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Spreading Dynamics of a Word-of-Mouth Model on Scale-Free Networks

WENJIN LIU, TAO LI, XIONGDING LIU, AND HAO XU
National Demonstration Center for Experimental Electrical and Electronic Education, Yangtze University, Jingzhou 434023, China
School of Electronics and Information, Yangtze University, Jingzhou 434023, China

Corresponding author: Tao Li (taohust2008@163.com)

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ABSTRACT Nowadays, human behaviors are known to spread through social contact. Word-of-mouth (WOM) communication has also become a prevalent strategy for product sales on the internet and most consumers attribute a higher importance to WOM communication. Mathematical modeling is an important approach to study WOM communication in online social networks. In order to investigate the characteristics of WOM communication and heterogeneity of online social networks, a *SIALS* (susceptible-infected-acknowledged-loath) model is presented on scale-free networks. The spreading dynamics of WOM are analyzed in detail by using the theory of mean field. The basic reproductive number R_0 is calculated by the next generation matrix method and two equilibriums are derived. The theoretical analysis indicates that the basic reproductive number depends mainly on the transmission of WOM and the topology of the underlying networks. Furthermore, the global stability of the WOM-disappearance equilibrium is proved. The permanence of WOM information spreading and the global attractivity of the WOM-prevailing equilibrium are also studied in detail. Finally, numerical simulations confirm the analytical results.

INDEX TERMS *SIALS* model, heterogeneity, scale-free networks, stability, permanence.

I. INTRODUCTION

Social networks are social structures that consist of a set of social actors and social interactions between them [1]. The word-of-mouth (WOM) effect is a central instrument in viral marketing campaigns, while the transmission and adoption of opinions is crucial in understanding various political and social issues [2]. Promotion like WOM communication is a common form of product sales in social media. It is well known that more and more merchants tend to advertise through the form of WOM communication, which means that people may purchase based on others' comments. For example, one may determine to purchase when his or her friend makes good evaluations of the goods he bought. In fact, at least 80% of people who plan to make a purchase will look through online consumers' reviews before making their purchase decision. As compared to the third-party advertising, such as TV and newspaper, WOM communication has attracted a lot of people to choose shopping for its convenience and caught lots of merchants for its lower cost and much faster propagation, which means that the WOM marketing outperforms the traditional advertising marketing [3]–[12]. Especially shopping on the internet or the consumption of mobile payment, the spreading of WOM is a prevailing phenomenon. It has been found that satisfied and dissatisfied consumers tend to respectively spread acknowledged and loath comments on the items that they have purchased and used [8]–[14]. In fact, consumers tend to buy things that are given acknowledged comments, namely, acknowledged comments are more cognitive and more considered [15]. By contrast, an item given loath comments may also be bought according to the individual demand. Without a doubt, a friend's opinion or advice often can be a decisive argument for a purchase. So, network models considering social contacts and the individual heterogeneity about new products diffusion are relatively reasonable. In all words, WOM communication with the acknowledged and loath comments are more likely to influence the consumers' opinion and affect the purchase decision of potential consumers [4], [5]. With the increasing popularity of online social networks such as Facebook, Myspace, and Twitter, WOM communication has become the mainstream way of product sales [16], [17]. In short, WOM as a kind of information affects people's life.

We all know that WOM communication is trustworthy and effective in the market sales, which is demonstrated by both industry surveys and academic research.

WOM communication is believed as an even more effective "advertisement" [18]. So, it is of great significance to study the spreading of a WOM model.

Wilke and Knower first found that WOM communication had a significant impact on users' purchase decisions and this conclusion had been confirmed by many succeeding studies [19]. A number of dynamic models capturing the WOM spreading processes had been suggested previously. Besides studying three aspects of communication motivation, WOM communication effect and influencing factors, scholars also explored the online WOM communication model [19]. For example, Brown and others used social network analysis method to propose a new word-of-mouth model [20]. And it once just focused on the factor that consumers spread positive WOM in online consumer-opinion platforms. Cheung and Lee [7] identified a number of key motives of consumers' electronic WOM intention and developed an associated model based on the social psychology literature. Once a study examined how the WOM hosted by third-party websites (external WOM) and third-party free sampling influenced the feedback mechanism between internal WOM and retail sales [21]. Additionally, a number of dynamic models capturing the WOM spreading processes have been discussed [22]–[32]. Based on the network communication model, the individual information is exchanged [33].

Very recently, focusing on the modeling and analysis of the WOM marketing, the author considered multiple products instead of a single product, and established a new model in [11]. However, the model did not consider the heterogeneity of the nodes and ignored the power-law degree distribution $p(k) \sim k^{-\gamma}$ $(2 < \gamma \leq 3)$ on scale-free networks. In addition, the basic reproductive number was not obtained. Apparently, some relevant models were based on homogeneous network [23] or just researched in the small world network [24] rather than heterogeneous network. Nevertheless, previous studies have provided us with a large number of important properties of transmission process of WOM communication on social networks. It is well known that an important characteristic of social networks is their scale-free property [25]. And a large number of studies have also shown that the model on scale-free networks can truly reflect much more systems [34]. What's more, previous work hasn't proved the global attractivity of equilibrium and the permanence of spreading [25]. So, to further understand the spreading dynamics of the WOM, we propose the WOM model *SIALS* on scale-free networks considering the people's interest in evaluation of purchasing. The mean field method is the most concise method and the most widely used in epidemic disease the analysis [33], which is also applied to the paper.

