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# Fault-Tolerant Hamiltonian Connectivity of Twisted Hypercube-Like Networks *THLNs*

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**ABSTRACT** The twisted hypercube-like networks (THLNs) include some well-known hypercube variants. A graph $G$ is $k$-fault-tolerant Hamiltonian connected if $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$. This paper is concerned with the fault-tolerant Hamiltonian connectivity of an $n$-dimensional ($n$-D) THLN. Let $G_n$ be an $n$-D THLN ($n \geq 5$) and $F$ be a subset of $V(G_n) \cup E(G_n)$ with $|F| \leq n-2$. We show that for arbitrary vertex-pair $(u, v)$ in $G_n - F$, there exists a $(n-2)$-fault-tolerant Hamiltonian path joining vertices $u$ and $v$ except $(u, v)$ being a weak vertex-pair in $G_n - F$. The technical theorem proposed in this paper can be applied to several multiprocessor systems, including $n$-D crossed cubes $CQ_n$, $n$-D twisted cubes $TQ_n$ for odd $n$, $n$-D locally twisted cubes $LTQ_n$, and $n$-D Möbius cubes $MQ_n$.

**INDEX TERMS** Network topology, multiprocessor interconnection networks, hypercubes, twisted hypercube-like networks *THLNs*, computer network reliability, fault tolerance, Hamiltonian connectivity.

## I. INTRODUCTION

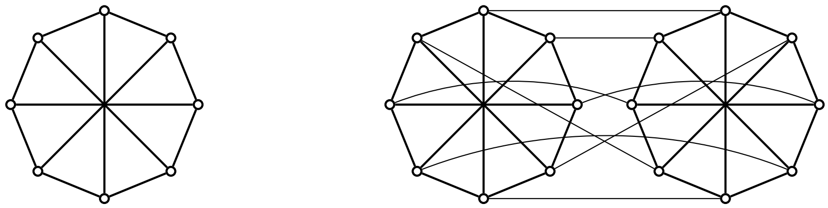
A topological structure of an interconnection network can be modeled by a graph $G = (V(G), E(G))$, where the vertex set $V(G)$ represents the set of processors and the edge set $E(G)$ represents the set of links joining processors. Investigating structures of $G$ is essential to design a suitable topology of interconnection network.

Several conflicting requirements must be considered in designing the topology of interconnection networks, such as regularity, low vertex degree, low diameter, existence of Hamiltonian paths, high fault tolerance, diagnosability, et al.

Noteworthily, graph embedding is a critical issue in evaluating an interconnection network, which concerns whether a host graph contains a guest graph as its subgraph. An embedding of one guest graph $G$ into another host graph $H$ is a one-to-one mapping $\phi$ from the vertex set of $G$ to the vertex set of $H$ [5]. An edge in $G$ corresponds to a path in $H$ under $\phi$. Graph embedding problem has many key applications, such as transplantation of parallel algorithm, concurrent processors allocation, architecture simulation, network expansion. In addition, the wiring problem on VLSI circuit boards can also be attributed to graph embedding problem [1].

As two common guest graphs, structures of paths (linear arrays) and cycles (rings) are fundamental topologies for parallel and distributed computation. They are suitable for designing simple algorithms with low communication costs [4]. Many efficient algorithms were originally designed based on linear arrays and rings for solving a variety of algebraic problems, graph problems and some parallel applications, such as those in image and signal processing [5], [31]. Thus, it is important to have an effective path and/or cycle embedding in a network. The path and/or cycle embedding properties of many interconnection networks have been widely studied for many well-known networks, see [9] for a detail survey on these topics.

A path (respectively, cycle) is a Hamiltonian path (respectively, hamiltonian cycle) if its vertices are distinct and span $V(G)$. A graph is hamiltonian if it has a hamiltonian cycle. One of the most difficult problems in graph theory is to find a necessary and sufficient condition for a graph to be Hamiltonian. A graph $G$ is Hamiltonian connected if for any pair of distinct vertices $u$ and $v$, there exists a Hamiltonian path $P_{uv}$. Hamiltonian paths and cycles can be applied to practical problems including online optimization of complex, flexible manufacturing systems [24], wormhole routing [25], [26], deadlock-free routing and broadcasting algorithms [40]. These applications motivated the study of embedding networks with hamiltonian paths.



(a)

(b)

**FIGURE 1.** Examples of 3*D THLN* and 4*D THLN*.

Fault-tolerance is an important index of the stability and reliability of the network. It is useful to consider faulty networks because node faults or link faults may occur in networks. A graph $G$ is $k$-fault-tolerant Hamiltonian connected if $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \leq k$.

The $n$-dimensional hypercube [2], is one of the most popular, efficient, versatile interconnection network, which possesses many excellent features such as high symmetry, relatively small degree, logarithmic diameter, recursive structure, linear bisection width, effective routing and broadcasting algorithms [29] and, thus, becomes the first choice for the topological structure of parallel processing and computing systems [4], [5]. In order to further improve some specific properties of hypercubes, a number of hypercube variants, such as the crossed cubes [21], the twisted cubes [28], the Möbius cubes [27] and the locally twisted cubes [37], have been suggested. Though both the hypercube variants and the ordinary hypercube have the same number of vertices and the same vertex degree, the diameter of the hypercube variants is approximately half that of the ordinary hypercube. The fault-tolerant hamiltonian path and cycle embedding problems of these hypercube variants and other well-known interconnection networks have received considerable research attention [6], [7], [15], [16], [23], [30], [32]–[35], [38], [39].

The hypercube-like networks(*HLNs*, for short) are a large class of network topologies [3], [11], [17]–[19], [22], [36], [41]. Among *HLNs* one may identify a subclass of networks, which in this paper are called the twisted hypercube-like networks (*THLNs*, for short). Yang *et al.* [35] first proposed the twisted hypercube-like networks(*THLNs*).

*Definition 1 [35]:* For $n \geq 3$, an $n$-dimensional ($n$-*D*, for short) twisted hypercube-like network (*THLN*, for short) is a graph defined recursively as follows.

(1) A 3-*D THLN* is isomorphic to the graph depicted in Fig.1(a).

(2) For $n \geq 4$, an $n$-*D THLN* $G_n$ is obtained from two vertex-disjoint $(n-1)$-*D THLNs*, denoted by $G^0_{n-1}$ and $G^1_{n-1}$, in this way:

$$\begin{aligned} V(G_n) &= V(G^0_{n-1}) \cup V(G^1_{n-1}), \\ E(G_n) &= E(G^0_{n-1}) \cup E(G^1_{n-1}) \cup \\ &\quad \{(u, \phi(u)) : u \in V(G^0_{n-1})\}, \end{aligned}$$

where $\phi : V(G^0_{n-1}) \rightarrow V(G^1_{n-1})$ is a bijective mapping. In the following, we will denote this graph $G_n$ as $G_n = \oplus_\phi(G^0_{n-1}, G^1_{n-1})$. Fig.1(b) plots a 4*D THLN*.

