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Global Well-Posedness and Dynamical Behavior of Delayed Reaction-Diffusion BAM Neural Networks Driven by Wiener Processes

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ABSTRACT This paper studies the global existence and uniqueness as well as asymptotic behavior of the reaction-diffusion bidirectional associative memory neural networks with S-type distributed delays and infinite dimensional Wiener processes. The conspicuous characteristics of this system are neurons in one layer interacting with neurons in another layer and the noise which disturbed this system has both time and spatial structure. First, several inequalities are proposed and proved for the preparation of future study. Then, the system is coped in the framework of semigroup theory and functional space. Furthermore, the local existence and uniqueness of mild solution for this system is proven by using the contraction mapping principle coupled with many functional inequalities such as Young inequality, Burkholder–Davis–Gundy inequality, Poincaré inequality. The global well-posedness are proven through a prior estimate from constructing appropriate Lyapunov–Krasovskii functional. Moreover, the existence of equilibrium is solved by using the topological degree theory and homotopy invariance. At last, the globally exponential stability of the equilibrium in the mean square sense is studied by constructing appropriate vector Lyapunov–Krasovskii functional and using an improved inequality proposed by us. The criteria of stability are given in the form of matrix form. It is easy to verify them in the computer and they will have a wider application. We give an example to examine the availability of our result, and the code is performed in Matlab. The approach used in this paper can also be extended to other systems.

INDEX TERMS Reaction-diffusion BAM neural network, existence and uniqueness, S-type delays, Lyapunov-Krasovskii functional, Wiener processes.

I. INTRODUCTION

Bidirectional associative memory neural networks (BAMNNs) are first proposed by Kosko [1], [2], which have two fully connected layers. This characteristic allows it to store multiple patterns and to search the goals in both forward and backward directions. Now BAMNNs have captured much more attention due to their practical application in pattern cognition and optimization [3]–[12]. The original BAMNNs are ideal models which are described by ordinary differential equations. However, much more factors must be considered in real operating surroundings in order to describe this system accurately. For example, Marcus and Westervelt [13] point out that it is necessary to consider delays in the signal transmission process, since BAMNNs with delays may be more preferable in theory and application. This type of BAMNNs is studied under the framework of functional differential equations. On the other hand, noise is unavoidable in the real world, since it can destabilize the BAMNNs, and cause oscillation and the bifurcation of this system [4], [14], [15]. The effect of reaction-diffusion is also inevitable in the operation when electrons are moving in an asymmetrical electromagnetic field [14]–[19]. Through above analysis, BAMNNs will be more precise if delay, diffusion and noise are incorporated into this model. BAMNNs with these factors are called stochastic delayed reaction-diffusion BAMNNs (SDRDBAMNNs) which have captured many attentions of the scholars [17]–[20]. Due to the complexity of this model, there is still little literature in this field.

Although this paper is motivated by previous paper in SDRDBAMNNs, it has several characteristics which are different from previous ones. Firstly, the space-time Wiener process is used rather than standard Wiener process since this type of noise not only depends on time t but also on space $\boldsymbol{x}$, which can describe the system more accurately in theory [14], [15], [21]. Unlike the noise studied previously, this type of noise is defined on an infinite dimensional Hilbert space instead of usual Euclid space. Roughly speaking, it can be expressed as the series of independent standard Wiener processes coupled with appropriate weights related to base function of Hilbert space. To the best of our knowledge, there is no article on DRDBAMNNs disturbed by this type of noise yet. Then, SDRDBAMNNs fall in the regime of stochastic partial functional differential equations (SPFDEs), they will be more powerful in theory if considered in an abstract Banach space. However, almost all the models in the existing literature are solved in the usual Euclidean space. Secondly, there is no general condition for existence and uniqueness for PDEs as the Lipschitz condition for ODEs, it is also true for the SDRDBAMNNs. So it is necessary to find and prove the criteria of existence and uniqueness of this system before we study the long-time behavior of RDBAMNNs driven by infinite-dimensional Wiener processes. However, it is not an easy work, since the spatial structure of this system will also pose a heavy burden on our research. For example, the mathematical theory of global well-posedness of Navier-Stokes equation in three dimensions is still not well-solved. As to our model, it can be regarded as a PDE coupled with delay and stochastic effect, and both the structure of them are complex, let alone different states variables interact with each other. No one has attacked the existence and uniqueness of this model yet. Although this work is the prerequisite of studying the quantitative and qualitative behavior of this system. Thirdly, we must also point out long-time behavior is based on the global well-posedness. It will be a challenge for us to prove the global well-posedness of this system since its complexity. New tools and new space must be introduced in this article to solve this problem. Fourthly, the application of BAMNNs depends on the stability of equilibrium of it in some sense, so it is important to study the stability of equilibrium of this system. However, we do not know whether the equilibrium exists or not in this new model. Due to complexity of this system, how to prove the stability of this system? What is the convergence rate of this system to the equilibrium? At last, we need criteria which are wide enough for application. The usual criteria of BAMNNs are based on the LMIs, but they can't be used in the system driven by infinite dimensional Wiener processes.

In addition, in the topological structure, BAMNNs contain two layers of neurons, X-layer and Y-layer. X-layer and Y-layer are fully connected to each other. The sizes of X-layer and Y-layer are different, which means the states variable must be treated in different functional spaces, and the harmony between these two spaces is also need to take in account in this paper. In other words, we need to use the Descartes product of two functional space. On the other hand, the weight matrix of BAMNNs is not square when compared with Hopfield neural networks. So the method based on the square matrix can't be used to solve this model, we must also pay attention to this point in application. These are the main difficulties we confronted when compared with delayed reaction diffusion Hopfield neural networks in our previous paper [15].

In mathematics, it has the following expression

$$\begin{aligned}
\mathrm{d}\boldsymbol{u} &= (\nabla\cdot(\boldsymbol{D^u}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) - \boldsymbol{C}\boldsymbol{u} \\
&\quad + \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0}\boldsymbol{v}(t\, \mathcal{C}\, s,\boldsymbol{x})\mathrm{d}\boldsymbol{\eta_v}(s)) + \boldsymbol{I})\mathrm{d}t + \boldsymbol{P}(\boldsymbol{v})\mathrm{d}\boldsymbol{B} \\
\mathrm{d}\boldsymbol{v} &= (\nabla\cdot(\boldsymbol{D^v}(\boldsymbol{x})\circ\nabla\boldsymbol{v}) - \boldsymbol{D}\boldsymbol{v} \\
&\quad + \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0}\boldsymbol{u}(t\, \mathcal{C}\, s,\boldsymbol{x})\mathrm{d}\boldsymbol{\eta_u}(s)) + \boldsymbol{J})\mathrm{d}t + \boldsymbol{Q}(\boldsymbol{u})\mathrm{d}\boldsymbol{B} \qquad (1)
\end{aligned}$$

where $\boldsymbol{u}(t,\boldsymbol{x}) = (u_1(t,\boldsymbol{x}),\ldots,u_m(t,\boldsymbol{x}))^{\mathrm{T}} \in \mathbb{R}^m$ represents the state vector of X-layer, and $\boldsymbol{v}(t,\boldsymbol{x}) = (v_1(t,\boldsymbol{x}),\ldots,v_n(t,\boldsymbol{x}))^{\mathrm{T}} \in \mathbb{R}^n$ denotes the state vector of Y-layer. Both of them depend on time t and space $\boldsymbol{x}$. $\boldsymbol{x}\in\mathcal{O}$, $\mathcal{O}$ is a connected compact set in $\mathbb{R}^l$ with smooth boundary. The superscript 'T' presents the transpose. $\boldsymbol{D^u} = (D^{\boldsymbol{u}}_{ij})_{m\times m}$, $\boldsymbol{D^v} = (D^{\boldsymbol{v}}_{ij})_{n\times n}$ are positive real matrices of diffusion coefficients of X-layer and Y-layer neurons respectively, which are determined by the Fick's law. Moreover, ∇ denotes the gradient operator, and $\boldsymbol{D^u}\circ\nabla\boldsymbol{u}$ is the Hadamard product of matrix $\boldsymbol{D^u}$ and $\nabla\boldsymbol{u}$, i.e. $\boldsymbol{D^u}\circ\nabla\boldsymbol{u} = (D^{\boldsymbol{u}}_{ij}\frac{\partial u_i}{\partial x_j})_{m\times l}$, we also have

$$\boldsymbol{D^v}\circ\nabla\boldsymbol{v} = (D^{\boldsymbol{v}}_{ij}\frac{\partial v_i}{\partial x_j})_{n\times l}.$$

$$\begin{aligned}
\nabla\cdot(\boldsymbol{D^u}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) &= (\sum_{j=1}^{l}\frac{\partial}{\partial x_j}(D^{\boldsymbol{u}}_{1j}(\boldsymbol{x})\frac{\partial u_1}{\partial x_j}),\ldots, \\
&\qquad\times\sum_{j=1}^{l}\frac{\partial}{\partial x_j}(D^{\boldsymbol{u}}_{mj}(\boldsymbol{x})\frac{\partial u_n}{\partial x_j}))^{\mathrm{T}}, \\
\nabla\cdot(\boldsymbol{D^v}(\boldsymbol{x})\circ\nabla\boldsymbol{v}) &= (\sum_{j=1}^{l}\frac{\partial}{\partial x_j}(D^{\boldsymbol{v}}_{1j}(\boldsymbol{x})\frac{\partial v_1}{\partial x_j}),\ldots, \\
&\qquad\times\sum_{j=1}^{l}\frac{\partial}{\partial x_j}(D^{\boldsymbol{v}}_{nj}(\boldsymbol{x})\frac{\partial v_n}{\partial x_j}))^{\mathrm{T}}.
\end{aligned}$$

$\boldsymbol{C} = diag(c_1, c_2, \cdots, c_m)$ and $\boldsymbol{D} = diag(d_1, d_2, \cdots, d_n)$ are positive real diagonal matrices which represent the rate of neurons can reset their potential to the resting state in isolation in X-layer and Y-layer respectively when disconnected from the networks and the external inputs of the Y-layer.

$\boldsymbol{W} = (w_{ij})_{m\times n}$ and $\boldsymbol{T} = (t_{ij})_{n\times m}$ are the connection weights matrix of X-layer and Y-layer respectively, it should be noted that the sizes of $\boldsymbol{W}$ and $\boldsymbol{T}$ are different in the usual condition, so they are not square matrix, it is also the difficulty in the study of this system. $\boldsymbol{I} = (I_1, I_2, \cdots, I_m)^{\mathrm{T}}$ is the external input of the X-layer and $\boldsymbol{J} = (J_1, J_2, \cdots, J_n)^{\mathrm{T}}$ denotes the external input of the Y-layer. The nonlinear diagonal functionals $\boldsymbol{f}(\boldsymbol{v}) = (f_1(v_1), \ldots, f_n(v_n))^{\mathrm{T}}$ and $\boldsymbol{g}(\boldsymbol{u}) = (g_1(u_1), \ldots, g_m(u_m))^{\mathrm{T}}$ are activation functionals of the X-layer and Y-layer respectively.

The delay r is represented in the Lebesgue-Stieljies integral form, which is caused by the neuron processing and signal delivery.

$$\boldsymbol{f}\Big(\int_{-r}^{0} \boldsymbol{v}(t\,\boldsymbol{C}\,s, \boldsymbol{x})\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{u}}(s)\Big)$$
$$= \Big(f_1\Big(\int_{-r}^{0} v_1(t+s, \boldsymbol{x})\mathrm{d}\eta_{v1}(s)\Big), f_2\Big(\int_{-r}^{0} v_2(t+s, \boldsymbol{x})\mathrm{d}\eta_{v2}(s)\Big), \cdots,$$
$$f_n\Big(\int_{-r}^{0} v_n(t+s, \boldsymbol{x})\mathrm{d}\eta_{vn}(s)\Big)\Big)^{\mathrm{T}}, \boldsymbol{g}\Big(\int_{-r}^{0} \boldsymbol{u}(t\,\boldsymbol{C}\,s, \boldsymbol{x})\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{u}}(s)\Big)$$
$$= \Big(g_1\Big(\int_{-r}^{0} u_1(t+s, \boldsymbol{x})\mathrm{d}\eta_{u1}(s)\Big), g_2\Big(\int_{-r}^{0} u_2(t+s, \boldsymbol{x})\mathrm{d}\eta_{u2}(s)\Big), \cdots,$$
$$g_m\Big(\int_{-r}^{0} u_m(t+s, \boldsymbol{x})\mathrm{d}\eta_{um}(s)\Big)\Big)^{\mathrm{T}}.$$

$\eta_{ui}(s), \eta_{vj}(s), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ are nondecreasing functions with bounded variation. It means there exists constants $K_i^u, K_j^v, K, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$ such that $\int_{-r}^{0} d\eta_{ui}(s) = K_i^u \leq K, \int_{-r}^{0} d\eta_{vj}(s) = K_j^v \leq K$. For convenience of study, we construct the matrix $\boldsymbol{K_u} = diag\{K_1^u, K_2^u, \cdots, K_m^u\}$, and $\boldsymbol{K_v} = diag\{K_1^v, K_2^v, \cdots, K_n^v\}$.

According to the theory of functional differential equations in Hale and Lunel Verduyn [22], delay can be categorized into discrete delay and distributed delay. And these delays are included in the Lebesgue-Stieljies form, so our model is more general than those studied previously.