The rest of this paper is organized as follows. In Section 2, we present a *SIALS* model on scale-free networks. In Section 3, the basic reproduction number and two equilibriums are obtained at first. Then, we analyze the globally asymptotic stability of WOM-disappearance equilibrium, the global attractivity of WOM-prevailing equilibrium and the permanence of WOM information spreading in detail. In Section 4, we present the simulation results of the proposed model. Finally, we conclude the paper in Section 5.

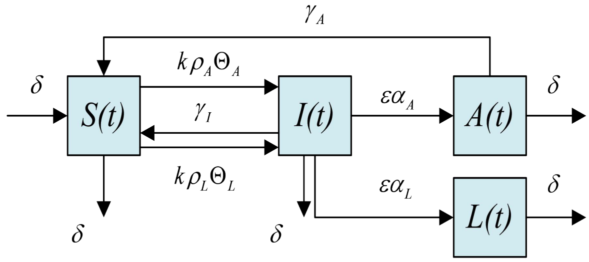

FIGURE 1. The transmission sketch of the ***SIALS*** model.

II. SYSTEM MODEL

Considering the population as a complex social network, each individual is abstracted as node and the edges are the direct connections between individuals along which the WOM may spread in population. In the dynamic transmission process of WOM communication, each node has four states: susceptible (S), which means that an individual hasn't bought any good recently but may purchase one; infected (I), which means that an individual has recently purchased one but hasn't made any comment on it yet; acknowledged (A), which means that an individual has recently purchased one and has made an acknowledged comment on it; loath (L), which means that an individual has recently purchased one and has made a loath comment on it. The transmission sketch is described in Figure 1.

During the spreading of WOM communication, a susceptible individual is connected to an acknowledged individual or a loath individual, then he will be infected with a probability of ρ_A or ρ_L, respectively, and becomes an infected individual. Considering the probability ε of people's interest in evaluation of purchasing, so when one expresses the feeling for the recently purchased goods, an infected individual becomes an acknowledged individual or a loath individual with probability of $\varepsilon\alpha_A$ or $\varepsilon\alpha_L$, respectively. In general, an infected individual or an acknowledged individual may purchase another thing according to demanding or the desire of shopping and becomes a susceptible individual at the average rate γ_I or γ_A, $\gamma_I \leq \gamma_A$. In order to make the total number of nodes remain time invariant, we assume that the entrance rate and leaving rate are both equal to δ. And all entrants occur into the susceptible class.

To account for the non-uniformity of each node, the proportion of above four nodes with k which represents the number of connections between one node and other nodes, are defined as $S_k(t)$, $I_k(t)$, $A_k(t)$, $L_k(t)$ at time t. According to the above model and the mean field theory, the dynamic mean-field reaction rate equations can be written as

$$\begin{cases} \frac{dS_k(t)}{dt} = \delta + \gamma_A A_k(t) + \gamma_I I_k(t) - \delta S_k(t) \\ \qquad -k\rho_A \Theta_A(t) S_k(t) - k\rho_L \Theta_L(t) S_k(t) \\ \frac{dI_k(t)}{dt} = k\rho_A \Theta_A(t) S_k(t) + k\rho_L \Theta_L(t) S_k(t) - \gamma_I I_k(t) \\ \qquad -\varepsilon\alpha_A I_k(t) - \varepsilon\alpha_L I_k(t) - \delta I_k(t) \\ \frac{dA_k(t)}{dt} = \varepsilon\alpha_A I_k(t) - \gamma_A A_k(t) - \delta A_k(t) \\ \frac{dL_k(t)}{dt} = \varepsilon\alpha_L I_k(t) - \delta L_k(t) \end{cases} \tag{1}$$

where $\Theta_A(t)$ denotes the probability of a contact pointing to an acknowledged individual and satisfies

$$\Theta_A(t) = \frac{1}{\langle k \rangle} \sum_k kP(k) A_k(t) \tag{2}$$

and $\Theta_L(t)$ denotes the probability of a contact pointing to a loath individual and satisfies

$$\Theta_L(t) = \frac{1}{\langle k \rangle} \sum_k kP(k) L_k(t) \tag{3}$$

Here, $\langle k \rangle$ denotes the mean degree values, i.e., $\langle k \rangle = \sum_k kP(k)$, and $P(k)$ describes the connectivity distribution. Let $A(t) = \sum_k P(k) A_k(t)$, which denotes the density of the acknowledged individuals, and $L(t) = \sum_k P(k) L_k(t)$, which denotes the density of the loath individuals in the whole network. For simplicity, given $\rho(t) = \rho_A \Theta_A(t) + \rho_L \Theta_L(t)$. Hence, the system (1) can be equivalent to the following model:

$$\begin{cases} \frac{dS_k(t)}{dt} = \delta + \gamma_A A_k(t) + \gamma_I I_k(t) - (k\rho(t) + \delta) S_k(t) \\ \frac{dI_k(t)}{dt} = k\rho(t) S_k(t) - (\gamma_I + \varepsilon(\alpha_A + \alpha_L) + \delta) I_k(t) \\ \frac{dA_k(t)}{dt} = \varepsilon\alpha_A I_k(t) - (\gamma_A + \delta) A_k(t) \\ \frac{dL_k(t)}{dt} = \varepsilon\alpha_L I_k(t) - \delta L_k(t) \end{cases} \tag{4}$$