For simply, we denote $G_n = L \oplus R$, where $L = G^0_{n-1}$ and $R = G^1_{n-1}$. For each vertex $x \in V(L)(or\ V(R))$, let $x^R(or\ x^L)$ be the unique adjacent vertex of $x$ in $R(or\ L)$, and $N_L(x)(or\ N_R(x))$ be the set of vertices adjacent to $x$ in $V(L)(or\ V(R))$. Let $E^C$ be the set of edges joining $L$ to $R$ and $E_L(x)(or\ E_R(x))$ be the set of edges that are incident to vertex $x$ in $L(or\ R)$.

*Proposition 2 [35]:* A *THLN* does not contain cycles of length 3.

*Proposition 3 [35]:* For any two distinct vertices $u$ and $v$ of a *THLN*, we have $|(N_{G_n}(u) \cup E_{G_n}(u)) \cap (N_{G_n}(v) \cup E_{G_n}(v))| \leq 2$.

*Theorem 4 [35]: Let $G_n$ be an n-D THLN($n \geqslant 7$) and $F \subseteq V(G_n) \cup E(G_n)$ with $|F| \leqslant 2n-9$. Then $G_n - F$ contains a hamiltonian cycle if $\delta(G_n - F) \geq 2$, and $G_n - F$ contains a near hamiltonian cycle if $\delta(G_n - F) \leq 1$.*

In particular, the above-mentioned hypercube variants are all *THLNs*. Park *et al.* [18] proved that all *n-D THLNs* are Hamiltonian with up to $n-2$ faulty elements and Hamiltonian connected with up to $n-3$ faulty elements.

A question arises naturally: What about the Hamiltonian connectivity in a *THLN* with $n-2$ faulty elements? This paper attempts to improve the bound of fault tolerance to $n-2$ excepting only a pair of vertices. Let $G_n$ be an *n-D THLN*($n \geq 5$) and $F \subset V(G_n) \cup E(G_n)$ with $|F| \leq n-2$, we will prove that for any two fault-free vertices $u$ and $v$ in $G_n$, there exists a $(n-2)$-fault-tolerant hamiltonian path $P_{uv}$ except $(u, v)$ being a weak vertex-pair in $G_n - F$. Our work extends some previously known results.

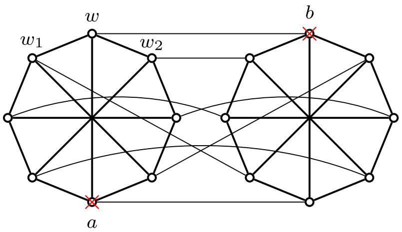
The remainder of this paper is organized as follows. Section II gives the definitions and properties of weak vertex-pair, and some useful lemmas in our proofs. Section III establishes the main result. Section IV describes applications of the proposed technical theorem to four popular multiprocessor systems. Finally, section V concludes the paper.

## II. WEAK VERTEX-PAIR AND SOME LEMMAS

In this section, we will give the definitions, some important properties of weak vertex-pair and some lemmas to be used in our proofs in Section III.

*Definition 5:* If $|F| = n-2$ and there exists a vertex $w$ with $N_{G_n-F}(w) = \{w_1, w_2\}$, then $w$ is called a weak 2-degree vertex and $(w_1, w_2)$ is called a $w$-weak vertex pair(or a weak vertex pair, for short).

For example, if $F = \{a, b\}$, then $w$ is a weak 2-degree vertex and $(w_1, w_2)$ is a $w$-weak vertex-pair in $G_4 - F$(See Figure 2).




$4D\ THLN(G_4)$

**FIGURE 2. Example of weak vertex-pair.**

Clearly, for the vertex-pair $(w_1, w_2)$, any fault-free path $P_{uv}$ of length $l$ with $l \geq 3$ can't contain vertex $w$. Thus, there exists no fault-free hamiltonian path joining vertices $w_1$ and $w_2$ in $G_n - F$. Fortunately, we proved that there exists at most one weak 2-degree vertex $w$ and at most one $w$-weak vertex-pair in $G_n - F(n \geq 5)$ for any $F \subset V(G_n) \cup E(G_n)$ with $|F| = n - 2$ in Proposition 8.

*Definition 6:* If $(w_1, w_2)$ is not a weak vertex-pair for arbitrary vertex $w \in V(G_n - F)$, then $(w_1, w_2)$ is called a normal vertex pair.

*Proposition 7:* If $xy \in E(G_n)$, then $(x, y)$ is a normal vertex pair in $G_n - F$ with $|F| \leq n - 2$.

*Proposition 8:* If $F \subset V(G_n) \cup E(G_n)$ with $|F| = n - 2$, then there exists at most one weak 2-degree vertex $w$ and at most one $w$-weak vertex-pair in $G_n - F(n \geq 5)$.

*Proof:* It is clear that if $G_n - F$ contains no weak 2-degree vertex, then $G_n - F$ contains no weak vertex-pair.

If $G_n - F$ contains a weak 2-degree vertex $w$, then $d_{G_n-F}(w) = 2$ and $F \subset N_{G_n}(w) \cup E_{G_n}(w)$. For arbitrary vertex $w' \in V(G_n) - \{w\}$, by Proposition 3, $|N_{G_n}(w) \cap N_{G_n}(w')| \leq 2$. Then $d_{G_n-F}(w') \geq n - 2 \geq 3(n \geq 5)$, vertex $w'$ is not a weak 2-degree vertex. Hence, $G_n - F$ contains an unique weak 2-degree vertex $w$. Let $N_{G_n-F}(w) = \{w_1, w_2\}$. Then there exists an unique $w$-weak vertex-pair $(w_1, w_2)$ in $G_n - F(n \geq 5)$. □

We use $P_{uv} = (u, u_1, \ldots, u_k, v)$ to denote a path from vertex $u$ to vertex $v$, where any two consecutive vertices are adjacent and $u_1, \ldots, u_k$ are all distinct. If $P_{uw} = (u, u_1, \cdots, u_s, w)$, $P_{wv} = (w, w_1, \cdots, w_t, v)$ and $V(P_{uw}) \cap V(P_{wv}) = \{w\}$, we use $P_{uw} + P_{wv}$ to denote the path $P_{uv} = (u, u_1, \cdots, u_s, w, w_1, \cdots, w_t, v)$, $P_{uv}(u_1, w_1)$ to denote the subpath of $P_{uv}$ which is from vertex $u_1$ to vertex $w_1$, $l_{uv}$ to denote the length of $P_{uv}$. If $u = v$, we use $C = (u, u_1, \ldots, u_k, u)$ to denote a cycle. In this paper, we only consider simple graphs. The terminologies and notations used but undefined in this paper can be found in [10].

*Definition 9 [20]:* A graph $G$ is called as $f$-fault-tolerant $k$-disjoint-path coverable ($f$-fault $k$-*DPC* for brevity) if, given any $k$ pairs of nodes $(u_1, v_1)$,$(u_2, v_2)$,…,$(u_k, v_k)$ in $G - F$, $k$ paths $P_{u_1v_1}$, $P_{u_2v_2}$,…, $P_{u_kv_k}$ exist in $G - F$ for any set of faulty elements $F$ for each $|F| \leq f$ such that $V(P_{u_iv_i}) \cap V(P_{u_jv_j}) = \varnothing$ for every $i, j \in \{1, 2, 3, \ldots, k\}$, $i \neq j$ and $V(P_{u_1v_1}) \cup V(P_{u_2v_2}) \cup \ldots \cup V(P_{u_kv_k}) = V(G - F)$. If $G$ is 0-fault $k$-*DPC*, then $G$ can directly be called as $k$-*DPC*.

