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Reliability Mathematical Models of Repairable Systems With Uncertain Lifetimes and Repair Times

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ABSTRACT In system reliability problems, the lifetimes and the repair times are usually indeterministic. When insufficient samples are available to estimate distribution functions of the lifetimes and the repair times, it is rational to invite some experts to evaluate the belief degrees. Uncertainty theory is a branch of mathematics for modelling belief degrees. In this paper, the lifetimes and the repair times of typical repairable systems are assumed to be uncertain variables. Based on the above-mentioned assumptions, the uncertain reliability mathematical models of simple repairable series systems, simple repairable parallel systems, simple repairable series-parallel systems, and simple repairable parallel–series systems are established, respectively. The reliability indices, such as steady state availability, steady state failure frequency, mean up time, and mean down time, are proposed. Finally, some numerical examples are given for illustration.

INDEX TERMS Uncertain system, reliability, availability, steady state failure frequency, mean up time, mean down time.

I. INTRODUCTION

The performance of repairable systems declines over time due to deterioration or damages during operation. Therefore, reliability modelling and evaluation for repairable systems is an essential topic, which has been discussed by many researchers. For instance, Barlow *et al.* [1] reviewed developments in mathematical reliability theory, especially those developments which had or were likely to have impact on engineering reliability problems. Barlow *et al.* [2] focused on probabilistic aspects of reliability and life testing. Ascher *et al.* [3] presented repairable system models in the nonhomogeneous poisson process, the renewal process, the piecewise exponential model and the modulated power law process. Kijima [4] developed general repair models for a repairable system by using the idea of the virtual age process of the system. Yañez *et al.* [5] offered generalized renewal process to model repairable systems. Crowder *et al.* [6] introduced statistical analysis of reliability data. Epstein [7] formulated the mean time between failures and repairs of repairable redundant systems. Gertsbakh [8] developed reliability theory with applications to preventive maintenance. Gnedenko *et al.* [9] presented mathematical methods of reliability theory in specific questions.

Although scholars have established various probability models to analyze the repairable system reliability. Probability hypothesis requires random variables such as lifetimes, repair times following certain probability distributions. Unfortunately, some information of variables are usually incomplete or represented by human language. In 1965, Zadeh [10] proposed fuzzy theory as a tool to deal with empirical data. After that, researchers began to analyze system reliability in fuzzy environment. Cai *et al.* [11] analyzed fuzzy variables as a basis for a theory of fuzzy reliability in the possibility context. Cai *et al.* [12] addressed fuzzy states as a basis for a theory of fuzzy reliability. Huang *et al.* [13] developed Bayesian reliability analysis for fuzzy lifetime data. Liu *et al.* [14] gave reliability analysis of repairable series system. Liu *et al.* [15] analyzed repairable parallel system with two non-identical components in random fuzzy situations. Hu and Su [16] introduced fuzzy availability assessment for a discrete time repairable multi-state series-parallel system. Utkin [17] studied fuzzy time-dependent availability and unavailability in the possibility context.

In the past few decades, some progress has been achieved in fuzzy set. However, in 2012, Liu [18] declared that fuzzy set theory fails to model belief degree under this situation via a counter example about the strength of a bridge. Similar counter examples are also found in the field of reliability. For example, if the lifetime of a component is "about 10 hours", which is regarded as a triangular fuzzy variable (8,10,12). Based on the possibility measure, it can be concluded that the possibility measure of "the lifetime of component being exactly 10 hours" is 1 and the possibility measure of "the lifetime of component not being 10 hours" is 1. According to people's common sense, the possibility of "the lifetime being exactly 10 hours" is almost zero. On the other hand, "exactly 10 hours" and "not 10 hours" have the same possibility measure. This paradoxical conclusion also shows that the lifetime of component is not suitable to be measured by the possibility measure.

In order to measure belief degree, an uncertainty theory was founded by Liu [19] in 2007 and subsequently refined it by Liu [20] based on normality, duality, subadditivity and product measure axioms. Liu [21] discussed a tool of uncertain reliability analysis. Zeng *et al.* [22], [23] proposed a new reliability metric by quantifying the performance margin and epistemic uncertainty, then used uncertainty theory for its evaluation. Therefore, uncertainty theory has become a new mathematical tool to describe human uncertainty, which has a wide application both in theory and engineering. It has been successfully employed for dealing with many uncertain situations, such as fault tree algorithms [24], redundant systems [25], block repalcement policies [26], facility location problems [27], optimal assignment problems [28], transportation problems [29] and so on.

The advantage of this paper is that uncertain variables are used to represent the lifetimes and the repair times of simple repairable systems. Meanwhile, the reliability indices are redefined by uncertain measure, which has the superiority of duality. The remainder of this paper is organized as follows. In section 2, some concepts of uncertainty variables are recalled. In section 3, definitions of reliability indexes under uncertain theory are introduced and some important theorems are proposed. In section 4, the reliability mathematical models of simple repairable series systems, simple repairable parallel systems, simple repairable series-parallel systems, simple repairable parallel-series systems with uncertain lifetimes and repair times are established, respectively. Furthermore, reliability analysis of each repairable system is given by reliability characteristics, including steady state availability, steady state failure frequency, mean up time and mean down time. Some numerical examples are given accordingly.

II. PRELIMINARIES

Let Γ be a nonempty set and Ł a σ-algebra over Γ. Each element $\Lambda \in$ Ł is called an event. Then a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicates the belief degree with which we believe Λ will occur. The uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degree, Liu [19] suggested the following four axioms:

Axiom 1. ***(Normality)*** $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \leq \Lambda_2$.

Axiom 3. (Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 4. (Countable Subadditivity) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{+\infty} \Lambda_i\right\} \leq \sum_{i=1}^{+\infty} \mathcal{M}\{\Lambda_i\}.$$

Definition 1 (Liu [19]): The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the four axioms.

