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High Performance Short Polar Codes: A Concatenation Scheme Using Spinal Codes as the Outer Code

XIAO XU, SHAOHUA WU, DAN DONG, JIAN JIAO, AND QINYU ZHANG
Communication Engineering Research Center, Harbin Institute of Technology (Shenzhen), Guangdong 518055, China

Corresponding author: Shaohua Wu (hitwush@hit.edu.cn)

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ABSTRACT An innovative scheme for concatenating the polar codes with the recently proposed spinal codes is presented in this paper. The primary objective is to improve the performance of the polar codes in the finite-length regime. One effective used way is to concatenate the polar codes with short outer codes. Essentially, the short outer codes that have as high performance as possible is required for concatenation. To this end, the newly invented spinal codes are used as the outer codes, which are characterized by its great error correcting capability even when the message length is short. First, the proposed codes, named spinal-polar, is implemented through an interleaved concatenation scheme. A joint iterative decoding algorithm is proposed accordingly, and the decoding complexity of the algorithm is also analyzed. Second, in order to reduce the decoding complexity of the full concatenation while maintaining the BER performance, the partial concatenation scheme is presented. Finally, we propose a rate allocation algorithm to further improve the BER performance of the designed concatenated codes. Extensive simulations results indicate that the proposed spinal-polar codes have significant performance improvement over the stand-alone polar codes, and the other improvement strategies mentioned earlier have also been proved to be effective.

INDEX TERMS Polar codes, interleaved concatenation, spinal codes, joint iterative decoding, rate allocation, partial concatenation.

I. INTRODUCTION

Polar codes, proposed by Arikan [1] in 2009, is a major breakthrough in coding theory recently. Polar codes are known for the ability to achieve the capacity of symmetric binary-input discrete memoryless channels (BDMC) as the block length goes to infinity, at the same time with low encoding and decoding complexity. The construction of Polar codes is based on channel polarization, including the operations of channel combining and channel splitting. By transmitting the information bits over those polarized noiseless channels and frozen bits over the other parts, Polar codes can achieve the symmetric capacity with the increase of the block length. However, the performance of Polar codes does not compete with the state-of-the-art coding schemes in moderate block length, such as the low-density parity check (LDPC) codes and Turbo codes. One reason is that the successive cancellation (SC) decoder is weak and susceptible to error propagation. The other is the poor minimum Hamming distance property of Polar codes. Despite the unsatisfactory finite-length codes performance, showing a series of remarkable characteristics, such as channel capacity accessibility and low decoding and decoding complexity, indicates that Polar codes still have the potential of getting into the ranks of the first-rate encoding techniques. Therefore, the aim of this paper is to provide schemes for improving the finite length performance of Polar codes, while preserving the low encoding and decoding complexity.

Considering more advanced decoders is one of the most efficient and direct ways. Belief propagation (BP) decoding of Polar codes is presented in [2]. Despite the better BER performance and soft output, the BP decoder requires a lot of memory space and the improvement is not significant. The linear programming (LP) decoder and successive cancellation list (SCL) decoding are presented

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in [3] and [4], respectively. However, LP decoder can only be used in binary erasure channel (BEC). For the SCL decoder, as the list size grows the memory requirement and the decoding complexity increase linearly. Hence, in [5], the authors have developed another improved decoding algorithm called successive cancellation stack (SCS), which has lower decoding complexity than the SCL decoder. After that, combining the principles of SCL and SCS, a new decoding algorithm called the successive cancellation hybrid (SCH) is proposed in [6], providing a flexible configuration when the time and space complexities are limited. Regarding external enhancement of the decoder, a survey of constructing the CRC-aided cancellation list/stack (CA-SCL/SCS) decoding of Polar codes is proposed in [7], which indicated that the CA-SCL decoder can even outperform the turbo codes without sacrificing the decoding complexity. Later in [8], the authors proposed the adaptive successive cancellation list (ASCL) decoding to reduce the decoding complexity of the SCL decoder.

Concatenation is another reasonable path enabling Polar codes to achieve performance improvements while maintaining the low encoding and decoding complexity. The interleaved concatenation of Polar codes with outer Reed-Solomon (RS) codes is proposed in [9], which can make the error decay rate asymptotically to be $2^{-N^{1-\varepsilon}}$ for any $\varepsilon > 0$, while the bound of error decay rate of Arikan's Polar codes is $2^{-N^{0.5-\varepsilon}}$. However, the major performance constraint of concatenated Polar codes is the error correction capability of RS codes. Similarly, in [10], the scheme that concatenating Polar codes with outer LDPC codes is provided; by comparison, concatenated codes have modest performance improvement according to the simulation results compared with stand-alone Polar codes. However, there only exists good error correction performance improvement in long block length with high complexity [11]. In [12], concatenating Polar codes with outer BCH and convolutional codes is introduced, and simulation results have shown that the BER performance of Conv-Polar codes is better than RS-Polar codes. Nevertheless, the error correcting capability of the proposed concatenated codes is limited by the constraint length [13].

Despite the published benefits, these concatenation schemes mentioned are either limited in the performance improvement compared with original Polar codes, or complex in the construction process on account of using long-length outer codes to get the performance improvement. In essence, the primary constraint of such schemes is the short codeword length performance of the outer codes. From this, there is a need for considering outer codes with impressive short code length performance to concatenate with Polar codes. Thanks to the research of J Perry, the latest invented Spinal codes [14] is well qualified to be the outer codes, which can carry out high performance with message length as short as $n = 16$. Proposed in 2012, Spinal codes are a new class of rateless codes that can achieve the capacity over both additive white Gaussian noise (AWGN) and binary symmetric channel (BSC) channels, achieving good decoding performance in short code length. The core of Spinal codes is the adoption of the hash function to encode the information bits randomly, making the codes have the good robustness to interference and noise.

In this work, we present the interleaved concatenation scheme of Spinal-Polar codes with joint iterative decoding to achieve BER performance improvements of Polar codes with short codeword length. The effectiveness of this concatenation scheme has been roughly demonstrated in our previous conference paper [15].

The advanced rate allocation scheme and the partial concatenation scheme are additionally proposed to further improve the BER performance and reduce the complexity, respectively. Simulation results show that the proposed concatenation scheme can outperform the stand-alone Polar codes significantly. The proposed rate allocation and the partial concatenation scheme are also proved to achieve their design objectives.