From a practical perspective, the initial conditions for the system (4) satisfy:

$$0 \le S_k(0), I_k(0), A_k(0), L_k(0) \le 1, \quad S_k(t) + I_k(t) + A_k(t) + L_k(t) = 1, \quad \rho(0) > 0. \tag{5}$$

III. STABILITY ANALYSIS OF MODEL

In this section, the dynamic propagation process of *SIALS* model is analyzed.

Theorem 1: Define the basic reproduction number

$$R_0 = \frac{< k^2 >}{< k >} \frac{\rho_A \varepsilon\alpha_A \delta + \rho_L \varepsilon\alpha_L (\delta + \gamma_A)}{\delta(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)}.$$

There always exists a WOM-disappearance equilibrium $E_0(1, 0, 0, 0)$ for the system (4). When $R_0 > 1$, the system (4) has a unique WOM-prevailing equilibrium $E_+(S_k^*, I_k^*, A_k^*, L_k^*)$.

Proof: Since the initial conditions for the system (4) satisfies $S_k(t) + I_k(t) + A_k(t) + L_k(t) = 1$, we have $S_k(t) = 1 - I_k(t) - A_k(t) - L_k(t)$. The system (4) can be equivalent to the following model:

$$\begin{cases} \frac{dI_k(t)}{dt} = k\rho(t)(1 - I_k(t) - A_k(t) - L_k(t)) \\ \qquad -(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k(t) \\ \frac{dA_k(t)}{dt} = \varepsilon\alpha_A I_k(t) - (\delta + \gamma_A) A_k(t) \\ \frac{dL_k(t)}{dt} = \varepsilon\alpha_L I_k(t) - \delta L_k(t) \end{cases} \tag{6}$$

According to the next generation matrix method [35], the system (6) can be written as following:

$$\frac{dx}{dt} = f(x) - v(x),$$

where

$$x = (I_k, A_k, L_k)^T,$$

$$f(x) = \begin{pmatrix} k\rho(t)(1 - I_k - A_k - L_k) \\ 0 \\ 0 \end{pmatrix},$$

$$v(x) = \begin{pmatrix} (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k \\ (\delta + \gamma_A) A_k - \varepsilon\alpha_A I_k \\ \delta L_k - \varepsilon\alpha_L I_k \end{pmatrix}.$$

The Jacobian matrices of $f(x)$ and $v(x)$ at the $E_0(0, 0, 0)$ are as following:

$$F = Df(E_0) = \begin{pmatrix} 0 & F_{12} & F_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$V = Dv(E_0) = \begin{pmatrix} V_{11} & 0 & 0 \\ V_{21} & V_{22} & 0 \\ V_{31} & 0 & V_{33} \end{pmatrix},$$

where

$$F_{12} = \frac{\rho_A}{\langle k \rangle} \begin{pmatrix} P(1) & 2P(2) & \cdots & nP(n) \\ 2P(1) & 2^2P(2) & \cdots & 2nP(n) \\ \vdots & \vdots & \ddots & \vdots \\ nP(1) & 2nP(2) & \cdots & n^2P(n) \end{pmatrix},$$

$$F_{13} = \frac{\rho_L}{\langle k \rangle} \begin{pmatrix} P(1) & 2P(2) & \cdots & nP(n) \\ 2P(1) & 2^2P(2) & \cdots & 2nP(n) \\ \vdots & \vdots & \ddots & \vdots \\ nP(1) & 2nP(2) & \cdots & n^2P(n) \end{pmatrix},$$

and

$$V_{11} = (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I, \; V_{21} = -\varepsilon\alpha_A I,$$
$$V_{22} = (\delta + \gamma_A) I, \; V_{31} = -\varepsilon\alpha_L I, \; V_{33} = \delta I,$$

where I represents the identity matrix. Then the basic reproduction number is denoted by following:

$$R_0 = \rho\left(FV^{-1}\right) = \frac{< k^2 >}{< k >} \frac{\rho_A \varepsilon\alpha_A \delta + \rho_L \varepsilon\alpha_L (\delta + \gamma_A)}{\delta(\delta+\gamma_A)(\delta+\gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)},$$

where $\langle k^2 \rangle = \sum_k k^2 P(k)$.