Definition 2 (Liu [19]): Let Γ be a nonempty set, Ł a σ-algebra over Γ, and $\mathcal{M}$ an uncertain measure. Then the triplet $(\Gamma, Ł, \mathcal{M})$is called an uncertainty space.

Definition 3 (Liu [19]): An uncertain variable is a function ξ from an uncertainty space $(\Gamma, Ł, \mathcal{M})$ to the set of real numbers such that $\{\xi \in \mathcal{B}\}$ is an event for any Borel set $\mathcal{B}$.

Definition 4 (Liu [19]): The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Definition 5 (Liu [20]): An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{(x-a)}{(b-a)}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x \geq c \end{cases}$$

denoted by $\mathcal{L}(a, b)$, where a, b are real numbers with $a < b$.

Definition 6 (Liu [20]): An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ \dfrac{(x-a)}{2(b-a)}, & \text{if } a \leq x \leq b \\ \dfrac{(x+c-2b)}{2(c-b)}, & \text{if } b \leq x \leq c \\ 1, & \text{if } x \geq c \end{cases}$$

denoted by $\mathcal{Z}(a, b, c)$, where a, b, c are real numbers with $a < b < c$.

Definition 7 (Liu [20]): Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Definition 8 (Liu [30]): The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in \mathcal{B}_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in \mathcal{B}_i\}$$

for any Borel sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ of real numbers.

Definition 9 (Liu [30]): The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in \mathcal{B}_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in \mathcal{B}_i\}$$

for any Borel sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ of real numbers.

Theorem 1 (Liu [31]): Let $(\Gamma_k, Ł_k, \mathcal{M}_k)$ be uncertainty spaces and $\Lambda_k \in Ł_k$, for $k = 1, 2, \ldots, n$. Then the events

$$\Gamma_1 \times \cdots \times \Gamma_{k-1} \times \Lambda_k \times \Gamma_{k+1} \times \cdots \times \Gamma_n, k = 1, 2, \ldots n$$

are always independent in the product uncertainty spaces. That is, the events

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_n$$

are always independent if they are from different uncertainty spaces.

Theorem 2 (Liu [20]): Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\ldots,x_n)=x} \min_{1\le i\le n} \Phi_i(x_i).$$

Definition 10 (Liu [20]): Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.

Definition 11 (Liu [20]): An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x\to-\infty} \Phi(x) = 0, \quad \lim_{x\to+\infty} \Phi(x) = 1.$$

Definition 12 (Liu [19]): Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\xi \le x\}dx$$

provided that at least one of the two integrals is finite.

Theorem 3 (Liu [20]): Let ξ be an uncertain variable with uncertainty distribution Φ. Then

$$E[\xi] = \int_{-\infty}^{+\infty} x d\Phi(x).$$

Theorem 4 (Liu and Ha [32]): Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distribution $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an expected value

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))d\alpha.$$

Theorem 5 (Yao and Gao [33]): Let a sequence of iid uncertain variables $\tau_1, \tau_2, \ldots,$ denote $S_n = \tau_1 + \tau_2 + \cdots \tau_n$ for $n \ge 1$. Then the S_n/n converges in distribution to τ_1.

III. RELIABILITY INDEXES BASED ON UNCERTAINTY THEORY

In this section, the reliability indices are redefined by uncertain measure, including availability, steady state failure frequency, mean up time and mean down time. Then an important theorem is proposed, which is useful in the reliability analysis of repairable systems.

Definition 13: Let $S(t)$ be the state of the uncertain repairable system. If the system is in operating state at time t, then $S(t) = 1$, else $S(t) = 0$.

(i) The availability at time t, denoted by $A(t)$, is given by

$$A(t) = \mathcal{M}\{S(t) = 1\}.$$

(ii) The steady state availability A is given by $\lim_{t\to+\infty} A(t)$ when it exists.

(iii) The average availability in $[0, T]$, denoted by $A_{av}(T)$, is given by

$$A_{av}(T) = \frac{1}{T}\int_0^T A(t)dt.$$

(iv) The limiting average availability, denoted by A_{av}, is given by

$$A_{av} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T A(t)dt.$$

Remark 1: Let U_T denote the total operating time of system before T. It is easy to see that

$$U_T = \int_0^T S(t)dt.$$

Then

$$\begin{aligned}\frac{E[U_T]}{T} &= \frac{1}{T}E\left[\int_0^T S(t)dt\right] \\ &= \frac{1}{T}\int_0^T E[S(t)]dt \\ &= \frac{1}{T}\int_0^T A(t)dt = A_{av}(T).\end{aligned}$$

We can also arrive at

$$A_{av} = \lim_{T\to+\infty} \frac{E[U_T]}{T}.$$

Theorem 6: If $\lim_{t\to+\infty} A(t) = A$ exists, then $A_{av} = A$.

Proof: Given $\varepsilon > 0$, there exists T' sufficiently large so that $|A(t) - A| < \varepsilon$ for all $t > T'$. For $T > T'$, we have

$$\begin{aligned} \frac{1}{T}\int_0^{T'} A(t)dt + \frac{1}{T}\int_{T'}^{T} (A-\varepsilon)dt &\le \frac{1}{T}\int_0^{T} A(t)dt \\ &\le \frac{1}{T}\int_0^{T'} A(t)dt + \frac{1}{T}\int_{T'}^{T} (A+\varepsilon)dt. \end{aligned}$$

We let $T \to +\infty$, then

$$A - \varepsilon \le \lim_{T\to+\infty} \frac{1}{T}\int_0^{T} A(t)dt \le A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, so $A_{av} = A$.

The theorem is proved.

