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Discharging Approach for Double Roman Domination in Graphs

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ABSTRACT The discharging method is most well-known for its central role in the proof of the Four Color Theorem. This proof technique was extensively applied to study various graph coloring problems, in particular on planar graphs. In this paper, we show that suitably altered discharging technique can also be used on domination-type problems. The general discharging approach for domination-type problems is illustrated on a specific domination-type problem, the double Roman domination on some generalized Petersen graphs. By applying this approach, we first prove that $\gamma_{dR}(G) \geq (3n/\Delta(G)+1)$ for any connected graph G with $n \geq 2$ vertices. As examples, we also determine the exact values of the double Roman domination numbers of the generalized Petersen graphs $P(n, 1)$ and the double generalized Petersen graphs $DP(n, 1)$. The obtained results imply that $P(n, 1)$ is double Roman if and only if $n \not\equiv 2 \pmod 4$ and $DP(n, 1)$ is double Roman if and only if $n \equiv 0 \pmod 4$.

INDEX TERMS Discharging approach, double Roman domination, generalized Petersen graph, double generalized Petersen graph, double Roman graph.

I. INTRODUCTION

For a vertex v in a graph G, the *open neighborhood* of v in G is denoted by $N(v)$, i.e. $N(v) = \{u|uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ of v in G is defined as $N[v] = \{v\} \cup N(v)$. The *degree* of a vertex v is denoted by $d(v)$, i.e. $d(v) = |N(v)|$, and the *maximum degree* of a graph G by $\Delta(G)$. A graph is called k*-regular* if each vertex has degree k. We denote by P_n and C_n the path and the cycle of n vertices, respectively. For a positive integer n, we write $[n] = \{1, 2, \cdots, n\}$. We also use "iff" to denote "if and only if".

A subset W of vertices of a graph G is said to be a *dominating set* if each vertex outside W has at least one neighbor in W. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set with cardinality $\gamma(G)$ is also called a γ-set of G.

An *efficient dominating set* in G is a subset W of vertices such that W is independent and each vertex outside W has exactly one neighbor in W [9]. Let $f : V(G) \to \{0, 1\}$ be a function such that $f(v) = 1$ iff $v \in D$. We say f is *a dominating (resp. an efficient dominating) function* respect to D of G iff D is a dominating (resp. an efficient dominating) set of G.

The domination of graphs and its variations have attracted considerable attention in the past [7], [16]. Among many domination type graph invariants, Roman domination and double Roman domination have recently been studied extensively [1], [2], [4], [8], [11]–[14]. While the original motivation, defending the Roman Empire, may be a popular motivation [17], there are many practical problems related to Roman domination, for example positioning emergency services such as fire brigades, in optimal way when limited resources are available. In the original problem, a province may be defended by using one of the two armies from a neighboring province, thus possibly covering the empire with less armies than provinces. When a huge fire emerges, more than one fire brigade may be needed in the proximity, hence double or even multiple Roman domination may be a relevant model.

A *double Roman dominating function* (DRDF) on a graph G is a mapping $f : V(G) \to \{0, 1, 2, 3\}$ such that: (1) every vertex assigned 0 has at least one neighbor assigned 3 or two neighbors assigned 2, and (2) every vertex u assigned 1 has at least one neighbor assigned 2 or 3. The *weight* $\omega(f)$ of a DRDF f is the value $\omega(f) = \sum_{u\in V(G)} f(u)$. The minimum weight over all DRDFs of a graph G is called the *double Roman domination number* $\gamma_{dR}(G)$ of G. A DRDF f of G with $\omega(f) = \gamma_{dR}(G)$ is said to be a *γ_{dR}-function* of G.

Beeler *et al.* [2] initiated the investigation of the double Roman domination. They demonstrated that $\gamma_{dR}(G)$ lies between $2\gamma(G)$ and $3\gamma(G)$, and defined the concept of a *double Roman* graph G that it satisfies $\gamma_{dR}(G) = 3\gamma(G)$, and proved

Proposition 1 [2]: In a DRDF with weight $\gamma_{dR}(G)$, no vertex needs to be labeled 1.

Using Proposition 1, we will later be able to restrict attention only to DRDFs in which no vertex assigned 1.

Given a DRDF f of a graph G, let $(V_0^f, V_1^f, V_2^f, V_3^f)$ be the partition such that $V_t^f = \{x \mid f(x) = t\}$. It can be seen that there will be a 1-1 mapping between f and $(V_0^f, V_1^f, V_2^f, V_3^f)$. Hence, we can simplify $f = (V_0^f, V_1^f, V_2^f, V_3^f)$ as $f = (V_0, V_1, V_2, V_3)$.

Petersen graphs are among the most interesting examples when considering nontrivial graph invariants. The domination and its variations on generalized Petersen graphs have attracted considerable attention, see [3], [5], [6], [11], [15], [18]–[21]. Let n and k be integers, and $n \geq 2k + 1 \geq 3$. The vertex set of the generalized Petersen graph $P(n, k)$ is $\{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$. The edges of $P(n, k)$ are $u_iu_{i+1}, u_iv_i, v_iv_{i+k}$ for $1 \leq i \leq n$, where subscripts are computed modulo n (see [20] and e.g. $(n, k) \in \{(9, 3), (10, 1)\}$ in Fig. 1). Note that generalized Petersen graphs are 3-regular graphs.