Next, we can easily find that $E_0(1,0,0,0)$ is always an equilibrium of the system (4). In order to obtain the equilibrium $E_+\left(S_k^*, I_k^*, A_k^*, L_k^*\right)$, we let the right side of the system (4) to be equal to zero. Here are the equations as follows:

$$\begin{cases} \delta + \gamma_A A_k(t) + \gamma_I I_k(t) - (k\rho(t) + \delta) S_k(t) = 0 \\ k\rho(t) S_k(t) - (\gamma_I + \varepsilon(\alpha_A + \alpha_L) + \delta) I_k(t) = 0 \\ \varepsilon\alpha_A I_k(t) - (\gamma_A + \delta) A_k(t) = 0 \\ \varepsilon\alpha_L I_k(t) - \delta L_k(t) = 0 \end{cases}$$

From the above equations, we can calculate

$$\begin{cases} A_k(t) = \dfrac{\varepsilon\alpha_A}{\gamma_A + \delta} I_k(t) \\ L_k(t) = \dfrac{\varepsilon\alpha_L}{\delta} I_k(t) \\ S_k(t) = \dfrac{\gamma_I + \varepsilon(\alpha_A + \alpha_L) + \delta}{k\rho(t)} I_k(t) \end{cases} \tag{7}$$

Then, according to the normalization condition $S_k^*(t) + I_k^*(t) + A_k^*(t) + L_k^*(t) = 1$, we can obtain

$$\begin{cases} I_k^*(t) = \dfrac{k\rho(t)\delta(\delta + \gamma_A)}{B_k} \\ S_k^*(t) = \dfrac{\delta(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)}{B_k} \\ A_k^*(t) = \dfrac{k\rho(t)\delta\varepsilon\alpha_A}{B_k} \\ L_k^*(t) = \dfrac{k\rho(t)\varepsilon\alpha_L(\delta + \gamma_A)}{B_k} \end{cases} \tag{8}$$

where

$$B_k = k\rho(t)\left[\delta\varepsilon\alpha_A + \varepsilon\alpha_L(\delta + \gamma_A)\right] + \delta(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L).$$

Because of $\rho(t) = \sum_k kP(k)(\rho_A A_k(t) + \rho_L L_k(t))/\langle k\rangle$, and from the system (8), we easily know that $0 < S_k^* < 1$, $0 < I_k^* < 1$, $0 < A_k^* < 1$, $0 < L_k^* < 1$. Therefore, we can find when $\rho(t) = 0$, $S_k(t) = 1$, $I_k(t) = 0$, $A_k(t) = 0$, $L_k(t) = 0$ is the WOM-disappearance equilibrium, and the WOM-prevailing equilibrium $E_+\left(S_k^*, I_k^*, A_k^*, L_k^*\right)$ is well defined. Hence, when $R_0 > 1$, a unique positive equilibrium $E_+\left(S_k^*, I_k^*, A_k^*, L_k^*\right)$ exists. This completes the proof.

Theorem 2: When $R_0 < 1$, the WOM-disappearance equilibrium E_0 is global asymptotically stable. If $R_0 > 1$, the system (4) is permanent, i.e., there exists a $\eta > 0$, such that $\liminf_{t\to\infty}\{I_k(t), A_k(t), L_k(t)\}_{k=1}^n \geq \eta$, where $(I_k(t), A_k(t), L_k(t))$ is any solution of the system (4) and $A_k(0) > 0$ or $L_k(0) > 0$.

Proof: For simplicity, it is denoted that $P_i = iP(i)/\langle k\rangle$ and $n = k_{\max}$ in this paper. The Jacobian matrix of the positive equilibrium of the system (6), a $3n \times 3n$ matrix, can be written as follows:

$$G = \begin{pmatrix} U_{11} & \cdots & U_{1n} \\ \vdots & \ddots & \vdots \\ U_{n1} & \cdots & U_{nn} \end{pmatrix},$$

where

$$U_{11} = \begin{pmatrix} -(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) & \rho_A P_1 & \rho_L P_1 \\ \varepsilon\alpha_A & -(\delta + \gamma_A) & 0 \\ \varepsilon\alpha_L & 0 & -\delta \end{pmatrix},$$

$$U_{1n} = \begin{pmatrix} 0 & \rho_A P_n & \rho_L P_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{n1} = \begin{pmatrix} 0 & n\rho_A P_n & n\rho_L P_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_{nn} = \begin{pmatrix} -(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) & n\rho_A P_n & n\rho_L P_n \\ \varepsilon\alpha_A & -(\delta + \gamma_A) & 0 \\ \varepsilon\alpha_L & 0 & -\delta \end{pmatrix}.$$

By mathematical introduction method, the characteristic polynomial of the WOM-disappearance equilibrium E_0 can be calculated as following form:

$$(\lambda + \delta)^{n-1}(\lambda + \delta + \gamma_A)^{n-1}(\lambda + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)^{n-1} \times \left(\lambda^3 + s\lambda^2 + p\lambda + q\right) = 0,$$

where $s = \gamma_I + \gamma_A + \varepsilon\alpha_A + \varepsilon\alpha_L + 3\delta$, and

$$p = \delta(2\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) + (\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) - (\varepsilon\alpha_A\rho_A + \varepsilon\alpha_L\rho_L)\sum_{i=1}^n iP_i,$$

$$q = \delta(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) - \left[\delta\varepsilon\alpha_A\rho_A + \varepsilon\alpha_L\rho_L(\delta + \gamma_A)\right]\sum_{i=1}^n iP_i.$$