Definition 14: Let $N(t)$ denote the number of failure that have occurred in $(0, t]$. The steady state failure frequency, denoted by M, is defined by

$$M = \lim_{t\to+\infty} \frac{EN(t)}{t},$$

if $\lim_{t\to+\infty} \frac{EN(t)}{t}$ exists.

Remark 2: The concept of "steady state failure frequency" is first proposed by classical reliability theory and based on probability theory, which is considered as an important reliability index. In order to express the parallel concept in random case, we reuse this concept in uncertain environments, but it is not the frequentist interpretation of probability.

Definition 15: The mean up time of repairable system is defined as

$$MUT = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} E\varsigma_i,$$

where $\varsigma_i, i = 1, 2, \ldots$ are uncertain operating times of the system.

Definition 16: The mean down time of repairable system is defined as

$$MDT = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} E\tau_i,$$

where $\tau_i, i = 1, 2, \ldots$ are uncertain repair times of the system.

Theorem 7 (Yao and Li [34]): Let uncertain on-times $\varsigma_1, \varsigma_2, \ldots$ be independent and identically distributed and uncertain off-times $\tau_1, \tau_2, \ldots$ be independent and identically distributed. Assume $(\varsigma_1, \varsigma_2, \ldots)$ and $(\tau_1, \tau_2, \ldots)$ are independent uncertain vectors. Then

$$\lim_{t\to+\infty} \frac{EU_t}{t} = E\left[\frac{\varsigma_1}{\varsigma_1 + \tau_1}\right],$$

where U_t is the total time at which the system is on up to time t.

Theorem 8 (Liu [20]): Let $N(t)$ be a renewal process with independent and identically distributed uncertain interarrival times $\varsigma_1, \varsigma_2, \ldots$ Then

$$\lim_{t\to+\infty} \frac{EN(t)}{t} = E\left[\frac{1}{\varsigma_1}\right].$$

Theorem 9: Let $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ be independent and identically distributed uncertain variables defined on the uncertainty space $\left(\Gamma_i^{(1)}, \text{Ł}_i^{(1)}, \mathcal{M}_i^{(1)}\right), i = 1, 2, \ldots, n$ and let $\tau_1, \tau_2, \ldots, \tau_n$ be independent and identically distributed uncertain variables defined on the uncertainty space $\left(\Gamma_i^{(2)}, \text{Ł}_i^{(2)}, \mathcal{M}_i^{(2)}\right), i = 1, 2, \ldots, n$. Then $\varsigma_1 + \tau_1, \varsigma_2 + \tau_2, \ldots, \varsigma_n + \tau_n$ are independent and identically distributed uncertain variables.

Proof: By Theorem 1, it is clear that the $\varsigma_i + \tau_i, i = 1, 2, \ldots, n$ are always independent if they are defined on different uncertainty spaces $(\Gamma_i, \text{Ł}_i, \mathcal{M}_i)$, where $\Gamma_i = \Gamma_i^{(1)} \times \Gamma_i^{(2)}$, $\text{Ł}_i = \text{Ł}_i^{(1)} \times \text{Ł}_i^{(2)}$, $\mathcal{M}_i = \mathcal{M}_i^{(1)} \times \mathcal{M}_i^{(2)}, i = 1, 2, \ldots, n$.

On the other hand, if $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ and $\tau_1, \tau_2, \ldots, \tau_n$ are identically distributed uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. By Theorem 2, we have

$$Z(x) = \sup_{x_1+x_2=x} \Phi(x_1) \wedge \Psi(x_2).$$

That is, the distribution function is irrelavent to the index i. Hence $\varsigma_i + \tau_i, i = 1, 2, \ldots, n$ have the identical uncertainty distribution. Thus $\varsigma_i + \tau_i, i = 1, 2, \ldots, n$ are independent and identically distributed uncertain variables.

The theorem is proved.

IV. RELIABILITY MATHEMATICAL MODELS OF UNCERTAIN REPAIRABLE SYSTEMS

A. SIMPLE REPAIRABLE SERIES SYSTEMS

Consider a simple repairable series system consisting of n components and enough repairmen. The lifetime of component i is assumed to be an uncertain variable ξ_i defined on the uncertainty space $(\Gamma_i, \text{Ł}_i, \mathcal{M}_i), i = 1, 2, \ldots, n$. Let $\xi_i, i = 1, 2, \ldots, n$ be independent uncertain variables with regular uncertainty distributions $\Phi_i, i = 1, 2, \ldots, n$. At the initial time, all components are new and once the series system breaks down, the failed series system is repaired immediately. When the series system has been repaired, it returns to normal working condition. Suppose the whole series system is as good as new after repair. Then the operating times of the simple repairable series system $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed uncertain variables. Let $\tau_k, k = 1, 2, \ldots$ be the repair times of the series system on the single uncertainty space $(\Gamma', \text{Ł}', \mathcal{M}')$, and $\tau_k, k = 1, 2, \ldots$ be independent and identically distributed uncertain variables with regular uncertainty distribution Ψ. By Theorem 9, the sequence of uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. For simplicity, the problem is discussed on the product uncertainty space

$(\Gamma, Ł, \mathcal{M})$, where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n \times \Gamma'$, $Ł = Ł_1 \times \cdots \times Ł_n \times Ł'$, $\mathcal{M} = \mathcal{M}_1 \wedge \cdots \wedge \mathcal{M}_n \wedge \mathcal{M}'$.

