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An Effective Numerical Algorithm Based on Stable Recovery for Partial Differential Equations With Distributed Delay

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ABSTRACT This paper is concerned with the numerical approximation of a nonlinear convection–reaction–diffusion equation with distributed delay. Using the stable recovery, we convert the original equation into nonlinear reaction–diffusion equation with distributed delay. Then, we propose a fully discrete numerical scheme to approximate the reduced equation. We investigate solvability, convergence, and stability of the method. Besides, we present a numerical example to verify the effectiveness of the method.

INDEX TERMS Convection-reaction-diffusion equation with distributed delay, compact difference scheme, convergence, stability.

I. INTRODUCTION

We are interested in proposing an effective numerical scheme for solving the following nonlinear convection-reaction-diffusion equation with distributed delay

$$\begin{aligned}\frac{\partial v}{\partial t}+a\frac{\partial v}{\partial x}-b\frac{\partial^2 v}{\partial x^2} &= f\left(t,x,v(t,x),\int_{t-\tau}^{t} g\big(t,x,s,v(s,x)\big)ds\right),\\ &\qquad (t,x)\in(0,T]\times(\alpha,\beta),\end{aligned} \tag{1}$$

$$v(t,x)=\phi(t,x),\quad t\in[-\tau,0],\ x\in[\alpha,\beta], \tag{2}$$

$$v(t,\alpha)=v_\alpha(t),\quad v(t,\beta)=v_\beta(t),\ t>0, \tag{3}$$

where a denotes the constant speed of convection, $b>0$ is diffusion coefficient, and τ is a positive constant.

Ordinary and partial differential equations with memory or delay (1) are widely used to model various phenomena in physics, engineering control systems, population dynamics, economics, epidemiology [1]–[11]. One remarkable feature is the integral term or delay in the models. Generally speaking, the solutions of delay partial differential equations cannot be expressed explicitly due to the terms. Most researchers turn to study the models' evolution by using numerical approximations [12]–[15]. For instance, Sun and Zhang [16] obtained convergence of a linearized Crank-Nicolson compact difference scheme for the nonlinear delay reaction-diffusion equation. Pang and Wang [17] considered the existence and attractivity of periodic solutions for nonlinear delay diffusive Nicholson blowflies equation. Li *et al.* [18] studied delay-dependent stability of compact finite difference scheme for the reaction-diffusion equation with delay. Zhang and Xiao [19] used one-leg θ-method to investigate the exact and numerical stability analysis of reaction-diffusion equation with distributed delay. Later on, they applied implicit-explicit multistep finite element methods to study the nonlinear convection-diffusion-reaction equation with time delay [20], and obtained the L^2-norm error estimates with handling the breaking point and stability results. Most related works focus on the problem with $a=0$ or the small convection term. For more details about the topic, we refer readers to [21]–[36].

As is known, the solutions of convection-diffusion equation have sharp layers when convection term dominates diffusion term [37]. In such cases, most numerical schemes fail to approximate the model effectively. Our goal is to develop an effective numerical scheme for solving the nonlinear convection-reaction-diffusion equation with distributed delay. Our scheme is based on a stable recovery, with which the original equation is transformed to the nonlinear reaction-diffusion equation with distributed delay. Thus, we can stably recover the numerical solution of the original equation from the numerical solution of the reduced equation. Then, we propose a fully discrete numerical scheme to solve the reduced equation. We also investigate solvability, convergence, and stability of the method. We show that the convergence order of the proposed scheme is $\mathcal{O}(\Delta t^2 + h^4)$ in L^∞ norm for the nonlinear convection-reaction-diffusion equation with distributed delay. While in the previous work, the researchers mainly studied the error estimates of different numerical schemes for the linear problems [38]–[40]. We remark that although the recovery is widely used in [41] and [42], we focus on the different models and the numerical analysis is more technical.

The rest of the paper is organized as follows. In Section II, equations (1)-(3) are reduced to a type of nonlinear reaction-diffusion equation with distributed delay by using the stable recovery. Section III is devoted to the construction of the fully discrete scheme. In Section IV, solvability, convergence, and stability of the scheme are investigated. In Section V, a numerical example is carried out to illustrate the theoretical results. Finally, some concluding remarks are summarized in Section VI.

II. THE EQUIVALENT CONVERSION

In this section, we present a stable recovery to reduce equations (1)-(3) to a type of nonlinear reaction-diffusion equation with distributed delay.