The main work and contributions of this thesis can be summarized as follows:

- We innovatively employ the latest Spinal codes as the outer codes to concatenate with inner Polar codes with interleaved structure, in which way each bit of the Spinal codeword can be scattered into different Polar codewords.
- We propose the partial concatenation scheme to reduce the complexity of the full concatenation while minimizing the performance loss.
- We design a rate allocation scheme of outer Spinal codes so that the polarized bit channels can have different levels of protection, which will further improve the performance of short Polar codes.
- In consideration of the susceptibility to error propagation of SC decoding, we design a joint iterative decoding algorithm for the concatenated Spinal-Polar codes. And it proved that this algorithm leads to another source of performance improvement.

The rest of the paper proceeds as follows. In Section II, the preliminaries are introduced. The proposed Spinal-Polar concatenation scheme is presented in Section III. Section IV introduces the proposed decoding method for Spinal-Polar codes. In Section V, simulation results are provided, followed by conclusion in Section VI.

II. PRELIMINARIES

In this section, we review Arikan's Polar codes and the basics of Spinal codes. The essential progress of Polar codes, i.e., construction of generator matrix and SC decoding algorithm, are presented, followed by the brief introduction of Spinal codes.

A. POLAR CODES

The construction of Polar codes is based on channel polarization. After polarization, part of the channels gradually become noiseless, whereas others become completely noisy.

Then we use the noiseless channels to transmit the information bits and the noisy channels to transmit frozen bits.

Specifically, Polar codes are constructed recursively based on the generator matrix $G_N = B_N F_2^{\otimes n}$, where $N = 2^n$ is the code length, B_N is an $N \times N$ bit-reversal permutation matrix, $F_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\otimes$ is the Kronecker product, leading to N synthesized bit channels. Polar codes can be generated by

$$x_1^N = u_1^N G_N = u_1^N B_N F_2^{\otimes n}, \tag{1}$$

where $u_1^N = (u_1, u_2, \ldots, u_N)$ is the bits to be encoded, and $x_1^N = (x_1, x_2, \ldots, x_N)$ is the encoded bits. We define $W(y|x)$ as the transition probability, and $y_1^N = (y_1, y_2, \ldots, y_N)$ as the received bits.

When it comes to the decoding progress, SC decoding is the basic decoding algorithm, with which Polar codes can achieve the channel capacity when N is large enough, and the decoding complexity is $O(N \log N)$. Here are the specific steps of SC decoding. For $i = 1, 2, \ldots, N$, the SC decoder calculates the likelihood ratio (LR) $L_N^{(i)}$ of u_i by

$$L_N^{(i)}(y_1^N, \hat{u}_1^{i-1}) = \frac{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|0)}{W_N^{(i)}(y_1^N, \hat{u}_1^{i-1}|1)}. \tag{2}$$

The likelihood ratio $L_N^{(i)}$ can be calculated recursively using the formulas:

$$L_N^{(2i-1)}(y_1^N, \hat{u}_1^{2i-2}) = \frac{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}) + 1}{L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2})}, \tag{3}$$

$$L_N^{(2i)}(y_1^N, \hat{u}_1^{2i-1}) = L_{N/2}^{(i)}(y_1^{N/2}, \hat{u}_{1,o}^{2i-2} \oplus \hat{u}_1^{2i-2})^{1-2\hat{u}_{2i-1}} \cdot L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}). \tag{4}$$

After that, SC decoder make decisions as follows:

$$u_i = \begin{cases} 0, & i \notin A \\ 0, & L_N^{(i)} \geq 1, i \in A \\ 1, & L_N^{(i)} < 1, i \in A \end{cases}, \tag{5}$$

where A denotes the information set.

B. SPINAL CODES

The encoding structure of Spinal codes is shown in Fig. 1. In this subsection, the encoding and decoding method of Spinal codes will be presented. The core idea of Spinal codes is the hash function and the random number generator (RNG), which can successively generate the pseudo-random bits.

Let $M = (m_1, m_2, \ldots, m_n)$ be the input message bits, where n denotes the number of bits. The spine value is represented as s_i and it can be obtained through the hash function. Finally, all the n/k spine values are sent to the RNG producing encoded sequences.

After the encoding, we will briefly introduce the maximum likelihood (ML) decoder of Spinal codes. The main idea of the ML Spinal decoding is searching for the encoded message that has the least difference from the received signal. Then we reuse the same initial spine value s_0, the hash function and RNG to reproduce the decoding tree completely. Finally, the result of the corresponding ML decoding for the AWGN channel is presented as follows

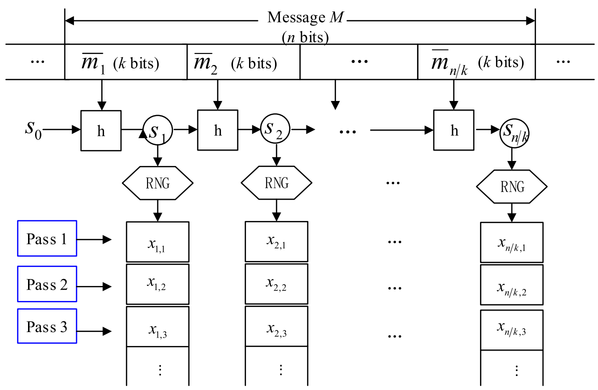

FIGURE 1. The encoding process of Spinal codes.

$$\hat{M} = \mathop{\arg\min}_{M' \in (0,1)^n} \left\| \bar{y} - \bar{x}(M') \right\|, \tag{6}$$

where $\bar{y}$ denotes the received vector and $\bar{x}(M')$ denotes the encoded vector.

When it comes to the BSC channel, the only difference is that the Euclidean distance is replaced into the hamming distance.

To decrease the decoding complexity, another improved decoder of Spinal codes named as bubble decoder, designed in [14], is an approximate ML decoding rule based on the decoding tree prune. The central concept is to keep only B nodes at each depth to get the minimum cost, and then the decoder expands $B \cdot 2^k$ possible nodes at the next depth of the tree.