Theorem 10: The steady state availability of the simple repairable series system is

$$A = \int_0^1 \frac{\min\limits_{1\le i\le n} \Phi_i^{-1}(\alpha)}{\min\limits_{1\le i\le n} \Phi_i^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed and uncertain variables $\tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 13, Theorem 4, Theorem 6 and Theorem 7 that

$$\begin{aligned} A &= \lim_{T\to+\infty} \frac{E[U_T]}{T} \\ &= E\left[\frac{\varsigma_1}{\varsigma_1 + \tau_1}\right] \\ &= E\left[\frac{\min\limits_{1\le i\le n} \xi_i}{\min\limits_{1\le i\le n} \xi_i + \tau_1}\right] \\ &= \int_0^1 \frac{\min\limits_{1\le i\le n} \Phi_i^{-1}(\alpha)}{\min\limits_{1\le i\le n} \Phi_i^{-1}(\alpha) + \Psi^{-1}(1-\alpha)} d\alpha. \end{aligned}$$

The theorem is proved.

Theorem 11: The steady state failure frequency of the simple repairable series system is

$$M = \int_0^1 \frac{1}{\min\limits_{1\le i\le n} \Phi_i^{-1}(1-\alpha) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Proof: Since uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 14, Theorem 4 and Theorem 8 that

$$\begin{aligned} M &= \lim_{t\to+\infty} \frac{EN(t)}{t} \\ &= E\left[\frac{1}{\varsigma_1 + \tau_1}\right] \\ &= E\left[\frac{1}{\min\limits_{1\le i\le n} \xi_i + \tau_1}\right] \\ &= \int_0^1 \frac{1}{\min\limits_{1\le i\le n} \Phi_i^{-1}(1-\alpha) + \Psi^{-1}(1-\alpha)} d\alpha. \end{aligned}$$

The theorem is proved.

Theorem 12: The mean up time of the simple repairable series system is given by

$$MUT = \int_0^{+\infty} \min_{1\le i\le n} (1 - \Phi_i(t)) dt.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 8, Definition 12, Definition 15 and Theorem 5 that

$$\begin{aligned} MUT &= \lim_{s\to+\infty} \frac{1}{s} \sum_{k=1}^{s} E[\varsigma_k] \\ &= E[\varsigma_1] \\ &= E\left[\min_{1\le i\le n} \xi_i\right] \\ &= \int_0^{+\infty} \mathcal{M}\left\{\min_{1\le i\le n} \xi_i > t\right\} dt \\ &= \int_0^{+\infty} \mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i > t)\right\} dt \\ &= \int_0^{+\infty} \min_{1\le i\le n} \mathcal{M}\{\xi_i > t\}\, dt \\ &= \int_0^{+\infty} \min_{1\le i\le n} (1 - \Phi_i(t)) dt. \end{aligned}$$

The theorem is proved.

Theorem 13: The mean down time of the simple repairable series system is given by

$$MDT = \int_0^{+\infty} (1 - \Psi(t))\, dt.$$

Proof: Since uncertain variables $\tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 12, Definition 16 and Theorem 5 that

$$\begin{aligned} MDT &= \lim_{s\to+\infty} \frac{1}{s} \sum_{k=1}^{s} E[\tau_k] \\ &= E[\tau_1] \\ &= \int_0^{+\infty} (1 - \Psi(t)) dt. \end{aligned}$$

The theorem is proved.

Example 1: Consider a simple repairable series system consisting of two components. Assume that the lifetimes of components are uncertain variables ξ_1 and ξ_2 with linear uncertainty distributions $\mathcal{L}(2, 5)$ and $\mathcal{L}(1, 6)$, respectively. The repair time of the series system is uncertain variable η with zigzag uncertainty distribution $\mathcal{Z}(1, 1.6, 2)$. Then we will give the reliability indices of the repairable series system. Firstly, by Definition 5, we can compute the uncertainty distributions of the lifetimes of components, namely

$$\Phi_1(t) = \begin{cases} 0, & \text{if } 0 \le t \le 2 \\ \dfrac{(t-2)}{3}, & \text{if } 2 \le t \le 5 \\ 1, & \text{if } t \ge 5 \end{cases}$$

and

$$\Phi_2(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1 \\ \dfrac{(t-1)}{5}, & \text{if } 1 \le t \le 6 \\ 1, & \text{if } t \ge 6, \end{cases}$$

respectively.

The inverse uncertainty distributions of the lifetimes of components are

$$\Phi_1^{-1}(\alpha) = 2 + 3\alpha, \quad \alpha \in [0, 1]$$

and

$$\Phi_2^{-1}(\alpha) = 1 + 5\alpha, \quad \alpha \in [0, 1],$$

respectively.

It is easy to arrive at

$$\min_{1\le i\le 2}\Phi_i^{-1}(\alpha)=\begin{cases}1+5\alpha, & \text{if } 0\le\alpha\le 0.5\\ 2+3\alpha, & \text{if } 0.5\le\alpha\le 1\end{cases}$$

and

$$\min_{1\le i\le 2}\Phi_i^{-1}(1-\alpha)=\begin{cases}5-3\alpha, & \text{if } 0\le\alpha\le 0.5\\ 6-5\alpha, & \text{if } 0.5\le\alpha\le 1.\end{cases}$$

By Definition 6, the uncertainty distribution of the repair time of the series system is

$$\Psi(t)=\begin{cases}0, & \text{if } 0\le t\le 1\\ \dfrac{5(t-1)}{6}, & \text{if } 1\le t\le 1.6\\ \dfrac{5(t-1.2)}{4}, & \text{if } 1.6\le t\le 2\\ 1, & \text{if } t\ge 2.\end{cases}$$

Then we can arrive at

$$\Psi^{-1}(\alpha)=\begin{cases}1+1.2\alpha, & \text{if } \alpha<0.5\\ 1.2+0.8\alpha, & \text{if } \alpha\ge 0.5\end{cases}$$

and

$$\Psi^{-1}(1-\alpha)=\begin{cases}2-0.8\alpha, & \text{if } \alpha\le 0.5\\ 2.2-1.2\alpha, & \text{if } \alpha>0.5.\end{cases}$$