The initial and adiabatic boundary conditions of (1) are given by

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial \nu}|_{\partial\mathcal{O}} \triangleq \Big(\dfrac{\partial u_1}{\partial \nu}, \dfrac{\partial u_2}{\partial \nu}, \ldots, \dfrac{\partial u_m}{\partial \nu}\Big)^{\mathrm{T}} = \boldsymbol{0} \\ \dfrac{\partial \boldsymbol{v}}{\partial \nu}|_{\partial\mathcal{O}} \triangleq \Big(\dfrac{\partial v_1}{\partial \nu}, \dfrac{\partial v_2}{\partial \nu}, \ldots, \dfrac{\partial v_n}{\partial \nu}\Big)^{\mathrm{T}} = \boldsymbol{0} \\ \boldsymbol{u}(s, \boldsymbol{x}) = \boldsymbol{\phi}(s, \boldsymbol{x}), \boldsymbol{v}(s, \boldsymbol{x}) = \boldsymbol{\psi}(s, \boldsymbol{x}) \end{cases} \tag{2}$$

where $-r \leq s \leq 0$, $\frac{\partial u_i}{\partial \nu} = (\frac{\partial u_i}{\partial x_1}, \frac{\partial u_i}{\partial x_2}, \cdots, \frac{\partial u_i}{\partial x_l})^{\mathrm{T}}$, $\frac{\partial v_j}{\partial \nu} = (\frac{\partial v_j}{\partial x_1}, \frac{\partial v_j}{\partial x_2}, \cdots, \frac{\partial v_j}{\partial x_l})^{\mathrm{T}}$, $i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$. $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$ with Lebesgue measure 0. $\boldsymbol{\phi} = (\phi_1(s, \boldsymbol{x}), \phi_2(s, \boldsymbol{x}), \cdots, \phi_m(s, \boldsymbol{x}))^{\mathrm{T}}$ and $\boldsymbol{\psi} = (\psi_1(s, \boldsymbol{x}), \psi_2(s, \boldsymbol{x}), \cdots, \psi_n(s, \boldsymbol{x}))^{\mathrm{T}}$ are bounded and continuous functions in $[-r, 0] \times \mathcal{O}$.

Random fluctuation phenomenon is described though the Wiener process $\boldsymbol{B}(t, \boldsymbol{x}) = (B_1(t, \boldsymbol{x}), B_2(t, \boldsymbol{x}), \cdots, B_s(t, \boldsymbol{x}))^{\mathrm{T}}$. Compared with standard Wiener process, it not only depends on the time t, but also the space variable $\boldsymbol{x}$, and will be described clearly in the later chapter. The intensified noise $\boldsymbol{P}\mathrm{d}\boldsymbol{B}$ and $\boldsymbol{Q}\mathrm{d}\boldsymbol{B}$ represent response of the neuron to a current impulse associates with the local potential.

II. MODEL DESCRIPTION AND PRELIMINARIES

Now we list some notation which will be used in the following sections.

- $X \times Y$ is the Cartesian product between Banach spaces X and Y;
- If $L^2(\mathcal{O})$ is the space of real Lebegue measurable functions on $\mathcal{O}$, it is a Banach space when equipped with the L_2-norm $||u(t)||_2 = (\int_{\mathcal{O}} |u(t, \boldsymbol{x})|^2 \mathrm{d}x)^{\frac{1}{2}}$;
- $U \triangleq \{L^2(\mathcal{O})\}^m$, when equipped with the norm $||\boldsymbol{u}||_U = (\int_{\mathcal{O}} \|\boldsymbol{u}\|_{\mathbb{R}^m}^2 dx)^{\frac{1}{2}}$, it becomes a Banach space, where $|\boldsymbol{u}|_{\mathbb{R}^m}^2$ denotes the usual Euclid norm of $\mathbb{R}^m$. When endowed with the inner product $(\boldsymbol{u}, \boldsymbol{v})_U = \int_{\mathcal{O}} (\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^m} dx$, it becomes a Hilbert space, where $(\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^m}$ denotes the usual inner product of $\mathbb{R}^m$;
- $\mathcal{U} \triangleq \{\boldsymbol{u} | \boldsymbol{u} \in \{L^2(\mathcal{O})\}^m, \frac{\partial \boldsymbol{u}}{\partial \nu}|_{\partial\mathcal{O}} = 0\}$, when equipped with the norm as that in U, it also becomes a Hilbert space;
- $\overline{\mathcal{U}} \triangleq \{\boldsymbol{u} | \boldsymbol{u} \in U, \nabla \boldsymbol{u} \in U, \frac{\partial \boldsymbol{u}}{\partial \nu}|_{\partial\mathcal{O}} = 0\}$, when equipped with the norm $|||\boldsymbol{u}|||_U = \|\nabla \boldsymbol{u}\|_U$, it also becomes a Hilbert space;
- $V \triangleq \{L^2(\mathcal{O})\}^n$ denotes the Hilbert space, equipped with the norm $||\boldsymbol{v}||_V = (\int_{\mathcal{O}} \|\boldsymbol{v}\|_{\mathbb{R}^n}^2 dx)^{\frac{1}{2}}$, and inner product $(\boldsymbol{u}, \boldsymbol{v})_V = \int_{\mathcal{O}} (\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^n} dx$;
- $\mathcal{V} \triangleq \{\boldsymbol{v} | \boldsymbol{v} \in \{L^2(\mathcal{O})\}^n, \frac{\partial \boldsymbol{v}}{\partial \nu}|_{\partial\mathcal{O}} = 0\}$;
- $\overline{\mathcal{V}} \triangleq \{\boldsymbol{v} | \boldsymbol{v} \in V, \ \nabla \boldsymbol{v} \in V, \ \frac{\partial \boldsymbol{v}}{\partial \nu}|_{\partial\mathcal{O}} = 0\}$;
- $C(U) \triangleq C([-r, 0], U)$ is the set of all continuous mappings from interval $[-r, 0]$ to Banach space U with the topology of uniform convergence, when equipped with the sup-norm $\|\boldsymbol{\varphi}\|_{C(U)} = \sup_{-r \leq s \leq 0} \|\boldsymbol{\varphi}(s)\|_U$, it becomes a Banach space;
- $BC([-r, 0], U)$ is the Banach space of all bounded and continuous operators from $[-r, 0]$ to U, and it is clear that $BC([-r, 0], U) \subset C(U)$;
- For $\boldsymbol{u} \in U, C(U) \ni \boldsymbol{u}_t(s, \boldsymbol{x}) \triangleq \boldsymbol{u}(t + s, \boldsymbol{x}), \forall s \in [-r, 0]$;
- For $\boldsymbol{w} = (u, v)^{\mathrm{T}}, \boldsymbol{w}^+ \triangleq (|u|, |v|)^{\mathrm{T}}$;
- For $\boldsymbol{w}_1 = (u_1, v_1)^{\mathrm{T}}, \boldsymbol{w}_2 = (u_2, v_2)^{\mathrm{T}}$, if $u_1 \leq u_2, v_1 \leq v_2$, we say $\boldsymbol{w}_1 \leq \boldsymbol{w}_2$;
- Let $\boldsymbol{W} = (w_{ij})_{m \times n}, \|\boldsymbol{W}\|_F \triangleq (\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij}^2)^{\frac{1}{2}}$ denotes the Frobenius norm of matrix $\boldsymbol{W}$;
- A matrix $\boldsymbol{W} = (w_{ij})_{m \times n}$ is said to be positive, if $w_{ij} > 0, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$;
- A vector $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ is also said to be positive, if $u_i > 0, i = 1, 2, \cdots, n$.

$\boldsymbol{B}(t, \boldsymbol{x}) = \boldsymbol{B}(t, \boldsymbol{x})(\omega)$ is a Wiener process with values in the separable Hilbert space $\mathbb{K}$ with expectation $\mathrm{E}\boldsymbol{B} = \boldsymbol{0}$ and covariance $\mathrm{E}(\boldsymbol{B}(t), \boldsymbol{u})_{\mathbb{K}}(\boldsymbol{B}(s), \boldsymbol{v})_{\mathbb{K}} = (t \wedge s)(\boldsymbol{Q}\boldsymbol{u}, \boldsymbol{v})_{\mathbb{K}}, \forall s, t \geq 0, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{K}$, $(\cdot, \cdot)_{\mathbb{K}}$ denotes the inner product of $\mathbb{K}$, where $t \wedge s = \min\{t, s\}$. Let $\{\boldsymbol{e}_n\}_{n=1}^{\infty}$ is an orthogonal basis of $\mathbb{K}$, Q is a Hilbert-Schmidt operator such that $Q\boldsymbol{e}_n = \alpha_n \boldsymbol{e}_n$, which means Q is a positive definite, symmetric, self-adjoint operator having a finite trace $\mathrm{tr}Q \triangleq \sum_{n=1}^{\infty} \alpha_n < +\infty$, $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of mutually independent standard Wiener processes in $(\Omega, \mathcal{F}, \mathbb{P})$, with these notations $\boldsymbol{B}$ can be written as $\boldsymbol{B} = \sum_{n=1}^{\infty} \sqrt{\alpha_n}\beta_n(t)\boldsymbol{e}_n(\boldsymbol{x})$. $\mathscr{L}_2(\mathbb{K}_0, U)$ is the space of all Hilbert-Schmidt operators from $\mathbb{K}_0 \triangleq Q^{\frac{1}{2}}(\mathbb{K})$ into U, when equipped with the norm $\|\Phi\|_2 \triangleq \sqrt{\mathrm{tr}(\Phi Q \Phi^*)}$ it becomes a Hilbert space, where $\Phi \in \mathscr{L}_2(\mathbb{K}_0, U)$, Φ^* denotes the adjoint of Φ. For more information, please see [15], [22].

Three inequalities are introduced in this section for later use.

Lemma 1: Let $\boldsymbol{w} = (u, v)^{\mathrm{T}}$, with $u, v \geq 0$, and let it satisfy

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \leq -\boldsymbol{A}\boldsymbol{w} + \boldsymbol{B} \sup\limits_{-r\leq\theta\leq 0} \boldsymbol{w}(t+\theta) \\ \boldsymbol{w}(\theta) = (\phi(\theta), \psi(\theta))^{\mathrm{T}}, \phi, \psi \in BC([-r, 0], \mathbb{R}) \end{cases} \tag{3}$$

where $\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$,$\boldsymbol{B} = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$. If $a, b, c, d, t, r > 0$, and $\alpha - \beta > 0$, with $\alpha = \min\{a, b\}, \beta = \max\{c, d\}$, then there exists a positive vector $\boldsymbol{k} = (\bar{k}, \bar{k})^{\mathrm{T}}$ and positive constant ς, such that

$$\boldsymbol{w} \leq \exp\{-\varsigma t\}\boldsymbol{k}. \tag{4}$$

Proof: Let $z = u + v$, from (3), we can see that

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &\leq -au - bv + c \sup_{-r\leq\theta\leq 0} v(t+\theta) + d \sup_{-r\leq\theta\leq 0} u(t+\theta) \\ &\leq -\alpha z + \beta \sup_{-r\leq\theta\leq 0} z(t+\theta) \end{aligned} \tag{5}$$

with $\alpha = \min\{a, b\} > 0$, $\beta = \max\{c, d\} > 0$.

Let

$$\begin{cases} \dfrac{\mathrm{d}z}{\mathrm{d}t} = -\alpha z + \beta \sup\limits_{-r\leq\theta\leq 0} z(t+\theta) \\ z(\theta) = \phi(\theta) + \psi(\theta), \phi, \psi \in BC([-r, 0], \mathbb{R}). \end{cases} \tag{6}$$

We suppose the solution of (6) has the form $z(t) = k\exp\{-\lambda t\}$, substitute it into (6), we then get

$$-\lambda = -\alpha + \beta\exp\{\lambda r\}. \tag{7}$$

Defining

$$f(x) \triangleq x - \alpha + \beta\exp\{xr\}. \tag{8}$$

From (8), we have $f(x) < 0$, if $x < 0$. $f(0) = -\alpha + \beta < 0$, and $\lim_{x\to+\infty} f(x) = +\infty$, from the intermediate theorem, the system (7) has a solution in $\mathbb{R}^+$. On the other hand, $f'(x) = 1 + \beta r\exp\{xr\} > 0$. According to the implicit theorem, the system (7) has a unique zero point in $\mathbb{R}^+$.

It follows that if $\alpha - \beta > 0$, then there exists a constant $\varsigma > 0$ such that $z(t) = \widetilde{k}_1\exp\{-\varsigma t\}$ is a solution of equation (6), with $\widetilde{k}_1 = |\phi(0)| + |\psi(0)|$.

By the comparison principle, we can deduce from (5) that there exists positive constants $\bar{k}$ and ς such that

$$z \leq \exp\{-\varsigma t\}\bar{k} \tag{9}$$

with $\bar{k} = (\|\phi\|_C + \|\psi\|_C)$.

Since $z = u + v$, $u, v \geq 0$, we also have

$$\boldsymbol{w} = \begin{pmatrix} u \\ v \end{pmatrix} \leq \begin{pmatrix} z \\ z \end{pmatrix} \leq \exp\{-\varsigma t\}\boldsymbol{k} \tag{10}$$

with $\boldsymbol{k} = (\bar{k}, \bar{k})^{\mathrm{T}}$. ∎

Lemma 2: Let $\boldsymbol{W} = (w_{ij})_{m\times n}$, $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)^{\mathrm{T}}$, $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)^{\mathrm{T}}$, $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{W}\boldsymbol{g}(\boldsymbol{u})$, $\boldsymbol{g} = (g_1(u_1), g_2(u_2), \cdots, g_n(u_n))^{\mathrm{T}}$, diagonal maps g_i satisfies the global Lipschitz condition $|g_i(u_i) - g_i(v_i)| \leq \sigma_i|u_i - v_i|$, $\sigma_i > 0, \forall i = 1, 2, \cdots, n$, then $\|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})\|_{\mathbb{R}^m} \leq \sqrt{n}\sigma_M\|\boldsymbol{W}\|_F\|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbb{R}^n}$, where $\|\boldsymbol{u}\|_{\mathbb{R}^m}$ represents the usual norm of $\mathbb{R}^m$, $\|\boldsymbol{u}\|_{\mathbb{R}^n}$ represents the usual norm of $\mathbb{R}^n$, $\sigma_M = \max\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$.

