4k+1}|W) = (4k+3, 4k+2, 2)$, $r(v_{i+2n-1}|W) = (1, 4, 4k+2)$ and in tables 5(a), 5(b) and tables 6(a), 6(b).

It can be seen that no two vertices of SM_n lying in column 1 of tables 5(a), 5(b) and tables 6(a), 6(b) have the same code with respect to W, which yields that W is a resolving set for $V(SM_n)$. Hence $\beta(SM_n) \le 3$ when $n \equiv 4 \pmod 8$. □

Lemma 5: If $n \equiv 6 \pmod 8$ and SM_n is the barycentric subdivision of a Mӧbius ladder, then $\beta(SM_n) \le 3$.

Proof: We can write $n = 8k+6$ where $k \ge 1$. We will show that for a chosen index i such that $0 \le i \le 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+2k+2}\}$ is a resolving set for $V(SM_n)$, where $k = \frac{n-6}{8}$. The codes of the vertices in $V(SM_n) \setminus W$ with respect to W are: $r(v_{i+n}|W) = (2, 1, 4k+3)$, $r(v_{i+n+4k+3}|W) = (4k+3, 4k+4, 2)$, $r(v_{i+2n-1}|W) = (1, 4, 4k+2)$, $r(u_i|W) = (1, 2, 4k+4)$ and in tables 7(a), 7(b) and tables 8(a), 8(b).

It can be seen that all the vertices lying in the first column of tables 7(a), 7(b) and tables 8(a), 8(b) have distinct codes with respect to W implying that $\beta(SM_n) \le 3$ when $n \equiv 6 \pmod 8$. □

Lemma 6: If n is even and SM_n is the barycentric subdivision of a Mӧbius ladder, then $\beta(SM_n) \ge 3$.

Proof: Suppose that $\beta(SM_n) = 2$, then the following three possibilities arise.

TABLE 7. (a) Codes for the outer vertices of SM_n. (b) Codes for the outer vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1}: 0 \le j \le 4k$	$j+1$	$j+4$
$v_{i+4k+j+2}: 0 \le j \le 2$	$4k+j+2$	$4k-j+3$
$v_{i+n-j-2}: 0 \le j \le 4k-1$	$j+4$	$j+1$
$v_{i+n+j+1}: 0 \le j \le 4k+1$	$j+3$	$j+2$
$v_{i+2n-j-2}: 0 \le j \le 4k$	$j+2$	$j+3$

(a)

$d(.,.)$	u_{i+2k+2}
$v_{i+j+1}: 0 \le j \le 4k$	$4k-j+4$
$v_{i+4k+j+2}: 0 \le j \le 2$	$3-j$
$v_{i+n-j-2}: 0 \le j \le 4k-1$	$4k-j+1$
$v_{i+n+j+1}: 0 \le j \le 4k+1$	$4k-j+4$
$v_{i+2n-j-2}: 0 \le j \le 4k$	$4k-j+1$

(b)

TABLE 8. (a) Codes for the inner vertices of SM_n. (b) Codes for the inner vertices of SM_n.

$d(.,.)$	v_i	v_{i+n-1}
$u_{i+j+1}: 0 \le j \le 2k$	$2j+3$	$2j+4$
$u_{i+2k+j+3}: 0 \le j \le 2k-1$	$4k-2j+1$	$4k-2j$

(a)

$d(.,.)$	u_{i+2k+2}
$u_{i+j+1}: 0 \le j \le 2k$	$4k-2j+4$
$u_{i+2k+j+3}: 0 \le j \le 2k-1$	$2j+4$

(b)

(1). Both vertices belong to the set $\{v_i : 0 \le i \le 2n-1\}$. Without loss of generality, we can suppose that $W = \{v_i, v_{i+j}\}$ is a resolving set where $1 \le j \le 2n-1$. But then we get

- *If* $1 \le j \le \frac{n}{2}$, then $r(u_i|W) = r(v_{i+2n-1}|W) = (1, j+1)$.
- *If* $\frac{n}{2}+1 \le j \le n-2$, then $r(u_i|W) = r(v_{i+2n-1}|W) = (1, n-j+1)$.
- *If* $j = n-1$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 4)$.
- *If* $n \le j \le n+1$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, j-n+3)$.
- *If* $n+2 \le j \le \frac{3n}{2}$, then $r(v_{i+1}|W) = r(u_i|W) = (1, j-n+1)$.
- *If* $\frac{3n}{2}+1 \le j \le 2n-1$, then $r(v_{i+1}|W) = r(u_i|W) = (1, 2n-j+1)$.

This is a contradiction.

(2). Both vertices belong to the set $\{u_i : 0 \le i \le \frac{n}{2}-1\}$. Without loss of generality, we can suppose that the resolving set is $W = \{u_i, u_{i+j}\}$ where $1 \le j \le \frac{n}{2}-1$. However, we have in this case

- *If* $1 \le j \le \frac{n}{4}$ *when* $n \equiv 0, 4 \pmod 8$ *and* $1 \le j \le \frac{n-2}{4}$ *when* $n \equiv 2, 6 \pmod 8$, then $r(v_i|W) = r(v_{i+n}|W) = (1, 2j+1)$.
- *If* $\frac{n}{4}+1 \le j \le \frac{n}{2}-1$ *when* $n \equiv 0, 4 \pmod 8$ *and* $\frac{n+2}{4} \le j \le \frac{n}{2}-1$ *when* $n \equiv 2, 6 \pmod 8$, then $r(v_i|W) = r(v_{i+n}|W) = (1, n-2j+1)$.

This is a contradiction.

(3). One vertex belongs to the set $\{v_i : 0 \le i \le 2n-1\}$ and another belongs to the set $\{u_i : 0 \le i \le \frac{n}{2}-1\}$. Without loss of generality, we can choose the resolving set as $W = \{v_i, u_{i+j}\}$ where $1 \leq j \leq \frac{n}{2} - 1$.