After all the decoding progress, for a spinal code with message length n and segment length k, we present the complexity of bubble decoder as $O\left(n \cdot B \cdot 2^k \cdot (v + k + \log B)\right)$, where v denotes the length of hash spine, and B denotes the pruning width, i.e., the number of nodes reserved in each depth. Obviously, the complexity is linear to the message length n and has exponent relation to k.

III. THE PROPOSED SPINAL-POLAR CONCATENATION SCHEME

In this section, the interleaved concatenation scheme of the Polar codes with outer Spinal codes is introduced at first, followed by the partial concatenation and the rate allocation scheme of Spinal-Polar codes.

A. INTERLEAVED CONCATENATION OF SPINAL-POLAR CODES

Before the introduction of the proposed concatenation scheme, the process of Spinal encoding is first introduced more specifically again.

As depicted in Fig. 1, the encoding procedure of Spinal codes can be described by the following three steps. First, the input message bits is divided into n/k blocks $(\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_{n/k})$ with k bits in each block. Secondly, the hash function is applied in each block to generate the corresponding spine value sequentially as follows:

$$s_i = h\left(s_{i-1}, \bar{m}_i\right), s_0 = o^v, \quad (7)$$

where s_0 is the initial spine value, known to both the encoder and the decoder. Finally, all the n/k spine values are send to the RNG. Specifically, RNG is a function which can transform a v bits spine seed to a c bits symbol as the pseudo-random sequence:

$$RNG : \{0, 1\}^v \times N \to \{0, 1\}^c. \quad (8)$$

Through the steps above, the encoder outputs a series of passes, each of which consists of n/k symbols as depicted in Fig. 1.

Then the interleaved concatenation scheme of Polar codes with outer Spinal codes is shown in Fig. 2.

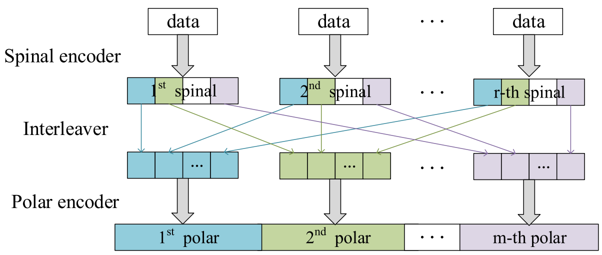

FIGURE 2. The proposed concatenation scheme of Spinal-Polar codes.

Since we adopt Spinal codes as the outer codes, the encoding process of Spinal codes is carried out earlier and then the encoding of Polar codes. Explicitly, the encoded bits of the Spinal codes corresponding to the first symbol are sequentially encoded into the first Polar codeword. Similarly, the information bits of the second Polar codeword is produced from the second symbol of the encoded Spinal codes. Consequently, the encoding progress of Polar codes and Spinal codes proceed parallelly to reduce the encoding latency, in which way can the burst errors occur in the Polar decoding progress be discretized, ensuring that the error symbols number in one codeword does not exceed the error correction capability of Spinal codes. The specific steps are all presented in Algorithm 1.

Note that the total length of the concatenation scheme is $N{=}m \cdot N$, so the rate of each inner Polar code is $R_I = r/N$. Similarly, the rate of each outer Spinal code is $R_o = n/m$. Hence, it turns out that the total rate of the concatenated codes is $R = R_I \cdot R_o$.

After confirmation of the respective decoding algorithm, we investigated the original and traditional decoding method for the interleaved Spinal-Polar codes, i.e., the separate multi-level decoding algorithm. The general course is to decode all the inner Polar codewords, followed by the de-interleaving

Algorithm 1 The Interleaved Concatenation Encoding Algorithm

Require:
The block number r;
The number of pass p;
The length of each Spinal codes m;
The ith block in each $n$bits spinal codes $\overline{m}_i$

Ensure:
Encoded Spinal-Polar matrix $\boldsymbol{U}$

```
Initialization: the input message M is divided into r blocks with n bits in each block; and the each n bits block are divided into n/k blocks (m̄_1 ∼ m̄_{n/k}) for Spinal encoding as shown in Fig.1, in each Spinal codes: (n/k)·p = m
for i ← 1 to n/k do
    for j ← 1 to r do
        //Spinal encoding on the jth (1 ∼ r) block
        u(j, i(p − 1) + 1 : ip) = Spinalencoder(m̄_i)
    end for
    //Polar encoding on the string of all the (i(p − 1) + 1) th ∼ (ip) th symbol bits
    k = u(:, i(p − 1) + 1 : ip)^T
    U(i(p − 1) + 1 : ip, :) = Polarencoder(k)
    //obtain the finial encoding matrix
end for
return Encoded Spinal-Polar matrix U(m × N)
```

operation, and then the decoded bits of Polar codes are transmitted to the Spinal decoder. The concatenated codes can be considered as a multilevel coding system, and the errors occur at the previous level (inner Polar codes) can be corrected by the next level (outer Spinal codes).

B. PARTIAL CONCATENATION OF SPINAL-POLAR CODES

In this sub-section, the partial concatenation scheme of Spinal-Polar codes is investigated to mainly reduce the complexity of full concatenation, which can achieve a better trade-off between performance and complexity compared with the full concatenation scheme.

For the finite length Polar codes, especially for short Polar codes, due to the low channel polarization rate, its BER performance is not so good compared to LDPC and Turbo codes. As shown in Fig. 3, after channel polarization, only part of the sub-channels can be completely polarized to the noiseless or noisy channel, meaning that the capacity of which is close to 1 or 0. Hence, there are still some not fully polarized bit channels whose capacity between 0 and 1. Since the number of fully polarized channels is limited, we have to use these not fully polarized bit channels to transmit the information bits, and these poor channels can significantly influence the BER performance of Polar codes. Therefore, we can use outer Spinal codes to provide further protection to those not fully polarized bit channels.

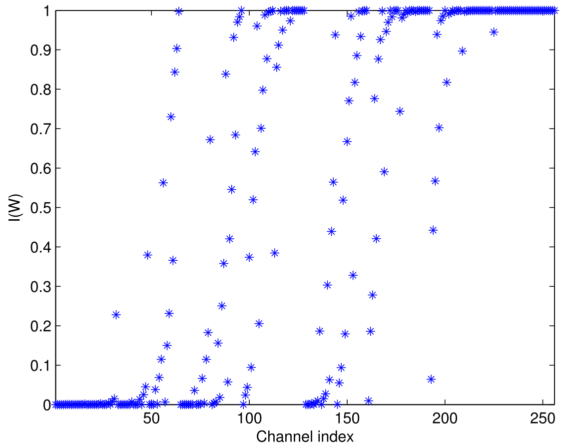

FIGURE 3. Channel polarization for a Polar code with length N=256 and rate R=0.5.