Kutnar and Petecki [10] proposed the double generalized Petersen graphs and studied their automorphisms and structural properties.

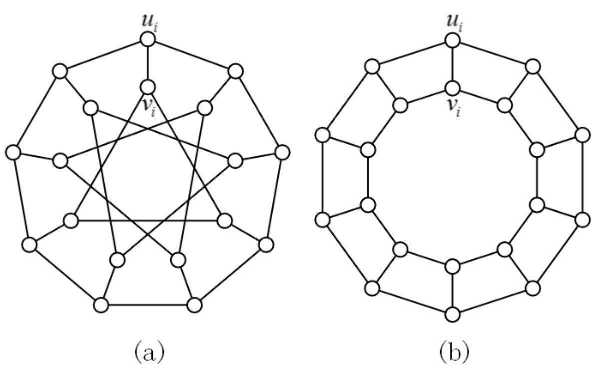

FIGURE 1. (a) $P(9, 3)$; (b) $P(10, 1)$.

Given an integer $n \geq 3$ and $1 \leq k \leq n - 1$, the double generalized Petersen graph $DP(n, k)$ has vertex set

$$V(DP(n, k)) = \{x_i, y_i, u_i, v_i \mid i \in [n]\}$$

and its edge set is the union $E(DP(n, k)) = E_1 \cup E_2 \cup E_3$, where

$$\begin{aligned} E_1 &= \{\{y_i, y_{i+1}\}, \{x_i, x_{i+1}\} | i \in [n]\}, \\ E_2 &= \{\{x_i, u_i\}, \{y_i, v_i\} | i \in [n]\}, \\ E_3 &= \{\{v_i, u_{i+k}\}, \{u_i, v_{i+k}\} | i \in [n]\}, \end{aligned}$$

and the subscripts are reduced modulo n (see e.g. $(n, k) \in \{(8, 3), (10, 1)\}$ in Fig. 2).

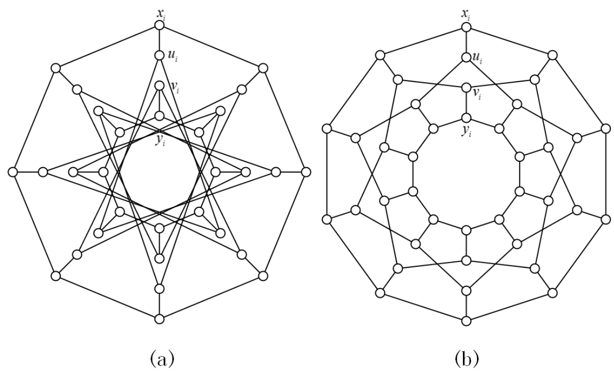

FIGURE 2. (a) $DP(8, 3)$; (b) $DP(10, 1)$.

In graph theory, the discharging approach was first proposed to successfully attack the famous Four-Color Theorem with the aid of computers. Since then it was extensively studied and applied to study various graph coloring problems on planar graphs. In this paper, we provide a general discharging method for domination type problems which is much different from that for graph coloring problems. By using the discharging method, we first prove that $\gamma_{dR}(G) \geq \frac{3n}{\Delta(G)+1}$ for any connected graph G with at least two vertices. Moreover, it is shown that if $\gamma_{dR}(G) = \frac{3n}{\Delta(G)+1}$, then either $G \in \{P_n, C_n\}$ or G is a double Roman graph. In addition, the exact values of the double Roman domination numbers of $P(n, 1)$ and $DP(n, 1)$ are determined. It is also shown that that $P(n, 1)$ is double Roman iff $n \not\equiv 2 \pmod 4$ and $DP(n, 1)$ is double Roman iff $n \equiv 0 \pmod 4$.

II. SOME PROPERTIES OF DOUBLE ROMAN GRAPHS

Lemma 1: A graph G is *double Roman* iff it has a γ_{dR}-function $(V_0, \emptyset, V_2, V_3)$ with $|V_2| = 0$.

Proof: ($\Leftarrow$) Suppose that $f = (V_0, \emptyset, V_2, V_3)$ is a γ_{dR}-function on G such that $|V_2| = 0$. Then $\gamma_{dR}(G) = 3|V_3|$ and V_3 is a dominating set of G. So we have $\gamma(G) \leq |V_3| = \gamma_{dR}(G)/3$. Since $\gamma_{dR}(G) \leq 3\gamma(G)$, we can obtain $\gamma_{dR}(G) = 3\gamma(G)$. Hence, G is a double Roman graph.

($\Rightarrow$) Suppose to the contrary that for any γ_{dR}-function $(V_0, \emptyset, V_2, V_3)$ of G we have $|V_2| > 0$.

Let $f = (V_0, \emptyset, V_2, V_3)$ be a γ_{dR}-function of G with minimum $|V_2|$. Then we have

$$\gamma_{dR}(G) = 2|V_2| + 3|V_3|. \quad (1)$$

Furthermore, we have

Claim 1: $V_2 \cup V_3$ is a γ-set of G.