First, applying the transformation

$$v(t,x) = \exp(px + qt)u(t,x)$$

to equation (1), we have

$$\begin{aligned}\frac{\partial u}{\partial t} &+ (a-2bp)\frac{\partial u}{\partial x} - b\frac{\partial^2 u}{\partial x^2} + (q+ap-bp^2)u(t,x)\\ &= \exp(-px-qt)f\Big(t, x, \exp(px+qt)u(t,x),\\ &\quad \int_{t-\tau}^{t} g\big(t,x,s,\exp(px+qs)u(s,x)\big)ds\Big). \end{aligned} \tag{4}$$

Setting

$$\begin{cases} a - 2bp = 0,\\ q + ap - bp^2 = 0,\end{cases} \tag{5}$$

and solving it, we obtain $p = \frac{a}{2b}$, $q = -\frac{a^2}{4b}$. Then, we eliminate the convection term $\frac{\partial u}{\partial x}$, and obtain the equivalent equations as follows

$$\begin{cases} \frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = \exp(-px-qt)f\Big(t,x,\exp(px+qt)u(t,x),\\ \int_{t-\tau}^{t} g\big(t,x,s,\exp(px+qs)u(s,x)\big)ds\Big),\\ u(t,x) = \exp(-px-qt)\phi(t,x),\\ u(t,\alpha) = \exp(-px-qt)v_\alpha(t),\\ u(t,\beta) = \exp(-px-qt)v_\beta(t). \end{cases} \tag{6}$$

Now, we need the following two steps to solve equations (1)-(3):

- Solve equation (6), and obtain the numerical solution of $u(t,x)$;
- Substitute the value of $u(t,x)$ to the transformation $v(t,x) = \exp(px+qt)u(t,x)$, and obtain the numerical solution of $v(t,x)$.

Equations (1)-(3) and equation (6) are equivalent. It is noted that although the scheme proposed in this article aims to solve the nonlinear reaction-diffusion equation with distributed delay, it actually works for the nonlinear convection-reaction-diffusion equation with distributed delay as well. For the sake of conciseness, we rewrite equation (6) as follows

$$\begin{cases} \frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = F\Big(t,x,u(t,x),\int_{t-\tau}^{t} G\big(t,x,s,u(s,x)\big)ds\Big),\\ u(t,x) = \varphi(t,x), \quad t\in[-\tau,0],\ x\in[\alpha,\beta],\\ u(t,\alpha) = u_\alpha(t), \quad u(t,\beta) = u_\beta(t),\ t>0. \end{cases} \tag{7}$$

We assume F and G satisfy the following assumptions. Suppose F has the first-order continuous derivative with respect to the third and fourth components in the ε_0-neighborhood of the solution, where $\varepsilon_0 > 0$ is a constant. We also assume that, for $x \in [\alpha,\beta]$, $0 < t < T$, $s \geq -\tau$,

$$\|F(t,x,u,z) - F(t,x,\bar{u},\bar{z})\| \leq c_1\|u-\bar{u}\| + c_2\|z-\bar{z}\|, \tag{8}$$

$$\|G(t,x,s,u) - G(t,x,s,\bar{u})\| \leq L\|u-\bar{u}\|, \tag{9}$$

where $u, \bar{u}, z, \bar{z} \in \mathbb{R}$, c_1, c_2, L are positive constants.

III. CONSTRUCTION OF LINEARIZED COMPACT SCHEME

In this section, we construct the linearized compact scheme to solve equation (7).

Set $\Delta t = \frac{\tau}{m}$, $h = \frac{1}{M}$, where m and M are two positive integers. Let $t_k = k\Delta t$, $t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1})$, denote $\Omega_{\Delta t} = \{t_k | -m \leq k \leq K\}$, where $K = [\frac{T}{\Delta t}]$. Set $x_i = ih$, $\Omega_h = \{x_i | 0 \leq i \leq M\}$. Let $\mathcal{W}_h = \{w_i^k |\ i = 0,1,2,\cdots,M, k = -m,-m+1,\cdots,0,1,\cdots,K\}$ be grid functions defined on $\Omega_{\Delta t}\times\Omega_h$. We introduce the following notations

$$w_i^{k+\frac{1}{2}} = \frac{w_i^{k+1}+w_i^k}{2}, \quad \delta_t w_i^{k+\frac{1}{2}} = \frac{w_i^{k+1}-w_i^k}{\Delta t},$$

$$\delta_x w_{i+\frac{1}{2}}^k = \frac{w_{i+1}^k - w_i^k}{h}, \quad \delta_x^2 w_i^k = \frac{\delta_x w_{i+\frac{1}{2}}^k - \delta_x v_{i-\frac{1}{2}}^k}{h}.$$

For $v, w \in \mathcal{W}_h$, we define the following inner products and corresponding norms

$$(v,w) = h\sum_{i=1}^{M-1} v_i w_i, \quad \|w\| = \sqrt{h\sum_{i=1}^{M-1} w_i^2},$$

$$|w|_1 = \sqrt{h\sum_{i=0}^{M-1}(\delta_x w_{i+\frac{1}{2}})^2}, \quad \|w\|_\infty = \max_{0\leq i\leq M}|w_i|.$$

We introduce the compact difference operator as follows

$$\mathcal{A}w_i^k = \begin{cases} \frac{w_{i-1}^k + 10w_i^k + w_{i+1}^k}{12}, & 1 \leq i \leq M-1,\\ w_i^k & i = 0 \text{ or } M. \end{cases}$$

Next, we present the construction of the linearized compact scheme and the derivation of the local truncation error.