Let us investigate some useful properties of $G_n$.

*Lemma 10 [20]:* $G_n$ is $(n - 4)$-fault 2-*DPC*.

*Lemma 11 [17]:* $G_n(n \geq 3)$ is $(n - 3)$-fault-tolerant Hamiltonian connected and $(n - 2)$-fault-tolerant Hamiltonian.

*Lemma 12:* If $F \subset L$ with $|F| = n - 2(n \geq 6)$, then for arbitrary vertex-pair $(x, y)$ with $x, y \in V(L)$, there exists a faulty element $f_1 \in F$ such that $(x, y)$ is a normal vertex-pair in $L - (F - \{f_1\})$.

*Proof:* We prove the result according to two cases as follows.

*Case 1:* $(x, y)$ is a $w$-weak vertex pair in $L - F$. Then $|N_{L-F}(w)| = 2$ and $|F \cap (N_L(w) \cup E_L(w))| = n - 3$.

There exists a faulty element $f_1 \in F \cap (N_L(w) \cup E_L(w))$. Let $F_1 = F - \{f_1\}$. Then $|N_{L-F_1}(w)| = 3$, *i.e.*, $(x, y)$ is not a $w$-weak vertex pair in $L - F_1$.

*Case 2:* $(x, y)$ is a normal vertex pair in $L - F$. Let vertex $z \in L - F$ with the least degree. If $\delta(L - F) \geq 2$, then $(x, y)$ is a normal vertex-pair in $L - (F - \{f_1\})$ for arbitrary faulty element $f_1$ of $F$. Thus, we only need to consider the situation of $\delta(L - F) = 1$. In this situation, $F \subset N_L(z)$, $|N_{L-F}(z)| = 1$, and $|N_{L-F}(z) \cap \{x, y\}| \leq 1$. Consider the following two cases.

*Case 2.1:* $|N_{L-F}(z) \cap \{x, y\}| = 1$. Without loss of generality, assume $N_{L-F}(z) = \{x\}$. If $y = z$, then $N_{L-F}(y) = \{x\}$. By Proposition 7, $(x, y)$ is a normal vertex-pair in $L - (F - \{f_1\})$ for arbitrary faulty element $f_1$ of $F$. We only need to consider the case that $y \neq z$.

If $zy \notin E(L)$, then $(x, y)$ is a normal vertex-pair in $L - (F - \{f_1\})$ for arbitrary faulty element $f_1$ of $F$.

If $zy \in E(L)$, then $zy \in F$. Let $za \in F \backslash \{zy\}$ and $F_1 = F - \{za\}$. Then $N_{L-F_1}(z) \cup E_{L-F_1}(z) = \{x, a, za, zx\}$, $(x, y)$ is a normal vertex-pair in $L - F_1$.

*Case 2.2:* $|N_{L-F}(z) \cap \{x, y\}| = 0$. Then $(x, y)$ is not a $z$-weak vertex-pair in $L - F$.

By Proposition 8, for any vertex $z_1 \in L - F - \{z\}$, $|(N_L(z) \cup E_L(z)) \cap (N_L(z_1) \cup E_L(z_1))| \leq 2$. Since $F \subset N_L(z)$, $|N_{L-F}(z_1)| \geq n - 1 - 2 \geq 3$. Then $(x, y)$ is a normal vertex-pair in $L - (F - \{f_1\})$ for arbitrary faulty element $f_1$ of $F$.

The Lemma holds. □

## III. MAIN RESULT

In this section, we will prove the main result of this paper. We format the theorem as follows.

*Theorem 13: Let $F \subset V(G_n) \cup E(G_n)$ with $|F| \leq n - 2(n \geq 5)$. Then for arbitrary vertex-pair $(u, v)$ in $G_n - F$, there exists a $(n - 2)$-fault-tolerant hamiltonian path $P_{uv}$ excepting $(u, v)$ being a weak vertex-pair in $G_n - F$.*

*Proof:* By Lemma 11, the theorem holds for $|F| \leq n - 3$. Thus, we only need to consider $|F| = n - 2$.

Denote $F^L = F \cap L$, $F^R = F \cap R$, $F^C = F \cap E^C$. We have $F = F^L \cup F^R \cup F^C$.

Without loss of generality, assume that $|F^R| \leq |F^L|$. Then $|F^R| \leq \lfloor \frac{n-2}{2} \rfloor \leq n-4$. Notice that $|N_R(x)| = n-1$ for any vertex $x \in R$, we have $|N_{R-F^R}(x)| \geq 3$. Then there exists no weak vertex-pair in $R - F^R$.

Now, we prove the theorem by induction on $n \geq 5$. The induction basis for $n = 5$ holds by developing a computer program using depth first searching technique combining with backtracking and branch and bound algorithm. Assume that the theorem holds for $n-1$ with $n \geq 6$, then we must show the theorem holds for $n$. According to the distribution of faulty elements, the following cases are considered.

*Case 1:* $|F^L| \leq n-3$.

*Case 1.1:* $u, v \in V(L - F^L)$ or $u, v \in V(R - F^R)$. Without loss of generality, assume that $u, v \in V(L - F^L)$.

*Case 1.1.1:* $(u, v)$ is a $w$-weak vertex-pair in $L - F^L$, *i.e.*, $|N_{L-F^L}(w)| = 2$ and $N_{L-F^L}(w) = \{u, v\}$. Notice that $N_L(w) = n-1$, we have $|F^L| = n-3$ and $|F^R| + |F^C| = 1$. Since $(u, v)$ is a normal vertex-pair in $G_n - F$, $w^R \notin F$. By Proposition 8, $(w, v)$ is a normal vertex-pair in $L - F^L$.

Since $|F^L| = n-3$, by induction hypothesis, there exists a hamiltonian path $P_{wv}$ in $L - F^L$. Since $N_{L-F^L}(w) = \{u, v\}$, $u \in N_{P_{wv}}(w)$. Let $a \in N_{P_{wv}}(u)$ with $a \neq w$ and $b \in N_{P_{wv}}(v)$. Since $|F^R| + |F^C| = 1$, $|\{a^R, aa^R, b^R, bb^R\} \cap F| \leq 1$.

For $a^R, aa^R \notin F$, since $|F^R| \leq 1$, by induction hypothesis, there exists a hamiltonian path $P_{w^R a^R}$ in $R - F^R$. Thus, $P_{uv} = uw + ww^R + P_{w^R a^R} + a^R a + P_{wv}(a, v)$ is a desired hamiltonian path in $G_n - F$(see Fig.3(a)).

For $b^R, bb^R \notin F$, since $|F^R| \leq 1$, by induction hypothesis, there exists a hamiltonian path $P_{w^R b^R}$ in $R - F^R$. Thus, $P_{uv} = P_{wv}(u, b) + bb^R + P_{b^R w^R} + w^R w + wv$ is a desired hamiltonian path in $G_n - F$(see Fig.3(b)).