We define the bit channels whose capacity is close to 1 as the good channels, the bit channels whose capacity is close to 0 as the bad channels, and the bit channels whose capacity is between 0 and 1 as the intermediate channels. The main idea of the partial concatenation scheme is to only provide protection to those intermediate bit channels rather than protecting all the bit channels in full concatenation scheme.

In the partial concatenation scheme, the channel polarization process is performed firstly. In this work, we adopt the Bhattacharyya Parameter [1]. We use the separated encoding instead of the parallel encoding in Algorithm 1 for the reason that only part of message bits are encoded by the outer Spinal codes. Then the polarized bit channels are sorted according to the reliability and divided into three types as defined above. The message bits corresponding to the intermediate channels are first encoded by the outer Spinal codes and note that *u_spinal*. Then we splice *u_spinal* and other message bits corresponding to good channels and bad channels to a new sequence, followed by the Polar encoding. The detailed progress is shown in the following Algorithm 2:

In this algorithm, to be more clear, we divided message bits into $M1$ and $M2$ sequentially. The transmission of different message bits on different subchannels is instantiated in the Polar encoding progress.

Then the code rate is taken into account. Let f_{good}, f_{inter} and f_{bad} be the corresponding channel index sets of the three types bit channels, defined as follows:

$$
\begin{aligned}
f_{good} &= \left\{ i \in \{1, 2, \cdots, N\}, \delta_1 \leq I\left(W_N^i\right) \leq 1 \right\} \\
f_{inter} &= \left\{ i \in \{1, 2, \cdots, N\}, \delta_2 \leq I\left(W_N^i\right) \leq \delta_1 \right\} \\
f_{bad} &= \left\{ i \in \{1, 2, \cdots, N\}, 0 \leq I\left(W_N^i\right) \leq \delta_2 \right\}, \qquad (9)
\end{aligned}
$$

where N is the code length of Polar codes, and $f_{good} \cup f_{inter} \cup f_{bad} = \{1, 2, \cdots, N\}$. According to the definition above, the rate of inner Polar codes is

$$R_I = \frac{|f_{good}| + |f_{inter}|}{N}, \qquad (10)$$

Algorithm 2 The Partial Concatenation Encoding Algorithm

Require:
The input message M; The length of inner Polar codes N; The rate of codes R; The ratio of Partial concatenation p; The number of Polar codes m; Channel variance σ;

Ensure:
Encoded Partial concatenated Spinal-Polar matrix $\boldsymbol{P}$

Initialization:
The length of the input message M is $Nk = N \cdot R$; set $Nk_1 = Nk \cdot p$ as the length of protected part, $Nk_2 = Nk \cdot (1 - p)$ as the length of the part no need to be protected by outer Spinal codes.
m is also the length of each Spinal codes(Algorithm 1)

```
Inds = Z (N, σ)
//Channel polarization process: the Bhattacharyya parameters are obtain by function Z and arranged in descending order, smaller parameters means higher reliability
frozen_positions = Inds(1 : N − Nk)
//the positions for bits of frozen channels
free_positions1 = sort(Inds(N − Nk + 1 : N − Nk + Nk1))
//the positions for bits of intermediate channels
free_positions2 = sort(Inds(N − Nk + Nk1 + 1 : N))
//the positions for bits of good channels
M1 = M(1 : Nk1)
M2 = M(Nk1 + 1 : Nk)
//Classified message bits
for i ← 1 to Nk1 do
    //Spinal encoding on the Nk1 bits of message
    u_spinal (i, :) = Spinalencoder (M(i))
end for
for j ← 1 to m do
    //Polar encoding
    k1 = u_spinal(; , j)^T
    u(free_positions1) = K1
    u(free_positions2) = M2
    //Splice the Spinal encoded bits and origin message bits to the Nk sequence for Polar encoding
    P(i, Nk + 1 : N) = Polarencoder(u)
end for
return Encoded Partial concatenated Spinal-Polar matrix P(m × N)
```

where $|f_{good}|$ is the cardinality of set f_{good}, so is $|f_{inter}|$. And the overall rate of the partial concatenation scheme is

$$R = \frac{|f_{good}| + |f_{inter}| \cdot R_o}{N}, \qquad (11)$$

where R_o is the code rate of outer Spinal codes.

C. RATE ALLOCATION SCHEME OF SPINAL-POLAR CODES

Generally speaking, due to the channel polarization process of Polar codes, different bit channels would have different reliability. That is to say, in the Spinal-Polar concatenation scheme, different bit channel needs to have different degrees of protection. Therefore, a flexible choice over the rate of outer Spinal codes is required to further improve the BER performance of Spinal-Polar codes. Obviously, the rate allocation scheme can be seen as the general case of partial concatenation scheme.

The classification of different reliability in different channels is executed by the Monte Carlo experiment or Bhattacharyya Parameter. For the bit channels with high reliability, the Spinal codes with larger code rate are allocated to them, while for the bit channels with low reliability, the Spinal codes with lower code rate are allocated to provide higher levels of protection to them. Meanwhile, the overall code rate of Spinal-Polar codes remains the same, and the error correction capacity of the whole system will be increased.

After obtaining the reliability valued by the error probability or other parameters, the general process is as follows:

According to the reliability of each bit channel, we sort the information bits in a certain order and divide them into several groups and each group has an equal number of information bits. When the average error probability of the group is high, it means that the bit channels in this group have low reliability, and the Spinal codes with lower rate were allocated to the group to provide higher levels of protection to those bit channels. While the low error probability indicates that the bit channels in this group have high reliability, and the Spinal codes with higher rate were allocated to the group to provide relatively lower levels of protection to those bit channels.