Considering euqation (7) at the point $(t_{k+\frac{1}{2}}, x_i)$, we have

$$\frac{\partial u}{\partial t}(t_{k+\frac{1}{2}}, x_i) - b\frac{\partial^2 u}{\partial x^2}(t_{k+\frac{1}{2}}, x_i) = F\Big(t_{k+\frac{1}{2}}, x_i, u(t_{k+\frac{1}{2}}, x_i), \tilde{z}(t_{k+\frac{1}{2}}, x_i)\Big), \quad (10)$$

where

$$\tilde{z}(t_{k+\frac{1}{2}}, x_i) = \int_{t_{k+\frac{1}{2}-m}}^{t_{k+\frac{1}{2}}} G\big(t_{k+\frac{1}{2}}, x_i, s, u(s, x_i)\big)ds.$$

Setting $U_i^k = u(t_k, x_i)$, Taylor expansion yields

$$\frac{\partial u}{\partial t}(t_{k+\frac{1}{2}}, x_i) = \delta_t U_i^{k+\frac{1}{2}} - \frac{\Delta t^2}{24}\frac{\partial^3 u}{\partial t^3}(\xi_i^k, x_i), \quad (11)$$

$$\frac{\partial^2 u}{\partial x^2}(t_{k+\frac{1}{2}}, x_i) = \frac{1}{2}\Big[\frac{\partial^2 u}{\partial x^2}(t_k, x_i) + \frac{\partial^2 u}{\partial x^2}(t_{k+1}, x_i)\Big] - \frac{\Delta t^2}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}(\eta_i^k, x_i), \quad (12)$$

where $t_k < \xi_i^k, \eta_i^k < t_{k+1}$.

Applying the compound trapezoidal formula to the integral term $\tilde{z}(t_{k+\frac{1}{2}}, x_i)$, we have

$$\tilde{z}(t_{k+\frac{1}{2}}, x_i) = \int_{t_{k+\frac{1}{2}-m}}^{t_{k+\frac{1}{2}}} G\big(t_{k+\frac{1}{2}}, x_i, s, u(s, x_i)\big)ds = Z_i^{k+\frac{1}{2}} - \frac{\tau}{12}\Delta t^2 \frac{\partial^2 G}{\partial t^2}\big(t_{k+\frac{1}{2}}, x_i, \phi^k, u(\phi^k, x_i)\big),$$

where

$$Z_i^{k+\frac{1}{2}} = \Delta t \sum_{j=0}^{m} c_j G(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}-j}, U_i^{k+\frac{1}{2}-j}),$$

$$c_0 = c_m = \frac{1}{2},$$

$$c_j = 1 (j = 1, \cdots, m-1), \quad \phi^k \in (t_{k+\frac{1}{2}-m}, t_{k+\frac{1}{2}}).$$

Then, applying Taylor expansion to the right hand side of equation (10), it holds that

$$\begin{aligned} &F\big(t_{k+\frac{1}{2}}, x_i, u(t_{k+\frac{1}{2}}, x_i), \tilde{z}(t_{k+\frac{1}{2}}, x_i)\big) \\ &= F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + \big(u(t_{k+\frac{1}{2}}, x_i) \\ &\quad - \tfrac{3}{2}U_i^k + \tfrac{1}{2}U_i^{k-1}\big)F_1(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k) \\ &\quad + \big(\tilde{z}(t_{k+\frac{1}{2}}, x_i) - Z_i^{k+\frac{1}{2}}\big)F_2(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k) \\ &= F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + \tfrac{3}{8}\Delta t^2 \frac{\partial^2 u}{\partial t^2}(\sigma_i^k, x_i) \\ &F_1(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k) - \frac{\tau}{12}\Delta t^2 \frac{\partial^2 G}{\partial t^2}\big(t_{k+\frac{1}{2}}, x_i, \phi^k, u(\phi^k, x_i)\big) \\ &F_2(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k) \\ &= F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + \Delta t^2 \psi_i^k, \end{aligned} \quad (13)$$

where

$$\psi_i^k = \frac{3}{8}\frac{\partial^2 u}{\partial t^2}(\sigma_i^k, x_i)F_1(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k) - \frac{\tau}{12}\frac{\partial^2 G}{\partial t^2}\big(t_{k+\frac{1}{2}}, x_i, \phi^k, u(\phi^k, x_i)\big)F_2(t_{k+\frac{1}{2}}, x_i, \theta_i^k, \rho_i^k),$$

and θ_i^k is between $u(t_{k+\frac{1}{2}}, x_i)$ and $\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}$, ρ_i^k is between $z(t_{k+\frac{1}{2}}, x_i)$ and $Z_i^{k+\frac{1}{2}}$, $\sigma_i^k \in (t_{k-1}, t_{k+\frac{1}{2}})$.

Plugging equations (11)-(13) into equation (10), we obtain

$$\delta_t U_i^{k+\frac{1}{2}} - \frac{b}{2}\Big[\frac{\partial^2 u}{\partial x^2}(t_k, x_i) + \frac{\partial^2 u}{\partial x^2}(t_{k+1}, x_i)\Big] = F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + \Delta t^2 r_i^k, \quad (14)$$

where

$$r_i^k = \frac{1}{24}\frac{\partial^3 u}{\partial t^3}(\xi_i^k, x_i) - \frac{b}{8}\frac{\partial^4 u}{\partial x^2 \partial t^2}(\eta_i^k, x_i) + \psi_i^k.$$

Acting operator $\mathcal{A}$ on both sides of equation (14), we have

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}} - \frac{b}{2}\Big[\mathcal{A}\frac{\partial^2 u}{\partial x^2}(t_k, x_i) + \mathcal{A}\frac{\partial^2 u}{\partial x^2}(t_{k+1}, x_i)\Big] = \mathcal{A}F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + \Delta t^2 \mathcal{A} r_i^k. \quad (15)$$

Now, we recall a lemma in [16] to obtain an estimate for the operator $\mathcal{A}$.