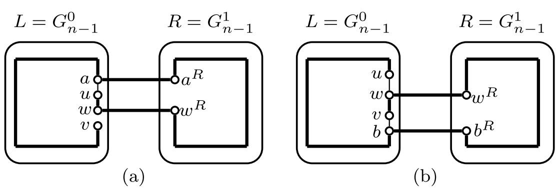



**FIGURE 3.** Illustrations of proofs of Case 1.1.1 of Theorem 13.

*Case 1.1.2:* $(u, v)$ is a normal vertex-pair in $L - F^L$.

Since $|F^L| \leq n-3$, by induction hypothesis, there exists a hamiltonian path $P_{uv}$ in $L - F^L$. Since $l_{uv} = \lfloor \frac{2^{n-1} - f_v^L}{2} \rfloor - (n-2) \geq 10$, there exists an edge $ab \in E(P_{uv})$ such that $\{a^R, b^R, aa^R, bb^R\} \cap F = \varnothing$. Since there exists no weak vertex-pair in $R - F^R$. By induction hypothesis, there exists a hamiltonian path $P_{a^R b^R}$ in $R - F^R$. Thus, $P^1_{uv} = P_{uv}(u, a) + aa^R + P_{a^R b^R} + b^R b + P_{uv}(b, v)$ is a desired hamiltonian path in $G_n - F$(see Fig.4(a)).

*Case 1.2:* $u \in V(L - F^L)$ and $v \in V(R - F^R)$.

According to the definition of $G_n$, $|E^C| = 2^{n-1}$. Since $|E^C| - |F| = 2^{n-1} - (n-2) \geq 28 (n \geq 6)$, there exists a fault-free edge $ab \in E^C$ such that $a, b \notin \{u, v\}$, $a, b \notin F$ and $(u, a)$ is a normal vertex-pair in $L - F^L$. Since there exists no weak vertex-pair in $R - F^R$, $(b, v)$ is a normal vertex-pair in

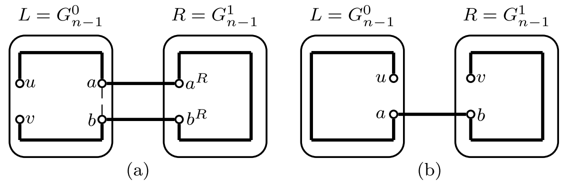



**FIGURE 4.** Illustrations of proofs of Case 1.1.2 and Case 1.2 of Theorem 13.

$R - F^R$. By induction hypothesis, there exists a hamiltonian path $P_{ua}$ in $L - F^L$ and a hamiltonian path $P_{bv}$ in $R - F^R$. Thus, $P_{uv} = P_{ua} + ab + P_{bv}$ is a desired hamiltonian path in $G_n - F$(see Fig.4(b)).

*Case 2:* $|F^L| = n-2$. Then $|F^R| = |F^C| = 0$.

*Case 2.1:* $u, v \in V(L - F^L)$.

By Lemma 12, there exists a faulty element $f_1 \in F^L$ such that $(u, v)$ is a normal vertex-pair in $L - (F^L - \{f_1\})$. Since $|F^L - \{f_1\}| = n-3$, by induction hypothesis, there exists a hamiltonian path $P_{uv}$ in $L - (F^L - \{f_1\})$.

If $P_{uv}$ contains $f_1$ and $f_1 \in E(L) \cap F^L$, say $f_1 = ab$, let $P_{uv} = P_{uv}(u, a) + ab + P_{uv}(b, v)$; if $P_{uv}$ contains $f_1$ and $f_1 \in V(L) \cap F^L$, let $P_{uv} = P_{uv}(u, a) + af_1 + f_1 b + P_{uv}(b, v)$. Since $|F^R| = 0$, by induction hypothesis, there exists a hamiltonian path $P_{a^R b^R}$ in $R$. Thus, $P^1_{uv} = P_{uv}(u, a) + aa^R + P_{a^R b^R} + b^R b + P_{uv}(b, v)$ is a desired hamiltonian path in $G_n - F$(see Fig.5(a) for $f_1 \in E(L) \cap F^L$, see Fig.5(b) for $f_1 \in V(L) \cap F^L$).

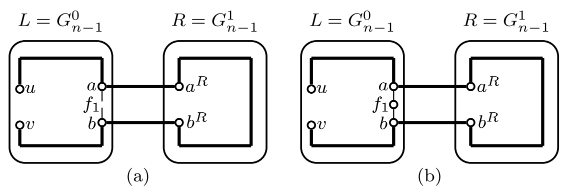



**FIGURE 5.** Illustrations of proofs of Case 2.1 of Theorem 13.

However, if $P_{uv}$ does not contain $f_1$, then an arbitrary edge $a_1 b_1$ can be chosen from $P_{uv}$. By replacing $a, b, a^R$ and $b^R$ with $a_1, b_1, a_1^R$ and $b_1^R$, respectively, we obtain $P^1_{uv} = P_{uv}(u, a_1) + a_1 a_1^R + P_{a_1^R b_1^R} + b_1^R b_1 + P_{uv}(b_1, v)$ is a desired hamiltonian path in $G_n - F$(see Fig. 6).

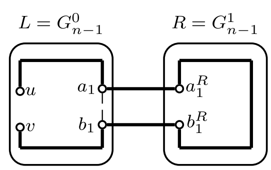



**FIGURE 6.** Illustrations of proofs of Case 2.1 of Theorem 13.

*Case 2.2:* $u \in V(L - F^L)$ and $v \in V(R)$.

A faulty element $f$ can be chosen from $F^L$ such that $f \notin N_L(v^L) \cup E_L(v^L)$. Otherwise, $F^L \subset N_L(v^L) \cup E_L(v^L)$. Consider the following two cases.

*Case 2.2.1:* $F^L \subset N_L(v^L) \cup E_L(v^L)$, *i.e.*, $|F^L \cap (N_L(v^L) \cup E_L(v^L))| = n-2$. Notice that $N_L(v^L) = n-1$, we have

$|N_{L-F^L}(v^L)| = 1$. Since $(u, v)$ is a normal vertex-pair in $G_n - F$, $N_{L-F^L}(v^L) \neq \{u\}$. Choose a faulty element $f$ from $F^L$, then $f \in N_L(v^L) \cup E_L(v^L)$. Since $|F^L - \{f\}| = n-3$, by Lemma 11, there exists a hamiltonian cycle $C$ in $L-(F^L - \{f\})$. Since $|N_{L-F^L}(v^L)| = 1$, we have that $C$ contains the faulty element $f$.

If $f \in N_L(v^L)$, let $C = (u, P_{ua}, a, f, v^L, P_{v^Lb}, b, u)$; if $f \in E_L(v^L)$, say $f = av^L$, let $C = (u, P_{ua}, a, v^L, P_{v^Lb}, b, u)$. Since $|F^R + \{v\}| = 1 \leq n-4$, by Lemma 11, there exists a hamiltonian path $P_{a^Rb^R}$ in $R - (F^R + \{v\})$. Thus, $P_{uv} = P_{ua} + aa^R + P_{a^Rb^R} + b^Rb + P_{bv^L} + v^Lv$ is a desired hamiltonian path in $G_n - F$(see Fig.7(a) for $f \in E_L(v^L)$, see Fig.7(b) for $f \in N_L(v^L)$).