Algorithm 3 The Joint Iterative Decoding Algorithm

Require: Receive matrix $\{y_j\}_{j=1}^{m}$; information set A; Polar code length N; Spinal decoding parameters of pass, k and B;

Ensure: Decoding matrix $\hat{u}$

Initialize $n \triangleq \log_2(N)$ and L is an $N \times (n+1)$ matrix
for $i \leftarrow 1$ to r **do**
  for $j \leftarrow 1$ to m **do**
    //decode the *ith* information bit of the *jth* Polar codeword
    $L_0(j,i) = \frac{-2y_{j,i}}{\sigma^2}$
    $L(:, n+1) = L_0$
    $c_i(j) = SCdecoder(N, L, A, u)$
  end for
  //decode the *ith* Spinal codeword
  $m_i = FSDdecoder(pass, k, B, c_i)$
  $\hat{u}_i = m_i$
  //update information obtained from Spinal decoder back to the SC decoder
  $s_i = Spinalencoder(m_i, k, pass)$
  for $j \leftarrow 1$ to m **do**
    $u_i(j) = s_i(j)$
  end for
end for
return The decoding matrix $\{\hat{u}_i\}_{i=1}^{r}$

IV. THE PROPOSED DECODING METHOD FOR SPINAL-POLAR CODES

A. JOINT ITERATIVE DECODING OF SPINAL-POLAR CODES

As mentioned above in section III-A, the traditional separate multilevel decoding algorithm is unable to deal with the error propagation problem of SC decoding well. On account of this, in this subsection, we present the joint iterative decoding algorithm of Spinal-Polar codes.

During the SC decoding of Polar codes, once an error occurs, the decoding of subsequent bits will be affected by the error propagation. Therefore, it is very important to make sure that the previous bit decoded by SC decoder is correct and then proceed to decode the next bits. In the proposed Spinal-Polar codes scheme, errors in the Polar decoded bits can be corrected by Spinal codes. By adopting the joint iterative decoding, the error propagation problem of SC decoder can be alleviated, thus bringing out the performance improvement of Polar codes in finite length regime.

Specifically, we employ the SC decoder as the decoding algorithm for inner Polar codes during the concatenated codes decoding scheme. For outer Spinal codes, as presented in section II-B, the decoding complexity of bubble decoder is $O\left(n \cdot B \cdot 2^k \cdot (v + k + \log B)\right)$, where n denotes the length of message bits. However, despite the reduction of complexity compared with the original ML decoder, the bubble decoder exhibits a crucial drawback, i.e., the exponential complexity, which still makes it difficult to apply in practice. Consequently, the forward stack decoding (FSD) algorithm is presented in [16] with the ability to decrease the decoding complexity of bubble decoder. The FSD algorithm performs a hierarchical operation on the decoding tree and searches the optimal decoding path on each layer until the bottom of the decoding tree is reached. Unlike the bubble decoder, the FSD algorithm just selects some portion of the nodes on each layer, hence the complexity has been reduced to a certain extent. While for the decoding performance, the FSD algorithm remains almost the same as the bubble decoding.

After the decoder is selected, the proposed joint iterative decoding algorithm is operated as follows.

To begin with, the SC decoder receives and decodes the first information bits of the whole m Polar codes parallelly, and then the decoded bits form an outer Spinal codeword, which is next delivered to the FSD decoder. The decoded information of Spinal codeword proceeds to conduct the encoding process of Spinal codes to form the corrected Spinal codeword. Afterward, the update information obtained from the corrected Spinal codeword are transmitted back to each of the SC decoders. That is, the update information will work as the prior information to decode the second information bit of each Polar codeword. Finally, the iteration procedure is repeated until all the information bits of inner Polar codes are decoded. The steps mentioned above are all presented in Algorithm 3.

B. SUCCESSIVE CANCELLATION LIST DECODING OF SPINAL-POLAR CODES

In order to further improve the decoding performance of Spinal-Polar codes, we propose to adopt the SCL decoding as the decoder of inner Polar codes. SCL decoding was proposed in [4] that can bridge the gap between SC decoding and maximum-likelihood decoding of Polar codes. The SCL decoding algorithm is an improved SC decoding algorithm that store the most possible sequence of decoding path with a list of size L. As with SC decoding, SCL decoding decodes the input bits one-by-one, but the L decoding paths are considered simultaneously in each decoding phase, while the SC decoding only searches the optimal path in one layer. Specifically, the SCL decoder divides the decoding path of each information bit u_i into two branches of $u_i = 0$ and $u_i = 1$, and then find the L most likely paths from the candidate paths layer by layer. At the end of the decoding procedure, the L most likely decoding paths are selected as the decoder output. Therefore, the SCL decoding will increase the decoding complexity. The decoding complexity of SCL decoding is given by $O(Ln\log n)$, where n denotes the code length of inner polar codes.

Furthermore, the CRC detector is considered to assist and enhance the SCL decoding. The SCL decoder outputs the L candidate path to the CRC detector, then the decoder outputs the first path that can pass the CRC detection as the decoding sequence. Compared with SC decoding, the CRC-aided SCL decoding can significantly improve the performance of Polar codes.

Due to the adoption of the SCL decoding algorithm, the decoding performance of inner polar codes can be improved. Since the SCL decoder calculates the probability of each path and chooses the path with the highest reliability, the decoding information of Polar codes is passed to Spinal decoder, which will bring out the performance improvement of Spinal-Polar codes.

C. DECODING COMPLEXITY OF SPINAL-POLAR CODES

In this sub-section, the decoding complexity of the proposed Spinal-Polar codes is demonstrated. When adopting the SC decoding for inner Polar codes, the decoding complexity of the concatenated codes mainly consists of two parts: SC decoding of inner Polar codes and FSD decoding of outer Spinal codes. The decoding complexity of the inner and outer codes is first considered, respectively.

Let C_P, C_S and C denote the complexity of the SC decoding of inner Polar codes, the FSD decoding of outer Spinal codes and the total complexity of Spinal-Polar codes, respectively. Note that for a Polar code of n length, the decoding complexity of SC decoder is given by $O(n\log n)$. When it comes to the proposed concatenation scheme, there are m Polar codes, thus the decoding complexity of Polar codes is given by

$$C_P = O(mn\log n) = O(N\log n), \tag{12}$$

where N denotes the total length of Spinal-Polar codes.