Lemma 1 [16]: Assume that $\varrho(x) \in C^6[x_{i-1}, x_{i+1}]$. Then

$$\frac{1}{12}\big[\varrho''(x_{i-1}) + 10\varrho''(x_i) + \varrho''(x_{i+1})\big] - \frac{1}{h^2}\big[\varrho(x_{i-1}) - 2\varrho(x_i) + \varrho(x_{i+1})\big] = \frac{h^4}{240}\varrho^{(6)}(\omega_i),$$

where $\omega_i \in (x_{i-1}, x_{i+1})$.

Applying the above lemma, we get

$$\mathcal{A}\frac{\partial^2 u}{\partial x^2}(t_k, x_i) = \delta_x^2 U_i^{k+\frac{1}{2}} + \frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(t_k, \gamma_i^k), \quad (16)$$

where $\gamma_i^k \in (x_{i-1}, x_{i+1})$.

Substituting equation (16) into equation (15), we have

$$\mathcal{A}\delta_t U_i^{k+\frac{1}{2}} - b\delta_x^2 U_i^{k+\frac{1}{2}} = \mathcal{A}F\big(t_{k+\frac{1}{2}}, x_i, \tfrac{3}{2}U_i^k - \tfrac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\big) + R_i^k, \quad (17)$$

where the truncation error

$$R_i^k = \Delta t^2 \mathcal{A} r_i^k + \frac{bh^4}{480}\Big[\frac{\partial^6 u}{\partial x^6}(t_k, \gamma_i^k) + \frac{\partial^6 u}{\partial x^6}(t_{k+1}, \gamma_i^{k+1})\Big],$$

and there exists a constant C_R such that

$$|R_i^k| \le C_R(\Delta t^2 + h^4).$$

Noticing the initial and boundary conditions in (7), we have

$$U_i^k = \varphi(t_k, x_i), \quad -m \le k \le 0, \ 0 \le i \le M, \quad (18)$$

$$U_0^k = u_\alpha(t_k), \quad U_M^k = u_\beta(t_k), \ 1 \le k \le K. \quad (19)$$

Denote u_i^k as the numerical approximation to $u(t_k, x_i)$. We omit the small term R_i^k and replace U_i^k with u_i^k, and develop the linearized compact scheme for problem (7)

$$\mathcal{A}\delta_t u_i^{k+\frac{1}{2}} - b\delta_x^2 u_i^{k+\frac{1}{2}} = \mathcal{A}F\left(t_{k+\frac{1}{2}}, x_i, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, z_i^{k+\frac{1}{2}}\right),\quad 0 \le k \le K-1,\quad 1 \le i \le M-1, \tag{20}$$

with the discrete initial and boundary conditions

$$u_i^k = \varphi(t_k, x_i),\quad -m \le k \le 0,\ 0 \le i \le M, \tag{21}$$
$$u_0^k = u_\alpha(t_k),\quad u_M^k = u_\beta(t_k),\ 1 \le k \le K, \tag{22}$$

where

$$\begin{aligned} z_i^{k+\frac{1}{2}} &= \Delta t \sum_{j=0}^{m} c_j G(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}-j}, u_i^{k+\frac{1}{2}-j}) \\ &= \Delta t c_0 G(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}) \\ &\quad + \Delta t \sum_{j=1}^{m} c_j G(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}-j}, \frac{1}{2}u_i^{k+1-j} + \frac{1}{2}u_i^{k-j}). \end{aligned}$$

Remark: If we employ the central finite difference scheme to approximate the diffusion term, then we obtain the following scheme

$$\begin{cases} \delta_t u_i^{k+\frac{1}{2}} - b\delta_x^2 u_i^{k+\frac{1}{2}} = F\left(t_{k+\frac{1}{2}}, x_i, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, z_i^{k+\frac{1}{2}}\right), \\ 0 \le k \le K-1, \quad 1 \le i \le M-1, \\ u_i^k = \varphi(t_k, x_i), \quad -m \le k \le 0,\ 0 \le i \le M, \\ u_0^k = u_\alpha(t_k), \quad u_M^k = u_\beta(t_k),\ 1 \le k \le K. \end{cases} \tag{23}$$

However, the convergence order of the scheme (23) in spatial direction is 2.

IV. NUMERICAL ANALYSIS OF THE COMPACT SCHEME

In this section, we focus on numerical analysis for the fully discrete scheme.

A. SOLVABILITY

Theorem 1: The compact scheme (20)-(22) has a unique solution.