Suppose that there is a γ-set D of G such that $|D| < |V_2 \cup V_3|$. Let $f' = (V'_0, \emptyset, V'_2, V'_3)$ with $V'_2 = \emptyset$ and $V'_3 = D$. Then $f' = (V'_0, \emptyset, V'_2, V'_3)$ is a DRDF of G and

$$|V'_3| = |D| = \gamma(G) \quad (2)$$

Since G is double Roman, we have $3\gamma(G) = \gamma_{dR}(G)$. Then by Eqs. (1) and (2),

$$3|V'_3| = 3\gamma(G) = \gamma_{dR}(G) = 2|V_2| + 3|V_3|$$

So we have $\omega(f) = \omega(f') = \gamma_{dR}(G)$, but $V'_2 = \emptyset$. That is to say, we find a γ_{dR}-function $(V'_0, \emptyset, V'_2, V'_3)$ with $V'_2 = \emptyset$, a contradiction. Hence $V_2 \cup V_3$ is a γ-set, as claimed.

Thus we have $\gamma(G) = |V_2| + |V_3|$ and so

$$3(|V_2| + |V_3|) = 3\gamma(G) = \gamma_{dR}(G) = 2|V_2| + 3|V_3|$$

implying $|V_2| = 0$, which is a contradiction. □

Now we introduce the discharging approach.

Discharging Procedure A: Let f be a DRDF of a graph G. The *initial charge* of every vertex $x \in V(G)$ is set to be $s(x) = f(x)$. We apply the discharging procedure defined by applying the following rules:

R1: Every vertex x with $s(x) = 3$ sends $\frac{3}{\Delta(G)+1}$ charge to each adjacent vertex in V_0.

R2: Every vertex x with $s(x) = 2$ sends $\frac{3}{2(\Delta(G)+1)}$ charge to each adjacent vertex in V_0.

Theorem 1: For any connected graph G with $n \geq 2$ vertices, we have

$$\gamma_{dR}(G) \geq \frac{3n}{\Delta(G)+1}.$$

Moreover, if $\gamma_{dR}(G) = \frac{3n}{\Delta(G)+1}$, then $G \in \{P_n, C_n\}$ or G is *double Roman.*

Proof: Let $f = \{V_0, \emptyset, V_2, V_3\}$ be a γ_{dR}-function of G. We will prove that $\omega(f) \geq \frac{3n}{\Delta(G)+1}$. If $\Delta(G) = 1$, then $n = 2$ and $\gamma_{dR}(G) = 3 = \frac{3n}{\Delta(G)+1}$. Obviously, G is *double Roman*. Now we assume $\Delta(G) \geq 2$ and proceed the proof by using Discharging Procedure A for f on G. Then we have:

- For each vertex $v \in V_3$, since it sends charges to at most $d_G(v)$ vertices, by **R1** we can obtain that the final charge $s'(v) \geq s(v) - d_G(v) \cdot \frac{3}{\Delta(G)+1} \geq 3 - \frac{3\Delta(G)}{\Delta(G)+1} = \frac{3}{\Delta(G)+1}$;
- For each vertex $v \in V_2$, since it sends charges to at most $d_G(v)$ vertices, by **R2** we can obtain that the final charge $s'(v) \geq s(v) - d_G(v) \cdot \frac{3}{2(\Delta(G)+1)} \geq 2 - \Delta(G) \cdot \frac{3}{2(\Delta(G)+1)} \geq \frac{3}{\Delta(G)+1}$;
- For each vertex $v \in V_0$, by the definition of double Roman domination, v has a neighbor u assigned 3 or two vertices assigned 2. Hence, by **R1** or **R2**, v receives $\frac{3}{\Delta(G)+1}$ charge from u or from u_1 and u_2. So $s'(v) \geq \frac{3}{\Delta(G)+1}$.

From the above analysis, we know that $s'(v) \geq \frac{3}{\Delta(G)+1}$ for any $v \in V(G)$. Because the discharging procedure does not change the total value of charge in G, we get

$$\sum_{x \in V(G)} s(x) = \sum_{x \in V(G)} s'(x) \geq \sum_{x \in V(G)} \frac{3}{\Delta(G)+1} = \frac{3n}{\Delta(G)+1}.$$

Hence, we have $\gamma_{dR}(G) = \omega(f) \geq \frac{3n}{\Delta(G)+1}$.