It is easy to find that $s > 0$. Note that $R_0 < 1$ is equivalent to $q > 0$, and it also implies

$$\delta(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) > \left[\delta\varepsilon\alpha_A\rho_A + \varepsilon\alpha_L\rho_L(\delta + \gamma_A)\right]\sum_{i=1}^n iP_i,$$

that is,

$$(\delta + \gamma_A)(\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) > \left[\varepsilon\alpha_A\rho_A + \varepsilon\alpha_L\rho_L\left(1 + \frac{\gamma_A}{\delta}\right)\right]\sum_{i=1}^n iP_i,$$

which apparently means $p > 0$. Therefore, the real eigenvalues of U are all negative if $R_0 < 1$, otherwise, if and only if $R_0 > 1$, there is a unique positive eigenvalue λ of U. According to the Perron-Frobenius theorem, this suggests that the maximal real part of all eigenvalues of λ is positive only if $R_0 > 1$. So, we obtained the results of this theorem through a theorem of Lajmanovich and Yorke [36], which completes the proof.

Theorem 3: Suppose that $(I_k(t), A_k(t), L_k(t))$ is a solution of the system (6), satisfying equation (5) with $A_k(0) > 0$ or $L_k(0) > 0$. If $R_0 > 1$, then $\lim_{t\to\infty}(I_k(t), A_k(t), L_k(t)) =$

$\left(I_k^*, A_k^*, L_k^*\right)$, where $\left(I_k^*, A_k^*, L_k^*\right)$ is the positive equilibrium of the system (6) for $k = 1, 2, \cdots, n$.

Proof: Given that k is suitable for any integer in $\{1, 2, \cdots, n\}$ as follows. There exists a positive constant $0 < \xi < 1/3$ and a large enough constant $T > 0$ such that $A_k(t) \geq \xi$ and $L_k(t) \geq \xi$ for $t > T$, according to Theorem 2. So, $\rho(t) > \xi(\rho_A + \rho_L)$ for $t > T$. Replacing the first equation of the system (6) with these shows

$$\frac{dI_k(t)}{dt} \leq k(\rho_A(t) + \rho_L(t))(1 - I_k(t)) - (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k(t)$$

for $t > T$.

By the standard comparison theorem of differential equations in the theory, for any given positive constant

$$0 < \xi_1 < \frac{\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L}{2[k(\rho_A(t) + \rho_L(t)) + (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)]},$$

there exists a $t_1 > T$, such that $I_k(t) \leq X_k^{(1)} - \xi_1$ for $t > t_1$, where

$$X_k^{(1)} = \frac{k(\rho_A(t) + \rho_L(t))}{k(\rho_A(t) + \rho_L(t)) + (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)} + 2\xi_1 < 1.$$

On the basic of the second equation of the system (6), it shows that

$$\frac{dA_k(t)}{dt} \leq \varepsilon\alpha_A(1 - A_k(t)) - (\gamma_A + \delta) A_k(t)$$

for $t > t_1$.

Therefore, as follows any given constant

$$0 < \xi_2 < \min\left\{1/2, \xi_1, (\delta + \gamma_A)[2(\delta + \gamma_A + \varepsilon\alpha_A)]^{-1}\right\},$$

there exists a $t_2 > t_1$, such that $A_k(t) \leq Y_k^{(1)} - \xi_2$ for $t > t_2$, where

$$Y_k^{(1)} = \frac{\varepsilon\alpha_A}{\delta + \gamma_A + \varepsilon\alpha_A} + 2\xi_2 < 1.$$

Similarly, the third equation of the system (6) gives that

$$\frac{dL_k(t)}{dt} \leq \varepsilon\alpha_L(1 - L_k(t)) - \delta L_k(t)$$

for $t > t_2$.

Therefore, as follows any given constant

$$0 < \xi_3 < \min\left\{1/3, \xi_2, \delta[2(\delta + \varepsilon\alpha_L)]^{-1}\right\},$$

there exists a $t_3 > t_2$, such that $L_k(t) \leq Z_k^{(1)} - \xi_3$ for $t > t_3$, where

$$Z_k^{(1)} = \frac{\varepsilon\alpha_L}{\delta + \varepsilon\alpha_L} + 2\xi_3 < 1.$$

On the other hand, replacing $A_k(t) \geq \xi$, $L_k(t) \geq \xi$ and $\rho(t) > \xi(\rho_A + \rho_L)$ into the first equation of the system (6), we calculate

$$\begin{aligned}\frac{dI_k(t)}{dt} &\geq k\xi(\rho_A + \rho_L)(1 - I_k(t) - A_k(t) - L_k(t)) - (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k(t) \\ &= k\xi(\rho_A + \rho_L)(1 - A_k(t) - L_k(t)) - [k\xi(\rho_A + \rho_L) + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L] I_k(t) \\ &\geq k\xi(\rho_A + \rho_L)\left(1 - Y_k^{(1)} - Z_k^{(1)}\right) - [k\xi(\rho_A + \rho_L) + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L] I_k(t)\end{aligned}$$

for $t > T$.