Proof: Let $u^k = \{u_i^k | \ 0 \le i \le M\}$. According to the initial condition (21), we obtain the values of u^k $(-m \le k \le 0)$. Now, suppose the solution of $u^l (l > 0)$ has been determined. By using (20), we derive the solution of u^{l+1}. We can check that the coefficient matrix of scheme (20) is symmetric positive definite. Therefore, the solution of u^{l+1} is obtained uniquely. By using mathematical induction, we derive the existence and uniqueness of the solution of the difference system (20)-(22). ■

Some lemmas are listed below, which will play an important role for the analysis of convergence and stability of the linearized compact scheme.

Lemma 2 ([16], [43]): For any grid function $w \in \mathcal{W}_h$, it holds that

$$||w||_\infty \le \frac{\sqrt{\beta - \alpha}}{2}|w|_1, \tag{24}$$
$$||w|| \le \frac{\beta - \alpha}{\sqrt{6}}|w|_1, \tag{25}$$
$$\frac{2}{3}\|w\|^2 \le (\mathcal{A}w, w) \le \|w\|^2. \tag{26}$$

Lemma 3 ([16], [44]): Suppose $\{H^k | \ k \ \ge \ 0\}$ be non-negative sequence, and satisfy

$$H^{k+1} \le A + B\Delta t \sum_{l=1}^{k} H^l, \quad k = 0, 1, \cdots$$

Then

$$H^{k+1} \le A \exp(Bk\Delta t), \quad k = 0, 1, 2, \cdots,$$

where A and B are non-negative constants.

In order to prepare the following convergence analysis, we need a proposition below.

Proposition 1: ([45]) The compound trapezoidal formula is convergent if and only if there exists a finite constant η, independent of m, such that

$$\Delta t \sum_{j=0}^{m} |c_j| < \eta, \quad \textit{with}\ m\Delta t = \tau. \tag{27}$$

And we need a slightly stronger condition than (27) for the convergence analysis

$$\Delta t \sqrt{(m+1)\sum_{j=0}^{m} |c_j|^2} < \eta, \quad \text{with}\ m\Delta t = \tau. \tag{28}$$

B. CONVERGENCE

Theorem 2: Let $\{u(t, x)| - \tau \le t \le T, \alpha \le x \le \beta\}$ be the solution of problem (7), and $\{u_i^k | -m \le k \le K, 0 \le i \le M\}$ be the numerical solution of the difference scheme (20)-(22). Denote

$$e_i^k = u(t_k, x_i) - u_i^k, \quad -m \le k \le K, \quad 0 \le i \le M.$$

Under the conditions of assumption (8) and (9), we have

$$||e^k||_\infty \le \tilde{C}(\Delta t^2 + h^4), \quad 0 \le k \le K, \tag{29}$$

where $\tilde{C}$ is a positive constant independent of Δt and h.

Proof: Subtracting equations (20)-(22) from equations (17)-(19), we get the error equation

$$\begin{aligned} &\mathcal{A}\delta_t e_i^{k+\frac{1}{2}} - b\delta_x^2 e_i^{k+\frac{1}{2}} \\ &= \mathcal{A}\Big[F\left(t_{k+\frac{1}{2}}, x_i, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, Z_i^{k+\frac{1}{2}}\right) \\ &\quad - F\left(t_{k+\frac{1}{2}}, x_i, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, z_i^{k+\frac{1}{2}}\right)\Big] + R_i^k, \\ &\qquad 0 \le k \le K-1,\ 1 \le i \le M-1, \end{aligned} \tag{30}$$

$$e_i^k = 0, \quad -m \le k \le 0,\ 0 \le i \le M, \tag{31}$$
$$e_0^k = 0, \quad e_M^k = 0,\ 1 \le k \le K. \tag{32}$$

Taking the inner product of (30) with $\delta_t e_i^{k+\frac{1}{2}}$ yields

$$\begin{aligned}(\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) - (b\delta_x^2 e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) &= \Big(\mathcal{A}\big[F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}U^k - \frac{1}{2}U^{k-1}, Z^{k+\frac{1}{2}}\big) \\ &\quad - F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}u^k - \frac{1}{2}u^{k-1}, z^{k+\frac{1}{2}}\big)\big], \delta_t e^{k+\frac{1}{2}}\Big) \\ &\quad + (R^k, \delta_t e^{k+\frac{1}{2}}). \end{aligned} \tag{33}$$

Now, we prove this theorem by using mathematical induction. It is obviously that (29) holds for $-m \le k \le 0$. Suppose that (29) holds for $k = 0, \ldots, l$. Next, we prove that (29) is also valid when $k = l + 1$.