According to Theorem 10 that

$$\begin{aligned}A&=\int_0^1\frac{\min_{1\le i\le 2}\Phi_i^{-1}(\alpha)}{\min_{1\le i\le 2}\Phi_i^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}d\alpha\\ &=\int_0^{0.5}\left(\frac{1+5\alpha}{1+5\alpha+2-0.8\alpha}\right)d\alpha\\ &\quad+\int_{0.5}^1\left(\frac{2+3\alpha}{2+3\alpha+2.2-1.2\alpha}\right)d\alpha\\ &\approx 0.6523.\end{aligned}$$

Following Theorem 11 that

$$\begin{aligned}M&=\int_0^1\frac{1}{\min_{1\le i\le 2}\Phi_i^{-1}(1-\alpha)+\Psi^{-1}(1-\alpha)}d\alpha\\ &=\int_0^{0.5}\left(\frac{1}{5-3\alpha+2-0.8\alpha}\right)d\alpha\\ &\quad+\int_{0.5}^1\left(\frac{1}{6-5\alpha+2.2-1.2\alpha}\right)d\alpha\\ &\approx 0.2343.\end{aligned}$$

In order to obtain the mean up time, we have to compute

$$\min_{1\le i\le 2}(1-\Phi_i(t))=\begin{cases}1, & \text{if } 0\le t\le 1\\ \dfrac{6-t}{5}, & \text{if } 1\le t\le 3.5\\ \dfrac{5-t}{3}, & \text{if } 3.5\le t\le 5\\ 0, & \text{if } t\ge 5.\end{cases}$$

It follows from Theorem 12 that

$$\begin{aligned}MUT&=\int_0^{+\infty}\min_{1\le i\le 2}(1-\Phi_i(t))dt\\ &=\int_0^1 1dt+\int_1^{3.5}\frac{6-t}{5}dt+\int_{3.5}^5\frac{5-t}{3}dt+\int_5^{+\infty}0dt\\ &=3.2500.\end{aligned}$$

Meanwhile, by Theorem 13, we can obtain

$$\begin{aligned}MDT&=\int_0^{+\infty}(1-\Psi(t))\,dt\\ &=\int_0^1(1-0)\,dt+\int_1^{1.6}\left(1-\frac{5(t-1)}{6}\right)dt\\ &\quad+\int_{1.6}^2\left(1-\frac{5(t-1.2)}{4}\right)dt+\int_2^{+\infty}(1-1)\,dt\\ &=1.5500.\end{aligned}$$

B. SIMPLE REPAIRABLE PARALLEL SYSTEMS

Consider a simple repairable parallel system consisting of n components and enough repairmen. The lifetime of component i is assumed to be an uncertain variable ξ_i defined on the uncertainty space $(\Gamma_i, \text{Ł}_i, \mathcal{M}_i)$, $i=1,2,\ldots,n$. Let ξ_i, $i=1,2,\ldots,n$ be independent uncertain variables with regular uncertainty distributions Φ_i, $i=1,2,\ldots,n$. At the initial time, all components are new and once the parallel system breaks down, the failed parallel system is repair immediately. When the parallel system has been repaired, it returns to normal working condition. Suppose the whole parallel system is as good as new after repair. Then the operating times of the repairable parallel system ς_k, $k=1,2,\ldots$ are independent and identically distributed uncertain variables. Let τ_k, $k=1,2,\ldots$ be the repair times of the parallel system on the single uncertainty space $(\Gamma', \text{Ł}', \mathcal{M}')$, and τ_k, $k=1,2,\ldots$ be independent and identically distributed uncertain variables with regular uncertainty distribution Ψ. By Theorem 9, the sequence of uncertain variables $\varsigma_k+\tau_k$, $k=1,2,\ldots$ are independent and identically distributed. For simplicity, the problem is discussed on the product uncertainty space $(\Gamma, \text{Ł}, \mathcal{M})$, where $\Gamma=\Gamma_1\times\cdots\times\Gamma_n\times\Gamma'$, $\text{Ł}=\text{Ł}_1\times\cdots\times\text{Ł}_n\times\text{Ł}'$, $\mathcal{M}=\mathcal{M}_1\wedge\cdots\wedge\mathcal{M}_n\wedge\mathcal{M}'$.

Theorem 14: The steady state availability of the simple repairable parallel system is

$$A=\int_0^1\frac{\max_{1\le i\le n}\Phi_i^{-1}(\alpha)}{\max_{1\le i\le n}\Phi_i^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}d\alpha.$$

Proof: Since uncertain variables ς_k, $k=1,2,\ldots$ are independent and identically distributed and uncertain variables τ_k, $k=1,2,\ldots$ are independent and identically distributed. It follows from Definition 13, Theorem 4, Theorem 6 and Theorem 7 that

$$A=\lim_{t\to+\infty}\frac{E[U_T]}{T}$$

$$= E\left[\frac{\varsigma_1}{\varsigma_1+\tau_1}\right] = E\left[\frac{\max\limits_{1\le i\le n}\xi_i}{\max\limits_{1\le i\le n}\xi_i+\tau_1}\right] = \int_0^1 \frac{\max\limits_{1\le i\le n}\Phi_i^{-1}(\alpha)}{\max\limits_{1\le i\le n}\Phi_i^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}d\alpha.$$

The theorem is proved.

Theorem 15: The steady state failure frequency of the simple repairable parallel system is

$$M = \int_0^1 \frac{1}{\max\limits_{1\le i\le n}\Phi_i^{-1}(1-\alpha)+\Psi^{-1}(1-\alpha)}d\alpha.$$

Proof: Since uncertain variables $\varsigma_k+\tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 14, Theorem 4 and Theorem 8 that

$$M = \lim_{t\to+\infty}\frac{EN(t)}{t} = E\left[\frac{1}{\varsigma_1+\tau_1}\right] = E\left[\frac{1}{\max\limits_{1\le i\le n}\xi_i+\tau_1}\right] = \int_0^1 \frac{1}{\max\limits_{1\le i\le n}\Phi^{-1}(1-\alpha)+\Psi^{-1}(1-\alpha)}d\alpha.$$

The theorem is proved.