Proof:

$$\begin{aligned} &(\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v}), \boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v}))_{\mathbb{R}^m} \\ &= (\boldsymbol{W}(\boldsymbol{g}(\boldsymbol{u}) - \boldsymbol{g}(\boldsymbol{v})), \boldsymbol{W}(\boldsymbol{g}(\boldsymbol{u}) - \boldsymbol{g}(\boldsymbol{v})))_{\mathbb{R}^m} \\ &= \sum_{i=1}^{m}(\sum_{j=1}^{n} w_{ij}(g_j(u_j) - g_j(v_j))^2 \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} w_{ij}w_{ik}(g_j(u_j) - g_j(v_j))(g_k(u_k) - g_k(v_k)) \\ &\leqslant \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}(w_{ij}^2(g_j(u_j) - g_j(v_j))^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n} w_{ik}^2(g_k(u_k) - g_k(v_k))^2) \\ &= n\sum_{i=1}^{m}\sum_{j=1}^{n}(w_{ij}^2(g_j(u_j) - g_j(v_j))^2) \\ &\leqslant n(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^2)(\sum_{j=1}^{n}(g_j(u_j) - g_j(v_j))^2) \\ &\leqslant n(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^2)(\sum_{j=1}^{n}\sigma_j^2(u_j - v_j)^2) \\ &\leqslant n\sigma_M^2(\sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij}^2)(\sum_{j=1}^{n}(u_j - v_j)^2) \\ &= n\sigma_M^2\|\boldsymbol{W}\|_F^2\|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbb{R}^n}^2. \end{aligned}$$

The proof is finished. ∎

We can infer from Lemma 2 that

Lemma 3: If $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{W}\boldsymbol{g}(\boldsymbol{u})$, by using the same assumption and notations as that in Lemma 2, we have $\|\boldsymbol{f}(\boldsymbol{u})\|_{\mathbb{R}^m}^2 \leq 2n\|\boldsymbol{W}\|_F^2\|\boldsymbol{g}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 + 2n\sigma_M^2\|\boldsymbol{W}\|_F^2\|\boldsymbol{u}\|_{\mathbb{R}^n}^2$.

Proof: Let $\boldsymbol{v} = \boldsymbol{0}$ in Lemma 2, by using the triangle inequality, we have

$$\|\boldsymbol{f}(\boldsymbol{u})\|_{\mathbb{R}^m} \leq \|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^m} + \sqrt{n}\sigma_M\|\boldsymbol{W}\|_F\|\boldsymbol{u}\|_{\mathbb{R}^n}.$$

By using the basic inequality $2ab \leq a^2 + b^2$, and definition of $\boldsymbol{f}$, we have

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{u})\|_{\mathbb{R}^m}^2 &\leq 2\|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^m}^2 + 2n\sigma_M^2\|\boldsymbol{W}\|_F^2\|\boldsymbol{u}\|_{\mathbb{R}^n}^2 \\ &\leq 2n\|\boldsymbol{W}\|_F^2\|\boldsymbol{g}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 + 2n\sigma_M^2\|\boldsymbol{W}\|_F^2\|\boldsymbol{u}\|_{\mathbb{R}^n}^2. \end{aligned}$$

∎

III. MAIN RESULTS

Let us first define the linear unbounded operator as follows:

$$\mathfrak{A}_1 : \mathcal{D}(\mathfrak{A}_1) \in U \to U$$

$$\mathfrak{A}_1 \boldsymbol{u} = \nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}), \boldsymbol{u} \in \mathcal{D}(\mathfrak{A}_1) \tag{11}$$
$$\mathfrak{A}_2 : \mathcal{D}(\mathfrak{A}_2) \in V \to V$$
$$\mathfrak{A}_2 \boldsymbol{v} = \nabla \cdot (\boldsymbol{D}^{\boldsymbol{v}}(\boldsymbol{x}) \circ \nabla \boldsymbol{v}), \boldsymbol{v} \in \mathcal{D}(\mathfrak{A}_2) \tag{12}$$

and $\mathcal{D}(\mathfrak{A}_1) = \{\boldsymbol{u} | \nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}) \in U\} \bigcap \mathcal{U}$, $\mathcal{D}(\mathfrak{A}_2) = \{\boldsymbol{v} | \nabla \cdot (\boldsymbol{D}^{\boldsymbol{v}}(\boldsymbol{x}) \circ \nabla \boldsymbol{v}) \in V\} \bigcap \mathcal{V}$.

Then, the Nemytskii operator is difined as follows:

$$\begin{cases} \boldsymbol{f}(\int_{-r}^{0} \boldsymbol{v}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{v}}(s)) = \boldsymbol{f}(\int_{-r}^{0} \boldsymbol{v}(t+s, \boldsymbol{x})\mathrm{d}\eta_{\boldsymbol{v}}(s)) \\ \boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(t+s)\mathrm{d}\eta_{\boldsymbol{u}}(s)) = \boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(t+s, \boldsymbol{x})\mathrm{d}\eta_{\boldsymbol{u}}(s)). \end{cases}$$

With these tools, we can rewrite the system (1) in the more abstract form

$$\begin{cases} \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \mathfrak{A}_1 \boldsymbol{u} - \boldsymbol{C}\boldsymbol{u} \\ +\boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0} \boldsymbol{v}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{v}}(s)) + \boldsymbol{I} + \boldsymbol{P}(\boldsymbol{v}(t))d\boldsymbol{B} \\ \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathfrak{A}_2 \boldsymbol{v} - \boldsymbol{D}\boldsymbol{v} \\ +\boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(t+s)\mathrm{d}\eta_{\boldsymbol{u}}(s)) + \boldsymbol{J} + \boldsymbol{Q}(\boldsymbol{u}(t))d\boldsymbol{B} \\ \boldsymbol{u}(s) = \boldsymbol{\phi}(s), \boldsymbol{v}(s) = \boldsymbol{\psi}(s), -r \le s \le 0, \\ (\boldsymbol{\phi}(s), \boldsymbol{\psi}(s)) \in BC([-r, 0], U) \times BC([-r, 0], V). \end{cases} \tag{13}$$

The following assumptions are needed in this paper.

H1 The diagonal activation function f_i, g_i and the noise intensifying function $\boldsymbol{P}, \boldsymbol{Q}$ satisfy the global Lipschitz condition

$$\begin{aligned} |f_i(x) - f_i(y)| &\le \sigma_i |x - y| \\ |g_j(x) - g_j(y)| &\le \gamma_j |x - y| \\ \|\boldsymbol{P}(\boldsymbol{v_1}) - \boldsymbol{P}(\boldsymbol{v_2})\|_V &\le k_1 \|\boldsymbol{v_1} - \boldsymbol{v_2}\|_V \\ \|\boldsymbol{Q}(\boldsymbol{u_1}) - \boldsymbol{P}(\boldsymbol{u_2})\|_U &\le k_2 \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_U \end{aligned} \tag{14}$$

with $\gamma_i, \sigma_i, k_1, k_2 \ge 0, \forall x, y \in \mathbb{R}, \boldsymbol{v_1}, \boldsymbol{v_2} \in V$, $\boldsymbol{u_1}, \boldsymbol{u_2} \in U, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$;

H2 There exists a positive constant α, such that $D_{ij}^{\boldsymbol{u}} \ge \frac{\alpha}{ml}$ and $D_{kj}^{\boldsymbol{v}} \ge \frac{\alpha}{nl}$, where $i = 1, 2, \cdots, m$, $k = 1, 2, \cdots, n, j = 1, 2, \cdots, l$;

H3 $(2k_{14} + 2\alpha L^2 - k_2^2 - 1) \wedge (2k_{15} + 2\alpha L^2 - k_1^2 - 1) > (n\|\boldsymbol{W}\|_F^2 \sigma_M^2 K^2) \vee (m\|\boldsymbol{T}\|_F^2 \gamma_M^2 K^2)$, where $a \vee b = \max\{a, b\}, a \wedge b = \min\{a, b\}$, $k_{14} = min\{c_1, c_2, \cdots, c_m\} > 0$, $k_{15} = min\{d_1, d_2, \cdots, d_n\} > 0$.

Theorem 4: From H1, we have

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{v})\|_V^2 &\le 2\|\boldsymbol{f}(\boldsymbol{0})\|_V^2 + 2\sigma^2 \|\boldsymbol{v}\|_V^2 \\ \|\boldsymbol{g}(\boldsymbol{u})\|_U^2 &\le 2\|\boldsymbol{g}(\boldsymbol{0})\|_U^2 + 2\gamma^2 \|\boldsymbol{u}\|_U^2 \end{aligned} \tag{15}$$

where $\sigma_M = \max\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$, $\gamma_M = \max\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$.

Proof: From H1, we can see that $\forall v_i \in \mathbb{R}$

$$|f_i(v_i)| \le |f_i(0)| + |f_i(v_i) - f_i(0)| \le |f_i(0)| + \sigma_i |v_i|. \tag{16}$$

By using the Cauchy inequality, we have

$$|f_i(v_i)|^2 \le 2|f_i(0)|^2 + 2\sigma_i^2 |v_i|^2, \quad \forall v_i \in \mathbb{R}. \tag{17}$$

Adding (17) from 1 to n and integrating above inequality on $\mathcal{O}$, we have

$$\|\boldsymbol{f}(\boldsymbol{v})\|_V^2 \le 2\|\boldsymbol{d}(\boldsymbol{0})\|_V^2 + 2\sigma_M^2 \|\boldsymbol{v}\|_V^2 \tag{18}$$

where $\sigma_M = \max\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$.

Following the same procedure, we also get

$$\|\boldsymbol{g}(\boldsymbol{u})\|_U^2 \le 2\|\boldsymbol{g}(\boldsymbol{0})\|_U^2 + 2\gamma_M^2 \|\boldsymbol{u}\|_U^2 \tag{19}$$

where $\gamma_M = \max\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$. ∎

By using this theorem, Lemma 2, Lemma 3, and H1, we have

Theorem 5:

$$\begin{aligned} \|\boldsymbol{W}\boldsymbol{f}(\boldsymbol{v_1}) - \boldsymbol{W}\boldsymbol{f}(\boldsymbol{v_2})\|_V &\le \sqrt{n}\sigma_M \|\boldsymbol{W}\|_F \|\boldsymbol{v_1} - \boldsymbol{v_2}\|_V \\ \|\boldsymbol{T}\boldsymbol{g}(\boldsymbol{u_1}) - \boldsymbol{T}\boldsymbol{g}(\boldsymbol{u_2})\|_U &\le \sqrt{m}\gamma_M \|\boldsymbol{T}\|_F \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_U \\ \|\boldsymbol{W}\boldsymbol{f}(\boldsymbol{v})\|_V^2 &\le p_1 \|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O}) + p_1 \sigma_M^2 \|\boldsymbol{v}\|_V^2 \\ \|\boldsymbol{T}\boldsymbol{g}(\boldsymbol{u})\|_U^2 &\le p_2 \|\boldsymbol{g}(\boldsymbol{0})\|_{\mathbb{R}^m}^2 mes(\mathcal{O}) + p_2 \gamma_M^2 \|\boldsymbol{u}\|_U^2 \end{aligned}$$

where $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u} \in U, \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v} \in V$, $p_1 = 2n\|\boldsymbol{W}\|_F^2$, $p_2 = 2m\|\boldsymbol{T}\|_F^2$.

Theorem 6: Defining the bilinear operators as $a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u}) = -(\mathfrak{A}_1 \boldsymbol{u}, \boldsymbol{u})_U$, $a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v}) = -(\mathfrak{A}_2 \boldsymbol{v}, \boldsymbol{v})_V$, $\boldsymbol{u} \in \mathcal{D}(\mathfrak{A}_1)$, $\boldsymbol{v} \in \mathcal{D}(\mathfrak{A}_2)$, then both $a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u})$ and $a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v})$ are coercive. Furthermore $a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u}) \geqslant \alpha \||\boldsymbol{u}\||_U^2$, $a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v}) \geqslant \alpha \||\boldsymbol{v}\||_V^2$.

Proof: $\forall \boldsymbol{u} \in \mathcal{D}(\mathfrak{A}_1)$

$$\begin{aligned} a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u})_U &= -(\nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}), \boldsymbol{u})_U \\ &= -\int_{\mathcal{O}} (\nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}), \boldsymbol{u})_{\mathbb{R}^m} \mathrm{d}x \\ &= -\sum_{i=1}^{m} \int_{\mathcal{O}} \nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u})_i u_i \mathrm{d}x. \end{aligned}$$

By the conclusion of [12] and [13], and Gaussian formula and the boundary condition, we get

$$\int_{\mathcal{O}} \boldsymbol{G}(\boldsymbol{u}) \nabla \cdot (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u}) dx = -\int_{\mathcal{O}} (\boldsymbol{D}^{\boldsymbol{u}}(\boldsymbol{x}) \circ \nabla \boldsymbol{u} \circ \nabla \boldsymbol{u}) \boldsymbol{E} \mathrm{d}x$$

where $\boldsymbol{G}(\boldsymbol{u}) = Diag(u_1, u_2, \cdots, u_m)$, $\boldsymbol{E} = (1, 1, \cdots, 1)^T$. By the definition of the Hadamard product and H2, we get $a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u}) \geqslant \alpha \||\boldsymbol{u}\||_U^2$. So the bilinear operator $a_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{u})$ is coercive.

Following the same method, we can get the bilinear operator $a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v})$ is also coercive, and $a_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v}) \geqslant \alpha \||\boldsymbol{v}\||_V^2$. The proof is finished. ∎

We can also prove the following Poincaré inequality by using the method in [7].

Lemma 7 (Poincaré Inequality): Let $\mathcal{O}$ be an open bounded domain with smooth boundary in $\mathbb{R}^l$, then there exists a positive constant L, such that

$$\begin{aligned} L\|\boldsymbol{\phi}\|_U &\le \||\boldsymbol{\phi}\||_U \\ L\|\boldsymbol{\psi}\|_V &\le \||\boldsymbol{\psi}\||_V, \quad \forall \boldsymbol{\phi} \in \overline{\mathcal{U}}, \boldsymbol{\psi} \in \overline{\mathcal{V}} \end{aligned}$$

and L depends on the size of the given domain $\mathcal{O}$.

If $\mathfrak{A}_1, \mathfrak{A}_2$ are the infinitesimal generators of continuous semigroups $S_1(t), S_2(t)$. Then we have the following properties.

Proposition 8: Let us consider the following equations

$$\begin{cases} \mathrm{d}\boldsymbol{u} = \mathfrak{A}_1 \boldsymbol{u} \\ \boldsymbol{u}(0) = \boldsymbol{\phi}, \boldsymbol{\phi} \in \mathcal{D}(\mathfrak{A}_1) \end{cases} \tag{20}$$

$$\begin{cases} \mathrm{d}\boldsymbol{v} = \mathfrak{A}_2 \boldsymbol{v} \\ \boldsymbol{v}(0) = \boldsymbol{\psi}, \boldsymbol{\psi} \in \mathcal{D}(\mathfrak{A}_2). \end{cases} \tag{21}$$

Let $\boldsymbol{u}(t) = S_1(t)\boldsymbol{\phi}, \boldsymbol{v}(t) = S_2(t)\boldsymbol{\psi}$ denote the solution to (20) and (21) respectively, then both $S_1(t)$ and $S_2(t)$ are contraction maps in U and V.