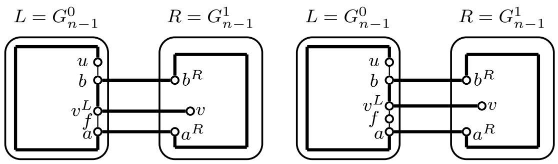



**FIGURE 7.** **Illustrations of proofs of Case 2.2.1 of Theorem 13.**

*Case 2.2.2:* There exists a faulty element $f \in F^L$ such that $f \notin N_L(v^L) \cup E_L(v^L)$. Since $|F^L - \{f\}| = n-3$, by Lemma 2.2, there exists a hamiltonian cycle $C$ in $L - (F^L - \{f\})$.

If $C$ does not contain $f$, let $C = (u, a, P_{ab}, b, u)$. Since $|\{a^R, b^R\} \cap \{v\}| \leq 1$, without loss of generality, assume that $b^R \neq v$. Since $|F^R| = 0$, by induction hypothesis, there exists a hamiltonian path $P_{b^Rv}$ in $R$. Thus, $P_{uv} = ua + P_{ab} + bb^R + P_{b^Rv}$ is a desired hamiltonian path in $G_n - F$(see Fig. 8).

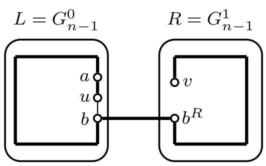



**FIGURE 8.** **Illustrations of proofs of Case 2.2.2 of Theorem 13.**

If $C$ contains $f$, the proof is divided into two parts: (1) $f \in N_C(u) \cup E_C(u)$ and (2) $f \notin N_C(u) \cup E_C(u)$.

*Case 2.2.2.1:* $f \in N_C(u) \cup E_C(u)$. If $f \in E_C(u)$, say $f = au$, let $C = (u, P_{ua}, a, u)$; if $f \in N_C(u)$, let $C = (u, P_{ua}, a, f, u)$.

If $a^R = v$, choose an edge $a_1b_1$ from $C$ such that $a_1, b_1 \notin \{u, a\}$. Since $|F^R + \{v\}| = 1 \leq n-4$, by Lemma 11, there exists a hamiltonian path $P_{a_1^Rb_1^R}$ in $R-(F^R+\{v\})$. Thus, $P_{uv} = P_{ua}(u, a_1) + a_1a_1^R + P_{a_1^Rb_1^R} + b_1^Rb_1 + P_{ua}(b_1, a) + av$ is a desired hamiltonian path in $G_n - F$(see Fig.9(a) for $f \in E_C(u)$, see Fig.9(b) for $f \in N_C(u)$).

If $a^R \neq v$. Since $|F^R| = 0$, by Lemma 11, there exists a hamiltonian path $P_{a^Rv}$ in $R$. Thus, $P_{uv} = P_{ua} + aa^R + P_{a^Rv}$ is a desired hamiltonian path in $G_n - F$(see Fig.10(a) for $f \in E_C(u)$, see Fig.10(b) for $f \in N_C(u)$).

*Case 2.2.2.2:* $f \notin N_C(u) \cup E_C(u)$. If $f \in E(L) \cap F^L$, say $f = a_1b_1$, let $C = (u, P_{ua_1}, a_1, b_1, P_{b_1a}, a, u)$; if $f \in V(L) \cap F^L$, let $C = (u, P_{ua_1}, a_1, f, b_1, P_{b_1a}, a, u)$.

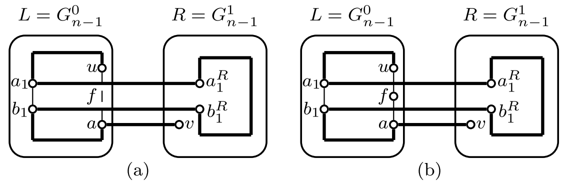



**FIGURE 9.** **Illustrations of proofs of Case 2.2.2.1 of Theorem 13.**

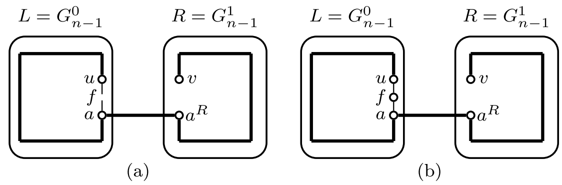



**FIGURE 10.** **Illustrations of proofs of Case 2.2.2.1 of Theorem 13.**

For $a^R = v$. Since $|F^R + \{v\}| = 1 \leq n-4$, by Lemma 11, there exists a hamiltonian path $P_{a_1^Rb_1^R}$ in $R-(F^R+\{v\})$. Thus, $P_{uv} = P_{ua_1} + a_1a_1^R + P_{a_1^Rb_1^R} + b_1^Rb_1 + P_{b_1a} + av$ is a desired hamiltonian path in $G_n - F$(see Fig.11(a) for $f \in E(L) \cap F^L$, see Fig.11(b) for $f \in V(L) \cap F^L$).

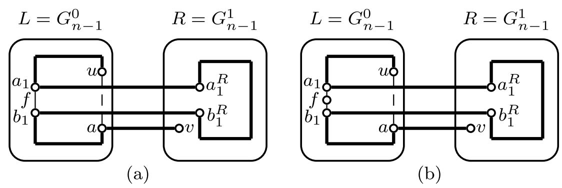



**FIGURE 11.** **Illustrations of proofs of Case 2.2.2.2 of Theorem 13.**

For $a^R \neq v$. Since $|F^R| = 0$, by Lemma 10, there exist two disjoint path $P_{a^Rv}$,$P_{a_1^Rb_1^R}$ such that $V(P_{a^Rv}) \cup V(P_{a_1^Rb_1^R}) = V(R)$. Thus, $P_{uv} = P_{ua_1} + a_1a_1^R + P_{a_1^Rb_1^R} + b_1^Rb_1 + P_{b_1a} + aa^R + P_{a^Rv}$ is a desired hamiltonian path in $G_n - F$(see Fig.12(a) for $f \in E(L) \cap F^L$, see Fig.12(b) for $f \in V(L) \cap F^L$).

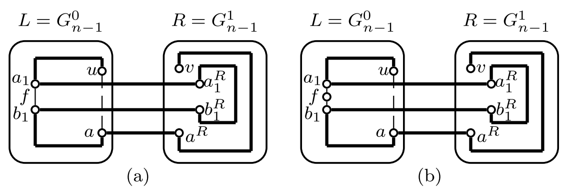



**FIGURE 12.** **Illustrations of proofs of Case 2.2.2.2 of Theorem 13.**

*Case 2.3:* $u, v \in V(R)$.

Let $f$ be a faulty element in $F^L$. Then $|F^L - \{f\}| = n-3$. By Lemma 11, there exists a hamiltonian cycle $C$ in $L-(F^L - \{f\})$.