Moreover, if $\omega(f) = \frac{3n}{\Delta(G)+1}$, then $s'(v) = \frac{3}{\Delta(G)+1}$ for any $v \in V(G)$. We will prove that $G \in \{P_n, C_n\}$ or G is a double Roman graph. If $f(v) \neq 2$ for any $v \in V(G)$, then by Lemma 1 we obtain that G is a double Roman graph. If there is a vertex $w \in V(G)$ assigned 2, then by **R2**, $s'(w) \geq 2 - \frac{3\Delta(G)}{2(\Delta(G)+1)} = \frac{3}{\Delta(G)+1} + \frac{\Delta(G)-2}{2(\Delta(G)+1)}$. Hence, $\frac{\Delta(G)-2}{2(\Delta(G)+1)} = 0$ and $\Delta(G) = 2$, namely, $G \in \{P_n, C_n\}$. □

III. DOUBLE ROMAN DOMINATION IN 3-REGULAR GRAPHS

In the following, for convenience we write $f(\cdot) = q^+$ instead of $f(\cdot) \geq q$. We call a path $v_1 v_2 \cdots v_k$ to be a path of *type* $c_1 - c_2 - \cdots - c_k$ if $f(v_i) = c_i$ for $i \in [k]$. Let H be a subgraph of a graph G induced by four vertices u_1, u_2, u_3, u_4 with $u_1u_2, u_1u_3, u_1u_4 \in E(H)$, satisfying $f(u_1) = 0, f(u_2) = a, f(u_3) = b$ and $f(u_4) = c$, then we call H to be a subgraph of *type* $0 - (a, b, c)$.

By Theorem 1, we have

Corollary 1: Suppose G is a 3-regular graph, then $\gamma_{dR}(G) \geq \lceil \frac{3|V(G)|}{4} \rceil$.

By using Discharging Procedure A, we get

Lemma 2: Let $f = (V_0, \emptyset, V_2, V_3)$ be a γ_{dR}-function of a 3-regular graph G. If we use Discharging Procedure A for f on G, then

a) if $\exists v \in V(G)$ such that $f(v) = 2$, then $s'(v) - \frac{3}{4} \geq \frac{1}{8}$,
b) if $\exists P$ of type $2 - 2$, then $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq 1$,
c) if $\exists P$ of type $2^+ - 3$, then $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \frac{5}{4}$,
d) if $\exists P$ of type $2 - 2 - 2^+$, then $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \frac{15}{8}$,
e) if $\exists P$ of type $2 - 0 - 3$, then $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \frac{1}{2}$,
f) if $\exists P$ of type $3 - 0 - 3$, then $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \frac{3}{4}$,
g) if $\exists H$ of type $0 - (2^+, 2^+, 2^+)$, then $\sum_{v \in V(H)}(s'(v) - \frac{3}{4}) \geq \frac{3}{4}$,
h) if $\exists H$ of type $0 - (2, 3, 3)$, then $\sum_{v \in V(H)}(s'(v) - \frac{3}{4}) \geq \frac{5}{4}$.

Proof: (a) Since v has at most 3 neighbors in V_0, by **R2**, we obtain that the final charge $s'(v) \geq 2 - 3 \times \frac{3}{8} = \frac{7}{8}$. Hence, $s'(v) - \frac{3}{4} \geq \frac{1}{8}$.

(b) Let $P = xy$. Since each $v \in \{x, y\}$ has at most 2 neighbors in V_0, by **R2**, we have $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \sum_{v \in V(P)}(2 - 2 \times \frac{3}{8} - \frac{3}{4}) = 1$.

(c) If P is a path of type $2 - 3$, then by **R1** and **R2**, $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq (2 - 2 \times \frac{3}{8} - \frac{3}{4}) + (3 - 2 \times \frac{3}{4} - \frac{3}{4}) = \frac{5}{4}$. If P is a path of type $3 - 3$, then by **R1**, $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq \sum_{v \in V(P)}(3 - 2 \times \frac{3}{4} - \frac{3}{4}) = \frac{3}{2} > \frac{5}{4}$.

(d) Let $P = xyz$. Since y has at most one neighbor in V_0 and each vertex in $\{x, z\}$ has at most 2 neighbors in V_0, by **R1** and **R2**, we have $\sum_{v \in V(P)}(s'(v) - \frac{3}{4}) \geq (2 - 2 \times \frac{3}{8} - \frac{3}{4}) + (2 - \frac{3}{8} - \frac{3}{4}) + \min\{2 - 2 \times \frac{3}{8} - \frac{3}{4}, 3 - 2 \times \frac{3}{4} - \frac{3}{4}\} = \frac{15}{8}$.

(e) By **R1** and **R2**, we have $\sum_{v\in V(P)}(s'(v)-\frac{3}{4}) \geq (2-2\times\frac{3}{8}-\frac{3}{4})+(\frac{3}{8}+\frac{3}{4}-\frac{3}{4})+(3-3\times\frac{3}{4}-\frac{3}{4})=\frac{1}{2}$.

(f) By **R1**, we have $\sum_{v\in V(P)}(s'(v)-\frac{3}{4}) \geq (3-3\times\frac{3}{4}-\frac{3}{4})+(\frac{3}{4}+\frac{3}{4}-\frac{3}{4})+(3-3\times\frac{3}{4}-\frac{3}{4})=\frac{3}{4}$.

(g) By **R1** and **R2**, we have $\sum_{v\in V(H)}(s'(v)-\frac{3}{4}) \geq 3\times\min\{2-2\times\frac{3}{8}-\frac{3}{4},3-2\times\frac{3}{4}-\frac{3}{4}\}-\frac{3}{4}=\frac{3}{4}$.