For a single pass of Spinal codes, the decoding complexity of FSD decoder can be upper bounded as $O\left(\frac{m'}{k}\cdot B\cdot 2^k\cdot(v+k+\log B)\right)$, where $m' = m\cdot R_o$ is the message length. Now that r and L denote the number of Spinal codes and the number of passes in each Spinal codes,respectively. And considering the rate is $R_o = k/L$, the decoding complexity of outer Spinal codes is upper bounded as

$$C_S \le O\left(r\cdot m\cdot B\cdot 2^k\cdot(v+k+\log B)\right). \tag{13}$$

Hence, the overall complexity of Spinal-Polar codes is upper bounded as

$$C \le O\left(r\cdot m\cdot B\cdot 2^k\cdot(v+k+\log B)+N\log n\right). \tag{14}$$

For the purpose of improving the decoding accuracy of Spinal codes, the pruning length B cannot be too small, and then usually B is not less than 256. Obviously, the complexity of the outer Spinal codes is the major determinant of the decoding complexity of Spinal-Polar codes. We have known that rm denotes the length of outer Spinal codes, which is significantly less than the total length of concatenated codes N, therefore the decoding complexity is substantially smaller than the stand-alone Spinal codes with the same code length and rate, as shown below:

$$\begin{aligned} C &\le O\left(r\cdot m\cdot B\cdot 2^k\cdot(v+k+\log B)+N\log n\right)\\ &\approx O\left(r\cdot m\cdot B\cdot 2^k\cdot(v+k+\log B)\right)\\ &\le O\left(N\cdot B\cdot 2^k\cdot(v+k+\log B)\right) = C_{spinal}. \end{aligned} \tag{15}$$

However, the overall complexity of Spinal-Polar codes is higher than that of stand-alone Polar codes with the same code length and rate, as shown below:

$$C \ge C_{polar} = O(N\log N), \tag{16}$$

where C_{spinal} and C_{polar} denote the complexity of pure Spinal codes and pure Polar codes with same length and rate as Spinal-Polar codes, respectively. As a result, we have

$$C_{polar} \le C \le C_{spinal}, \tag{17}$$

then the actual complexity of the concatenated codes is between the decoding complexity of the pure Polar codes and pure Spinal codes.

When adopting the SCL decoding instead of the SC decoder for inner Polar codes, the decoding complexity of SCL decoder is given as $O(L\cdot n\log n)$, the complexity of which is L times that of SC decoding. Hence, the total complexity of Spinal-Polar codes with SCL decoding is bounded as

$$C \le O\left(r\cdot m\cdot B\cdot 2^k\cdot(v+k+\log B)+L\cdot N\log n\right). \tag{18}$$

In this case, we cannot decide whether the Spinal codes or Polar codes have a higher decoding complexity. But what we can be sure of is that adopting SCL decoding instead of SC decoder is a method to achieve high BER performance at the expense of decoding complexity.

V. SIMILATION RESULTS

In this section, extensive simulations are carried out to demonstrate the advantages of the proposed Spinal-Polar concatenation scheme as well as the other modified techniques based on the proposed scheme mentioned above.

A. THE BER PERFORMANCE OF SPINAL CODES OVER BSC CHANNEL

Since we have known that the performance of concatenation scheme is significantly depended on the error correcting capability of the Spinal codes over the equivalent BSC channel, a fixed-rate of 1/4 Spinal codes transmission over BSC channel with FSD decoding is analyzed, where the message length of the designed Spinal codes is 16, $k = 4$, $c = 1$ and $B = 256$. The BER performance of Spinal codes over BSC channel is shown in Fig. 4.

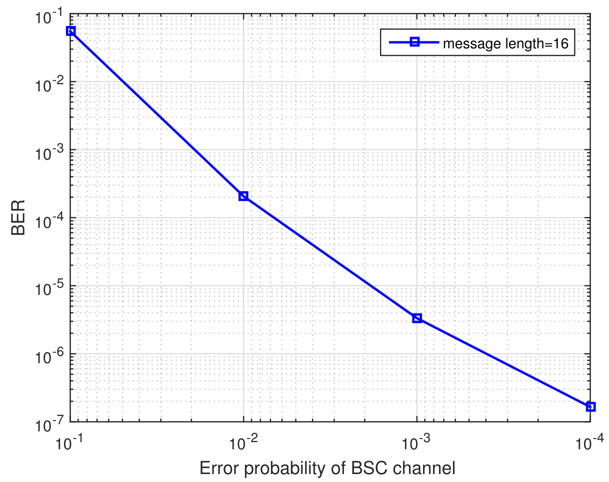

FIGURE 4. BER performance of the Spinal codes over BSC channel.

It is clear that Spinal codes have good error correction performance even in short code length. More specifically, when the error probability of BSC channel is 10^{-3}, the BER performance of Spinal codes can reach to 10^{-6}. Consequently, the proposed Spinal-Polar concatenation scheme is feasible because of the impressive error correction capability of Spinal codes as shown in Fig. 4.

B. THE BER PERFORMANCE OF THE PROPOSED SPINAL-POLAR CODES

In this sub-section, the simulation results of the BER performance of the proposed Spinal-Polar concatenation scheme are carried out compared with the stand-alone Polar codes and RS-Polar codes. Fig. 5 depicts the contrast of the BER performance of the Spinal-Polar codes, RS-Polar codes, and the pure Polar codes with SC decoding. Specifically, we construct the proposed scheme with inner Polar codes (256,128) and outer Spinal codes (64,16) over the AWGN channel with BPSK modulation, which leads to the fact that the overall code rate is 1/8. To ensure the fairness of the comparison, simulation results are carried out with the code rate of the pure Polar codes and RS-Polar codes equal to the concatenated scheme.

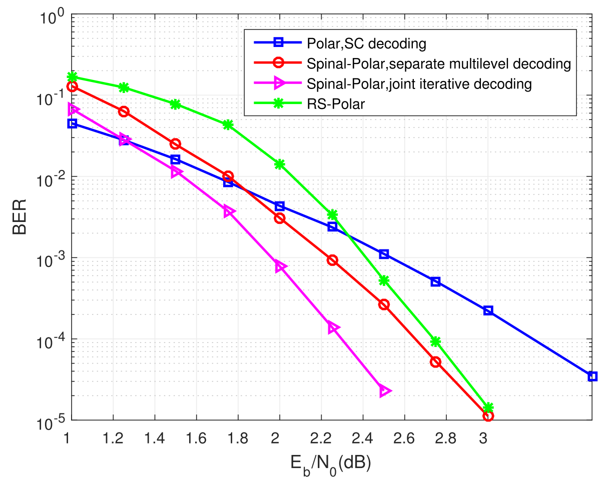

FIGURE 5. BER performance of RS-Polar codes and Spinal-Polar codes with separate multilevel decoding, joint iterative decoding over AWGN channel .