If $C$ contains $f$ and $f \in E(L) \cap F^L$, say $f = ba$, let $C = (a, P_{ab}, b, a)$; if $C$ contains $f$ and $f \in V(L) \cap F^L$, let $C = (a, P_{ab}, b, f, a)$. If $C$ does not contain $f$, let $C = (a, P_{ab}, b, a)$. Consider the following three cases.

*Case 2.3.1:* $|\{a^R, b^R\} \cap \{u, v\}| = 0$. Since $F^R = 0$, by Lemma 10, there exist two disjoint path $P_{ua^R}$,$P_{b^Rv}$ such that $V(P_{ua^R}) \cup V(P_{b^R v}) = V(R)$. Thus, $P_{uv} = P_{ua^R} + a^R a + P_{ab} + bb^R + P_{b^R v}$ is a desired hamiltonian path in $G_n - F$ (see Fig.13(a) for $f \in E(L) \cap F^L$, see Fig.13(b) for $f \in V(L) \cap F^L$).

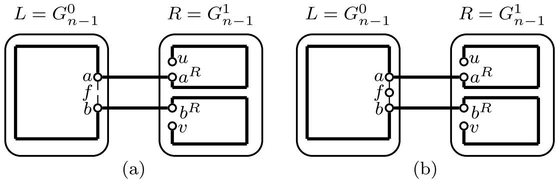



**FIGURE 13.** Illustrations of proofs of Case 2.3.1 of Theorem 13.

*Case 2.3.2:* $|\{a^R, b^R\} \cap \{u, v\}| = 2$. Without loss of generality, assume that $a^R = u$ and $b^R = v$. Let $a_1 b_1$ be an edge of $C$ with $a_1, b_1 \notin \{a, b\}$. Since $|F^R + \{u, v\}| = 2 \le n - 4$, by Lemma 11, there exists a hamiltonian path $P_{a_1^R b_1^R}$ in $R - (F^R + \{u, v\})$. Thus, $P_{uv} = ua + P_{ab}(a, a_1) + a_1 a_1^R + P_{a_1^R b_1^R} + b_1^R b_1 + P_{ab}(b_1, b) + bv$ is a desired hamiltonian path in $G_n - F$ (see Fig.14(a) for $f \in E(L) \cap F^L$, see Fig.14(b) for $f \in V(L) \cap F^L$).

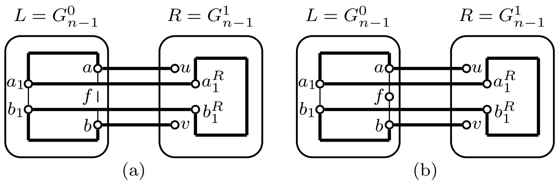



**FIGURE 14.** Illustrations of proofs of Case 2.3.2 of Theorem 13.

*Case 2.3.3:* $|\{a^R, b^R\} \cap \{u, v\}| = 1$.

*Case 2.3.3.1:* $u \in \{a^R, b^R\}$. Without loss of generality, assume that $u = a^R$. Since $|F^R + \{u\}| = 1 \le n - 4$, by Lemma 11, there exists a hamiltonian path $P_{b^R v}$ in $R - (F^R + \{u\})$. Thus, $P_{uv} = ua + P_{ab} + bb^R + P_{b^R v}$ is a desired hamiltonian path in $G_n - F$ (see Fig.15(a) for $f \in E(L) \cap F^L$, see Fig.15(b) for $f \in V(L) \cap F^L$).

*Case 2.3.3.2:* $v \in \{a^R, b^R\}$. Without loss of generality, assume that $v = a^R$. Since $|F^R + \{v\}| = 1 \le n - 4$, by Lemma 11, there exists a hamiltonian path $P_{ub^R}$ in $R - (F^R + \{v\})$. Thus, $P_{uv} = P_{ub^R} + b^R b + P_{ba} + av$ is a desired hamiltonian path in $G_n - F$ (see Fig.16(a) for $f \in E(L) \cap F^L$, see Fig.16(b) for $f \in V(L) \cap F^L$).

Combining the above cases, we completed the proof of the Theorem 13. □

## IV. APPLICATION TO MULTIPROCESSOR SYSTEMS

This section describes the application of the theorem presented in Section III to several popular multiprocessor systems. The theorem can also be applied to many other useful systems.

The $n$-dimensional crossed cube $CQ_n$ was first proposed by Efe [21]. $CQ_n$ has $2^n$ vertices labeled by $n$-bit strings and can be recursively defined as follows:

(1) $CQ_1$ is a complete graph with two vertices labeled by 0 and 1.

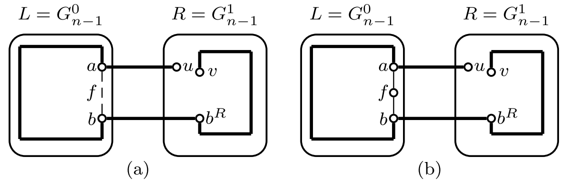



**FIGURE 15.** Illustrations of proofs of Case 2.3.3.1 of Theorem 13.

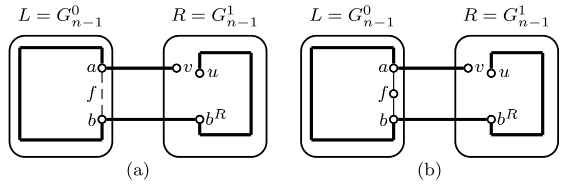



**FIGURE 16.** Illustrations of proofs of Case 2.3.3.2 of Theorem 13.

(2) For $n \ge 2$, $CQ_n$ is obtained by taking two copies of $CQ_{n-1}$, denoted by $0CQ_{n-1}$ with vertex-set $V(0CQ_{n-1}) = \{0x_{n-2} \dots x_1 x_0 : x_i = 0 \text{ or } 1, i = 0, 1, \dots, n-2\}$ and $1CQ_{n-1}$ with vertex-set $V(1CQ_{n-1}) = \{1y_{n-2} \dots y_1 y_0 : y_i = 0 \text{ or } 1, i = 0, 1, \dots, n-2\}$, respectively, and adding an edge joining $0x_{n-2} \dots x_1 x_0 \in V(0CQ_{n-1})$ and $1y_{n-2} \dots y_1 y_0 \in V(1CQ_{n-1})$ if and only if

(a) $x_{n-2} = y_{n-2}$ if $n$ is even, and

(b) $x_{2i+1}x_{2i} \sim y_{2i+1}y_{2i}$ for $0 \le i \le \lfloor \frac{n-1}{2} \rfloor$, where two binary strings $x = x_1 x_0$ and $y = y_1 y_0$ are pair-related, denoted as $x \sim y$, if and only if $(x, y) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$.

According to the definition, there is a perfect matching between $V(0CQ_{n-1})$ and $V(1CQ_{n-1})$. Moreover, $CQ_n$ belongs to *THLNs* and can be represented as $CQ_n = \oplus_\phi(0CQ_{n-1}, 1CQ_{n-1})$. Fig. 17 illustrates $CQ_3$ and $CQ_4$.

Hence, by Theorem 13, we can obtain the Corollary 14 as follows.