- *If* $j = 0$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 2)$.
- *If* $1 \leq j \leq \frac{n}{4}$ *when* $n \equiv 0, 4 \pmod 8$ *and* $1 \leq j \leq \frac{n-2}{4}$ *when* $n \equiv 2, 6 \pmod 8$, then $r(v_{i+2j+1}|W) = r(v_{i+n+2j-1}|W) = (2j+1, 2)$.
- *If* $\frac{n}{4} + 1 \leq j \leq \frac{n}{2} - 1$ *when* $n \equiv 0, 4 \pmod 8$ *and* $\frac{n+2}{4} \leq j \leq \frac{n}{2} - 1$ *when* $n \equiv 2, 6 \pmod 8$, then $r(v_{i+2j+1}|W) = r(v_{i+n+2j-1}|W) = (n - 2j + 1, 2)$.

Again a contradiction is obtained.

Hence in all cases we have $\beta(SM_n) = 3$. □

IV. THE METRIC DIMENSION OF GENERALIZED PETERSEN MULTIGRAPHS $P(2n, n)$

The *generalized Petersen graphs* $P(n, m)$, $n \geq 4$, $1 \leq m \leq \frac{n}{2}$ form an important class of 3-regular graphs with $2n$ vertices and $3n$ edges having vertex set

$$V(P(n, m)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

and edge set

$$E(P(n, m)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+m} : 1 \leq i \leq n\}$$

where all indices are taken modulo n.

For $m = 1$, the generalized Petersen graph $P(n, 1)$ is called a prism, denoted by D_n. In [4], it was shown that

$$\beta(D_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{otherwise.} \end{cases}$$

So prisms constitute a family of cubic graphs with bounded metric dimension. In [15], Javaid *et al.* proved that the generalized Petersen graphs $P(n, 2)$ are a family of graphs with constant metric dimension 3 for every integer $n \geq 5$. Imran *et al.* [11] considered the generalized Petersen graphs $P(n, 3)$ and deduced the following results:

Theorem 3 [11]: For generalized Petersen graphs $P(n, 3)$, we have

(a) $\beta(P(n, 3)) = 4$ for $n \equiv 0 \pmod 6$ and $n \geq 24$.
(b) $\beta(P(n, 3)) = 3$ for $n \equiv 1 \pmod 6$ and $n \geq 25$.
(c) $\beta(P(n, 3)) \leq 5$ for $n \equiv 2 \pmod 6$ and $n \geq 8$.
(d) $\beta(P(n, 3)) \leq 4$ for $n \equiv 3, 4, 5 \pmod 6$ and $n \geq 17$.

Javaid *et al.* [14] proved that

$$\beta(P(2n+1, n)) = \begin{cases} 2, & \text{if n=1;} \\ 3, & \text{otherwise.} \end{cases}$$

It was proved in [16] that the generalized Petersen graphs $P(2n, n-1)$ constitute a family of graphs with constant metric dimension 3 when n is odd, and metric dimension 4 when n is even.

The generalized Petersen graphs $P(2n, n)$ are in fact multigraphs and in this section, we study their metric dimension. The generalized Petersen multigraphs $P(2n, n)$ have vertex set $V(P(2n, n)) = \{v_i, u_i : 0 \leq i \leq 2n-1\}$ and edge set $E(P(2n, n)) = \{v_iv_{i+1}, v_iu_i, u_iu_{i+n} : 0 \leq i \leq 2n-1\}$, where indices are taken modulo $2n$. We call the vertices $v_0, \ldots, v_{2n-1}$ outer vertices, numbered clockwise, and $u_0, \ldots, u_{2n-1}$ inner vertices.

Note that in the generalized Petersen graphs $P(2n, n)$, the vertices u_i and u_{i+n} are joined by parallel edges, but since we are interested in finding the metric dimension of $P(2n, n)$ these parallel edges play no role. For this reason, we view the vertices of the set $\{u_i : 0 \leq i \leq 2n-1\}$ as vertices of degree two. Now after this observation, we can obtain the simple generalized Petersen graph $P(2n, n)$ from the graph of the Möbius ladder M_{2n} by subdividing twice those edges that join the vertices v_i and v_{i+n}, where $0 \leq i \leq 2n-1$ and indices are taken modulo $2n$. Two different views of $P(12, 6)$ are shown in Figure 2.

We view the $P(12, 6)$ as particular subdivision (as mentioned above) of a prism with one twisted edge. In the next theorem, we extend the study to the metric dimension of generalized Petersen graphs $P(2n, n)$. Note that the choice of appropriate basis vertices is the core of the problem.

Lemma 7: If $n \equiv 0 \pmod 4$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \leq 3$.

Proof: We can write $n = 4k$ where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+n+2k}\}$ is a resolving set for $P(2n, n)$, where $k = \frac{n}{4}$. For $n = 4$, the codes of the vertices in $P(2n, n) \setminus W$ with respect to $W = \{v_0, v_3, u_6\}$ are in Table 9 and Table 10.

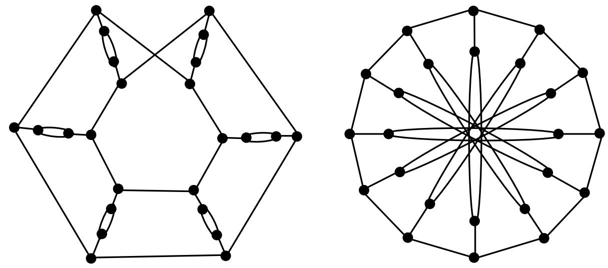

FIGURE 2. Two views of $P(12, 6)$.

TABLE 9. Codes for the outer vertices of $P(8, 4)$.

$d(.,.)$	v_0	v_3	u_6
v_1	1	2	3
v_2	2	1	2
v_4	3	1	3
v_5	3	2	2
v_6	2	3	1
v_7	1	3	2

TABLE 10. Codes for the inner vertices of $P(8, 4)$.