Proof: We recall that the solution to (21) is $\boldsymbol{u}(t) = e^{\mathfrak{A}_1 t}\boldsymbol{\phi}$, so $S_1(t) = e^{\mathfrak{A}_1 t}$. Now we take the inner product of (21) with $\boldsymbol{u}(t)$ in U, by employing the Gauss formula and condition H2 as well as Theorem 6, we get $(\mathfrak{A}_1 \boldsymbol{u}, \boldsymbol{u})_U \le -\alpha |||\boldsymbol{u}|||_U^2, \boldsymbol{u} \in \mathcal{D}(\mathfrak{A}_1)$, and we also have

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_U^2 + \alpha \,|||\, \boldsymbol{u}(t)|||_U^2 \le 0$$

thanks to the Lemma 7, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|_U^2 + 2\alpha\beta^2\|\boldsymbol{u}(t)\|_U^2 \le 0$$

by using the Gronwall inequality, we have

$$\|\boldsymbol{u}(t)\|_U^2 \le e^{-2\alpha\beta^2\, t}\|\boldsymbol{\phi}\|_U^2 \le \|\boldsymbol{\phi}\|_U^2$$

so $S_1(t)$ is a contraction map.

If we define $\boldsymbol{v}(t) = e^{\mathfrak{A}_2 t}\boldsymbol{\psi}$, and $S_2(t) = e^{\mathfrak{A}_2 t}$, following the same procedure as above, we can get $S_2(t)$ is also a contraction map. ■

With the semigroups defined in Proposition 8, we rewrite (1) as

$$(\boldsymbol{u}_t(\boldsymbol{\phi}), \boldsymbol{v}_t(\boldsymbol{\psi}))^{\mathrm{T}} = \begin{cases} S_1(t)\boldsymbol{\phi}(0) + \\ \int_0^t S_1(t-s)(-\boldsymbol{C}\boldsymbol{u} + \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{\boldsymbol{0}} \boldsymbol{v}(\boldsymbol{s}\ \mathcal{C}\ \boldsymbol{\theta})\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta})) + \boldsymbol{I})\mathrm{d}s \\ + \int_0^t S_1(t-s)\boldsymbol{P}(\boldsymbol{u}(s))\mathrm{d}\boldsymbol{B} \\ S_2(t)\boldsymbol{\psi}(0) + \\ \int_0^t S_2(t-s)(-\boldsymbol{D}\boldsymbol{v} + \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(s+\theta)\mathrm{d}\eta_{\boldsymbol{u}}(\theta)) + \boldsymbol{J})\mathrm{d}s \\ + \int_0^t S_2(t-s)\boldsymbol{Q}(\boldsymbol{u}(s))\mathrm{d}\boldsymbol{B}, t > 0 \\ (\phi(t), \boldsymbol{\psi}(t))^T, t \in [-r, 0]. \end{cases} \tag{22}$$

The solution of (22) is called the mild solution of (1).

A. EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION

In order to prove the local existence and uniqueness of the mild solution of (1), we introduce the following function spaces.

- $C(U) \times C(V) \triangleq C([-r, 0], U) \times C([-r, 0], V)) = \{\boldsymbol{\phi}, \boldsymbol{\psi})|\boldsymbol{\phi} \in C([-r, 0], U), \boldsymbol{\psi} \in C([-r, 0], V)\}$, when endowed with the norm $\|(\boldsymbol{\phi}, \boldsymbol{\psi})\|_{C(U)\times C(V)} = \|\boldsymbol{\phi}\|_{C(U)} + \|\boldsymbol{\psi}\|_{C(V)}$, it becomes a Banach space;
- Let $X_T^U \triangleq L^2(\Omega, C([-r, T] \times \mathcal{O}, U))$, when equipped with the norm $\|\boldsymbol{u}\|_{X_T^U} = (\sup_{t\in[-r,T]} \mathrm{E}\,\|\boldsymbol{u}(t)\|^2)^{\frac{1}{2}}$, it becomes a Banach space. By calculation, we have $\|\boldsymbol{u}\|_{X_T^U}^2 = \sup_{t\in[-r,T]} \mathrm{E}\,\|\boldsymbol{u}(t)\|_U^2 = \sup_{t\in[0,T]}\sup_{\theta\in[-r,0]} \mathrm{E}\,\|\boldsymbol{u}(t+\theta)\|_U^2 = \sup_{t\in[0,T]} \mathrm{E}\,\|\boldsymbol{u}_t\|_{C(U)}^2$;
- $\Sigma(a, T, U) = \{\boldsymbol{u}(t) \in X_T^U : \|\boldsymbol{u}\|_{X_T^U} \le a\}$;
- $\Sigma(a, T, V) = \{\boldsymbol{v}(t) \in X_T^V : \|\boldsymbol{v}\|_{X_T^V} \le a\}$;
- Let us construct the function space $X_T = X_T^U \times X_T^V$, when endowed with the norm $\|\boldsymbol{u}\|_{X_T^U} + \|\boldsymbol{v}\|_{X_T^V}$, it also becomes a Banach space.

Theorem 9: For any $\|\boldsymbol{\phi}\|_{C(U)}^2 + \|\boldsymbol{\psi}\|_{C(V)}^2 < \frac{m}{5}$, if the system (1) satisfies H1 and H2, then there exists $T > 0$, such that (1) has a unique mild solution in $\Sigma(a, T, U) \times \Sigma(a, T, V)$.

Proof: Taking $(\boldsymbol{u}, \boldsymbol{v}) \in \Sigma(a, T, U) \times \Sigma(a, T, V)$ and defining the map $\mathcal{T}(\boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{u}, \boldsymbol{v}) = (\Xi, \Pi)$, with

$$\begin{cases} \Xi(t) = S_1(t)\boldsymbol{\phi}(0) + \\ \int_0^t S_1(t-s)(-\boldsymbol{C}\boldsymbol{u} + \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{\boldsymbol{0}} \boldsymbol{v}(\boldsymbol{s}\ \mathcal{C}\ \boldsymbol{\theta})\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta})) + \boldsymbol{I})\mathrm{d}s \\ + \int_0^t S_1(t-s)\boldsymbol{P}(\boldsymbol{v}(s))\mathrm{d}\boldsymbol{B} \\ \Pi(t) = S_2(t)\boldsymbol{\psi}(0) + \\ \int_0^t S_2(t-s)(-\boldsymbol{D}\boldsymbol{v} + \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(s+\theta)\mathrm{d}\eta_{\boldsymbol{u}}(\theta)) + \boldsymbol{J})\mathrm{d}s \\ + \int_0^t S_2(t-s)\boldsymbol{Q}(\boldsymbol{u}(s))\mathrm{d}\boldsymbol{B} \end{cases} \tag{23}$$

taking expectation on both sides of the first equality of (23) and applying the inequality $(\sum_{i=1}^n a_i)^p \le n^{(p-1)} \sum_{i=1}^n a_i^p$, for $p \ge 2$, we have

$$\begin{aligned} &\mathrm{E}\,\|\,\Xi(t)\|_U^2 \\ &\le 5\,\mathrm{E}\,\|S_1(t)\boldsymbol{\phi}(0)\|_U^2 + 5\,\mathrm{E}\,\|\int_0^t -\boldsymbol{C}S_1(t-s)\boldsymbol{u}\mathrm{d}s\|_U^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_1(t-s)\boldsymbol{I}\mathrm{d}s\|_U^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_1(t-s)\boldsymbol{W}\boldsymbol{f}(\int_{-\boldsymbol{r}}^{\boldsymbol{0}} \boldsymbol{v}(\boldsymbol{s}\ \mathcal{C}\ \boldsymbol{\theta})\mathrm{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta}))\mathrm{d}s\|_U^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_1(t-s)\boldsymbol{P}(\boldsymbol{v}(s))\mathrm{d}\boldsymbol{B}\|_U^2 \\ &\triangleq I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Because of the contraction of the map $\mathfrak{A}_1(t)$ and the definition of $\|\cdot\|_{C(U)}$, we find

$$I_1 \le 5\,\mathrm{E}\,\|S_1(t)\boldsymbol{\phi}\|_{C(U)}^2 \le 5\,\mathrm{E}\,\|\boldsymbol{\phi}\|_{C(U)}^2. \tag{24}$$

With Proposition 8, Lemma 2, and the Hölder inequality, it follows that

$$\begin{aligned} I_2 &\le 5 \sup_{t\in[0,T]} (t\,\mathrm{E}\int_0^t \|-\boldsymbol{C}S_1(t-s)\boldsymbol{u}\|_U^2\mathrm{d}s) \\ &\le 5m\|\boldsymbol{C}\|_F^2 T\,\mathrm{E}\int_0^T \|S_1(t-s)\boldsymbol{u}\|_U^2\mathrm{d}s \\ &\le 5m\|\boldsymbol{C}\|_F^2\, T\,\mathrm{E}\int_0^T \|\boldsymbol{u}\|_U^2\mathrm{d}s \\ &\le 5m\|\boldsymbol{C}\|_F^2 T^2\|\boldsymbol{u}\|_{X_T^U}^2. \end{aligned} \tag{25}$$

Using the Hölder inequality and Proposition 8, we have

$$\begin{aligned} I_3 &\le 5\,T \sup_{t\in[0,T]} \mathrm{E}\int_0^t \|\boldsymbol{I}\|_U^2\mathrm{d}s \\ &\le 5T^2\|\boldsymbol{I}\|_{\mathbb{R}^m}^2 mes(\mathcal{O}) \end{aligned} \tag{26}$$

where $mes(\mathcal{O})$ stands for the Lebesgue measure of the domain $\mathcal{O}$.

By virtue of Lemma 2, Lemma 3, Theorem 5, Hölder inequality, the total boundedness of Stieljies integral and H1, we get

$$\begin{aligned} I_4 &\leq 5 \sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|\boldsymbol{W}\boldsymbol{f}(\int_{-r}^{\mathbf{0}} \boldsymbol{v}(s\,\mathcal{C}\,\boldsymbol{\theta})\mathbf{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta}))\|_V^2 \mathrm{d}s \\ &\leq 5\, nT\|\boldsymbol{W}\|_F^2\,\mathrm{E}\int_0^T \|\boldsymbol{f}(\int_{-r}^{\mathbf{0}} \boldsymbol{v}(s\,\mathcal{C}\,\boldsymbol{\theta})\mathbf{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta}))\|_V^2 \mathrm{d}s \\ &\leq p_3\,\mathrm{E}\int_0^T (\|\boldsymbol{f}(\mathbf{0})\|_V^2 + \sigma_M^2\|\int_{-r}^{\mathbf{0}} \boldsymbol{v}(s\,\mathcal{C}\,\boldsymbol{\theta})\mathbf{d}\boldsymbol{\eta}_{\boldsymbol{v}}(\boldsymbol{\theta})\|_V^2)\mathrm{d}s \\ &\leq p_3\,\mathrm{E}\int_0^T (\|\boldsymbol{f}(\mathbf{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O}) + \sigma_M^2 K^2\|\boldsymbol{v}_s\|_{C(V)}^2)\mathrm{d}s \\ &\leq p_3\,T(\|\boldsymbol{f}(\mathbf{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O}) + \sigma_M^2 K^2\|\boldsymbol{v}\|_{X_T^V}^2) \end{aligned} \tag{27}$$

where $p_3 = 10\, nT\|\boldsymbol{W}\|_F^2$.

With the Burkholder-Davis-Gundy inequality, H1, and Proposition 8, there exists a constant $k_3 > 0$, such that

$$\begin{aligned} I_5 &\leq 5k_3\,\mathrm{E}\sup_{t\in[0,T]}\int_0^t \|S_1(t-s)\boldsymbol{P}(\boldsymbol{v}(s))\|_2^2 \mathrm{d}s \\ &\leq 5 \sup_{t\in[0,T]}\int_0^t \mathrm{E}\,\|\boldsymbol{P}(\boldsymbol{v}(s))\|_2^2 \mathrm{d}s \\ &\leq 10\int_0^T (\|\boldsymbol{P}(\mathbf{0})\|_2^2 + k_1^2\,\mathrm{E}\,\|\boldsymbol{v}(s)\|_V^2)\mathrm{d}s \\ &\leq 10k_1^2 T\|\boldsymbol{v}\|_{X_T^V}^2 + 10\|\boldsymbol{P}(\mathbf{0})\|_2^2 T. \end{aligned} \tag{28}$$

It follows from (23)-(28) that

$$\|\Xi(t)\|_{X_T^U}^2 \leq 5\|\boldsymbol{\phi}\|_{C(U)}^2 + k_4T^2 + k_5T \tag{29}$$

where $k_4 = 5\|\boldsymbol{I}\|_{\mathbb{R}^m}^2 mes(\mathcal{O}) + 5m\|C\|_F^2\|\boldsymbol{u}\|_{X_T^U}^2 + 10n\|\boldsymbol{W}\|_F^2\|\boldsymbol{f}(\mathbf{0})\|_{\mathbb{R}^n}^2\ mes(\mathcal{O}) + 10n\sigma_M^2K^2\|\boldsymbol{v}\|_{X_T^V}^2$, $k_5 = 10k_3\ k_1^2\|\boldsymbol{v}\|_{X_T^V}^2 + 10k_3\|\boldsymbol{P}(\mathbf{0})\|_2^2$. Hence $\|\Xi\|_{X_T} \leq a$, provided T is sufficiently small, such that

$$5\|\boldsymbol{\phi}\|_C^2 + k_4T^2 + k_5T < a. \tag{30}$$

Taking expectation on both sides of the second equality of (23), we have

$$\begin{aligned} &\mathrm{E}\,\|\Pi(t)\|_V^2 \\ &\leq 5\,\mathrm{E}\,\|S_2(t)\boldsymbol{\psi}(0)\|_V^2 + 5\,\mathrm{E}\,\|\int_0^t -\boldsymbol{D}S_2(t-s)\boldsymbol{v}\mathrm{d}s\|_V^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_2(t-s)\boldsymbol{J}\mathrm{d}s\|_V^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_2(t-s)\boldsymbol{T}\boldsymbol{g}(\int_{-r}^{\mathbf{0}} \boldsymbol{u}(s\,\mathcal{C}\,\boldsymbol{\theta})\mathbf{d}\boldsymbol{\eta}_{\boldsymbol{u}}(\boldsymbol{\theta}))\mathrm{d}s\|_V^2 \\ &\quad + 5\,\mathrm{E}\,\|\int_0^t S_2(t-s)\boldsymbol{Q}(\boldsymbol{u}(s))\mathrm{d}\boldsymbol{B}\|_V^2 \\ &\triangleq I_6 + I_7 + I_8 + I_9 + I_{10}. \end{aligned}$$