First, let us estimate the right hand side terms of equation (33). Let

$$\zeta^{k+\frac{1}{2}} = Z^{k+\frac{1}{2}} - z^{k+\frac{1}{2}}.$$

By using conditions (8)-(9) and ε-inequality, we get

$$\begin{aligned}&\Big(\mathcal{A}\big[F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}U^k - \frac{1}{2}U^{k-1}, Z^{k+\frac{1}{2}}\big) \\ &\quad - F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}u^k - \frac{1}{2}u^{k-1}, z^{k+\frac{1}{2}}\big)\big], \delta_t e^{k+\frac{1}{2}}\Big) \\ &\le \Big(\mathcal{A}(c_1|\frac{3}{2}e^k - \frac{1}{2}e^{k-1}| + c_2|\zeta^{k+\frac{1}{2}}|), \delta_t e^{k+\frac{1}{2}}\Big) \\ &\le \frac{1}{2\varepsilon_1}(2c_1^2\|\frac{3}{2}e^k - \frac{1}{2}e^{k-1}\|^2 + 2c_2^2\|\zeta^{k+\frac{1}{2}}\|^2) + \frac{\varepsilon_1}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2 \\ &\le \frac{c_1^2}{\varepsilon_1}\Big(\frac{9}{2}\|e^k\|^2 + \frac{1}{2}\|e^{k-1}\|^2\Big) \\ &\quad + \frac{c_2^2}{\varepsilon_1}\|\zeta^{k+\frac{1}{2}}\|^2 + \frac{\varepsilon_1}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2, \end{aligned} \tag{34}$$

where ε_1 is a positive constant, which will be explicitly given later.

By condition (9), (28), and the Cauchy inequality, we obtain the estimate for $\|\zeta^{l+\frac{1}{2}}\|^2$.

$$\begin{aligned}\|\zeta^{k+\frac{1}{2}}\|^2 &\le \Big[\Delta t \sum_{j=0}^{m} |c_j| \|G(t_{k+\frac{1}{2}}, x, t_{k+\frac{1}{2}-j}, U^{k+\frac{1}{2}-j}) \\ &\qquad - G(t_{k+\frac{1}{2}}, x, t_{k+\frac{1}{2}-j}, u^{k+\frac{1}{2}-j})\|\Big]^2 \\ &\le \Big[\Delta t L \sum_{j=0}^{m} |c_j| \|e^{k+\frac{1}{2}-j}\|\Big]^2 \\ &\le (\Delta t L)^2 \sum_{j=0}^{m} |c_j|^2 \sum_{j=0}^{m} \|e^{k+\frac{1}{2}-j}\|^2 \\ &\le \frac{L^2\eta^2}{m+1} \sum_{j=0}^{m} \|e^{k+\frac{1}{2}-j}\|^2 \\ &\le \frac{L^2\eta^2}{2(m+1)}\Big[9\|e^k\|^2 + \|e^{k-1}\|^2 \\ &\qquad + \sum_{j=1}^{m}\big(\|e^{k+1-q}\|^2 + \|e^{k-j}\|^2\big)\Big] \\ &\le 5\frac{L^2\eta^2}{m+1} \sum_{j=1}^{m}\big(\|e^{k+1-j}\|^2 + \|e^{k-j}\|^2\big). \end{aligned}$$

Plugging the estimation of $\|\zeta^{l+\frac{1}{2}}\|^2$ into inequality (34) yields

$$\begin{aligned}&\Big(\mathcal{A}\big[F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}U^k - \frac{1}{2}U^{k-1}, Z^{k+\frac{1}{2}}\big) \\ &\quad - F\big(t_{k+\frac{1}{2}}, x, \frac{3}{2}u^k - \frac{1}{2}u^{k-1}, z^{k+\frac{1}{2}}\big)\big], \delta_t e^{k+\frac{1}{2}}\Big) \\ &\le \frac{c_1^2}{\varepsilon_1}\Big(\frac{9}{2}\|e^k\|^2 + \frac{1}{2}\|e^{k-1}\|^2\Big) + \frac{5c_2^2L^2\eta^2}{\varepsilon_1(m+1)} \sum_{j=1}^{m}\big(\|e^{k+1-j}\|^2 \\ &\quad + \|e^{k-j}\|^2\big) + \frac{\varepsilon_1}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2. \end{aligned} \tag{35}$$

Applying ε-inequality gives

$$\begin{aligned}(R^k, \delta_t e^{k+\frac{1}{2}}) &\le \frac{||R^k||^2}{2\varepsilon_2} + \frac{\varepsilon_2}{2}||\delta_t e^{k+\frac{1}{2}}||^2 \\ &\le \frac{C_R(\Delta t^2 + h^4)^2}{2\varepsilon_2} + \frac{\varepsilon_2}{2}||\delta_t e^{k+\frac{1}{2}}||^2, \end{aligned} \tag{36}$$

where ε_2 is a positive constant, which will be explicitly given later.

Then, let us estimate the left hand side terms of (33). By using (26) of Lemma 2, we have

$$\frac{2}{3}\|\delta_t e^{k+\frac{1}{2}}\|^2 \le (\mathcal{A}\delta_t e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}). \tag{37}$$

In addition, we have

$$(\delta_x^2 e^{k+\frac{1}{2}}, \delta_t e^{k+\frac{1}{2}}) = -\frac{1}{2\Delta t}(|e^{k+1}|_1^2 - |e^k|_1^2). \tag{38}$$