$d(.,.)$	v_0	v_3	u_6
u_0	1	3	4
u_1	2	3	4
u_2	3	2	1
u_3	3	1	4
u_4	2	2	4
u_5	3	3	3
u_7	2	2	3

TABLE 11. (a) Codes for the outer vertices of $P(2n, n)$. (b) Codes for the outer vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1} : 0 \leq j \leq 2k-4$	$j+1$	$j+5$
$v_{i+2k+j-2} : 0 \leq j \leq 2$	$2k+j-2$	$2k-j+1$
$v_{i+n-j-2} : 0 \leq j \leq 2k-4$	$j+5$	$j+1$
$v_{i+n+j} : 0 \leq j \leq 2k-2$	$j+3$	$j+1$
$v_{i+n+2k+j-1} : 0 \leq j \leq 1$	$2k-j+1$	$2k+j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+1$	$j+3$

(a)

$d(.,.)$	u_{i+n+2k}
$v_{i+j+1} : 0 \leq j \leq 2k-4$	$2k-j+1$
$v_{i+2k+j-2} : 0 \leq j \leq 2$	$4-j$
$v_{i+n-j-2} : 0 \leq j \leq 2k-4$	$2k-j$
$v_{i+n+j} : 0 \leq j \leq 2k-2$	$2k-j+1$
$v_{i+n+2k+j-1} : 0 \leq j \leq 1$	$2-j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$2k-j$

(b)

TABLE 12. Codes for the inner vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$u_{i+j+1} : 0 \leq j \leq 2k-3$	$j+2$	$j+4$	$2k-j+2$
$u_{i+n-j-1} : 0 \leq j \leq 2k-2$	$j+3$	$j+1$	$2k-j+2$
$u_{i+n+j} : 0 \leq j \leq 2k-1$	$j+2$	$j+2$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+2$	$j+2$	$2k-j+1$

It can be seen that all the vertices in $V(P(8, 4)) \setminus W$ have distinct codes with respect to W. Now the codes of the vertices of $V(P(2n, n)) \setminus W$ when $n > 4$ are: $r(v_{i+2k+1}|W) = (2k+1, 2k-2, 3)$, $r(u_i|W) = (1, 3, 2k+2)$, $r(u_{i+2k-1}|W) = (2k, 2k+1, 4)$, $r(u_{i+2k}|W) = (2k+1, 2k, 1)$ and in tables 11(a), 11(b) and table 12.

It can be seen that no two vertices of $P(2n, n)$ lying in column 1 of the tables 11(b), 11(b) and table 12 have the same code with respect to W, this yields W is a resolving set for $P(2n, n)$. Hence $\beta(P(2n, n)) \leq 3$ when $n \equiv 0 \pmod 4$. □

Lemma 8: If $n \equiv 2 \pmod 4$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \leq 3$.

Proof: We can write $n = 4k + 2$, where $k \geq 1$. We will show that for a chosen index i such that $0 \leq i \leq 2n-1$, $W = \{v_i, v_{i+n-1}, u_{i+n+2k}\}$ is a resolving set for $P(2n, n)$, where $k = \frac{n-2}{4}$. The codes of the vertices in $V(P(2n, n)) \setminus W$ with respect to W are: $r(u_{i+2k}|W) = (2k+1, 2k+2, 1)$, $r(u_{i+2k+1}|W) = (2k+2, 2k+1, 4)$, $r(u_{i+n-1}|W) = (3, 1, 2k+3)$ and in tables 13(a), 13(b) and table 14.

Since all the vertices lying in the first column of above two tables have distinct codes with respect to W implying that $\beta(P(2n, n)) \leq 3$ when $n \equiv 2 \pmod 4$. □

Lemma 9: If $n \equiv 1 \pmod 4$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \leq 4$.

Proof: We can write $n = 4k + 1$, where $k \geq 1$. For a chosen index i such that $0 \leq i \leq 2n-1$, we show that $W = \{v_i, v_{i+n}, u_{i+n+2k-1}, u_{i+n+2k+1}\}$ is a resolving set for $P(2n, n)$, where $k = \frac{n-1}{4}$.

The codes of the vertices in $V(P(2n, n)) \setminus W$ with respect to W are: $r(v_{i+n-1}|W) = (4, 1, 2k+1, 2k+1)$, $r(u_i|W) = (1, 2, 2k+2, 2k+2)$, $r(u_{i+2k-1}|W) = (2k, 2k+1, 1, 5)$,

TABLE 13. (a) Codes for the outer vertices of $P(2n, n)$. (b). Codes for the outer vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n-1}
$v_{i+j+1} : 0 \leq j \leq 2k-3$	$j+1$	$j+5$
$v_{i+2k+j-1} : 0 \leq j \leq 1$	$2k+j-1$	$2k-j+2$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$2k+j+1$	$2k-j$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+5$	$j+1$
$v_{i+n+j} : 0 \leq j \leq 2k-1$	$j+3$	$j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$2k-j+2$	$2k+j+1$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$j+1$	$j+3$

(a)

$d(.,.)$	u_{i+n+2k}
$v_{i+j+1} : 0 \leq j \leq 2k-3$	$2k-j+1$
$v_{i+2k+j-1} : 0 \leq j \leq 1$	$3-j$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$j+3$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$2k-j+2$
$v_{i+n+j} : 0 \leq j \leq 2k-1$	$2k-j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$j+1$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$2k-j+2$

(b)

TABLE 14. Codes for the inner vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n-1}	u_{i+n+2k}
$u_{i+j} : 0 \leq j \leq 2k-1$	$j+1$	$j+3$	$2k-j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-2$	$j+4$	$j+2$	$2k-j+3$
$u_{i+n+j} : 0 \leq j \leq 2k-1$	$j+2$	$j+2$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k$	$j+2$	$j+2$	$2k-j+3$

TABLE 15. (a) Codes for the outer vertices of $P(2n, n)$. (b) Codes for the outer vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n}
$v_{i+j+1} : 0 \leq j \leq 2k-2$	$j+1$	$j+4$
$v_{i+2k+j} : 0 \leq j \leq 1$	$2k+j$	$2k-j+1$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+5$	$j+2$
$v_{i+n+j+1} : 0 \leq j \leq 2k-2$	$j+4$	$j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$2k-j+1$	$2k+j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+1$	$j+4$