By using the contraction of the map $S_2(t)$ and the definition of $\|\cdot\|_{C(V)}$, we find

$$I_6 \leq 5\,\mathrm{E}\,\|S_2(t)\boldsymbol{\psi}\|_{C(V)}^2 \leq 5\,\mathrm{E}\,\|\boldsymbol{\psi}\|_{C(V)}^2. \tag{31}$$

With Proposition 8, Lemma 2, and the Hölder inequality, it follows that

$$\begin{aligned} I_7 &\leq 5 \sup_{t\in[0,T]} (t\,\mathrm{E}\int_0^t \|-\boldsymbol{D}S_2(t-s)\boldsymbol{v}\|_V^2 \mathrm{d}s) \\ &\leq 5n\|\boldsymbol{D}\|_F^2 T\,\mathrm{E}\int_0^T \|S_2(t-s)\boldsymbol{v}\|_V^2 \mathrm{d}s \\ &\leq 5n\|\boldsymbol{D}\|_F^2 T^2\|\boldsymbol{v}\|_{X_T^V}^2. \end{aligned} \tag{32}$$

Using the Hölder inequality and Proposition 8, we have

$$\begin{aligned} I_8 &\leq 5\,T \sup_{t\in[0,T]} \mathrm{E}\int_0^t \|\boldsymbol{J}\|^2 \mathrm{d}s \\ &\leq 5T^2\|\boldsymbol{J}\|_{\mathbb{R}^n}^2 mes(\mathcal{O}). \end{aligned} \tag{33}$$

By virtue of Lemma 2, Lemma 3, Theorem 5, Hölder inequality, the total boundedness of Stieljies integral and H1, we get

$$\begin{aligned} I_9 &\leq 5\sup_{t\in[0,T]} t\,\mathrm{E}\textstyle\int_0^t \|\boldsymbol{T}\boldsymbol{g}(\int_{-r}^{\mathbf{0}} \boldsymbol{u}(s\,\mathcal{C}\,\boldsymbol{\theta})\mathbf{d}\boldsymbol{\eta}_{\boldsymbol{u}}(\boldsymbol{\theta}))\|_V^2 \mathrm{d}s \\ &\leq 10\, mT^2\|\boldsymbol{T}\|_F^2(\|\boldsymbol{g}(\mathbf{0})\|_{\mathbb{R}^m}^2 mes(\mathcal{O}) + \sigma_M^2K^2\|\boldsymbol{u}\|_{X_T^U}^2). \end{aligned} \tag{34}$$

With the Burkholder-Davis-Gundy inequality, H1-H2, and Proposition 8, there exists a constant $k_6 > 0$, such that

$$\begin{aligned} I_{10} &\leq 5k_6\,\mathrm{E}\sup_{t\in[0,T]}\int_0^t \|S_2(t-s)\boldsymbol{Q}(\boldsymbol{u}(s))\|_2^2 \mathrm{d}s \\ &\leq 5k_6 \sup_{t\in[0,T]}\int_0^t \mathrm{E}\,\|\boldsymbol{Q}(\boldsymbol{u}(s))\|_2^2 \mathrm{d}s \\ &\leq 10k_6\int_0^T (\|\boldsymbol{Q}(\mathbf{0})\|_2^2 + k_2^2\,\mathrm{E}\,\|\boldsymbol{u}(s)\|_U^2)\mathrm{d}s \\ &\leq 10k_6k_2^2T\|\boldsymbol{u}\|_{X_T^U}^2 + 10k_6\|\boldsymbol{Q}(\mathbf{0})\|_2^2 T. \end{aligned} \tag{35}$$

It follows from (31)-(35) that

$$\|\Pi(t)\|_{X_T^V}^2 \leq 5\|\boldsymbol{\psi}\|_{C(V)}^2 + k_7T^2 + k_8T \tag{36}$$

where $k_7 = 5n\|\boldsymbol{D}\|_F^2\|\boldsymbol{v}\|_{X_T^V}^2 + 5\|\boldsymbol{J}\|_{\mathbb{R}^n}^2 mes(\mathcal{O}) + 10m\|\boldsymbol{T}\|_F^2\|\boldsymbol{g}(\mathbf{0})\|_{\mathbb{R}^m}^2\ mes(\mathcal{O}) + 10m\gamma_M^2K^2\|\boldsymbol{u}\|_{X_T^U}^2$, $k_8 = 10\,k_6k_2^2\|\boldsymbol{u}\|_{X_T^U}^2 + 10k_6\|\boldsymbol{Q}(\mathbf{0})\|_2^2$.

Hence $\|\Pi\|_{X_T^V} \leq a$, provided T is sufficiently small, such that

$$5\|\boldsymbol{\phi}\|_C^2 + k_4T^2 + k_5T < a. \tag{37}$$

From (29), (30), (36), (37), we obtain that $(\Xi, \Pi) \in \Sigma(a, T, U) \times \Sigma(a, T, V)$.

Let $(\boldsymbol{u_1}, \boldsymbol{v_1}), (\boldsymbol{u_2}, \boldsymbol{v_2}) \in \Sigma(a, T, U) \times \Sigma(a, T, V)$ and for fixed $(\boldsymbol{\phi}, \boldsymbol{\psi})$, we construct $\mathcal{T}_i(\boldsymbol{u_i}, \boldsymbol{v_i}) = (\Xi_i, \Pi_i)$, $i = 1, 2$, and $\boldsymbol{\varpi} = (\Xi_1, \Pi_1) - (\Xi_2, \Pi_2)$. For $t \in [-r, 0]$, we have $\boldsymbol{\varpi} = \mathbf{0}$, and

$$\Xi_1 - \Xi_2(t) = -\int_0^t S_1(t-s)\boldsymbol{C}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\mathrm{d}s$$

$$+\int_0^t S_1(t-s)W(f(\int_{-r}^0 \boldsymbol{v_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta)) -f(\int_{-r}^0 \boldsymbol{v_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta)))\mathrm{d}s + \int_0^t S_1(t-s)(P(v_1(s))-P(v_2(s)))\mathrm{d}B$$

for $t\in[0,T]$.

Due to the Jensen inequality, we get

$$\begin{aligned}&\|\Xi_1(t)-\Xi_2(t)\|^2_{X_T^U}\\ &\le 3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_1(t-s)(-C(u_1-u_2))\mathrm{d}s\|_U^2\\ &\quad+3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_1(t-s)C(f(\int_{-r}^0 \boldsymbol{v_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta))\\ &\quad-f(\int_{-r}^0 \boldsymbol{v_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta)))\mathrm{d}s\|_U^2+\\ &\quad 3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_1(t-s)(P(v_1(s))-P(v_2(s)))\mathrm{d}B\|_U^2\\ &\triangleq I_{11}+I_{12}+I_{13}.\end{aligned}$$

By employing Proposition 8, Lemma 2, and the Hölder inequality, it follows that

$$\begin{aligned}I_{11}&\le 3\sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|S_1(t-s)C(u_1-u_2)\|_U^2\mathrm{d}s\\ &\le 3T\sup_{t\in[0,T]}\mathrm{E}\int_0^t \|C(u_1-u_2)\|_U^2\mathrm{d}s\\ &\le 3Tm\|C\|_F^2\int_0^T \mathrm{E}\,\|u_1-u_2\|_U^2\mathrm{d}s\\ &\le 3T^2m\|C\|_F^2\|u_1-u_2\|^2_{X_T^U}.\end{aligned}$$

With the contraction of $S_1(t)$, the total boundedness of Stieljies integral, Hölder inequality, and Theorem 5, we deduce that

$$\begin{aligned}I_{12}&\le 3\sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|S_1(t-s)W(f(\int_{-r}^0 \boldsymbol{v_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta))\\ &\quad-f(\int_{-r}^0 \boldsymbol{v_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta)))\|_U^2\mathrm{d}s\\ &\le 3\sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|W(f(\int_{-r}^0 \boldsymbol{v_1}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s))\\ &\quad-f(\int_{-r}^0 \boldsymbol{v_2}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s)))\|_U^2\mathrm{d}s\\ &\le 3n\|W\|_F^2T\int_0^T \mathrm{E}\,\|(f(\int_{-r}^0 \boldsymbol{v_1}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s))\\ &\quad-f(\int_{-r}^0 \boldsymbol{v_2}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s)))\|_V^2\mathrm{d}s\\ &\le 3n\|W\|_F^2T^2\sigma_M^2K^2\sup_{t\in[0,T]}\mathrm{E}\,\|(v_1-v_2)_t\|^2_{C(V)}\\ &=3n\|W\|_F^2T^2\sigma_M^2K^2\|v_1-v_2\|^2_{X_T^V}.\end{aligned}$$

Following the same procedure as that in (28), we get

$$\begin{aligned}I_{13}&\le 3k_1^2k_3\sup_{t\in[0,T]}\int_0^t \mathrm{E}\,\|u_1(s)-u_2(s)\|_U^2\mathrm{d}s\\ &\le 3k_1^2k_3T\|u_1-u_2\|^2_{X_T^U}\end{aligned}\tag{38}$$

which means

$$\|\Xi_1(t)-\Xi_2(t)\|^2_{X_T^U}\le k_9\|u_1-u_2\|^2_{X_T^U}+k_{10}\|v_1-v_2\|^2_{X_T^V}\tag{39}$$

where $k_9=3T^2m\|C\|_F^2+3\,k_1^2k_3T$, $k_{10}=3n\|W\|_F^2T^2\sigma_M^2K^2$.

We also have

$$\begin{aligned}\Pi_1-\Pi_2(t)=&-\int_0^t S_2(t-s)D(v_1-v_2)\mathrm{d}s\\ &+\int_0^t S_2(t-s)T(g(\int_{-r}^0 \boldsymbol{u_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta))\\ &-g(\int_{-r}^0 \boldsymbol{u_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)))\mathrm{d}s\\ &+\int_0^t S_2(t-s)(Q(u_1(s))-Q(u_2(s)))\mathrm{d}B\end{aligned}$$

for $t\in[0,T]$.

Due to the Jensen inequality, we get

$$\begin{aligned}&\|\Pi_1-\Pi_2(t)\|^2_{X_T^V}\\ &\le 3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_2(t-s)(-D(v_1-v_2))\mathrm{d}s\|_V^2\\ &\quad+3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_2(t-s)T(g(\int_{-r}^0 \boldsymbol{u_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta))\\ &\quad-g(\int_{-r}^0 \boldsymbol{u_2}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s)))\mathrm{d}s\|_V^2\\ &\quad+3\sup_{t\in[0,T]}\mathrm{E}\|\int_0^t S_2(t-s)(Q(u_1)-Q(u_2))\mathrm{d}B\|_V^2\\ &\triangleq I_{14}+I_{15}+I_{16}.\end{aligned}$$

By employing Proposition 2.1, Lemma 2, and the Hölder inequality, it follows that

$$\begin{aligned}I_{14}&\le 3\sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|S_2(t-s)D(v_1-v_2)\|_V^2\mathrm{d}s\\ &\le 3T\sup_{t\in[0,T]}\mathrm{E}\int_0^t \|D(v_1-v_2)\|_V^2\mathrm{d}s\\ &\le 3Tn\|D\|_F^2\int_0^T \mathrm{E}\,\|v_1-v_2\|_V^2\mathrm{d}s\\ &\le 3T^2n\|D\|_F^2\|v_1-v_2\|^2_{X_T^V}.\end{aligned}$$

With the contraction of $S_2(t)$, total boundedness of Stieljies integral, Theorem 5, and the Hölder inequality, we get

$$\begin{aligned}&I_{15}\\ &\le 3\sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|S_2(t-s)T(g(\int_{-r}^0 \boldsymbol{u_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta))\\ &\quad-g(\int_{-r}^0 \boldsymbol{u_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)))\|_U^2\mathrm{d}s\end{aligned}$$

$$\leq 3 \sup_{t\in[0,T]} t\,\mathrm{E}\int_0^t \|\boldsymbol{T}(\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)) - \boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u_2}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)))\|_U^2 \mathrm{d}s$$
$$\leq 3n\|\boldsymbol{T}\|_F^2 T\int_0^T \mathrm{E}\,\|(\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u_1}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)) - \boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u_2}(t\,\mathcal{C}\,s)\mathrm{d}\eta_u(s)))\|_U^2 \mathrm{d}s$$
$$\leq 3n\|\boldsymbol{T}\|_F^2 T^2\gamma_M^2 K^2 \sup_{t\in[0,T]} \mathrm{E}\,\|(\boldsymbol{u}_1-\boldsymbol{u}_2)_t\|_{C(U)}^2$$
$$= 3n\|\boldsymbol{T}\|_F^2 T^2\gamma_M^2 K^2 \|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{X_T^U}^2.$$

Following the same procedure as that in (28), we get

$$I_{16} \leq 3k_2^2k_3 \sup_{t\in[0,T]}\int_0^t \mathrm{E}\,\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_U^2\mathrm{d}s \leq 3k_2^2k_3T\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{X_T^U}^2 \tag{40}$$

which means

$$\|\Pi_1-\Pi_2(t)\|_{X_T^V}^2 \leq k_{11}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{X_T^U}^2 + k_{12}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{X_T^V}^2 \tag{41}$$

where $k_{11} = 3T^2n\|\boldsymbol{T}\|_F^2 + 3\,k_2^2k_3T$, $k_{12} = 3m\|\boldsymbol{T}\|_F^2T^2\gamma_M^2K^2$. From (39) and (41), we have

$$\|\Xi_1(t)-\Xi_2(t)\|_{X_T^U}^2 + \|\Pi_1(t)-\Pi_2(t)\|_{X_T^V}^2 \leq k_{13}(\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{X_T^U}^2 + \|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{X_T^V}^2) \tag{42}$$

where $k_{13} = \max\{k_9+k_{11}, k_{10}+k_{12}\}$. We take T sufficiently small, such that

$$k_{13} < 1. \tag{43}$$

So $\mathcal{T}$ is a strict contraction map on $\Sigma(a, T, U)\times\Sigma(a, T, V)$, which has a unique fixed point $(\boldsymbol{u}, \boldsymbol{v})$. This completes the proof of Theorem 9. ■

Remark 10: Although we use a more abstract method, which is different from that in previous paper, we find that global Lipschitz condition of activation function and positiveness of diffusion coefficient can ensure the existence and uniqueness of mild solution of the RDBAMNNs with S-delays driven by infinite dimensional Wiener processes. That means our result is credible and our method is believable. The boundedness restriction for activation function $\boldsymbol{f}$ and $\boldsymbol{g}$ is waived in this paper when compared with the results in Zhu and Cao [4] and Song *et al.* [19], let alone our model is much more complex than those proposed by them. That means our paper makes further improvement on the result of previous scholars in some sense.