(h) By **R1** and **R2**, we have $\sum_{v\in V(H)}(s'(v)-\frac{3}{4}) \geq (2-2\times\frac{3}{8}-\frac{3}{4})+2\times(3-2\times\frac{3}{4}-\frac{3}{4})-\frac{3}{4}=\frac{5}{4}$. □

A. DOUBLE ROMAN DOMINATION IN $P(n,1)$

In [5], it was proved that

Theorem 2: [5] Let $n \geq 3$. Then we have

$$\gamma(P(n,1)) = \begin{cases} \lceil\frac{n}{2}\rceil, & n \equiv 0,1,3 \pmod 4 \\ \frac{n}{2}+1, & n \equiv 2 \pmod 4 \end{cases}$$

An analogous result for double Roman domination is given by Theorem 3 that we will prove in this section.

Theorem 3: Let $n \geq 3$. Then we have

$$\gamma_{dR}(P(n,1)) = \begin{cases} \frac{3n}{2}, & n \equiv 0 \pmod 4 \\ \frac{3n+3}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{3n+4}{2}, & n \equiv 2 \pmod 4 \end{cases}$$

Together with Theorem 2, Theorem 3 directly implies

Corollary 2: Let $n \geq 3$. Then $P(n,1)$ is *double Roman* iff $n \not\equiv 2 \pmod 4$.

Now, we will prove Proposition 2 and several Lemmas that together imply Theorem 3.

Proposition 2: Let $n \geq 3$. Then we have

$$\gamma_{dR}(P(n,1)) \leq \begin{cases} \frac{3n}{2}, & n \equiv 0 \pmod 4 \\ \frac{3n+3}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{3n+4}{2}, & n \equiv 2 \pmod 4 \end{cases}$$

Proof: Since $\gamma_{dR}(G) \leq 3\gamma(G)$ for any G, together with Theorem 2, we have the desired upper bound if $n \not\equiv 2$ (mod 4).

Assume $n \equiv 2 \pmod 4$ and let

$$P = \begin{bmatrix} 0 & 0 & 3 & 0 & 0 & 2 \\ 3 & 0 & 0 & 0 & 3 & 0 \end{bmatrix},$$

where P defines f by

$$P = \begin{bmatrix} f(u_1) & f(u_2) & f(u_3) & f(u_4) & f(u_5) & f(u_6) \\ f(v_1) & f(v_2) & f(v_3) & f(v_4) & f(v_5) & f(v_6) \end{bmatrix}.$$

Then, by repeating the leftmost 4 columns of P, we obtain a DRDF with weight $\frac{3n+4}{2}$ and the desired upper bound is established. □

By Corollary 1, we have

Lemma 3: Let $n \geq 3$, $n \equiv 0 \pmod 4$ and f be a γ_{dR}-function of $P(n,1)$. Then $\omega(f) \geq \frac{3n}{2}$.

Lemma 4: Assume $n \geq 3$, $n \equiv 1,3 \pmod 4$ and f is a γ_{dR}-function of $P(n,1)$. Then we have $\omega(f) \geq \frac{3n+3}{2}$.

Proof: Suppose that there is a γ_{dR}-function f of $P(n,1)$ for which $\omega(f)=\frac{3n+1}{2}$. If we use Discharging Procedure A for f on $P(n,1)$, then we get

$$\begin{aligned} \sum_{x\in V(P(n,1))}(s'(x)-\frac{3}{4}) &= \sum_{x\in V(P(n,1))}(s(x)-\frac{3}{4}) \\ &= \omega(f)-\frac{3|V(P(n,1))|}{4} \\ &= \frac{1}{2} \end{aligned} \tag{3}$$

Claim 2: There is no path P of type 2^+-2^+ or $3-0-3$ and no subgraph H of type $0-(2^+,2^+,2^+)$.

The Claim 2 easily follows from Lemma 2 and Eq.(3).

Since $n \equiv 1,3 \pmod 4$, we can assume that $n=2k+1$, where $k \geq 1$. Then $\omega(f)=\frac{3n+1}{2}=3k+2$. Hence, there is at least one vertex $u_i \in V(P(n,1))$ for which $f(u_i)=2$. By Claim 2, we have $f(v_i)=f(u_{i\pm1})=0$. Since u_{i+1} must be double Roman dominated and by Claim 2, we have $(f(v_{i+1}),f(u_{i+2})) \in \{(2,0),(0,2^+),(3,0)\}$.

- If $(f(v_{i+1}),f(u_{i+2}))=(2,0)$, then we have $f(v_{i+2})=0$ and $f(u_{i+3})=3$. Since v_{i+2} must be double Roman dominated, we get $f(v_{i+3}) \geq 2$. Then $v_{i+3}u_{i+3}$ is a path of type 2^+-2^+, contradicting Claim 2.
- If $(f(v_{i+1}),f(u_{i+2}))=(0,2^+)$, then because v_{i+1} must be double Roman dominated, we obtain that $f(v_{i+2})=3$. Then $v_{i+2}u_{i+2}$ is a path of type 2^+-3, a contradiction.
- If $(f(v_{i+1}),f(u_{i+2}))=(3,0)$, then $u_iu_{i+1}v_{i+1}$ and $u_iv_iv_{i+1}$ are both paths of type $2-0-3$, and we have $\sum_{v\in\{v_i,u_{i+1}\}}(s'(v)-\frac{3}{4}) \geq \frac{3}{4}$, a contradiction with Eq. (3).