(a)

$d(.,.)$	$u_{i+n+2k-1}$	$u_{i+n+2k+1}$
$v_{i+j+1} : 0 \leq j \leq 2k-2$	$2k-j$	$2k-j+2$
$v_{i+2k+j} : 0 \leq j \leq 1$	$j+3$	$3-j$
$v_{i+n-j-2} : 0 \leq j \leq 2k-3$	$2k-j+2$	$2k-j$
$v_{i+n+j+1} : 0 \leq j \leq 2k-2$	$2k-j-1$	$2k-j+1$
$v_{i+n+2k+j} : 0 \leq j \leq 1$	$j+2$	$2-j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$2k-j+2$	$2k-j$

(b)

$r(u_{i+2k}|W) = (2k+1, 2k+2, 4, 4)$, $r(u_{i+2k+1}|W) = (2k+2, 2k+1, 5, 1)$, $r(u_{i+n+2k}|W) = (2k+2, 2k+1, 3, 3)$ and in the tables 15(a), 15(b) and tables 16(a), 16(b).

It can be verified that all the vertices of $V(P(2n, n)) \setminus W$ that are lying in the first column of the tables mentioned above have distinct codes with respect to W. This yields that $\beta(P(2n, n)) \leq 3$ for $n \equiv 1 \pmod 4$. □

Lemma 10: If $n \equiv 3 \pmod 4$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \leq 4$.

Proof: We write $n = 4k + 3$ where $k \geq 1$. For a chosen index i such that $0 \leq i \leq 2n-1$, we show that

TABLE 16. (a) Codes for the inner vertices of $P(2n, n)$. (b) Codes for the inner vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n}
$u_{i+j+1} : 0 \leq j \leq 2k-3$	$j+2$	$j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-3$	$j+4$	$j+3$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$3-j$	$2-j$
$u_{i+n+j+1} : 0 \leq j \leq 2k-3$	$j+3$	$j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$j+2$	$j+3$

(a)

$d(.,.)$	$u_{i+n+2k-1}$	$u_{i+n+2k+1}$
$u_{i+j+1} : 0 \leq j \leq 2k-3$	$2k-j+1$	$2k-j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-3$	$2k-j+3$	$2k-j+1$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$2k-j+2$	$2k+j+2$
$u_{i+n+j+1} : 0 \leq j \leq 2k-3$	$2k-j$	$2k-j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-2$	$2k-j+3$	$2k-j+1$

(b)

TABLE 17. (a) Codes for the outer vertices of $P(2n, n)$. (b) Codes for the outer vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n}
$v_{i+j+1} : 0 \leq j \leq 2k-1$	$j+1$	$j+4$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$2k+j+1$	$2k-j+2$
$v_{i+n-j-2} : 0 \leq j \leq 2k-2$	$j+5$	$j+2$
$v_{i+n+j+1} : 0 \leq j \leq 2k-1$	$j+4$	$j+1$
$v_{i+n+2k+j+1} : 0 \leq j \leq 1$	$2k-j+2$	$2k+j+1$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$j+1$	$j+4$

(a)

$d(.,.)$	u_{i+n+2k}	$u_{i+n+2k+2}$
$v_{i+j+1} : 0 \leq j \leq 2k-1$	$2k-j+1$	$2k-j+3$
$v_{i+2k+j+1} : 0 \leq j \leq 1$	$j+3$	$3-j$
$v_{i+n-j-2} : 0 \leq j \leq 2k-2$	$2k-j+3$	$2k-j+1$
$v_{i+n+j+1} : 0 \leq j \leq 2k-1$	$2k-j$	$2k-j+2$
$v_{i+n+2k+j+1} : 0 \leq j \leq 1$	$j+2$	$2-j$
$v_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$2k-j+3$	$2k-j+1$

(b)

$W = \{v_i, v_{i+n}, u_{i+n+2k}, u_{i+n+2k+2}\}$ is a resolving set for $P(2n, n)$ where $k = \frac{n-3}{4}$.

The codes of the vertices in $V(P(2n, n)) \setminus W$ with respect to W are: $r(v_{i+n-1}|W) = (4, 1, 2k+2, 2k+2)$,$r(u_i|W) = (1, 2, 2k+3, 2k+3)$, $r(u_{i+2k}|W) = (2k+1, 2k+2, 1, 5)$, $r(u_{i+2k+1}|W) = (2k+2, 2k+3, 4, 4)$, $r(u_{i+2k+2}|W) = (2k+3, 2k+2, 5, 1)$, $r(u_{i+n+2k+1}|W) = (2k+3, 2k+2, 3, 3)$ and in tables 17(a), 17(b) and tables 18(a), 18(b).

Since no two distinct vertices of $V(P(2n, n)) \setminus W$ lying in tables $17(a)$, $17(b)$ and tables $18(a)$, $18(b)$ have the same code. Thus we get $\beta(P(2n, n)) \leq 4$ when $n \equiv 3 \pmod 4$. □

Lemma 11: If $n \equiv 0, 2 \pmod 4$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \geq 3$.

Proof: Suppose that $\beta(P(2n, n)) = 2$, then the following three possibilities arise.

Case 1: Both vertices belong to the set $\{v_i : 0 \leq i \leq 2n-1\} \subset V(P(2n, n))$. For fixed i, suppose that resolving set is $W = \{v_i, v_{i+j}\}$. However then

- *If* $1 \leq j \leq \frac{n}{2}$, then $r(v_{i+2n-1}|W) = r(u_i|W) = (1, j+1)$.
- *If* $\frac{n}{2}+1 \leq j \leq n-1$, then $r(v_{i+2n-1}|W) = r(u_i|W) = (1, n-j+2)$.

TABLE 18. (a) Codes for the inner vertices of $P(2n, n)$. (b) Codes for the inner vertices of $P(2n, n)$.