Theorem 11: We suppose the conditions H1 and H2 hold, then the mild solution of (1) satisfies the following priori estimate

$$\mathrm{E}(\|\boldsymbol{u}\|_U^2 + \|\boldsymbol{v}\|_V^2) \leq c_7,\ t\in[0, T]$$

where $c_7 > 0$ is a constant, which depends on T.

Proof: Let $\boldsymbol{u}$ be the mild solution of (1), we define the Lyapunov-Krasovskii function as $V(t) = e^{-\lambda t}(\|\boldsymbol{u}\|_U^2 + \|\boldsymbol{v}\|_V^2)$, where λ is a sufficient large positive constant, which will be determined later. According to the Itô formula, we have

$$\mathrm{d}V = \mathcal{L}V\mathrm{d}t + 2e^{-\lambda t}(\boldsymbol{u}, \boldsymbol{P}\mathrm{d}\boldsymbol{B})_U + 2e^{-\lambda t}(\boldsymbol{v}, \boldsymbol{Q}\mathrm{d}\boldsymbol{B})_V \tag{44}$$

where the generator $\mathcal{L}V$ for the evolution of V is given by

$$\begin{aligned}\mathcal{L}V &= -\lambda e^{-\lambda t}(\|\boldsymbol{u}\|_U^2 + \|\boldsymbol{v}\|_V^2)\\ &\quad +2e^{\lambda t}((\boldsymbol{u}, \mathfrak{A}_1\boldsymbol{u})_U + (\boldsymbol{v}, \mathfrak{A}_2\boldsymbol{v}))_V\\ &\quad -2e^{-\lambda t}(\boldsymbol{u}, \boldsymbol{C}\boldsymbol{u})_U - 2e^{-\lambda t}(\boldsymbol{v}, \boldsymbol{D}\boldsymbol{v})_V\\ &\quad +2e^{-\lambda t}(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0}\boldsymbol{v}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s)))_V\\ &\quad +2e^{-\lambda t}(\boldsymbol{v}, \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0}\boldsymbol{u}(t\,\mathcal{C}\,s)\mathrm{d}\eta_u(s)))_U\\ &\quad +2e^{-\lambda t}(\boldsymbol{u}, \boldsymbol{I})_U + 2e^{-\lambda t}(\boldsymbol{v}, \boldsymbol{J})_V\\ &\quad +e^{-\lambda t}\mathrm{tr}(\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^\star) + e^{\lambda t}\mathrm{tr}(\boldsymbol{Q}\boldsymbol{Q}\boldsymbol{Q}^\star).\end{aligned}$$

Integrating on both sides of (44) between 0 and t and taking expectation on it, then using stochastic Fubini's theorem and the formula $2\,\mathrm{E}\int_0^t e^{-\lambda s}(\boldsymbol{u}, \boldsymbol{P}\mathrm{d}\boldsymbol{B})_U = 0$, $2\,\mathrm{E}\int_0^t e^{-\lambda s}(\boldsymbol{v}, \boldsymbol{Q}\mathrm{d}\boldsymbol{B})_V = 0$, we have

$$\begin{aligned}\mathrm{E}\,V(t) &= \mathrm{E}\,\|\boldsymbol{\phi}(0)\|_U^2 + \mathrm{E}\,\|\boldsymbol{\psi}(0)\|_V^2 - \lambda\int_0^t \mathrm{E}\,V(s)\mathrm{d}s\\ &\quad +2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{u}, \mathfrak{A}_1\boldsymbol{u})_U\mathrm{d}s + 2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{v}, \mathfrak{A}_2\boldsymbol{v})_V\mathrm{d}s\\ &\quad -2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{C}\boldsymbol{u}, \boldsymbol{u})_U\mathrm{d}s - 2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{D}\boldsymbol{v}, \boldsymbol{v})_V\mathrm{d}s\\ &\quad +2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0}\boldsymbol{v}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_v(\theta)))_U\mathrm{d}s\\ &\quad +2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{v}, \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0}\boldsymbol{u}(s\,\mathcal{C}\,\theta)\mathrm{d}\eta_u(\theta)))_V\mathrm{d}s\\ &\quad +2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{u}, \boldsymbol{I})_U\mathrm{d}s + 2\int_0^t e^{-\lambda s}\,\mathrm{E}(\boldsymbol{v}, \boldsymbol{J})_V\mathrm{d}s\\ &\quad +\int_0^t e^{-\lambda s}\,\mathrm{E}\,\|\boldsymbol{P}(\boldsymbol{v}(s))\|_2^2\mathrm{d}s + \int_0^t e^{-\lambda s}\,\mathrm{E}\,\|\boldsymbol{Q}(\boldsymbol{u}(s))\|_2^2\mathrm{d}s.\end{aligned}$$

According to the theory of functional differential equation, which is equivalent to the system

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{E}\,V(t)}{dt} &= -\lambda e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}(t)\|_U^2 - \lambda e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}(t)\|_V^2\\ &\quad +2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{u}, \mathfrak{A}_1\boldsymbol{u})_U + 2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{v}, \mathfrak{A}_2\boldsymbol{v})_V\\ &\quad -2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{u}, \boldsymbol{C}\boldsymbol{u})_U - 2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{v}, \boldsymbol{D}\boldsymbol{v})_V\\ &\quad +2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{u}, \boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0}\boldsymbol{v}(t\,\mathcal{C}\,s)\mathrm{d}\eta_v(s)))_U\\ &\quad +2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{v}, \boldsymbol{W}\boldsymbol{g}(\int_{-r}^{0}\boldsymbol{u}(t\,\mathcal{C}\,s)\mathrm{d}\eta_u(s)))_V\\ &\quad +2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{u}, \boldsymbol{I})_U + 2e^{-\lambda t}\,\mathrm{E}(\boldsymbol{v}, \boldsymbol{J})_V\end{aligned}$$

$$
\begin{aligned}
&+e^{-\lambda t}\,\mathrm{E}\,\mathrm{tr}(\boldsymbol{PQP}^{\star})\\
&+e^{-\lambda t}\,\mathrm{E}\,\mathrm{tr}(\boldsymbol{QQQ}^{\star})\\
\triangleq\ & l_1+l_2+l_3+l_4+l_5+l_6\\
&+l_7+l_8+l_9+l_{10}+l_{11}+l_{12} \qquad (45)
\end{aligned}
$$

with the initial data

$$\mathrm{E}\,V(0)=\mathrm{E}\,\|\boldsymbol{\phi}(0)\|_U^2+\mathrm{E}\,\|\boldsymbol{\psi}(0)\|_V^2.$$

By using the adiabatic boundary condition, poincaré formula (Lemma 7), Theorem 6 and H2, it follows that

$$
\begin{aligned}
l_3 &\le -2\alpha e^{-\lambda t}\,\mathrm{E}\,\||\boldsymbol{u}\||_U^2\\
&\le -2\alpha L^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2. \qquad (46)
\end{aligned}
$$

Following the same procedure, we have

$$l_4 \le -2\alpha L^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2. \qquad (47)$$

Then, taking into account the positiveness of c_i and d_j, $i=1,2,\cdots,m, j=1,2,\cdots,n$, we get

$$
\begin{aligned}
l_5 &\le -2k_{14}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2 \qquad (48)\\
l_6 &\le -2k_{15}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2 \qquad (49)
\end{aligned}
$$

where $k_{14}=min\{c_1,c_2,\cdots,c_m\}>0$, $k_{15}=min\{d_1,d_2,\cdots,d_n\}>0$.

By using the Young inequality, Lemma 2, and condition H1, we have

$$
\begin{aligned}
l_7 \le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&+\frac{1}{k_{17}}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{Wf}(\int_{-r}^{0}\boldsymbol{v}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_v(s))\|_U^2\\
\le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&+n\frac{1}{k_{17}}\|\boldsymbol{W}\|_F^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{f}(\int_{-r}^{0}\boldsymbol{v}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_v(s))\|_V^2\\
\le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&+\frac{2n}{k_{17}}\|\boldsymbol{W}\|_F^2\sigma_M^2 e^{-\lambda t}\|\int_{-r}^{0}\boldsymbol{v}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_v(s)\|_V^2\\
&+\frac{2}{k_{17}}ne^{-\lambda t}\|\boldsymbol{W}\|_F^2\|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O})\\
\le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&+\frac{2n}{k_{17}}\|\boldsymbol{W}\|_F^2\sigma_M^2\ K^2 e^{-\lambda t}\sup_{-r\le s\le 0}\ \mathrm{E}\,\|\boldsymbol{v}(t+s)\|_V^2\\
&+\frac{n}{k_{17}}\|\boldsymbol{W}\|_F^2\|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O}) \qquad (50)
\end{aligned}
$$

where $k_{17}=k_{16}+\alpha^2 L$, $k_{16}=k_{14}\wedge k_{15}$.

Following the same procedure, we often get

$$
\begin{aligned}
l_8 \le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2\\
&+\frac{1}{k_{17}}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{Tg}(\int_{-r}^{0}\boldsymbol{u}(t\ \mathcal{C}\ s)\mathrm{d}\boldsymbol{\eta}_u(s))\|_V^2\\
\le\ & k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2\\
&+\frac{2m}{k_{17}}\|\boldsymbol{T}\|_F^2\gamma_M^2\ K^2 e^{-\lambda t}\sup_{-r\le s\le 0}\ \mathrm{E}\,\|\boldsymbol{u}(t+s)\|_U^2\\
&+\frac{2m}{k_{17}}\|\boldsymbol{T}\|_F^2\|\boldsymbol{g}(\boldsymbol{0})\|_{\mathbb{R}^m}^2 mes(\mathcal{O}). \qquad (51)
\end{aligned}
$$

By using the Cauchy inequality, we obtain

$$
\begin{aligned}
l_9 &\le \frac{1}{k_{17}}e^{-\lambda t}\|\boldsymbol{I}\|_U^2+k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&\le \frac{1}{k_{17}}\|\boldsymbol{I}\|_{\mathbb{R}^m}^2 mes(\mathcal{O})+k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2 \qquad (52)
\end{aligned}
$$

following the procedure of (52), we have

$$
\begin{aligned}
l_{10} &\le \frac{1}{k_{17}}e^{-\lambda t}\|\boldsymbol{J}\|_V^2+k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2\\
&\le \frac{1}{k_{17}}\|\boldsymbol{J}\|_{\mathbb{R}^n}^2 mes(\mathcal{O})+k_{17}e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2. \qquad (53)
\end{aligned}
$$

Utilizing the Young inequality and condition H1, we obtain

$$
\begin{aligned}
l_{11} &\le \frac{1}{4k_{17}}e^{-\lambda t}\|\boldsymbol{P}(\boldsymbol{0})\|_2^2+k_{17}k_1^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2\\
&\le \frac{1}{4k_{17}}\|\boldsymbol{P}(\boldsymbol{0})\|_2^2+k_{17}k_1^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{v}\|_V^2. \qquad (54)\\
l_{12} &\le \frac{1}{4k_{17}}e^{-\lambda t}\|\boldsymbol{Q}(\boldsymbol{0})\|_2^2+k_{17}k_2^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2\\
&\le \frac{1}{4k_{17}}\|\boldsymbol{Q}(\boldsymbol{0})\|_2^2+k_{17}k_2^2 e^{-\lambda t}\,\mathrm{E}\,\|\boldsymbol{u}\|_U^2. \qquad (55)
\end{aligned}
$$

We can infer from (45) - (55) that

$$\frac{\mathrm{d}\,\mathrm{E}\,V(t)}{dt}\le -c_1\,\mathrm{E}\,V(t)+c_2\sup_{-r\le s\le 0}\ \mathrm{E}\,V(t+s)+c_3$$

where $c_1=\lambda-k_{17}(k_1^2\vee k_2^2)$, $c_2=\max\{\frac{2n}{k_{17}}\|\boldsymbol{W}\|_F^2\sigma_M^2\ K^2, \frac{2m}{k_{17}}\|\boldsymbol{T}\|_F^2\gamma_M^2\ K^2\}$, $c_3=\frac{1}{4k_{17}}\|\boldsymbol{P}(\boldsymbol{0})\|_2^2+\frac{1}{4k_{17}}\|\boldsymbol{Q}(\boldsymbol{0})\|_2^2+\frac{2m}{k_{17}}\|\boldsymbol{T}\|_F^2\|\boldsymbol{g}(\boldsymbol{0})\|_{\mathbb{R}^m}^2 mes(\mathcal{O})+\frac{2n}{k_{17}}\|\boldsymbol{W}\|_F^2\|\boldsymbol{f}(\boldsymbol{0})\|_{\mathbb{R}^n}^2 mes(\mathcal{O})+\frac{1}{k_{17}}\|\boldsymbol{I}\|_{\mathbb{R}^m}^2+\frac{1}{k_{17}}\|\boldsymbol{J}\|_{\mathbb{R}^n}^2$. That means we can choose a sufficiently large positive number λ such that $\lambda-k_{17}>0$ and $c_1-c_2>0$. By utilizing the generalized Hanalay inequality [20], there exist some positive constants c_4, c_5, c_6, such that

$$\mathrm{E}\,V(t)\le c_4 e^{-c_5\,t}+c_6\le c_4+c_6, t\in[0,T]$$

in other words

$$\mathrm{E}(\|\boldsymbol{u}\|_U^2+\|\boldsymbol{v}\|_V^2)\le(c_4+c_6)e^{\lambda t}\le(c_4+c_6)e^{\lambda T}, t\in[0,T]$$

we can conclude that, there is a constant $c_7=(c_4+c_6)e^{\lambda T}$ such that

$$\mathrm{E}\,\|\boldsymbol{u}\|_U^2+\mathrm{E}\,\|\boldsymbol{v}\|_V^2\le c_7, t\in[0,T].$$

The proof is finished. ■

IV. GLOBAL ASYMPTOTIC STABILITY OF EQUILIBRIUM

Definition 12: A fixed point $(\boldsymbol{u}^*,\boldsymbol{v}^*)=(u_1^*,\ldots,u_m^*, v_1^*,\ldots,v_n^*)$ is called the equilibrium point of system (1) if it satisfies

$$
\begin{cases}
-\boldsymbol{Cu}^*+\boldsymbol{Wf}(\boldsymbol{K}_v\boldsymbol{v}^*)+\boldsymbol{I}=\boldsymbol{0}\\
-\boldsymbol{Dv}^*+\boldsymbol{Tg}(\boldsymbol{K}_u\boldsymbol{u}^*)+\boldsymbol{J}=\boldsymbol{0}.
\end{cases} \qquad (56)
$$

Theorem 13: If the system (1) satisfies conditions H1 and $k_{16}-\|\boldsymbol{K}_w\|_F\varrho>0$, where $\varrho=\sigma_M\vee\gamma_M$ and $\boldsymbol{K}_w$ will be defined later, then system (1) has an equilibrium point.