As is demonstrated in Fig. 5, in lower SNR the concatenated Spinal-Polar codes have the worse BER performance compared with the non-concatenated Polar codes, but as SNR grows the BER performance of the proposed codes gradually improves. And here is the reason: After the interference in the channel transmission, the number of errors exceeds the correcting capability of Spinal codes, so the decoding progress might generate more errors which in turn leads to the worse BER performance than the stand-alone Polar codes. However, with the increase of SNR, the number of errors would be within the correcting capability of Spinal codes, thus the errors in the decoded bits can be corrected by outer Spinal codes, certainly leading to the BER performance improvement.

Obviously, the proposed Spinal-Polar codes outperform RS-Polar codes, which prove the performance advantages of the Spinal-Polar codes in short code length. When the SNR is greater than 1.5 dB, the BER performance of the proposed Spinal-Polar codes under joint iterative decoding significantly outperform that under traditional separate multilevel decoding as well as stand-alone Polar codes. At the BER of 10^{-4}, it can be observed that the proposed Spinal-Polar codes under joint iterative decoding has nearly 0.35 dB and 0.9 dB SNR gain over the separate multilevel decoding of Spinal-Polar codes and the non-concatenated Polar codes, respectively. In addition, the Spinal-Polar codes under separate multilevel decoding have about 0.55 dB gain compared with the non-concatenated Polar codes. Thus, the BER performance of Polar codes can be improved by concatenating with outer Spinal codes.

C. THE BER PERFORMANCE OF THE RATE ALLOCATION SCHEME AND THE PARTIAL CONCATENATION SCHEME

In Fig.6, we give the simulation result of Spinal-Polar codes with rate allocation and part concatenation scheme over the

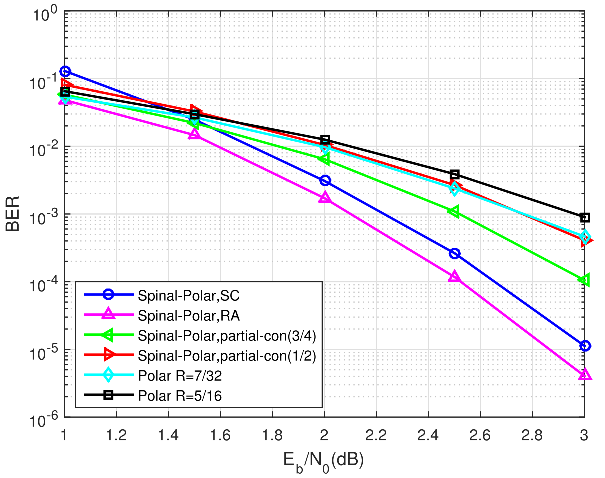

FIGURE 6. BER performance of Spinal-Polar codes with rate allocation and partial concatenation over AWGN channel.

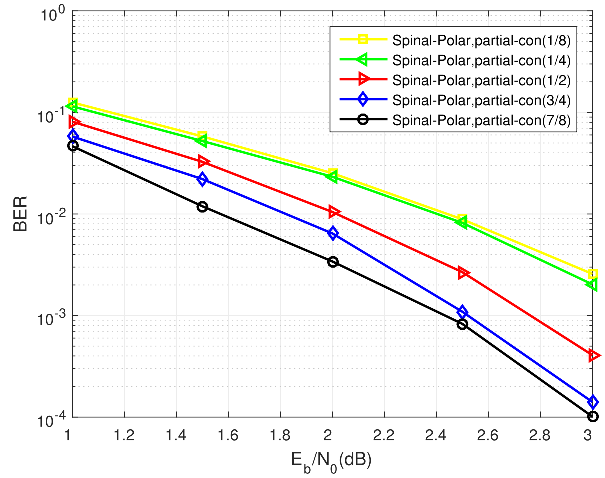

FIGURE 7. BER performance of the Spinal-Polar codes with partial concatenation in different ratios over AWGN channel.

AWGN channel. For simplicity, we use RA to denote the rate allocation scheme of Spinal-Polar codes, and partial-con denotes the partial concatenation scheme.

We construct Spinal-Polar codes with inner Polar codes of length 256, rate 1/2, and Spinal codes with the message length of 16, rate 1/4. The curve of Spinal-Polar, partial-con (3/4) represents that 3/4 of information bits of Polar codes are protected by outer Spinal codes, while the other 1/4 of information bits are only encoded by Polar codes with no extra protection. The overall rate of the Spinal-Polar codes with 3/4 partial concatenation scheme is 7/32. Similarly, the curve of Spinal-Polar, partial-con (1/2) means that 1/2 of information bits of Polar codes are protected by outer Spinal codes, while the other 1/2 of information bits are only encoded by Polar codes with no extra protection. The overall rate of the Spinal-Polar codes with 1/2 partial concatenation scheme is 5/16.

As shown in Fig. 6, we can observe that the Spinal-Polar codes with rate allocation perform better than the Spinal-Polar codes without rate allocation under SC decoding. For the partial concatenation scheme, under the same bit rate:7/32 and 5/16, the Spinal-Polar codes perform better than the non-concatenated Polar codes, whereas the BER performance is slightly worse than Spinal-Polar codes with full concatenation.

Moreover, more partial concatenation schemes with different ratios of protected bits by the outer Spinal codes to all the information bits of Polar codes are shown in Fig. 7. We can see that the performance is monotonically decreasing as the concatenation ratio goes down. Hence, with the increase of the proportion of protected information bits of inner Polar codes, i.e., the increase of the ratio, the performance of concatenated codes becomes better, which is due to the reason that higher ratio means more information bits protected by outer Spinal codes. Thus, the complexity reduction of the full concatenation scheme is achieved at the expense of certain loss of BER performance.