This assertion completes the proof of Lemma 4. □

Lemma 5: Suppose $n \geq 3$, $n \equiv 2 \pmod 4$ and f is a γ_{dR}-function, then we have $\omega(f) \geq \frac{3n+4}{2}$.

Proof: Suppose that there is a γ_{dR}-function f of $P(n,1)$ with $\omega(f)=\frac{3n+2}{2}$. We use Discharging Procedure A for f on $P(n,1)$. Then

$$\begin{aligned} \sum_{x\in V(P(n,1))}(s'(x)-\frac{3}{4}) &= \sum_{x\in V(P(n,1))}(s(x)-\frac{3}{4}) \\ &= \omega(f)-\frac{3|V(P(n,1))|}{4} \\ &= 1 \end{aligned} \tag{4}$$

Observe that the condition Eq.(4) and Lemma 2 directly imply Claim 3.

Claim 3: There is no path P of type 2^+-3 or $2-2-2^+$, and no subgraph H of type $0-(2,3,3)$.

Claim 4: There is no path of type $2-2$.

Suppose to the contrary that there exists a path $P=uv$ of type $2-2$. Then by Lemma 2 (b) and Eq.(4), we can obtain that $s'(x)-\frac{3}{4}=0$ for any $x \in V(P(n,1)) \setminus V(P)$. Obviously, by Lemma 2 (a), there exists no vertex $x \in V(P(n,1)) \setminus V(P)$ such that $f(x)=2$. Let $w \in N(u) \setminus \{v\}$. Since there is no path of type $2-2-2^+$, we have $f(w)=0$. Hence, in order to double Roman dominate w, we know that w has a neighbor y with $f(y)=3$, implying that $s'(w)-\frac{3}{4} \geq \frac{3}{8}$, a contradiction.

Since $n \equiv 2 \pmod 4$, we can assume that $n = 4k + 2$, where k is an integer with $k \geq 1$. Then $\omega(f) = \frac{3n+2}{2} = 3(2k+1)+1$. Hence, there exists at least one vertex $u_i \in V(P(n,1))$ such that $f(u_i) = 2$. By Claim 4, we have $f(v_i) = f(u_{i\pm1}) = 0$. Since u_{i+1} should be double Roman dominated and by Claim 3, we have $(f(v_{i+1}), f(u_{i+2})) \in \{(2,0), (0,2^+), (2,2), (3,0), (2,3), (3,2)\}$.
Now, we consider five cases.

Case 1: $(f(v_{i+1}), f(u_{i+2})) = (2,0)$.
Since there exists no path of type $2-2^+$, we have $f(v_{i+2}) = 0$. As u_{i+2} and v_{i+2} should be double Roman dominated, we have $f(u_{i+3}) = 3$ and $f(v_{i+3}) \geq 2$. Then there exists a path of type 2^+-3, contradicting Claim 3.

Case 2: $(f(v_{i+1}), f(u_{i+2})) = (0, 2^+)$.
Since v_{i+1} must be double Roman dominated, we have $f(v_{i+2}) = 3$. Then there is a path of type 2^+-3, contradicting Claim 3.

Case 3: $(f(v_{i+1}), f(u_{i+2})) = (2,2)$.
Since there exists no path of type $2-2^+$, we get $f(v_{i+2}) = f(u_{i+3}) = 0$. Then $f(v_{i+3}) = 0$, otherwise, if $f(v_{i+3}) \geq 2$ then $\sum_{v\in A}(s'(v) - \frac{3}{4}) \geq \frac{5}{4}$, where $A = \{v_i, v_{i+1}, v_{i+2}, v_{i+3}, u_i, u_{i+1}, u_{i+2}, u_{i+3}\}$, a contradiction with Eq. (4). Note that v_{i+3} and u_{i+3} need to be double Roman dominated, we have $f(v_{i+4}) = 3$ and $f(u_{i+4}) \geq 2$. Then there is a path of type 2^+-3, contradicting Claim 3.

Case 4: $(f(u_{i+2}), f(v_{i+1})) \in \{(2,3), (3,2)\}$.
In this case, let $A = \{v_i, v_{i+1}, v_{i+2}, u_i, u_{i+1}, u_{i+2}\}$. Then $\sum_{v\in A}(s'(v) - \frac{3}{4}) \geq 7 - \frac{3}{8} - \frac{3}{4} - 6\times\frac{3}{4} = \frac{11}{8}$, a contradiction with Eq. (4).

Case 5: $(f(v_{i+1}), f(u_{i+2})) = (3,0)$.
By the proofs of Cases 1-4 and the symmetry, we get $(f(v_{i-1}), f(u_{i-2})) = (3,0)$. Let $A = \{v_{i-1}, v_i, v_{i+1}, u_{i-1}, u_i, u_{i+1}\}$. Then $\sum_{v\in A}(s'(v) - \frac{3}{4}) \geq 8 - \frac{3}{4} - \frac{3}{4} - 6\times\frac{3}{4} = 2$, a contradiction with Eq. (4).