Proof: Let us construct the following block matrices $\boldsymbol{w} = \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}$, $\boldsymbol{R} = \begin{pmatrix} \boldsymbol{C} & 0 \\ 0 & \boldsymbol{D} \end{pmatrix} = diag\{c_1, c_2, \cdots, c_m, d_1, d_2, \cdots, d_n\}$, $Z = \begin{pmatrix} \boldsymbol{W} & 0 \\ \boldsymbol{o} & \boldsymbol{T} \end{pmatrix}$, $\boldsymbol{h} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{pmatrix}$, $\boldsymbol{K} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{J} \end{pmatrix}$, $\boldsymbol{K_w} = \begin{pmatrix} \boldsymbol{K_u} & 0 \\ 0 & \boldsymbol{K_v} \end{pmatrix}$, that means $\boldsymbol{h}(\boldsymbol{K_w w}) = diag\{f_1(K_1^v v_1), \cdots, f_n(K_n^v v_n), g_1(K_1^u u_1), \ldots, g_m(K_m^u u_m)\}$. With these notations, (56) is equivalent to

$$-\boldsymbol{R}\boldsymbol{w}^* + \boldsymbol{Z}\boldsymbol{h}(\boldsymbol{K_w}\boldsymbol{w}^*) + \boldsymbol{K} = \boldsymbol{0}. \tag{57}$$

Let us define

$$\mathfrak{h}(\boldsymbol{w}) = \boldsymbol{R}\boldsymbol{w} - \boldsymbol{Z}\boldsymbol{h}(\boldsymbol{K_w}\boldsymbol{w}) - \boldsymbol{K}. \tag{58}$$

It is obvious that the solutions to $\mathfrak{h}(\boldsymbol{w}) = 0$ are the equilibrium points of system (1). Let us define homotopic mapping as follows:

$$\mathfrak{H}(\boldsymbol{w}, \lambda) = \lambda\mathfrak{h}(\boldsymbol{K_w w}) + (1-\lambda)\boldsymbol{w}, \lambda \in [0, 1] \tag{59}$$

so we get

$$\begin{aligned} \|\mathfrak{H}\| &\geq \lambda\|\boldsymbol{R}\boldsymbol{w}\| - \lambda\|\boldsymbol{Z}\boldsymbol{h}(\boldsymbol{K_w w})\| - \lambda\|\boldsymbol{K}\| + (1-\lambda)\|\boldsymbol{w}\| \\ &\geq (\lambda k_{16} + 1 - \lambda)\|\boldsymbol{w}\| - \lambda\|\boldsymbol{Z}\|_F\|\boldsymbol{h}(\boldsymbol{K_w w})\| - \lambda\|\boldsymbol{K}\| \\ &\geq (\lambda k_{16} + 1 - \lambda)\|\boldsymbol{w}\| \\ &\quad - \lambda\|\boldsymbol{K_w}\|_F\varrho\|\boldsymbol{w}\| - \lambda(\|\boldsymbol{Z}\|_F\|\boldsymbol{h}(\boldsymbol{0})\| + \|\boldsymbol{K}\|) \end{aligned} \tag{60}$$

where $\varrho = \sigma_M \vee \gamma_M$. We must point out in the proof of this theorem, $\|\cdot\|$ means the product norm of $U \times V$.

By the assumption in this theorem, there exists a positive constant r such that $(k_{16} - \lambda\|\boldsymbol{K_w}\|_F\varrho)r > 0$ and $(k_{16} - \lambda\|\boldsymbol{K_w}\|_F\varrho)^{-1} > 0$.

Now, let us define

$$\mathfrak{U}(R_0) = \{\boldsymbol{w} | \|\boldsymbol{w}\| \leq R_0\} \tag{61}$$

where $R_0 = \frac{(\|\boldsymbol{Z}\|_F\|\boldsymbol{h}(\boldsymbol{0})\| + \|\boldsymbol{K}\|) + r}{k_{16} - \lambda\|\boldsymbol{K_w}\|_F}$. If $\boldsymbol{w} \in \partial\mathfrak{U}(R_0)$, then

$$\begin{aligned} \|\mathfrak{H}\| &\geq (1-\lambda)\|\boldsymbol{w}\| + \lambda(k_{16} - \|\boldsymbol{K_w}\|_F\varrho)\|\boldsymbol{w}\| \\ &\quad -\lambda(\|\boldsymbol{Z}\|_F\|\boldsymbol{h}(\boldsymbol{0})\| + \|\boldsymbol{K}\|) \\ &\geq (1-\lambda)\|\boldsymbol{w}\| + \lambda(k_{16} - \|\boldsymbol{K_w}\|_F\varrho)^{-1}\|\boldsymbol{w}\|r \\ &> 0 \end{aligned} \tag{62}$$

that means

$$\mathfrak{H}(\boldsymbol{w}, \lambda) \neq 0, \quad \forall \boldsymbol{w} \in \partial\mathfrak{U}(R_0), \ \lambda \in [0, 1]. \tag{63}$$

By homotopy invariance theorem, we have

$$\begin{aligned} deg(\mathfrak{h}(\boldsymbol{w}), \mathfrak{U}(R_0), 0) &= deg(\mathfrak{H}(\boldsymbol{w}, \lambda), \mathfrak{U}(R_0), 0) \\ &= deg(\mathfrak{H}(\boldsymbol{w}, 0), \mathfrak{U}(R_0), 0) = 1 \end{aligned} \tag{64}$$

where *deg* denotes the topological degree. By using the topological degree theory (see [10], [24]–[27]), we can conclude that (57) has a solution. In other words, system (1) has an equilibrium point. ∎

Remark 14: Our results are slightly different from previous results given by Song *et al.* s [19], the difference between us and him is that we study different models and use different tools However, our criteria are also easy to check in the computer.

Definition 15: The equilibrium point $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ of system (1) is said to be exponentially stable in the mean square sense, if there exists $\kappa_1 > 0$ and $\kappa_2 > 0$, such that any mild solution $(\boldsymbol{u}, \boldsymbol{v})$ of (1) satisfies

$$\mathrm{E}\,\|\boldsymbol{u} - \boldsymbol{u}^*\|_U^2 \vee \mathrm{E}\,\|\boldsymbol{v} - \boldsymbol{v}^*\|_V^2 \leq \kappa_1 e^{-\kappa_2(t)}, \quad t \geq 0 \tag{65}$$

where $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in BC([-r, 0], U) \times BC([-r, 0], V)$.

In this paper, we assume $\boldsymbol{P}(\boldsymbol{v}^*) = \boldsymbol{0}, \boldsymbol{Q}(\boldsymbol{u}^*) = \boldsymbol{0}$.

Let $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ be an equilibrium point of system (1), we can get the following equation by defining $\boldsymbol{\chi} = \boldsymbol{u} - \boldsymbol{u}^*$ and $\boldsymbol{\xi} = \boldsymbol{v} - \boldsymbol{v}^*$

$$\begin{cases} \mathrm{d}\boldsymbol{\chi} = (\mathfrak{A}_1\boldsymbol{\chi} - \boldsymbol{C}\boldsymbol{\chi} \\ +\boldsymbol{W}\boldsymbol{f}(\int_{-r}^{0} \boldsymbol{v}(t\,\mathcal{C}\,s)\mathrm{d}\boldsymbol{\eta}_v(s)) - \boldsymbol{W}\boldsymbol{f}(\boldsymbol{K_v}\boldsymbol{v}^*))\mathrm{d}t \\ +\boldsymbol{P}(\boldsymbol{v})\mathrm{d}\boldsymbol{B} \\ \mathrm{d}\boldsymbol{\xi} = (\mathfrak{A}_2\boldsymbol{\xi} - \boldsymbol{D}\boldsymbol{\xi} \\ +\boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(t\,\mathcal{C}\,s)\mathrm{d}\boldsymbol{\eta}_u(s)) - \boldsymbol{T}\boldsymbol{g}(\boldsymbol{K_u}\boldsymbol{u}^*))\mathrm{d}t \\ +\boldsymbol{Q}(\boldsymbol{u})\mathrm{d}\boldsymbol{B}, \quad t > 0 \\ \boldsymbol{\chi}(t) = \boldsymbol{\phi}(t) - \boldsymbol{u}^*, \boldsymbol{\xi}(t) = \boldsymbol{\psi}(t) - \boldsymbol{v}^*, t \in [-r, 0]. \end{cases} \tag{66}$$

Theorem 16: If the system (1) satisfies conditions H1-H3, then the equilibrium of system (1) is exponentially stable in the mean square sense.

Proof: Let $(\boldsymbol{\chi}, \boldsymbol{\xi})$ be the mild solution of (66), we define the vector Lyapunov-Krasovskii functional as:

$$\boldsymbol{V}(t) = (V_1, V_2)^T \triangleq (\|\boldsymbol{\chi}(t)\|_U^2, \|\boldsymbol{\xi}(t)\|_V^2)^{\mathrm{T}}. \tag{67}$$

By using the Itô formula and the isometry formula $\mathrm{E}\int_0^t(\boldsymbol{\xi}, \boldsymbol{P}\mathrm{d}\boldsymbol{B})_U = 0$, $\mathrm{E}\int_0^t(\boldsymbol{\chi}, \boldsymbol{Q}\mathrm{d}\boldsymbol{B})_V = 0$, we find that

$$\begin{cases} \mathrm{E}\,V_1 = \mathrm{E}\,V_1(0) + \int_0^t \mathrm{E}\,\mathcal{L}V_1(s)\mathrm{d}s \\ \mathrm{E}\,V_2 = \mathrm{E}\,V_2(0) + \int_0^t \mathrm{E}\,\mathcal{L}V_2(s)\mathrm{d}s. \end{cases} \tag{68}$$

With the hypothesis H1, H2, H4, H5, and (34), which means

$$\begin{cases} \frac{\mathrm{d}\,\mathrm{E}\,V_1(t)}{\mathrm{d}} = \mathrm{E}\,\mathcal{L}V_1(t) \\ \frac{\mathrm{d}\,\mathrm{E}\,V_2(t)}{\mathrm{d}} = \mathrm{E}\,\mathcal{L}V_2(t) \\ \mathrm{E}\,V_1(0) = \mathrm{E}\,\|\boldsymbol{\chi}(0)\|_U^2, \mathrm{E}\,V_2(0) = \mathrm{E}\,\|\boldsymbol{\xi}(0)\|_V^2. \end{cases} \tag{69}$$

The infinite generators $\mathcal{L}_1\ V_1$ and $\mathcal{L}_2\ V_2$ are given by

$$\begin{cases} \mathcal{L}V_1 = 2(\boldsymbol{\chi}, \mathfrak{A}_1\boldsymbol{\chi})_U - 2(\boldsymbol{\chi}, \boldsymbol{C}\boldsymbol{\chi})_U + \mathrm{tr}(\boldsymbol{P}\boldsymbol{Q}\boldsymbol{P}^\star) \\ +2(\boldsymbol{\chi}, \boldsymbol{W}(\boldsymbol{f}(\int_{-r}^{0} \boldsymbol{v}(t\,\mathcal{C}\,s)\mathrm{d}\boldsymbol{\eta}_v(s)) - \boldsymbol{f}(K_v\boldsymbol{v}^*))_U \\ \mathcal{L}V_2 = 2(\boldsymbol{\xi}, \mathfrak{A}_2\boldsymbol{\xi})_V - 2(\boldsymbol{\xi}, \boldsymbol{D}\boldsymbol{\xi})_V + \mathrm{tr}(\boldsymbol{Q}\boldsymbol{Q}\boldsymbol{Q}^\star) \\ +2(\boldsymbol{\xi}, \boldsymbol{T}\boldsymbol{g}(\int_{-r}^{0} \boldsymbol{u}(t\,\mathcal{C}\,s, x)\mathrm{d}\boldsymbol{\eta}_u(s)) - \boldsymbol{T}\boldsymbol{g}(K_u\boldsymbol{u}^*))_V. \end{cases} \tag{70}$$

By the positiveness of $\boldsymbol{C}, \boldsymbol{D}$, we have

$$-(\boldsymbol{\chi}, \boldsymbol{C}\boldsymbol{\chi})_U \leq -k_{14}\|\boldsymbol{\chi}\|_U^2 \tag{71}$$

$$-(\boldsymbol{\xi}, \boldsymbol{D}\boldsymbol{\xi})_V \leq -k_{15}\|\boldsymbol{\xi}\|_V^2. \tag{72}$$