Therefore, as follows any given constant

$$0 < \xi_4 < \min \times \left\{\frac{1}{4}, \xi_3, \frac{k\xi(\rho_A + \rho_L)\left(1 - Y_k^{(1)} - Z_k^{(1)}\right)}{2[k\xi(\rho_A + \rho_L) + (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)]}\right\},$$

there exists a $t_4 > t_3$, such that $I_k(t) \geq x_k^{(1)} + \xi_4$ for $t > t_4$, where

$$x_k^{(1)} = \frac{k\xi(\rho_A + \rho_L)\left(1 - Y_k^{(1)} - Z_k^{(1)}\right)}{k\xi(\rho_A + \rho_L) + (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L)} - 2\xi_4 > 0.$$

It follows that,

$$\frac{dA_k(t)}{dt} \geq \varepsilon\alpha_A x_k^{(1)} - (\delta + \gamma_A) A_k(t)$$

for $t > t_4$.

Therefore, as follows any given constant

$$0 < \xi_5 < \min\left\{\frac{1}{5}, \xi_4, \frac{\varepsilon\alpha_A x_k^{(1)}}{2(\delta + \gamma_A)}\right\},$$

there exists a $t_5 > t_4$, such that $A_k(t) \geq y_k^{(1)} + \xi_5$ for $t > t_5$, where

$$y_k^{(1)} = \frac{\varepsilon\alpha_A x_k^{(1)}}{\delta + \gamma_A} - 2\xi_5 > 0.$$

Similarly,

$$\frac{dL_k(t)}{dt} \geq \varepsilon\alpha_L x_k^{(1)} - \delta I_k(t)$$

for $t > t_5$.

Therefore, as follows any given constant

$$0 < \xi_6 < \min\left\{\frac{1}{6}, \xi_5, \frac{\varepsilon\alpha_L x_k^{(1)}}{2\delta}\right\},$$

there exists a $t_6 > t_5$, such that $L_k(t) \geq z_k^{(1)} + \xi_6$ for $t > t_6$, where

$$z_k^{(1)} = \frac{\varepsilon\alpha_L x_k^{(1)}}{\delta} - 2\xi_6 > 0.$$

Since ξ is a small constant, it holds that $0 < x_k^{(1)} < X_k^{(1)} < 1$, $0 < y_k^{(1)} < Y_k^{(1)} < 1$ and $0 < z_k^{(1)} < Z_k^{(1)} < 1$. Let

$$q^{(j)} = \sum_{i=1}^{n} P_i\left(\rho_A y_i^{(j)} + \rho_L z_i^{(j)}\right),$$

$$Q^{(j)} = \sum_{i=1}^{n} P_i\left(\rho_A Y_i^{(j)} + \rho_L Z_i^{(j)}\right) \quad \text{j=1, 2, \ldots}$$

From the above discussion, it is clear that

$$0 < q^{(1)} \leq \rho(t) \leq Q^{(1)} < \rho_A + \rho_L$$

and $t > t_6$.

Again, by the system (6), one has

$$\begin{aligned}\frac{dI_k(t)}{dt} &\leq kQ^{(1)}\left(1 - I_k(t) - y_k^{(1)} - z_k^{(1)}\right) \\ &\quad - (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k(t) \\ &= kQ^{(1)}\left(1 - y_k^{(1)} - z_k^{(1)}\right) \\ &\quad - \left(kQ^{(1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L\right) I_k(t)\end{aligned}$$

for $t > t_6$.

Hence, for any given constant $0 < \xi_7 < \min\{1/7, \xi_6\}$, there exists a $t_7 > t_6$, such that

$$\begin{aligned}I_k(t) \leq X_k^{(2)} &\min \\ &\times \left\{X_k^{(1)} - \xi_1, \frac{kQ^{(1)}\left(1 - y_k^{(1)} - z_k^{(1)}\right)}{kQ^{(1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L} + \xi_7\right\}\end{aligned}$$

for $t > t_7$.

Thus,

$$\frac{dA_k(t)}{dt} \leq \varepsilon\alpha_A X_k^{(2)} - (\delta + \gamma_A) A_k(t)$$

for $t > t_7$.

So, for any given constant $0 < \xi_8 < \min\{1/8, \xi_7\}$, there exists a $t_8 > t_7$, such that

$$A_k(t) \leq Y_k^{(2)} \min\left\{Y_k^{(1)} - \xi_2, \varepsilon\alpha_A X_k^{(2)} / (\delta + \gamma_A) + \xi_8\right\}$$

for $t > t_8$.

As a result, one gets that

$$\frac{dL_k(t)}{dt} \leq \varepsilon\alpha_L X_k^{(2)} - \delta L_k(t)$$

for $t > t_8$.

So, for any given constant $0 < \xi_9 < \min\{1/9, \xi_8\}$, there exists a $t_9 > t_8$, such that

$$L_k(t) \leq Z_k^{(2)} \min\left\{Z_k^{(1)} - \xi_3, \varepsilon\alpha_L X_k^{(2)} / \delta + \xi_9\right\}$$

for $t > t_9$.