This completes the proof of Lemma 5. □

B. DOUBLE ROMAN DOMINATION IN $DP(n, 1)$

Lemma 6: Let $n \geq 4$. Then

$$\gamma(DP(n,1)) = \begin{cases} n, & n \equiv 0 \pmod 4 \\ n+1, & \text{otherwise} \end{cases}$$

Proof: Suppose f is a γ-function of $DP(n,1)$ with $n \geq 4$. First, we will show that $\gamma(DP(n,1)) \geq n$. Since $\gamma(G) \geq \frac{|V(G)|}{\Delta+1}$, we have $\gamma(DP(n,1)) \geq \frac{|V(DP(n,1))|}{\Delta+1} = \frac{4n}{3+1} = n$, with equality iff $DP(n,1)$ has an efficient dominating set.

Next, we will demonstrate that $\gamma(DP(n,1)) \geq n+1$, for $n \not\equiv 0 \pmod 4$. Suppose that $\gamma(DP(n,1)) \leq n$, for $n \not\equiv 0 \pmod 4$. Then there exists an efficient dominating function f of $DP(n,1)$.

First observe that we must have $f(x_i) = 1$ for some $i \in [n]$. Otherwise, to dominate x_i, $f(u_i) = 1$ for $i \in [n]$. But n vertices u_i do not dominate any of y_i, for $i \in [n]$, and hence $|\{w \mid f(w) = 1\}| > n$, so f is not an efficient dominating function, a contradiction. So we may assume that $f(x_i) = 1$ and we have $f(x_{i-1}) = f(x_{i-2}) = f(u_{i-1}) = f(x_{i+1}) = f(x_{i+2}) = f(u_{i+1}) = f(u_i) = f(v_{i+1}) = f(v_{i-1}) = 0$. Now, to dominate v_i, we consider the following two cases.

- If $f(v_i) = 0$, we have $f(y_i) = 1$ and $f(y_{i-1}) = f(y_{i-2}) = f(v_{i-1}) = f(y_{i+1}) = f(y_{i+2}) = f(v_{i+1}) = 0$. To dominate $u_{i\pm1}$ and $v_{i\pm1}$, we have $f(u_{i\pm2}) = f(v_{i\pm2}) = 1$. It is now straightforward that $f(x_j) = f(x_{j+4})$, $f(u_j) = f(u_{j+4})$, $f(v_j) = f(v_{j+4})$ and $f(y_j) = f(y_{j+4})$ for any j. Therefore, an efficient dominating function f exists only if $n \equiv 0 \pmod 4$, a contradiction.
- If $f(v_i) = 1$, then analogous as before, the assumption that f is an efficient dominating function implies that $f(u_{i+2}) = 1$ and $f(y_{i+2}) = 1$, forcing in turn $f(x_{i+4}) = 1$ and $f(v_{i+4}) = 1$, leading to contradiction when $n \not\equiv 0 \pmod 4$.

Now we show the upper bounds, and we will use a pattern with 4 rows and n columns to represent a DRDF as follows.

$$f(V(DP(n,1))) = \begin{pmatrix} f(x_1) & f(x_2) & \cdots & f(x_n) \\ f(u_1) & f(u_2) & \cdots & f(u_n) \\ f(v_1) & f(v_2) & \cdots & f(v_n) \\ f(y_1) & f(y_2) & \cdots & f(y_n) \end{pmatrix}$$

For $\ell \geq 1$, let

$$f(V(DP(4\ell,1))) = \left(\begin{array}{c|cccc} & 1 & 0 & 0 & 0 \\ - & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 0 \end{array}\right),$$

$$f(V(DP(4\ell+1,1))) = \left(\begin{array}{c|cccc|c} & 1 & 0 & 0 & 0 & 1 \\ - & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$f(V(DP(4\ell+2,1))) = \left(\begin{array}{c|cccc|cc} & 1 & 0 & 0 & 0 & 1 & 0 \\ - & 0 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$f(V(DP(4\ell+3,1))) = \left(\begin{array}{c|cccc|ccc} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ - & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right),$$

where "–" means that we repeat the leftmost four columns of the corresponding pattern at most $\ell - 1$ times. Then f induces a dominating set of $DP(n,1)$. Therefore, we have $\gamma(DP(n,1)) \leq n$ if $n \equiv 0 \pmod 4$ and $\gamma(DP(n,1)) \leq n+1$, otherwise. This assertion completes the proof. □

Proposition 3: Let $n \geq 4$. Then

$$\gamma_{dR}(DP(n,1)) = \begin{cases} 3n, & n \equiv 0 \pmod 4 \\ 3n+2, & \text{otherwise} \end{cases}$$

Proof: We first show the upper bounds. If $n \equiv 0 \pmod 4$, we have $\gamma_{dR}(DP(n,1)) \leq 3\gamma(DP(n,1)) = 3n$, as desired.