Theorem 16: The mean up time of the simple repairable parallel system is given by

$$MUT = \int_0^{+\infty}\max_{1\le i\le n}(1-\Phi_i(t))dt.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 8, Definition 12, Definition 15 and Theorem 5 that

$$\begin{aligned} MUT &= \lim_{s\to+\infty}\frac{1}{s}\sum_{k=1}^{s}E[\varsigma_k] \\ &= E[\varsigma_1] \\ &= E\left[\max_{1\le i\le n}\xi_i\right] \\ &= \int_0^{+\infty}\mathcal{M}\left\{\max_{1\le i\le n}\xi_i > t\right\}dt \\ &= \int_0^{+\infty}\left(1-\mathcal{M}\left\{\max_{1\le i\le n}\xi_i \le t\right\}\right)dt \\ &= \int_0^{+\infty}\left(1-\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i\le t)\right\}\right)dt \\ &= \int_0^{+\infty}\left(1-\min_{1\le i\le n}\mathcal{M}\{\xi_i\le t\}\right)dt \\ &= \int_0^{+\infty}\left(1-\min_{1\le i\le n}\Phi_i(t)\right)dt \\ &= \int_0^{+\infty}\max_{1\le i\le n}(1-\Phi_i(t))dt. \end{aligned}$$

The theorem is proved.

Theorem 17: The mean down time of the simple repairable parallel system is given by

$$MDT = \int_0^{+\infty}(1-\Psi(t))\,dt.$$

Proof: The proof is the same as Theorem 13.

Example 2: Consider a simple repairable parallel system consisting of two components. Assume that the lifetimes of components are uncertain variables ξ_1 and ξ_2 with linear uncertainty distributions $\mathcal{L}(2, 5)$ and $\mathcal{L}(1, 6)$, respectively. The repair time of the parallel system is η with zigzag uncertainty distribution $\mathcal{Z}(1, 1.6, 2)$. Then we will give the reliability indices of the repairable parallel system.

The uncertainty distributions of the components' lifetimes and the uncertainty distribution of parallel system's repair time have derived in Example 1, respectively. Then we can arrive at

$$\max_{1\le i\le 2}\Phi_i^{-1}(\alpha) = \begin{cases} 2+3\alpha, & \text{if } 0\le\alpha\le 0.5 \\ 1+5\alpha, & \text{if } 0.5\le\alpha\le 1 \end{cases}$$

and

$$\max_{1\le i\le 2}\Phi_i^{-1}(1-\alpha) = \begin{cases} 6-5\alpha, & \text{if } 0\le\alpha\le 0.5 \\ 5-3\alpha, & \text{if } 0.5\le\alpha\le 1. \end{cases}$$

It follows from Theorem 14 that

$$\begin{aligned} A &= \int_0^1 \frac{\max\limits_{1\le i\le 2}\Phi_i^{-1}(\alpha)}{\max\limits_{1\le i\le 2}\Phi_i^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}d\alpha \\ &= \int_0^{0.5}\left(\frac{2+3\alpha}{2+3\alpha+2-0.8\alpha}\right)d\alpha + \int_{0.5}^1\left(\frac{1+5\alpha}{1+5\alpha+2.2-1.2\alpha}\right)d\alpha \\ &\approx 0.6907. \end{aligned}$$

By using Theorem 15, we can obtain

$$\begin{aligned} M &= \int_0^1 \frac{1}{\max\limits_{1\le i\le 2}\Phi_i^{-1}(1-\alpha)+\Psi^{-1}(1-\alpha)}d\alpha \\ &= \int_0^{0.5}\left(\frac{1}{6-5\alpha+2-0.8\alpha}\right)d\alpha + \int_{0.5}^1\left(\frac{1}{5-3\alpha+2.2-1.2\alpha}\right)d\alpha \\ &\approx 0.2035. \end{aligned}$$

In order to obtain the mean up time, we can arrive at

$$\max_{1\le i\le n}(1-\Phi_i(t)) = \begin{cases} 1, & \text{if } 0\le t\le 2 \\ \dfrac{5-t}{3}, & \text{if } 2\le t\le 3.5 \\ \dfrac{6-t}{5}, & \text{if } 3.5\le t\le 6 \\ 0, & \text{if } t\ge 6. \end{cases}$$

By Theorem 16 and Theorem 17, we have

$$\begin{aligned} MUT &= \int_0^{+\infty} \max_{1\le i\le n} (1-\Phi_i(t))dt \\ &= \int_0^2 1dt + \int_2^{3.5} \frac{5-t}{3}dt + \int_{3.5}^6 \frac{6-t}{5}dt + \int_6^{+\infty} 0dt \\ &= 3.7500 \end{aligned}$$

and

$$\begin{aligned} MDT &= \int_0^{+\infty} (1-\Psi(t))\,dt \\ &= \int_0^1 (1-0)\,dt + \int_1^{1.6} \left(1-\frac{5(t-1)}{6}\right) dt \\ &\quad + \int_{1.6}^2 \left(1-\frac{5(t-1.2)}{4}\right) dt + \int_2^{+\infty} (1-1)\,dt \\ &= 1.5500. \end{aligned}$$