$d(.,.)$	v_i	v_{i+n}
$u_{i+j+1} : 0 \leq j \leq 2k-2$	$j+2$	$j+3$
$u_{i+n-j-2} : 0 \leq j \leq 2k-2$	$j+4$	$j+3$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$3-j$	$2-j$
$u_{i+n+j+1} : 0 \leq j \leq 2k-2$	$j+3$	$j+2$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$j+2$	$j+3$

(a)

$d(.,.)$	u_{i+n+2k}	$u_{i+n+2k+2}$
$u_{i+j+1} : 0 \leq j \leq 2k-2$	$2k-j+2$	$2k-j+4$
$u_{i+n-j-2} : 0 \leq j \leq 2k-2$	$2k-j+4$	$2k-j+2$
$u_{i+n+j-1} : 0 \leq j \leq 1$	$2k-j+3$	$2k+j+3$
$u_{i+n+j+1} : 0 \leq j \leq 2k-2$	$2k-j+1$	$2k-j+3$
$u_{i+2n-j-1} : 0 \leq j \leq 2k-1$	$2k-j+4$	$2k-j+2$

(b)

- *If* $j = n$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 4)$. And for $k = 1$ when $n \equiv 0 \pmod 4$, we have $c_W(v_{i+1}) = c_W(v_{i+7}) = (1, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2}-1$, then $r(v_{i+1}|W) = r(u_i|W) = (1, j-n+2)$.
- *If* $\frac{3n}{2} \leq j \leq 2n-1$, then $r(v_{i+1}|W) = r(u_i|W) = (1, 2n-j+1)$.

Case 2: Both vertices belong to the set $\{u_i : 0 \leq i \leq 2n-1\} \subset V(P(2n, n))$. For fixed i, suppose the resolving set is $W = \{u_i, u_{i+j}\}$. However then

- *If* $1 \leq j \leq \frac{n}{2}-1$, then $r(v_{i+n}|W) = r(v_{i+2n-1}|W) = (2, j+2)$.
- *If* $\frac{n}{2} \leq j \leq n-1$, then $r(v_{i+n}|W) = r(v_{i+2n-1}|W) = (2, n-j+1)$.
- *If* $j = n$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (2, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2}$, then $r(v_{i+1}|W) = r(v_{i+n}|W) = (2, j-n+1)$.
- *If* $\frac{3n}{2}+1 \leq j \leq 2n-1$, then $r(v_{i+1}|W) = r(v_{i+n}|W) = (2, 2n-j+2)$.

Case 3: One vertex belongs to the set $\{v_i : 0 \leq i \leq 2n-1\} \subset V(P(2n, n))$ and another belongs to $\{u_i : 0 \leq i \leq 2n-1\} \subset V(P(2n, n))$, then two subcases arise:

Subcase(i): For fixed i, suppose $W = \{v_i, u_{i+j}\}$ is a resolving set. However, we have

- *If* $j = 0$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 2)$.
- *If* $1 \leq j \leq \frac{n}{2}-1$, then $r(v_{i+2n-1}|W) = r(u_i|W) = (1, j+2)$.
- *If* $\frac{n}{2} \leq j \leq n-2$, then $r(u_{i+n}|W) = r(u_{i+2n-1}|W) = (2, n-j+2)$.
- *If* $j = n-1$, then $r(v_{i+1}|W) = r(u_i|W) = (1, 4)$. And for $k = 1$ when $n \equiv 0 \pmod 4$, we have $r(v_{i+6}|W) = r(u_{i+4}|W) = (2, 3)$.
- *If* $j = n$, then $r(v_{i+1}|W) = r(v_{i+2n-1}|W) = (1, 3)$.
- *If* $n+1 \leq j \leq \frac{3n}{2}-2$, then $r(v_{i+2n-1}|W) = r(u_i|W) = (1, j-n+3)$.
- *If* $\frac{3n}{2}-1 \leq j \leq \frac{3n}{2}$, then $r(u_{i+1}|W) = r(u_{i+n}|W) = (2, j-n+2)$. For $k = 1$ when $n \equiv 0 \pmod 4$, then for

$j = \frac{3n}{2} - 1$ we have $r(v_{i+2}|W) = r(u_{i+4}|W) = (2, 3)$ and when $j = \frac{3n}{2}$, we get $r(u_{i+1}|W) = r(u_{i+4}|W) = (2, 4)$.
- *If* $\frac{3n}{2} + 1 \leq j \leq 2n - 1$, then $r(u_{i+1}|W) = r(u_{i+n}|W) = (2, 2n - j + 3)$.

Subcase(ii): For fixed i, suppose $W = \{u_i, v_{i+j}\}$ is a resolving set. But then

- *If* $0 \leq j \leq \frac{n}{2} - 2$, then $r(v_{i+j+2}|W) = r(u_{i+j+1}|W) = (j + 3, 2)$.
- *If* $j = \frac{n}{2} - 1$, then $r(u_{i+\frac{n}{2}}|W) = r(u_{i+\frac{3n}{2}-1}|W) = (\frac{n}{2} + 2, 2)$.
- *If* $\frac{n}{2} \leq j \leq n - 2$, then $r(u_{i+j+1}|W) = r(u_{i+j+n}|W) = (n - j + 2, 2)$.
- *If* $j = n-1$, then $r(v_{i+n+1}|W) = r(u_{i+2n-1}|W) = (3, 2)$.
- *If* $n \leq j \leq \frac{3n}{2} - 3$, then $r(v_{i+j+2}|W) = r(u_{i+j+1}|W) = (j - n + 4, 2)$.
- *If* $\frac{3n}{2} - 2 \leq j \leq \frac{3n}{2}$, then $r(u_{i+j-1}|W) = r(u_{i+j-n}|W) = (j - n + 2, 2)$. For $k = 1$ when $n \equiv 0 \pmod 4$, then for $j = \frac{3n}{2} - 2$ we have $r(u_{i+3}|W) = r(u_{i+5}|W) = (4, 2)$. If $j = \frac{3n}{2} - 1$, then $r(v_{i+3}|W) = r(u_{i+1}|W) = (3, 2)$ and when $j = \frac{3n}{2}$, we have $r(u_{i+2}|W) = r(u_{i+5}|W) = (4, 2)$.
- *If* $\frac{3n}{2} + 1 \leq j \leq 2n - 1$, then $r(u_{i+j-1}|W) = r(u_{i+j-n}|W) = (2n - j + 3, 2)$.