Now, substituting (35)-(38) into equation (33), we have

$$\begin{aligned}&\frac{2}{3}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{b}{2\Delta t}(|e^{k+1}|_1^2 - |e^k|_1^2) \\ &\le \frac{5c_2^2L^2\eta^2}{\varepsilon_1(m+1)} \sum_{j=1}^{m}\big(\|e^{k+1-j}\|^2 + \|e^{k-j}\|^2\big) \\ &\quad + \frac{\varepsilon_1}{2}\|\delta_t e^{k+\frac{1}{2}}\|^2 + \frac{c_1^2}{\varepsilon_1}\Big(\frac{9}{2}\|e^k\|^2 + \frac{1}{2}\|e^{k-1}\|^2\Big) \\ &\quad + \frac{C_R(\Delta t^2 + h^4)^2}{2\varepsilon_2} + \frac{\varepsilon_2}{2}||\delta_t e^{k+\frac{1}{2}}||^2. \end{aligned}$$

Choosing $\epsilon_1 = \epsilon_2 = \frac{2}{3}$, we eliminate the term $||\delta_t e^{k+\frac{1}{2}}||^2$. Then, we arrive at

$$\begin{aligned}&|e^{k+1}|_1^2 - |e^k|_1^2 \\ &\le \frac{3c_1^2\Delta t}{2b}\big(9\|e^k\|^2 + \|e^{k-1}\|^2\big) + \frac{15c_2^2L^2\eta^2\Delta t}{b(m+1)} \\ &\quad \sum_{j=1}^{m}\big(\|e^{k+1-j}\|^2 + \|e^{k-j}\|^2\big) + \frac{3\Delta t}{2b}C_R(\Delta t^2 + h^4)^2. \end{aligned} \tag{39}$$

Summing the above inequality (39) for k from 0 to l yields

$$|e^{l+1}|_1^2 \le \frac{3c_1^2\Delta t}{2b}\sum_{i=0}^{l}\left(9\|e^i\|^2+\|e^{i-1}\|^2\right) + \frac{15c_2^2L^2\eta^2\Delta t}{b(m+1)}\sum_{i=0}^{l}\sum_{j=1}^{m}\left(\|e^{i+1-j}\|^2+\|e^{i-j}\|^2\right) + \frac{3(l+1)\Delta t}{2b}C_R(\Delta t^2+h^4)^2. \quad (40)$$

Together with the facts that

$$\sum_{i=0}^{l}\sum_{j=1}^{m}\|e^{i+1-j}\|^2 \le m\sum_{i=0}^{l}\|e^i\|^2,$$
$$\sum_{i=0}^{l}\sum_{j=1}^{m}\|e^{i-j}\|^2 \le m\sum_{i=0}^{l}\|e^{i-1}\|^2,$$

inequality (40) can be rewritten as

$$|e^{l+1}|_1^2 \le \frac{15c_1^2\Delta t}{b}\sum_{i=0}^{l}\|e^i\|^2 + \frac{30c_2^2L^2\eta^2\Delta t}{b}\sum_{i=0}^{l}\|e^i\|^2 + \frac{3T}{2b}C_R(\Delta t^2+h^4)^2. \quad (41)$$

Letting $C=\max\{\frac{15c_1^2\Delta t}{b}, \frac{30c_2^2L^2\eta^2\Delta t}{b}\}$, we find that

$$|e^{l+1}|_1^2 \le \frac{3T}{2b}C_R(\Delta t^2+h^4)^2 + 2C\Delta t\sum_{i=0}^{l}\|e^i\|^2. \quad (42)$$

Applying Grönwall inequality (Lemma 3), we obtain

$$|e^{l+1}|_1^2 \le \frac{3T}{2b}C_R(\Delta t^2+h^4)^2\exp(2CT). \quad (43)$$

By using inequality (24) of Lemma 2, we finally have

$$\|e^{l+1}\|_\infty \le \frac{\sqrt{\beta-\alpha}}{2}|e^{l+1}|_1 \le \tilde{C}(\Delta t^2+h^4), \quad (44)$$

where

$$\tilde{C} = \frac{1}{2}\sqrt{\frac{3T}{2b}(\beta-\alpha)C_R\exp(2CT)}.$$

By inductive principle, it completes the proof. ∎

C. STABILITY

Assume there is a small perturbation of the initial condition when using a numerical method to perform computation, this will result in a perturbation for the numerical solution. Next, we analyze the stability of the linearized compact scheme (20)-(22).

Suppose $\{\omega_i^k | -m \le k \le K, 0 \le i \le M\}$ admit the following equations

$$\mathcal{A}\delta_t\omega_i^{k+\frac{1}{2}} - b\delta_x^2\omega_i^{k+\frac{1}{2}} = \mathcal{A}F\left(t_{k+\frac{1}{2}}, x_i, \frac{3}{2}\omega_i^k - \frac{1}{2}\omega_i^{k-1}, \hat{z}_i^{k+\frac{1}{2}}\right), \quad 0 \le k \le K-1, \quad 1 \le i \le M-1, \quad (45)$$

with the perturbed initial and boundary conditions

$$\omega_i^k = \varphi(t_k, x_i) + \varphi_i^k, \quad -m \le k \le 0, \ 0 \le i \le M, \quad (46)$$
$$\omega_0^k = u_\alpha(t_k), \quad \omega_M^k = u_\beta(t_k), \ 1 \le k \le K, \quad (47)$$