Therefore we can assume that $n > 4$ in the rest of the proof. For $\ell \geq 0$, let

$$f(V(DP(4\ell+5,1))) = \left(\begin{array}{c|cccc|ccccc} & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 \\ - & 3 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 \\ & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 3 & 0 \\ & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 \end{array}\right),$$

$$f(V(DP(4\ell+6,1))) = \left(\begin{array}{c|cccc|cccccc} & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 3 & 0 \\ - & 3 & 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 2 & 3 & 0 \\ & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \end{array}\right).$$

For $\ell \geq 1$, let

$$f(V(DP(4\ell+3,1))) = \left(\begin{array}{c|cccc|ccc} & 0 & 0 & 3 & 0 & 0 & 3 & 0 \\ - & 3 & 0 & 0 & 0 & 3 & 0 & 2 \\ & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ & 3 & 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right).$$

Then f induces a DRDF of $DP(n, 1)$ with weight $3n + 2$ if $n \not\equiv 0 \pmod 4$. Therefore, we have $\gamma_{dR}(DP(n, 1)) \leq 3n + 2$ if $n \not\equiv 0 \pmod 4$.

Now we will show the lower bounds. First, by Proposition 1, we have $\gamma_{dR}(DP(n, 1)) \geq 3n$. Hence, it is sufficient to prove $\gamma_{dR}(DP(n, 1)) \geq 3n + 2$ for $n \not\equiv 0 \pmod 4$. Suppose that $\gamma_{dR}(DP(n, 1)) \leq 3n + 1$ for $n \not\equiv 0 \pmod 4$ and we assume f is a DRDF with $\omega(f) = 3n + 1$. By applying Discharging Procedure A, we have $s'(v) \geq \frac{3}{4}$ for any $v \in V(DP(n, 1))$. Recalling Proposition 1 we have

$$\sum_{v \in V(DP(n,1))} (s'(v) - \frac{3}{4}) \leq 1. \tag{5}$$

Lemma 2 directly implies

Claim 5: There exists no path of type $2^+ - 3$ or $2 - 2 - 2^+$.

The next claim follows by an argument, similar as used in the proof of Claim 4.

Claim 6: There exists no path of type $2 - 2$.

Since $\omega(f) = 3n + 1 = 3|V_3| + 2|V_2|$, we have $|V_2| = 3k + 2$ for some $k \in \mathbb{Z}$. By Lemma 2(a), we have $|V_2| \leq 8$. So $|V_2| = 2$, 5 or 8. Now, we consider two cases.

Case 1: $|V_2| = 2$.

In this case, let $u_2 = \{u, v\}$. Then by Claim 6, u is not a neighbor of v. Hence, there would exist at least two paths of type $2 - 0 - 3$ started at u: $P_1 = ux_1y_1$ and $P_2 = ux_2y_2$, and at least two paths of type $2 - 0 - 3$ started at v: $P_3 = vx_3y_3$ and $P_4 = vx_4y_4$. Consequently,

$$\sum_{v \in V'} (s'(v) - \frac{3}{4}) \geq 2 \times (8 - \frac{3}{8} - 4 \times \frac{3}{4} - 5 \times \frac{3}{4}) = \frac{7}{4} > 1,$$

where $V' = V(P_4) \cup V(P_3) \cup V(P_2) \cup V(P_1)$, which is a contradiction with Eq. (5).

Case 2: $|V_2| = 5$ or 8.

In this case, there would exist at least three mutually disjoint paths P_1, P_2, P_3 of type $2 - 0 - 3$. Hence, $\sum_{v \in V(P_1) \cup V(P_2) \cup V(P_3)} (s'(v) - \frac{3}{4}) \geq 3 \times \frac{1}{2} > 1$, a contradiction with Eq. (5).

This completes the proof of proposition. $\square$

To provide a complete answer, we need the following fact that can easily be proved as an exercise

Fact 1: $\gamma_{dR}(DP(3, 1)) = 11$.

From Fact 1 and Propositions 1 and 3 we conclude

Theorem 4: Let $n \geq 3$.

$$\gamma_{dR}(DP(n, 1)) = \begin{cases} 3n, & n \equiv 0 \pmod 4 \\ 3n + 2, & \text{otherwise} \end{cases}$$

Corollary 3: If $n \geq 3$, then $DP(n, 1)$ is double Roman iff $n \equiv 0 \pmod 4$.

IV. CONCLUSION

The discharging method is most well-known for its central role in the proof of the Four Color Theorem. This proof technique was extensively applied to study various graph coloring problems, in particular on planar graphs. In this paper, we show that suitably altered discharging technique can also be used on domination type problems. The general discharging approach for domination type problems is illustrated on a specific domination type problem, the double Roman domination on some generalized Petersen graphs.

This discharging approach is promising and effective for processing other types of domination, such as Roman domination and independent domination, on many other interesting classes of graphs, e.g. regular graphs and Cartesion product of paths and cycles.

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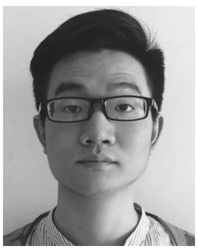

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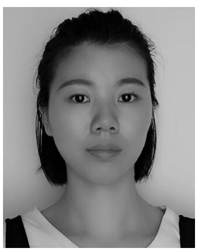

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