We get a contradiction in all above cases, which implies that no two vertices for $V(P(2n, n))$ serve as basis vertices. Hence $\beta(P(2n, n)) \geq 3$ when n is even. □

Now we prove that $\beta(P(2n, n)) \geq 4$ when n is odd. For this purpose, we need some more notions and definitions. Without loss of generality, we can suppose that the vertices $v_0, v_1, \ldots, v_{2n-1}$ of the outer cycle are labeled in clockwise direction. For any two vertices v_i and v_j $(i \neq j)$, the clockwise distance, $d^*(v_i, v_j)$, is the distance measured in clockwise direction from v_i to v_j in the subgraph induced by the outer cycle. For example, $d^*(v_0, v_{2n-1}) = 2n - 1$ and $d^*(v_{2n-1}, v_0) = 1$. This definition can be extended to any two vertices of $P(2n, n)$. The indices will be taken modulo $2n$.

Consider a vertex on the outer cycle, say v_0. A vertex u_i is called a good vertex for v_0 if $d(v_0, u_i) = d$, where $d \in \{d(v_0, u_{i+n-1}), d(v_0, u_{i-n+1})\}$; otherwise, u_i is called a bad vertex for v_0. And v_i is called a good vertex for v_0 if $d(v_0, v_i) = d$, where $d \in \{d(v_0, v_{i+n-3}), d(v_0, v_{i-n+3})\}$. This definition can be extended to any two inner vertices belonging to the set $\{u_0, \ldots, u_{2n-1}\}$. The vertex v_i is a good vertex for u_0 if $d(u_0, v_i) = d$, where $d \in \{d(u_0, v_{i+n-1}), d(u_0, v_{i-n+1})\}$; u_i is a good vertex for u_l, say $l = 0$, if $d(u_0, u_i) = d$, where $d \in \{d(u_0, u_{i+n-1}), d(u_0, u_{i-n+1}), d(u_0, u_{i+2n-2}), d(u_0, u_{i-2n+2}), d(u_l, v_l)\}$ and bad otherwise.

Figure 3, shows the graph of $P(10, 5)$. In this figure if we consider a vertex u_0, then it is easy to see that v_4, v_5 and v_6 are the associated good vertices for u_0.

It is important to note that the set of good vertices for u_0 can be obtained from the set of good vertices for v_0 by adding vertices v_1, v_2, v_{2n-2} and v_{2n-1}. Similarly a vertex u_j is good for the pair $\{v_0, v_i\}$ if it satisfies the above definition for a vertex u_j to be good for the outer vertices. If v_l is good for the pairs $\{v_0, v_i\}$ and $\{v_0, v_j\}$ then v_l is good for the

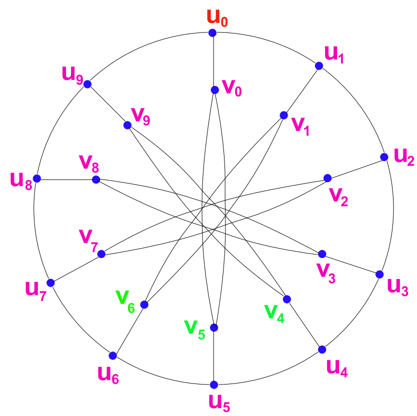

FIGURE 3. **The graph $P(10, 5)$ and the associated good vertices for the vertex u_0.**

triplet $\{v_0, v_i, v_j\}$. Due to rotational symmetry of the graph $P(2n, n)$ we deduce the following result:

Lemma 12: For any two vertices v_i and u_j of $P(2n, n)$ such that $v_i \neq u_j$, we have $d(v_i, u_j) = d(v_{i+r}, u_{i+r})$ for any $1 \leq r \leq 2n - 1$.

In order to find vertices that are good for pairs of vertices belonging to the outer cycle, the following lemmas will be useful.

Lemma 13: Let $0 \leq j \leq 2n - 3$. If u_i is good for v_0 and u_{i-j-1} is also good for v_0, then u_i is also good for the pair $\{v_0, v_{j+1}\}$.

Proof: By definition, $d(v_0, u_i) = d(v_0, u_{i+n-1})$ or $d(v_0, u_i) = d(v_0, u_{i-n+1})$ and $d(v_0, u_{i-j-1}) = d(v_0, u_{i-j+n-2})$ or $d(v_0, u_{i-j-1}) = d(v_0, u_{i-j-n})$. By Lemma 12, the last two equalities imply that $d(v_{1+j}, u_i) = d(v_{1+j}, u_{i+n-1})$ or $d(v_{1+j}, u_i) = d(v_{1+j}, u_{i-n+1})$. □

Lemma 14: Let $0 \leq j \leq 2n - 7$. If v_i is good for v_0 and v_{i-j-1} is also good for v_0, then v_i is also good for the pair $\{v_0, v_{j+1}\}$.

Clearly $\beta(P(2n, n)) > 1$ because paths are the only graphs with metric dimension 1, see [5].

Lemma 15: If $n \equiv 1, 3 \pmod 4$, $n \geq 7$ and $P(2n, n)$ is the generalized Petersen multigraph, then $\beta(P(2n, n)) \geq 3$.

Proof: We show that there is no resolving set of $V(P(2n, n))$ consisting of two vertices A and B. If both A and B belong to the outer cycle, we can suppose that $A = v_0$. Let $d^*(v_0, B) = j + 1$. Since the vertices $u_1, u_2, \ldots, u_{2n-1}$ and $v_3, v_4, \ldots, v_{2n-3}$ are good vertices for v_0. By using Lemma 13, we find that u_{2n-1} is a good vertex for all pairs $\{v_0, B\}$, where $B \in \{v_{j+1} : 0 \leq j \leq 2n - 3\}$ and if $B = v_{2n-1}$, then u_1 is a good vertex for the pair $\{v_0, B\}$. Similarly, by Lemma 14 we can find that v_{2n-3} is a good vertex for every pair $\{v_0, B\}$ such that $B \in \{v_{j+1} : 0 \leq j \leq 2n - 7\}$ and if $B \in \{v_{2n-5}, v_{2n-4}, \ldots, v_{2n-1}\}$, then v_{2n-8} is good for all pairs $\{v_0, B\}$.