By utilizing adiabatic boundary condition, poincaré formula (Lemma 7), Theorem 6 and H2, it follows that

$$(\boldsymbol{\chi}, \mathfrak{A}_1\boldsymbol{\chi})_U \leq -\alpha\|\|\boldsymbol{\chi}\|\|_U^2 \leq -\alpha L^2\|\boldsymbol{\chi}\|_U^2 \tag{73}$$

$$(\boldsymbol{\xi}, \mathfrak{A}_2\boldsymbol{\xi})_V \leq -\alpha\|\|\boldsymbol{\xi}\|\|_V^2 \leq -\alpha L^2\|\boldsymbol{\xi}\|_V^2. \tag{74}$$

With H1 and the Cauchy inequality, it follows that

$$\begin{aligned}
&2(\boldsymbol{\chi}, \boldsymbol{Wf}(\int_{-r}^{0} \boldsymbol{v}(t\,\mathcal{C}\,s, x)\mathbf{d}\boldsymbol{\eta_v}(s)) - \boldsymbol{Wf}(\boldsymbol{v}^*))_U \\
&\quad \leq \|\boldsymbol{\chi}\|^2 + \|\boldsymbol{Wf}(\int_{-r}^{0} \boldsymbol{v}(t\,\mathcal{C}\,s, x)\mathbf{d}\boldsymbol{\eta_v}) - \boldsymbol{Wf}(\boldsymbol{v}^*)\|_U^2 \\
&\quad \leq \|\boldsymbol{\chi}\|_U^2 + n\|\boldsymbol{W}\|_F^2\sigma_M^2 K^2 \sup_{-\tau\leq s\leq 0} \|\boldsymbol{\xi}(t+s)\|_V^2 \qquad (75)
\end{aligned}$$

and

$$\begin{aligned}
&2(\boldsymbol{\xi}, \boldsymbol{Tg}(\int_{-r}^{0} \boldsymbol{u}(t\,\mathcal{C}\,s, x)\mathbf{d}\boldsymbol{\eta_u}(s)) - \boldsymbol{Tg}(\boldsymbol{u}^*))_V \\
&\quad \leq \|\boldsymbol{\chi}\|_V^2 + \|\boldsymbol{Tg}(\int_{-r}^{0} \boldsymbol{u}(t\,\mathcal{C}\,s, x)\mathbf{d}\boldsymbol{\eta_u}(s)) - \boldsymbol{Tg}(\boldsymbol{u}^*)\|_V^2 \\
&\quad \leq \|\boldsymbol{\xi}\|_V^2 + m\|\boldsymbol{T}\|_F^2\gamma_M^2 K^2 \sup_{-\tau\leq s\leq 0} \|\boldsymbol{\chi}(t+s)\|_U^2. \qquad (76)
\end{aligned}$$

By utilizing condition H1, H4 and definition of $\|\cdot\|_2$, we have

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{PQP}^\star) &= \|\boldsymbol{P}(\boldsymbol{v}(t))\|_2^2 \\
&= \|\boldsymbol{P}(\boldsymbol{v}(t)) - \boldsymbol{P}(\boldsymbol{v}^*)\|_2^2 \leq k_1^2\|\boldsymbol{\xi}(t)\|_U^2 \qquad (77)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{QQQ}^\star) &= \|\boldsymbol{Q}(\boldsymbol{u}(t))\|_2^2 \\
&= \|\boldsymbol{Q}(\boldsymbol{u}(t)) - \boldsymbol{Q}(\boldsymbol{u}^*)\|_2^2 \leq k_2^2\|\boldsymbol{\chi}(t)\|_V^2. \qquad (78)
\end{aligned}$$

By virtue of (70)-(78), one obtains

$$\begin{cases}
\mathcal{L}V_1 \leq (-2k_{14} - 2\alpha L^2 + k_2^2 + 1)V_1 \\
+n\|\boldsymbol{W}\|_F^2\sigma_M^2 K^2 \sup_{-r\leq s\leq 0} V_2(t+s) \\
\mathcal{L}V_2 \leq (-2k_{15} - 2\alpha L^2 + k_1^2 + 1)V_2 \\
+m\|\boldsymbol{T}\|_F^2\gamma_M^2 K^2 \sup_{-r\leq s\leq 0} V_1(t+s)
\end{cases} \qquad (79)$$

from (68) and (79), we assert

$$\begin{aligned}
\frac{\mathrm{d}\,\mathrm{E}\,V_1(t)}{\mathrm{d}t} &\leq -(2k_{14} + 2\alpha L^2 - k_2^2 - 1)\,\mathrm{E}\,V_1(t) \\
&\quad + n\|\boldsymbol{W}\|_F^2\sigma_M^2 K^2 \sup_{-r\leq s\leq 0} \mathrm{E}\,V_2(t+s) \\
\frac{\mathrm{d}\,\mathrm{E}\,V_2(t)}{\mathrm{d}t} &\leq -(2k_{15} + 2\alpha L^2 - k_1^2 - 1)\,\mathrm{E}\,V_2(t) \\
&\quad + m\|\boldsymbol{T}\|_F^2\gamma_M^2 K^2 \sup_{-r\leq s\leq 0} \mathrm{E}\,V_2(t+s) \qquad (80)
\end{aligned}$$

with the Lemma 1 and H3, there is a positive vector $\boldsymbol{k} = (\bar{k}, \bar{k})^{\mathrm{T}}$ and a positive constant κ such that

$$\mathrm{E}\,\boldsymbol{V} \leq \boldsymbol{k}e^{-\kappa t}$$

in other words

$$\mathrm{E}\,\|\boldsymbol{u} - \boldsymbol{u}^*\|_U^2 \vee \mathrm{E}\,\|\boldsymbol{v} - \boldsymbol{v}^*\|_V^2 \leq \bar{k}e^{-\kappa t}, \quad t \geq 0.$$

So the equilibrium point of system (1) is exponentially stable in the mean square sense. ■

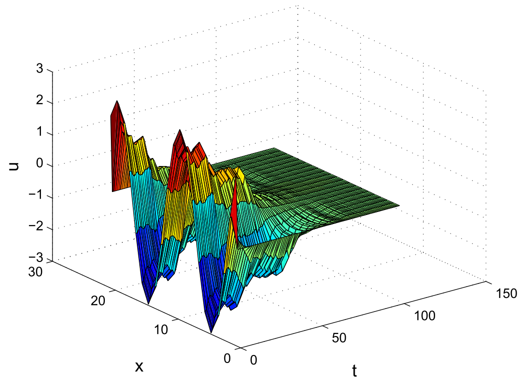

FIGURE 1. Surface of u of Example 17.

V. AN EXAMPLE

Example 17:

$$\begin{cases}
du = (\nabla\cdot(D^u(x)\nabla u - Cu \\
+Wf(\int_{-r}^{0} v(t+s,x)d\eta_v(s)) + I)\mathrm{d}t + P(v)\mathrm{d}B \\
\mathrm{d}v = (\nabla\cdot(D^v(x)\nabla v - Dv \\
+Tg(\int_{-r}^{0} u(t+s,x)d\eta_u(s)) + J)\mathrm{d}t + Q(u)\mathrm{d}B \\
\dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,20) = \dfrac{\partial v}{\partial x}(t,0) = \dfrac{\partial v}{\partial x}(t,20) = 0,\ t\geq 0 \\
u(t,x) = 3\cos(0.2\pi x) \\
v(t,x) = 3\cos(0.2\pi x),\ x\in\Omega = [0,20],\ t\in[-r,0]
\end{cases}$$

where $r = 1$ and

$$\eta_u(s) = \eta_v(s) = \begin{cases} 0, & -1\leq s < 0 \\ 1, & s = 0. \end{cases} \qquad (81)$$

By calculating the Leabesgue-Stieljies integral, we get $\int_{-1}^{0} u(t+s,x)d\eta_u(s) = u(t-1,x)$, $\int_{-1}^{0} v(t+s,x)d\eta_v(s) = v(t-1,x)$. In this case $f(v) = P(v) = \tanh(v)$, $g(u) = Q(u) = \tanh(u)$, $W = 2$, $T = -1$, $n = 1, m = 1$, $D^u(x) = 10$, $D^v(x) = 10$, such that $\nabla\cdot(D^u(x)\nabla u) = 10\Delta u$, $\nabla\cdot(D^v(x)\nabla v) = 10\Delta v$, Δ is the Laplacian operator. We can also get $\alpha = 1$, so H2 is satisfied. $\Omega = [0,20]$, $C = 5, D = 4$, $I = J = 1$, $U = L^2(\Omega)$, $V = H_0^1(\Omega)$, $|\tanh(u) - \tanh(v)| \leq |u - v|$, so $\sigma = \gamma = 1$ is chosen. We can also get $k_1 = k_2 = 1$. So H1 is fulfilled. $2k_{14} + 2\alpha L^2 - k_2^2 - 1 > 8 > 0$, $2k_{15} + 2\alpha L^2 - k_1^2 - 1 > 6 > 0$, $n\|W\|_F^2\sigma_M^2 K^2 = 4$, $m\|T\|_F^2\gamma_M^2 K^2 = 1$, it is easy to find that $(2k_{14} + 2\alpha L^2 - k_2^2 - 1)\wedge(2k_{15} + 2\alpha L^2 - k_1^2 - 1) > 6$, $(n\|W\|_F^2\sigma_M^2 K^2)\vee(m\|T\|_F^2\gamma_M^2 K^2) = 4$, so H3 is fulfilled in this example.

Let

$$e_n = \sqrt{\frac{1}{20}}\sin\frac{n\pi x}{20}, \quad x\in[0,20],\ n\in\mathbb{N}^+ \qquad (82)$$

$\{e_n\}_{n=1}^{\infty}$ forms a complete orthonormal basis of $\mathbb{K}$. B is a mean zero Wiener process with covariance operator $Q : U \to U$ satisfying $Qe_n = \frac{1}{n^2}e_n$, Since $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$, so Q is a Hilbert-Schmidt operator. In other words, $B = \sum_{n=1}^{\infty}\frac{1}{n}\beta_n(t)e_n(x)$, $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of the standard Wiener motions.

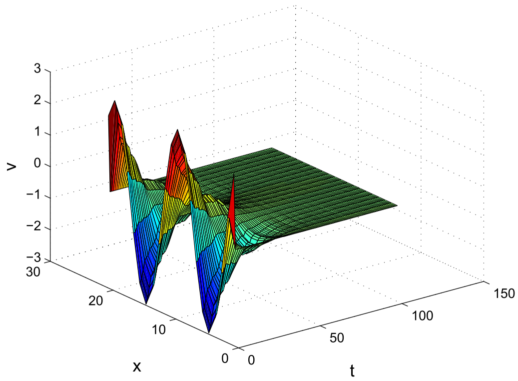

FIGURE 2. Surface of *v* in Example 17.

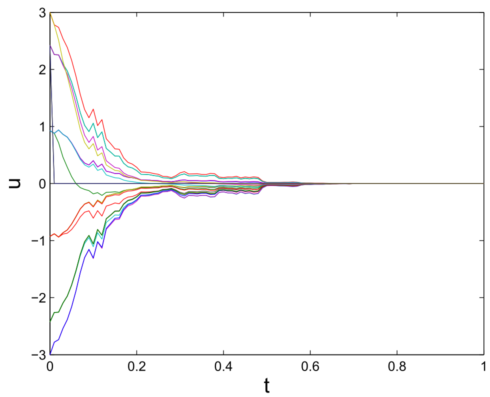

FIGURE 3. Frequency of *u* in Example 17.

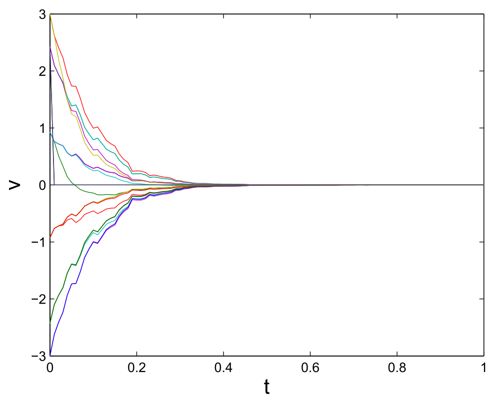

FIGURE 4. Frequency of *v* in Example 17.

We can see that $\mathbf{0}$ is an equilibrium of (81) and since H1-H3 are fulfilled for (81). By Theorem 16, the system has an exponentially stable equilibrium $\mathbf{0}$ in the mean square sense.

The finite difference method is used to discretize (81). The uniform mesh is used with space width $\Delta x = 1$ and time scale $\Delta t = 0.05$. The second order centered difference scheme is utilized to discrete the diffusion terms. The Runge-Kutta-Chebyshev is used to discrete the time. We use the Matlab to perform the code. For detailed information, please see Fig. 1-Fig. 4. From Fig.1-Fig. 2, it can be seen that as time t increases to the infinity, the surfaces of u and v converge to the equilibrium $\mathbf{0}$. It coincides with the result of Theorem 16. To present a more detailed information of u, v, we also give the trajectory of u, v for some chosen x in Fig. 3 and Fig. 4. It can be seen clearly that there is large oscillation due to the existence of random noise, especially at the beginning of the process. As the time increases, the trajectory becomes smooth and converges to the equilibrium quickly. This analysis falls in the expectation of our result, so the result is convincing and will have a wider application.

VI. CONCLUSION

We have discussed the global existence and uniqueness of reaction-diffusion BAMNNs driven by infinite dimensional Wiener processes. We can see that global Lipschitz condition, positiveness of diffusion coefficients, totally bounded variation of S-delays can ensure the existence and uniqueness of this system. It is a general condition and easy to check. As to the stability of this system, the criterion is given in the matrix norm. It is also easy to check through the computer, and will have a wider application.

The complex-valued neural networks (CVNNs) have been extensively studied in recent decades since they can solve problems which can't be solved by real-valued NNs [28]–[32]. The method used in this article can also be utilized in CVNNs after preliminary investigation. We plan to study this model in the future. By the way, the method used in this paper even can be extended to some biological models such as [33]–[40].

In the future, we will further relax the restriction for well-posedness of this system. The criteria of stability will be given in the form of LMIs as that in Zhang *et al.* [17], Song *et al.* [29], in order to use YALMIP in Matlab. At last, the asymptotic behavior will be studied by constructing different Lyapunov-Krasovskii functional.

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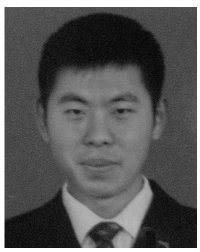

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