Turning back to the system (6), one gets

$$\begin{aligned}\frac{dI_k(t)}{dt} &\geq kq^{(1)}\left(1 - I_k(t) - Y_k^{(2)} - Z_k^{(2)}\right) \\ &\quad - (\delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L) I_k(t)\end{aligned}$$

for $t > t_9$.

So, for any given constant

$$0<\xi_{10}<\min\left\{1/10, \xi_9, \frac{kq^{(1)}\left(1 - Y_k^{(2)} - Z_k^{(2)}\right)}{2\left(kq^{(1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L\right)}\right\},$$

there exists a $t_{10} > t_9$, and $I_k(t) \geq x_k^{(2)} + \xi_{10}$, $t > t_{10}$, where

$$x_k^{(2)} = \max\left\{x_k^{(1)} + \xi_4, \frac{kq^{(1)}\left(1 - Y_k^{(2)} - Z_k^{(2)}\right)}{kq^{(1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L} - 2\xi_{10}\right\}.$$

Thus,

$$\frac{dA_k(t)}{dt} \geq \varepsilon\alpha_A x_k^{(2)} - (\delta + \gamma_A) A_k(t)$$

for $t > t_{10}$.

So, for any given constant

$$0 < \xi_{11} < \min\left\{1/11, \xi_{10}, \frac{\varepsilon\alpha_A x_k^{(2)}}{2(\delta + \gamma_A)}\right\},$$

there exists a $t_{11} > t_{10}$, and $A_k(t) \geq y_k^{(2)} + \xi_{11}$, $t > t_{11}$, where

$$y_k^{(2)} = \max\left\{y_k^{(1)} + \xi_5, \frac{\varepsilon\alpha_A x_k^{(2)}}{\delta + \gamma_A} - 2\xi_{11}\right\}.$$

Similarly,

$$\frac{dL_k(t)}{dt} \leq \varepsilon\alpha_L x_k^{(2)} - \delta L_k(t)$$

for $t > t_{11}$.

So, for any given constant

$$0 < \xi_{12} < \min\left\{1/12, \xi_{11}, \frac{\varepsilon\alpha_L x_k^{(2)}}{2\delta}\right\},$$

there exists a $t_{12} > t_{11}$, and $L_k(t) \geq z_k^{(2)} + \xi_{12}$, $t > t_{12}$, where

$$z_k^{(2)} = \max\left\{z_k^{(1)} + \xi_6, \frac{\varepsilon\alpha_L x_k^{(2)}}{\delta} - 2\xi_{12}\right\}.$$

What's more, we can carry out step $r (r = 3, 4, \cdots)$ of the calculation and obtain six sequences: $\{X_k^{(r)}\}$, $\{Y_k^{(r)}\}$, $\{Z_k^{(r)}\}$, $\{x_k^{(r)}\}$, $\{y_k^{(r)}\}$ and $\{z_k^{(r)}\}$. Because the first three sequences are monotone increasing and the last three sequences are strictly monotone decreasing, there exists a large positive integer M such that for $r \geq M$:

$$\begin{cases} X_k^{(r)} = \dfrac{kQ^{(r-1)}\left(1 - y_k^{(r-1)} - z_k^{(r-1)}\right)}{kQ^{(r-1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L} + \xi_{6r-5} \\ Y_k^{(r)} = \dfrac{\varepsilon\alpha_A X_k^{(r)}}{\delta + \gamma_A} + \xi_{6r-4} \\ Z_k^{(r)} = \dfrac{\varepsilon\alpha_L X_k^{(r)}}{\delta} + \xi_{6r-3} \\ x_k^{(r)} = \dfrac{kq^{(r-1)}\left(1 - Y_k^{(r)} - Z_k^{(r)}\right)}{kq^{(r-1)} + \delta + \gamma_I + \varepsilon\alpha_A + \varepsilon\alpha_L} - 2\xi_{6r-2} \\ y_k^{(r)} = \dfrac{\varepsilon\alpha_A x_k^{(r)}}{\delta + \gamma_A} - 2\xi_{6r-1} \\ z_k^{(r)} = \dfrac{\varepsilon\alpha_L x_k^{(r)}}{\delta} - 2\xi_{6r} \end{cases} \tag{9}$$

It is obvious that

$$\begin{cases} x_k^{(r)} \leq I_k(t) \leq X_k^{(r)} \\ y_k^{(r)} \leq A_k(t) \leq Y_k^{(r)} \\ z_k^{(r)} \leq L_k(t) \leq Z_k^{(r)}, \end{cases} \quad t > t_{6r} \tag{10}$$

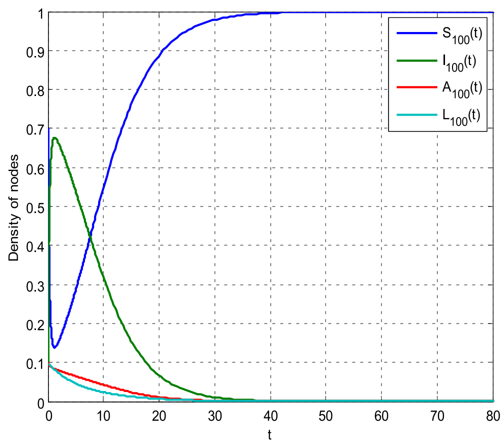

FIGURE 2. The time series and orbits of four states with $R_0 = 0.9480 < 1$ and initial values $S(0) = 0.7, I(0) = 0.1, A(0) = 0.1, L(0) = 0.1$.