where

$$\hat{z}_i^{k+\frac{1}{2}} = \Delta t\sum_{j=0}^{m}c_jG(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}-j}, \omega_i^{k+\frac{1}{2}-j}) = \Delta t c_0G(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}}, \frac{3}{2}\omega_i^k - \frac{1}{2}\omega_i^{k-1}) + \Delta t\sum_{j=1}^{m}c_jG(t_{k+\frac{1}{2}}, x_i, t_{k+\frac{1}{2}-j}, \frac{1}{2}\omega_i^{k+1-j} + \frac{1}{2}\omega_i^{k-j}),$$

and φ_i^k denote the small perturbation of $\varphi(t_k, x_i)$. Let

$$\tilde{u}_i^k = \omega_i^k - u_i^k, \quad -m \le k \le K, \ 0 \le i \le M.$$

If there exists certain bound on $\tilde{u}_i^k$ under certain conditions, then the numerical process will behave stably. Indeed, by using a similar method of the convergence, we can obtain the following stability result.

Theorem 3: Under the conditions of Theorem 2, we have

$$\|\tilde{u}^k\|_\infty \le \hat{C}\max_{-m\le j\le 0}|\varphi_i^j|, \quad 0 \le k \le K, \quad (48)$$

where $\hat{C}$ is a positive constant independent of Δt and h.

V. NUMERICAL SIMULATION

In this section, a numerical experiment is conducted to confirm the theoretical results, and all the computations are performed using Matlab. Let $\text{err}_\infty(\Delta t, h) = \max_{0\le i\le M, 0\le k\le K,} |e_i^k|$, where $e_i^k = v(t_k, x_i) - v_i^k(\Delta t, h)$ be the errors computed with different stepsizes Δt and h.

Consider the following nonlinear convection-reaction-diffusion equation with distributed delay

$$\frac{\partial v}{\partial t} + a\frac{\partial v}{\partial x} - b\frac{\partial^2 v}{\partial x^2} = v(t,x)\left(1 - \int_{t-\tau}^{t} v(s,x)ds\right) + h(t,x), \ (t,x) \in (\alpha,\beta)\times(0,T], \quad (49)$$

where the term $h(t,x)$, the initial condition $\phi(t,x)$, and boundary conditions $v_\alpha(t), v_\beta(t)$ are specified by the exact solution $v(t,x) = \exp(-t)\sin(x)$.

Now, we use the approach introduced in Section II to solve it. First, convert equation (49) into the following nonlinear reaction-diffusion equation with distributed delay,

$$\begin{cases} \frac{\partial u}{\partial t} - b\frac{\partial^2 u}{\partial x^2} = u(t,x)\left(1 - \int_{t-\tau}^{t}\exp(px+qs)u(s,x)ds\right) \\ +\exp(-px-qt)h(t,x), \quad (t,x) \in (0,T]\times(\alpha,\beta), \\ u(t,\alpha) = \exp\left(-p\alpha-(q+1)t\right)\sin(\alpha), \ t \in (0,T], \\ u(t,\beta) = \exp\left(-p\beta-(q+1)t\right)\sin(\beta), \ t \in (0,T], \\ u(t,x) = \exp\left(-px-(q+1)t\right)\sin(x), \ (t,x) \in [-\tau,0]\times(\alpha,\beta). \end{cases}$$

Then, we set the parameters $a = b = 1, \alpha = 0, \beta = 1, \tau = 1, T = 4$, and solve the problem with the proposed linearized scheme (20). Let $\Delta t = h^2$. Table 1 shows the errors in L^∞ norm and convergence orders, from which we obtain that the linearized compact scheme (20) is convergent with the order $\mathcal{O}(\Delta t^2 + h^4)$.

TABLE 1. Maximum norm errors and convergence orders for equation (49).

	Compact scheme (20)		Scheme (23)	
h	$\mathrm{err}_\infty(\Delta t, h)$	Orders	$\mathrm{err}_\infty(\Delta t, h)$	Orders
$\frac{1}{5}$	$2.4345e-05$	–	$2.7029e-04$	–
$\frac{1}{10}$	$1.5178e-06$	4.0036	$6.3764e-05$	2.0837
$\frac{1}{20}$	$9.5187e-08$	3.9951	$1.6169e-05$	1.9795
$\frac{1}{40}$	$5.9559e-09$	3.9984	$4.0606e-06$	1.9935
$\frac{1}{80}$	$3.7226e-10$	3.9999	$1.0187e-06$	1.9950

To show the advantage of the proposed linearized compact scheme (20), we compare it with the scheme (23), and set $\Delta t = h$ to solve the problem (49). The results are shown in Table 1. All results confirm the convergence of the two schemes. In addition, we know that by using the same spatial stepsizes, the linearized compact scheme (20) gives a better convergent result.

VI. CONCLUSION

In this paper, a linearized compact scheme is proposed for solving nonlinear convection-reaction-diffusion equation with distributed delay. We have introduced an exponential transformation to convert the equation into a type of nonlinear reaction-diffusion equation with distributed delay. The proposed scheme is of order 2 in temporal direction and of order 4 in spatial direction in the sense of L^∞ norm. We present a numerical experiment to show that the proposed scheme gives a better convergent result than the central finite difference scheme. In the future, we will extend this method to the problem with the variable coefficient case for the convection term.

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