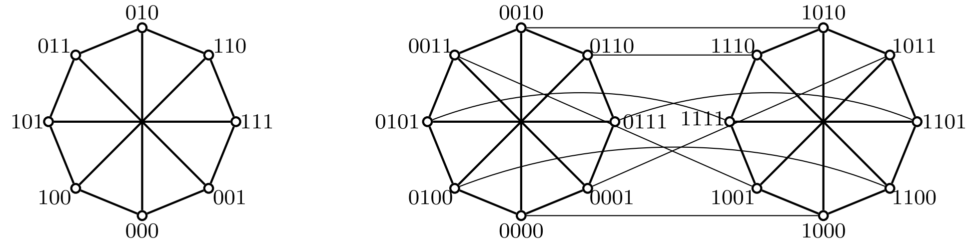
*Corollary 14:* Let $F \subset V(CQ_n) \cup E(CQ_n)$ with $|F| \le n-2$. Then for arbitrary vertex-pair $(u, v)$ in $CQ_n - F (n \ge 5)$, there exists a $(n-2)$-fault-tolerant hamiltonian path joining vertices $u$ and $v$ except $(u, v)$ being a weak vertex-pair in $CQ_n - F$.

The $n$-dimensional locally twisted cube $LTQ_n$ $(n \ge 2)$, proposed by Yang *et al.* [37], is defined recursively as follows:

(1) $LTQ_2$ is a graph isomorphic to $Q_2$.

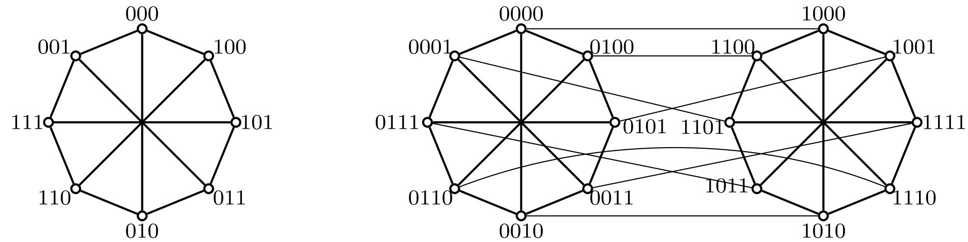
(2) For $n \ge 3$, $LTQ_n$ is built from two disjoint copies of $LTQ_{n-1}$ according to the following steps. Let $0LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of one copy of $LTQ_{n-1}$ with 0, let $1LTQ_{n-1}$ denote the graph obtained by prefixing the label of each vertex of the other copy $LTQ_{n-1}$ with 1, and connect each vertex $x = 0x_2 x_3 \dots x_n$ of $0LTQ_{n-1}$ with the vertex $1(x_2 + x_n)x_3 \dots x_n$ of $1LTQ_{n-1}$ by an edge, where $'+'$ represents the modulo 2 addition.

According to the preceding definition, there is a perfect matching between $V(0LTQ_{n-1})$ and $V(1LTQ_{n-1})$. Moreover, $LTQ_n$ belongs to *THLNs* and can be represented as




(a) (b)

**FIGURE 17.** **(a) $CQ_3$; (b) $CQ_4$.**




(a) (b)

**FIGURE 18.** **(a) $LTQ_3$; (b) $LTQ_4$.**

$LTQ_n = \oplus_\phi(0LTQ_{n-1}, 1LTQ_{n-1})$. The graphs shown in Fig. 18 are $LTQ_3$ and $LTQ_4$.

Hence, by Theorem 13, we can obtain the Corollary 15 as follows.

*Corollary 15:* Let $F \subset V(LTQ_n) \cup E(LTQ_n)$ with $|F| \leq n-2$. Then for arbitrary vertex-pair $(u, v)$ in $LTQ_n - F(n \geq 5)$, there exists a $(n-2)$-fault-tolerant hamiltonian path joining vertices $u$ and $v$ except $(u, v)$ being a weak vertex-pair in $LTQ_n - F$.

The $n$-dimensional Möbius cube $MQ_n$, proposed by Cull and Larson [27], is such an undirected graph, its vertex set is the same as the vertex set of $Q_n$, the vertex $X = x_1x_2 \cdots x_n$ connects to $n$ other vertices $Y_i$, $(1 \leq i \leq n)$, where each $Y_i$ satisfies one of the following equations:

$$Y_i = \begin{cases} x_1x_2 \cdots x_{i-1}\bar{x}_i x_{i+1} \cdots x_n & \text{when } x_{i-1} = 0; \\ x_1x_2 \cdots x_{i-1}\bar{x}_i \bar{x}_{i+1} \cdots \bar{x}_n & \text{when } x_{i-1} = 1. \end{cases}$$

From the above definition, $X$ connects to $Y_i$ by complementing the bit $x_i$ if $x_{i-1} = 0$ or by complementing all bits of $x_i, \cdots, x_n$ if $x_{i-1} = 1$. The connection between $X$ and $Y_1$ is undefined, so we can assume that $x_0$ is either equal to 0 or equal to 1, which gives us slightly different network topologies. If we assume $x_0 = 0$, we call the network a 0-Möbius cube; and if we assume $x_0 = 1$, we call the network a 1-Möbius cube, denoted by $MQ_n^0$ and $MQ_n^1$, respectively.

According to the above definition, it is not difficult to see that $MQ_n^0$ (respectively, $MQ_n^1$) can be recursively constructed from $MQ_{n-1}^0$ and $MQ_{n-1}^1$ by adding $2^{n-1}$ edges. $MQ_n^0$ is constructed by connecting all pairs of vertices that differ only in the 1-th bit, and $MQ_n^1$ is constructed by connecting all pairs of vertices that differ in the 1-th through the $n$-th bits. Therefore, $MQ_n = \oplus_\phi(MQ_{n-1}^0, MQ_{n-1}^1)$ and $MQ_n \in HTLNs$. Fig. 19 illustrates $MQ_3^0$, $MQ_4^0$, $MQ_3^1$ and $MQ_4^1$.

Hence, by Theorem 13, we can obtain the Corollary 16 as follows.

*Corollary 16:* Let $F \subset V(MQ_n) \cup E(MQ_n)$ with $|F| \leq n-2$. Then for arbitrary vertex-pair $(u, v)$ in $MQ_n - F(n \geq 5)$, there exists a $(n-2)$-fault-tolerant hamiltonian path joining vertices $u$ and $v$ except $(u, v)$ being a weak vertex-pair in $MQ_n - F$.

The $n$-dimensional twisted cube $TQ_n$ was first proposed by Hilbers *et al.* [28]. We denote the complement of $u_i$ by $\overline{u_i} = 1 - u_i$. To define $TQ_n$, first of all, a parity function $P_i(x)$ is introduced. Let $u = u_{n-1}u_{n-2}, \ldots, u_1u_0$ be a binary string. For $0 \leq i \leq n-1$, $P_i(u) = u_i \oplus u_{i-1} \oplus \ldots \oplus u_1 \oplus u_0$, where $\oplus$ is the exclusive-or operation. We give the recursive definition of the $TQ_n$ for any odd integer $n \geq 1$ as follows.

(1) $TQ_1$ is the complete graph with two nodes labeled by 0 and 1, respectively.