C. SIMPLE REPAIRABLE SERIES-PARALLEL SYSTEMS

Consider a simple repairable series-parallel system consisting of m series subsystems. Each subsystem consists of n parallel components. The series-parallel system is maintained by enough repairmen. The lifetime of component j in ith subsystem is assumed to be an uncertain variables ξ_{ij} defined on the uncertainty space $(\Gamma_{ij}, \text{Ł}_{ij}, \mathcal{M}_{ij})$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Let $\xi_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ be independent uncertain variables with regular uncertainty distributions $\Phi_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. At the initial time, all components are new and once the series-parallel system breaks down, the failed series-parallel is repaired immediately. When the series-parallel system has been repaired, it returns to normal working condition. Suppose the whole series-parallel system is as good as new after repair. Then the operating times of the simple repairable series-parallel system $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed uncertain variables. Let $\tau_k, k = 1, 2, \ldots$ be the repair times of the series-parallel system on the single uncertainty space $(\Gamma', \text{Ł}', \mathcal{M}')$, and $\tau_k, k = 1, 2, \ldots$ be independent and identically distributed uncertain variables with regular uncertainty distribution Ψ. By Theorem 9, the sequence of uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. For simplicity, the problem is discussed on the product uncertainty space $(\Gamma, \text{Ł}, \mathcal{M})$, where $\Gamma = \Gamma_{11} \times \cdots \times \Gamma_{mn} \times \Gamma'$, $\text{Ł} = \text{Ł}_{11} \times \cdots \times \text{Ł}_{mn} \times \text{Ł}'$, $\mathcal{M} = \mathcal{M}_{11} \wedge \cdots \wedge \mathcal{M}_{mn} \wedge \mathcal{M}'$.

Theorem 18: The steady state availability of the simple repairable series-parallel system is

$$A = \int_0^1 \frac{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(\alpha)\right)}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(\alpha)\right) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed and uncertain variables $\tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 13, Theorem 4, Theorem 6 and Theorem 7 that

$$\begin{aligned} A &= \lim_{t\to+\infty} \frac{E[U_T]}{T} \\ &= E\left[\frac{\varsigma_1}{\varsigma_1+\tau_1}\right] \\ &= E\left[\frac{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \xi_{ij}\right)}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \xi_{ij}\right) + \tau_1}\right] \\ &= \int_0^1 \frac{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(\alpha)\right)}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(\alpha)\right) + \Psi^{-1}(1-\alpha)} d\alpha. \end{aligned}$$

The theorem is proved.

Theorem 19: The steady state failure frequency of the simple repairable series-parallel system is

$$M = \int_0^1 \frac{1}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(1-\alpha)\right) + \Psi^{-1}(1-\alpha)} d\alpha.$$

Proof: Since uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 14, Theorem 4 and Theorem 8 that

$$\begin{aligned} M &= \lim_{t\to+\infty} \frac{EN(t)}{t} \\ &= E\left[\frac{1}{\varsigma_1+\tau_1}\right] \\ &= E\left[\frac{1}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \xi_{ij}\right) + \tau_1}\right] \\ &= \int_0^1 \frac{1}{\min\limits_{1\le i\le m}\left(\max\limits_{1\le j\le n} \Phi_{ij}^{-1}(1-\alpha)\right) + \Psi^{-1}(1-\alpha)} d\alpha. \end{aligned}$$

The theorem is proved.

Theorem 20: The mean up time of the simple repairable series-parallel system is given by

$$MUT = \int_0^{+\infty} \min_{1\le i\le m} \max_{1\le j\le n} \left(1-\Phi_{ij}(t)\right)dt.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 8, Definition 12, Definition 15 and Theorem 5 that

$$\begin{aligned} MUT &= \lim_{s\to+\infty} \frac{1}{s}\sum_{k=1}^{s} E\left[\varsigma_k\right] \\ &= E[\varsigma_1] \\ &= E\left[\min_{1\le i\le m}\left(\max_{1\le j\le n} \xi_{ij}\right)\right] \\ &= \int_0^{+\infty} \mathcal{M}\left\{\min_{1\le i\le m}\left(\max_{1\le j\le n} \xi_{ij}\right) > t\right\} dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} \mathcal{M}\left\{\bigcap_{i=1}^{m}\left(\max_{1\le j\le n} \xi_{ij} > t\right)\right\}dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \mathcal{M}\left\{\max_{1\le j\le n} \xi_{ij} > t\right\}dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \left(1-\mathcal{M}\left\{\max_{1\le j\le n} \xi_{ij} \le t\right\}\right)dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \left(1-\mathcal{M}\left\{\bigcap_{j=1}^{n}(\xi_{ij} \le t)\right\}\right)dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \left(1-\min_{1\le j\le n}\mathcal{M}\left\{\xi_{ij} \le t\right\}\right)dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \left(1-\min_{1\le j\le n}\Phi_{ij}(t)\right)dt \\
&= \int_0^{+\infty} \min_{1\le i\le m} \max_{1\le j\le n} \left(1-\Phi_{ij}(t)\right)dt.
\end{aligned}
$$

The theorem is proved.

Theorem 21: The mean down time of the simple repairable series-parallel system is given by

$$MDT = \int_0^{+\infty} (1-\Psi(t))dt.$$

Proof: The proof is the same as Theorem 13.