If $A, B \in \{u_i : 0 \leq i \leq 2n-1\}$, we can consider $A = u_0$ and $B = u_i$. This case can be reduced to the case when $A = v_0$ and $B = v_i$ because the set of good vertices for u_l also includes the set of good vertices for v_l for any $0 \leq l \leq 2n-1$. If $A = v_i$

and $B = u_i$, then any good vertex for v_i is also a good vertex for u_i, hence for the pair $\{A, B\}$. The remaining case when $A = v_i$ and $B = u_j$ $(i \neq j)$ can also be reduced to the case when $A = v_i$ and $B = v_j$. It follows that there is no resolving set containing two vertices in this case, which completes the proof. □

Lemma 16: If $n \equiv 1, 3 \pmod 4$, $n \geq 7$, then $\beta(P(2n, n)) \geq 4$.

Proof: Clearly, $\beta(P(2n, n)) \geq 3$, by Lemma 15. Now we have to show that there is no resolving set of $V(P(2n, n))$ consisting of three vertices A, B and C when $n \equiv 1, 3 \pmod 4$. By the same reasoning as in Lemma 15, it is enough to consider only the case when A, B and C belong to the outer cycle. Since the set of good vertices for u_0 can be obtained from the set of good vertices for v_0 by adding vertices v_0, v_2, v_{2n-2} and v_{2n-1}. Without loss of generality we suppose that $d^*(A, B) < d^*(A, C)$ and let $A = v_0$.

Consider $(d^*(v_0, B), d^*(v_0, C)) = (l + 1, j + 1)$ such that $B = v_{l+1}$ and $C = v_{j+1}$. Since $u_1, \ldots, u_{2n-1}$ and $v_3, \ldots, v_{2n-3}$ are good vertices for v_0. By applying Lemma 13, we find that u_{2n-1} is a good vertex for all the pairs $\{v_0, B\}$ and $\{v_0, C\}$, where $l, j = 0, 1, \ldots, 2n - 3$ and hence for all the triplets $\{v_0, B, C\}$. And if $B = v_{2n-2}$ and $C = v_{2n-1}$, then u_1 is a good vertex for the pairs $\{v_0, B\}$ and $\{v_0, C\}$ and hence for the triplet $\{v_0, B, C\}$. Now by Lemma 14, we find v_{2n-3} is a good vertex for all the pairs $\{v_0, B\}$ and $\{v_0, C\}$, where $l, j = 0, 1, \ldots, 2n - 7$ and hence for all the triplets $\{v_0, B, C\}$. When $B, C \in \{v_{2n-5}, \ldots, v_{2n-1}\}$, then v_{2n-8} is a good vertex for all pairs $\{v_0, B\}$ and $\{v_0, C\}$ and hence for all triplets $\{v_0, B, C\}$. It follows that there is no resolving set with three vertices in this case, which completes the proof. □

Theorem 4: Let $P(2n, n)$ denote the generalized Petersen multigraph. Then for every integer $n \geq 2$ we have

$$\beta(P(2n, n)) = \begin{cases} 3, & \text{if n is even;} \\ 4, & \text{otherwise.} \end{cases}$$

Proof:

Case 1: n is even.

By Lemmas 7, 8 and 11, $\beta(P(2n, n)) = 3$.

Case 2: n is odd.

By Lemmas 9, 10, 15 and 16, $\beta(P(2n, n)) = 4$. □

V. EXCHANGE PROPERTY FOR RESOLVING SETS

We have seen that a subset W of vertices of a graph G is a resolving set if every vertex in G is uniquely determined by its distances to the vertices of W. Resolving sets behave like bases in a *vector space* in that each vertex in the graph can be uniquely identified relative to the vertices of these sets. But though resolving sets do share some of the properties of bases in a vector space, they do not always have the exchange property from linear algebra. Resolving sets are said to have the *exchange property* in G if whenever S and R are minimal resolving sets for G and $r \in R$, then there exists an $s \in S$ so that $S - \{s\} \cup \{r\}$ is a minimal resolving set [2].

If the exchange property holds for a graph G, then every minimal resolving set for G has the same size and algorithmic methods for finding the metric dimension of G are more feasible. Thus to show that the exchange property does not hold in a given graph, it is sufficient to describe two minimal resolving sets of different size. However, since the converse is not true, knowing that the exchange property does not hold does not guarantee that there are minimal resolving sets of different size.

The following results concerning the exchange property for resolving sets were deduced in [2].

Theorem 5 [2]: The exchange property holds for resolving sets in trees.

Theorem 6 [2]: For $n \geq 8$, resolving sets do not have the exchange property in wheels W_n.

The exchange property for resolving sets of Möbius ladders M_n when $n \equiv 6 \pmod 8$ will be discussed in the next theorem.

Theorem 7: The exchange property for minimal resolving sets does not hold in Möbius ladders M_n when $n \equiv 6 \pmod 8$, where $n \geq 14$.

Proof: We can write $n = 8k + 6$, where $k \geq 1$. Since $W = \{v_1, v_2, v_{4k+3}\}$ is a metric basis (see [1]) and hence a minimal resolving set. Also $W^* = \{v_1, v_{2k+2}, v_{4k+3}, v_{4k+4}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_1$, then $r(v_1|S) = r(v_{4k+5}|S) = (2k + 1, 2, 1)$. When $w = v_{2k+2}$, we get $r(v_{2k+2}|S) = r(v_{6k+6}|S) = (2k + 1, 2k + 1, 2k + 2)$. If $w = v_{4k+3}$, then $r(v_{4k+3}|S) = r(v_{4k+5}|S) = (2, 2k + 1, 1)$ and when $w = v_{4k+4}$, we get $r(v_{4k+4}|S) = r(v_{8k+6}|S) = (1, 2k + 2, 1)$.