(2) For an odd integer $n \geq 3$, $TQ_n$ consists of four copies of $TQ_{n-2}$. We use $TQ_{n-2}^{ij}$ to denote a $(n-2)$-dimensional twisted cube which is a subgraph of $TQ_n$ induced by the nodes labeled by $iju_{n-3} \ldots u_0$, where $i, j \in \{0, 1\}$. Each node $u = u_{n-1}u_{n-2} \ldots u_1u_0 \in V(TQ_n)$ is adjacent to $\overline{u_{n-1}}u_{n-2} \ldots u_1u_0$ and $\overline{u_{n-1}}\overline{u_{n-2}} \ldots u_1u_0$ if $P_{n-3}(u) = 0$; and to $\overline{u_{n-1}}u_{n-2} \ldots u_1u_0$ and $u_{n-1}\overline{u_{n-2}} \ldots u_1u_0$ if $P_{n-3}(u) = 1$.

Therefore, $TQ_n = \oplus_\phi(\oplus_\phi(TQ_{n-2}^{00}, TQ_{n-2}^{10}), \oplus_\phi(TQ_{n-2}^{01}, TQ_{n-2}^{11}))$ and $TQ_n \in HTLNs$. The graphs shown in Fig. 20 are $TQ_3$ and $TQ_5$.

Hence, by Theorem 13, we can obtain the Corollary 17 as follows.

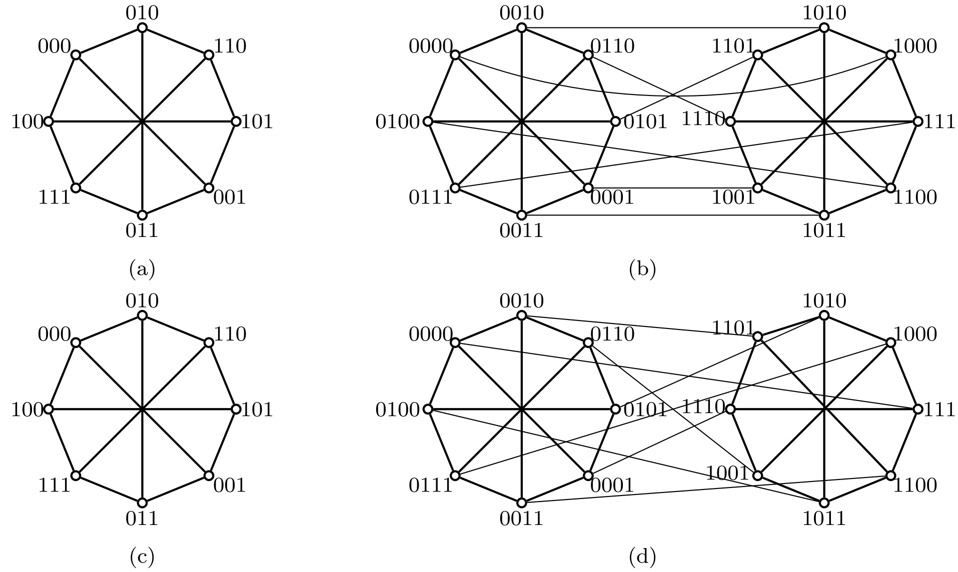



**FIGURE 19.** (a) $MQ_3^0$; (b) $MQ_4^0$; (c) $MQ_3^1$; (d) $MQ_4^1$.

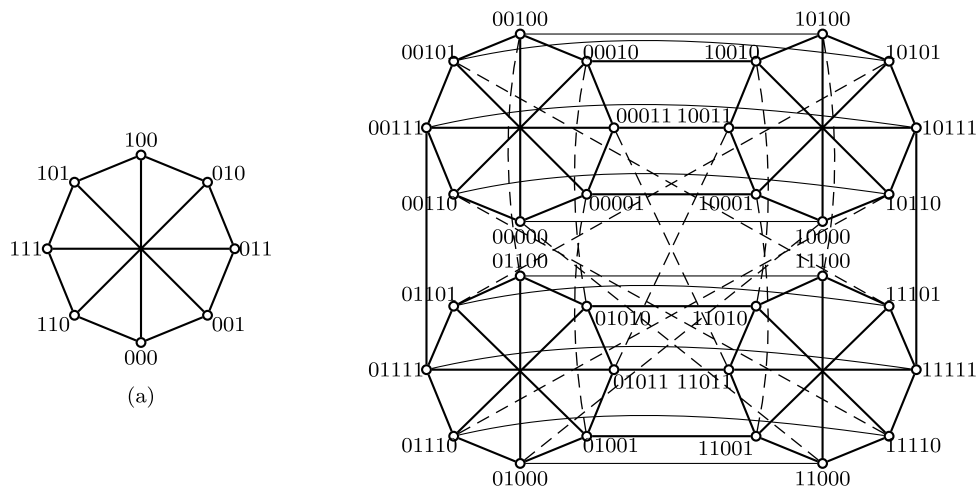



**FIGURE 20.** (a) $TQ_3$; (b) $TQ_5$.

*Corollary 17:* Let $F \subset V(TQ_n) \cup E(TQ_n)$ with $|F| \leq n-2$. Then arbitrary vertex-pair $(u, v)$ in $TQ_n - F$, there exists a $(n-2)$-fault-tolerant hamiltonian path joining vertices $u$ and $v$ for every odd $n \geq 5$ except $(u, v)$ being a weak vertex-pair in $TQ_n - F$.

## V. CONCLUDING REMARKS

As one of the most fundamental networks for parallel and distributed computation, paths are suitable for developing simple algorithms with low communication cost. Edge and/or vertex failures are inevitable when a large parallel computer system is put in use. Therefore, the fault-tolerant capacity of a network is a critical issue in parallel computing. The fault-tolerant hamiltonian connectivity of an interconnection network is a measure of its capability of implementing the longest path-structured parallel algorithms in a communication-efficient fashion in the presence of faults.

In view of the fact that the twisted hypercube-like networks *THLNs* have several properties superior to the hypercube, for example, the diameter is approximately half that of the ordinary hypercube with the same vertex degree. This shows that, when the twisted hypercube-like networks are used to model the topological structure of a large-scale parallel processing system, our result implies that the system has larger capability of implementing the hamiltonian path-structured parallel algorithms in a communication-efficient fashion in the hybrid presence of edge and vertex failures than one of the hypercube network.

This paper studied the longest path in an *n-D THLN*$(n \geq 5)$ with a set $F$ of up to $n-2$ faulty elements. We have proved that for arbitrary vertex-pair $(u, v)$ in $G_n - F$, there exists a $(n-2)$-fault-tolerant hamiltonian path joining vertices $u$ and $v$ except $(u, v)$ being a weak vertex-pair in $G_n - F$. The technical theorem proposed in this paper can be applied to several multiprocessor systems, including $n$-dimensional crossed cubes $CQ_n$, $n$-dimensional twisted cubes $TQ_n$ for odd $n$, $n$-dimensional locally twisted cubes $LTQ_n$, and $n$-dimensional Möbius cubes $MQ_n$.

In a future work, we will extend our strategy to other classes of networks.

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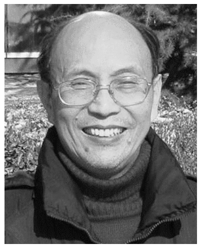

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