D. THE BER PERFORMANCE OF THE PROPOSED SPINAL-POLAR CODES WITH DIFFERENT DECODING ALGORITHM

In Fig.8, we compare the BER performance of the Spinal-Polar codes with SC decoding, CRC-aided SCL decoding with the different list size of L, and LDPC codes with the same rate under BP decoding with 50 iterations. Simulation results show that the Spinal-Polar codes when decoded with the SCL-CRC decoding algorithm can outperform LDPC codes with BP decoding. Moreover, we can see that as the list size L increases, the BER performance of the Spinal-Polar code improves.

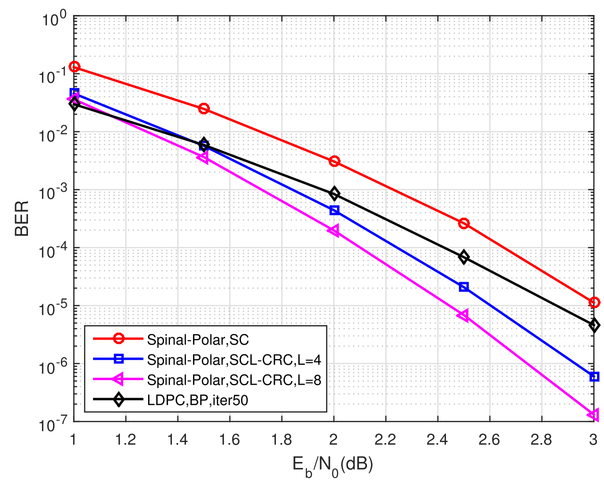

FIGURE 8. BER performance of the Spinal-Polar codes with SC decoding and SCL-CRC decoding over AWGN channel.

VI. CONCLUSION

As the improvement study of the Polar code-based concatenation scheme, in this paper, we have proved that by applying the proposed interleaved concatenation scheme of the Polar codes with the Spinal codes, a significant BER performance improvement of the Polar codes in finite-length regime compared with the non-concatenated Polar codes is feasible. The idea of partial concatenation was also taken into account to reduce the system complexity and minimizes the losses of performance as much as possible. Besides, the rate allocation scheme was next carried out to further boost the BER performance. Furthermore, the joint iterative decoding algorithm for the concatenation scheme was proposed to alleviate the error propagation problem of SC decoding, and we also adopted the SCL decoder to further improve the BER performance of short Polar codes. The decoding complexity analysis of the proposed Spinal-Polar codes was investigated meanwhile. Simulation results verified that the proposed concatenation scheme can provide a significant performance gain compared with stand-alone Polar codes, and other improvement schemes proposed above have also proved to be effective.

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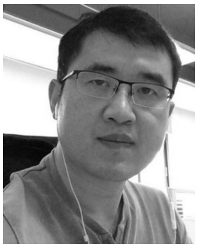
XIAO XU was born in Tai'an, Shandong, China, in1995. He received the B.S. degree from the Harbin Institute of Technology, Weihai, China, in 2017. He is currently pursuing the Harbin Institute of Technology, Shenzhen. His current research interests include advanced channel coding techniques and cross-layer optimizations.



SHAOHUA WU (S'07–M'11) received the Ph.D. degree in communication engineering from the Harbin Institute of Technology in 2009. From 2009 to 2011, he held a post-doctoral position with the Department of Electronics and Information Engineering, Shenzhen Graduate School, Harbin Institute of Technology, where he has been an Associate Professor since 2012. From 2014 to 2015, he was a Visiting Researcher with BBCR, University of Waterloo. His current research interests include wireless image/video transmission, deep space communication, advanced channel coding techniques, and 5G wireless transmission technologies. He has authored or co-authored over 70 papers in these fields and holds over 30 Chinese patents.

DAN DONG was born in Heze, Shandong, China, in 1992. She received the B.S. degree from the Harbin Institute of Technology, Weihai, China, in 2015. She is currently pursuing the degree with the Harbin Institute of Technology, Shenzhen. Her current research interests include advanced channel coding techniques and cross-layer transmission technologies.

JIAN JIAO (M'16) received the M.S. and Ph.D. degree in communication engineering from the Harbin Institute of Technology (HIT) in 2007 and 2011, respectively. From 2011 to 2015, he was a Post-Doctoral Research Fellow with the Communication Engineering Research Centre, Shenzhen Graduate School, HIT, Shenzhen, China. From 2016 to 2017, he was with the School of Electrical and Information Engineering, University of Sydney, Sydney, Australia, as a China Scholarship Council Visiting Scholar. Since 2017, he has been an Assistant Professor with the Department of Electrical and Information Engineering, HIT Shenzhen Graduate School. He has authored or co-authored over 60 peer-reviewed papers, holds over 30 invention patents in his research fields. His current interests include error control codes, space information networks, random multiple access, and machine-to-machine communications.

QINYU ZHANG (M'08–SM'12) received the bachelor's degree in communication engineering from the Harbin Institute of Technology (HIT) in 1994 and the Ph.D. degree in biomedical and electrical engineering from the University of Tokushima, Japan, in 2003. From 1999 to 2003, he was an Assistant Professor with the University of Tokushima. From 2003 to 2005, he was an Associate Professor with the Shenzhen Graduate School, HIT, and was the Founding Director of the Communication Engineering Research Center, School of Electronic and Information Engineering. Since 2005, he has been a Full Professor and serves as the Dean of the EIE School. His research interests include aerospace communications and networks, wireless communications and networks, cognitive radios, signal processing, and biomedical engineering. He is on the Editorial Board of some academic journals, such as the *Journal on Communications*, the *KSII Transactions on Internet and Information Systems*, and the *Science China: Information Sciences*. He was the TPC Co-Chair of the IEEE/CIC ICCC'15, the Symposium Co-Chair of the IEEE VTC'16, an Associate Chair of the Finance of ICMMT'12, and the Symposium Co-Chair of CHINACOM'11. He has been a TPC Member of INFOCOM, ICC, GLOBECOM, WCNC, and other flagship conferences in communications. He was the Founding Chair of the IEEE Communications Society Shenzhen Chapter. He has received the National Science Fund for Distinguished Young Scholars, the Young and Middle-Aged Leading Scientist of China, and the Chinese New Century Excellent Talents in University, and received three scientific and technological awards from governments.