D. SIMPLE REPAIRABLE PARALLEL-SERIES SYSTEMS

Consider a simple repairable parallel-series system consisting of m parallel subsystems. Each subsystem consists of n components. The parallel-series system is maintained by enough repairmen. The lifetime of component j in ith subsystem is assumed to be an uncertain variable ξ_{ij} defined on the uncertainty space $(\Gamma_{ij}, Ł_{ij}, \mathcal{M}_{ij})$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Let $\xi_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ be independent uncertain variables with regular uncertainty distributions $\Phi_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. At the initial time, all components are new and once the parallel-series system breaks down, the failed parallel-series system is repaired immediately. When the parallel-series system has been repaired, it returns to normal working condition. Suppose the whole parallel-series system is as good as new after repair. Then the operating times of the simple repairable parallel-series system $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed uncertain variables. Let $\tau_k, k = 1, 2, \ldots$ be the repair times of the parallel-series system on the single uncertainty space $(\Gamma', Ł', \mathcal{M}')$, and $\tau_k, k = 1, 2, \ldots$ be independent and identically distributed uncertain variables with regular uncertainty distribution Ψ. By Theorem 9, the sequence of uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. For simplicity, the problem is discussed on the product uncertainty space $(\Gamma, Ł, \mathcal{M})$, where $\Gamma = \Gamma_{11} \times \cdots \times \Gamma_{mn} \times \Gamma'$, $Ł = Ł_{11} \times \cdots \times Ł_{mn} \times Ł'$, $\mathcal{M} = \mathcal{M}_{11} \wedge \cdots \wedge \mathcal{M}_{mn} \wedge \mathcal{M}'$.

Theorem 22: The steady state availability of the simple repairable parallel-series system is

$$A = \int_0^1 \frac{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(\alpha)\right)}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(\alpha)\right)+\Psi^{-1}(1-\alpha)}d\alpha.$$

Proof: Since uncertain variables $\varsigma_k, k = 1, 2, \ldots$ are independent and identically distributed and uncertain variables $\tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 13, Theorem 4, Theorem 6 and Theorem 7 that

$$
\begin{aligned}
A &= \lim_{t\to+\infty}\frac{E[U_T]}{T} \\
&= E\left[\frac{\varsigma_1}{\varsigma_1+\tau_1}\right] \\
&= E\left[\frac{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\xi_{ij}\right)}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\xi_{ij}\right)+\tau_1}\right] \\
&= \int_0^1 \frac{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(\alpha)\right)}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(\alpha)\right)+\Psi^{-1}(1-\alpha)}d\alpha.
\end{aligned}
$$

The theorem is proved.

Theorem 23: The steady state failure frequency of the simple repairable parallel-series system is

$$M = \int_0^1 \frac{1}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(1-\alpha)\right)+\Psi^{-1}(1-\alpha)}d\alpha.$$

Proof: Since uncertain variables $\varsigma_k + \tau_k, k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 14, Theorem 4 and Theorem 8 that

$$
\begin{aligned}
M &= \lim_{t\to+\infty}\frac{EN(t)}{t} \\
&= E\left[\frac{1}{\varsigma_1+\tau_1}\right] \\
&= E\left[\frac{1}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\xi_{ij}\right)+\tau_1}\right] \\
&= \int_0^1 \frac{1}{\max\limits_{1\le i\le m}\left(\min\limits_{1\le j\le n}\Phi_{ij}^{-1}(1-\alpha)\right)+\Psi^{-1}(1-\alpha)}d\alpha.
\end{aligned}
$$

The theorem is proved.

Theorem 24: The mean up time of the simple repairable parallel-series system is given by

$$MUT = \int_0^{+\infty} \max_{1\le i\le m}\min_{1\le j\le n}\left(1-\Phi_{ij}(t)\right)dt.$$

Proof: Since uncertain variables ς_k, $k = 1, 2, \ldots$ are independent and identically distributed. It follows from Definition 8, Definition 9, Definition 12, Definition 15 and Theorem 5 that

$$\begin{aligned}
MUT &= \lim_{s\to+\infty} \frac{1}{s}\sum_{k=1}^{s} E\left[\varsigma_k\right] \\
&= E[\varsigma_1] \\
&= E\left[\max_{1\le i\le m}\left(\min_{1\le j\le n} \xi_{ij}\right)\right] \\
&= \int_0^{+\infty} \mathcal{M}\left\{\max_{1\le i\le m}\left(\min_{1\le j\le n} \xi_{ij}\right) > t\right\} dt \\
&= \int_0^{+\infty} \mathcal{M}\left\{\bigcup_{i=1}^{m}\left(\min_{1\le j\le n} \xi_{ij} > t\right)\right\} dt \\
&= \int_0^{+\infty} \max_{1\le i\le m} \mathcal{M}\left\{\min_{1\le j\le n} \xi_{ij} > t\right\} dt \\
&= \int_0^{+\infty} \max_{1\le i\le m} \mathcal{M}\left\{\bigcap_{j=1}^{n}\left(\xi_{ij} \le t\right)\right\} dt \\
&= \int_0^{+\infty} \max_{1\le i\le m} \min_{1\le j\le n} M\left\{\xi_{ij} > t\right\} dt \\
&= \int_0^{+\infty} \max_{1\le i\le m}\left(1 - \max_{1\le j\le n} \mathcal{M}\left\{\xi_{ij} \le t\right\}\right) dt \\
&= \int_0^{+\infty} \max_{1\le i\le m}\left(1 - \max_{1\le j\le n} \Phi_{ij}(t)\right) dt \\
&= \int_0^{+\infty} \max_{1\le i\le m} \min_{1\le j\le n}\left(1 - \Phi_{ij}(t)\right) dt.
\end{aligned}$$

The theorem is proved.

Theorem 25: The mean down time of the simple repairable parallel-series system is given by

$$MDT = \int_0^{+\infty} (1 - \Psi(t))dt.$$

Proof: The proof is the same as Theorem 13.

V. CONCLUSION

Uncertainty theory is a branch of mathematics for modelling belief degrees. In this paper, the lifetimes and the repair times of systems are assumed to be uncertain variables. Then the reliability mathematical models of simple repairable series systems, simple repairable parallel systems, simple repairable series-parallel systems and simple repairable parallel-series systems are established, respectively. Furthermore, the expressions of reliability indexes such as steady state availability, steady state failure frequency, mean up time and mean down time are obtained. In addition, two numerical examples are given to illustrate the calculation methods for reliability indices. Other repairable system reliability models will be considered under uncertainty environments in the near future.

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