There are minimal resolving sets of different size. Hence the exchange property does not hold for resolving sets in M_n when $n \equiv 6 \pmod 8$. □

In the next theorem, we show that the exchange property does not hold for resolving sets of the barycentric subdivision of Möbius ladders denoted by SM_n for every even integer $n \geq 8$.

Theorem 8: For any even integer $n \geq 8$, resolving sets do not have the exchange property in the barycentric subdivisions of Möbius ladders denoted by SM_n.

Proof:

Case 1: We can write $n = 8k$ where $k \geq 1$. Without loss of generality we can choose $i = 0$. Then $W = \{v_0, v_{n-1}, u_{2k-1}\}$ is a metric basis (see Lemma 2) and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k-1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{4k+1}|S) = r(v_{12k+1}|S) = (4k, 4k, 4)$. When $w = v_1$, we get $r(v_{4k}|S) = r(v_{12k}|S) = (4k, 4k+1, 3)$. If $w = u_0$, then $r(v_{4k+1}|S) = r(u_{2k}|S) = (4k + 1, 4k, 4)$ and when $w = u_{2k-1}$, we get $r(v_{8k-1}|S) = r(v_{8k+1}|S) = (3, 4, 2)$.

Case 2: We can write $n = 8k + 2$ where $k \geq 1$. Without loss of generality we can choose $i = 0$. Then $W = \{v_0, v_{n-1}, u_{2k+1}\}$ is a metric basis (see Lemma 3) and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k+1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{8k+5}|S) = r(u_{4k}|S) = (4, 4, 4k)$. When $w = v_1$, then we get $r(v_3|S) = r(u_{4k}|S) = (3, 4, 4k)$. If $w = u_0$, then $r(v_{8k+5}|S) = r(u_2|S) = (5, 4, 4k)$ and when $w = u_{2k+1}$, then $r(v_{8k+1}|S) = r(v_{8k+3}|S) = (3, 4, 2)$.

Case 3: We can write $n = 8k+4$ where $k \geq 1$. Without loss of generality we choose $i = 0$. Then $W = \{v_0, v_{n-1}, u_{2k}\}$ is a metric basis (see Lemma 4) and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{4k+3}|S) = r(v_{12k+7}|S) = (4k+2, 4k+2, 4)$. When $w = v_1$, then we get $r(v_{4k+2}|S) = r(v_{12k+6}|S) = (4k+2, 4k+3, 3)$. If $w = u_0$, then $r(v_{8k+1}|S) = r(u_{4k}|S) = (5, 6, 4k+2)$ and when $w = u_{2k}$, we get $r(v_{8k+3}|S) = r(v_{8k+5}|S) = (3, 4, 2)$.

Case 4: We can write $n = 8k+6$ where $k \geq 1$. Without loss of generality we choose $i = 0$. Then $W = \{v_0, v_{n-1}, u_{2k+2}\}$ is a metric basis (see Lemma 5) and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, u_{2k+2}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then we have $r(v_{8k+11}|S) = r(u_{4k+1}|S) = (6, 6, 4k)$. When $w = v_1$, then $r(v_3|S) = r(u_{4k+2}|S) = (3, 4, 4k+2)$. If $w = u_0$, then $r(v_{8k+5}|S) = r(u_{4k+2}|S) = (3, 4, 4k+2)$ and when $w = u_{2k+2}$, then $r(v_{8k+5}|S) = r(v_{8k+7}|S) = (3, 4, 2)$.

In each case, there are minimal resolving sets of different size. Hence the exchange property does not hold in SM_n for any even integer $n \geq 8$. □

The following theorem shows that the exchange property does not hold for resolving sets of generalized Petersen multigraphs $P(2n, n)$ for any even integer $n \geq 4$.

Theorem 9: For any even integer $n \geq 4$, resolving sets do not have the exchange property in generalized Petersen multigraphs $P(2n, n)$.

Proof: We can write $n = 4k, 4k+2$ where $k \geq 1$. Without loss of generality we choose $i = 0$. Then $W = \{v_0, v_{n-1}, u_{n+2k}\}$ is a metric basis (see Lemma 7, 8) and hence a minimal resolving set. Also $W^* = \{v_0, v_1, u_0, v_{n-1}\}$ is a minimal resolving set. There is no $w \in W^*$ such that $S = W^* \setminus \{w\}$ is still a resolving set.

If $w = v_0$, then $r(v_{n+1}|S) = r(u_{2n-1}|S) = (3, 3, 2)$. When $w = v_1$, then $r(u_{n+1}|S) = r(u_{2n-2}|S) = (3, 4, 3)$. If $w = u_0$, then $r(u_n|S) = r(u_{2n-1}|S) = (2, 3, 2)$ and when $w = v_{n-1}$, then $r(v_2|S) = r(u_1|S) = (2, 1, 3)$.

There are minimal resolving sets of different size. Hence the exchange property does not hold in $P(2n, n)$ for any even integer $n \geq 4$. □

VI. CONCLUSION

The problem of determining whether $\beta(G) < k$ is an *NP*-complete problem. In this paper, we have studied the metric dimension of the subdivision of a Möbius ladder denoted by SM_n and generalized Petersen multigraphs $P(2n, n)$. We proved that only three vertices suffice to resolve all the vertices of SM_n. For the generalized Petersen graphs $P(2n, n)$ which are multigraphs, we proved that their metric dimension is 3 when n is even and 4 otherwise.

It is natural to ask for a characterization of graph classes with respect to the nature of their metric dimension. We have also studied the exchange property for minimal resolving sets of the subdivision of a Möbius ladder M_n. It has been shown that the exchange property of the bases in a vector space does not hold for minimal resolving sets of barycentric subdivisions of Möbius ladders and also does not hold for minimal resolving sets of generalized Petersen multigraphs $P(2n, n)$ when n is even. We close this section by raising a question that naturally arises from the text.

Open Problem: When does a nontrivial connected graph have the same metric dimension as its barycentric subdivision?

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