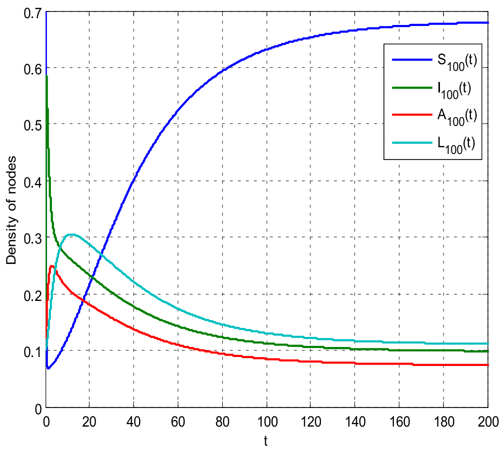

FIGURE 3. The time series and orbits of four states with $R_0 = 6.1433 > 1$ and initial values $S(0) = 0.7, I(0) = 0.1, A(0) = 0.1, L(0) = 0.1$.

Because of the sequential limits of the system (9), let $\lim_{t\to\infty} \Omega_k^{(r)} = \Omega_k$, where $\Omega_k \in \{X_k, Y_k, Z_k, x_k, y_k, z_k, Q_k, q_k\}$ and $\Omega_k^{(r)} \in \left\{X_k^{(r)}, Y_k^{(r)}, Z_k^{(r)}, x_k^{(r)}, y_k^{(r)}, z_k^{(r)}, Q_k^{(r)}, q_k^{(r)}\right\}$. Since $0 < \xi_r < 1/r$, one has $\xi_r \to 0$ as $r \to \infty$. Regarding, we can calculate the equation of the system (9) and get

$$\begin{cases} X_k = \dfrac{kQ(1-y_k-z_k)}{kQ+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L}, \quad Y_k = \dfrac{\varepsilon\alpha_A X_k}{\delta+\gamma_A}, \\ Z_k = \dfrac{\varepsilon\alpha_L X_k}{\delta}, \\ x_k = \dfrac{kq(1-Y_k-Z_k)}{kq+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L}, \quad y_k = \dfrac{\varepsilon\alpha_A x_k}{\delta+\gamma_A}, \\ z_k = \dfrac{\varepsilon\alpha_L x_k}{\delta}, \end{cases} \tag{11}$$

where

$$q = \sum_{i=1}^{n} P_i(\rho_A y_i + \rho_L z_i),\ Q = \sum_{i=1}^{n} P_i(\rho_A Y_i + \rho_L Z_i).$$

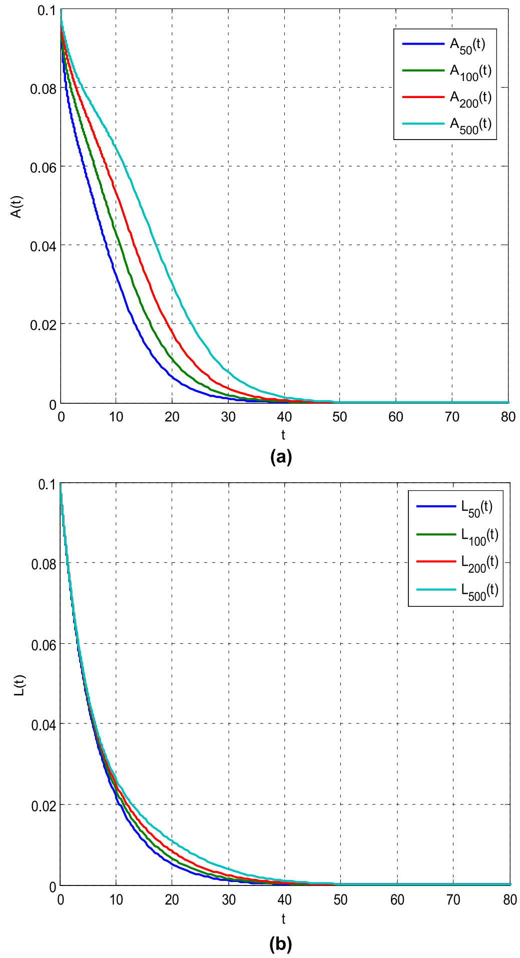

FIGURE 4. The time series and orbits of the acknowledged or the loath individuals with $R_0 < 1$ and initial values $A(0) = 0.1, L(0) = 0.1, k = 50, 100, 200, 500$. (a) Acknowledged individuals $A(t)$. (b) Loath individuals $L(t)$.

What's more,

$$\begin{cases} X_k = \dfrac{k^2Q\delta(\delta+\gamma_A)}{G_k} \\ \qquad \times \begin{bmatrix} \delta(\delta+\gamma_A)(kq+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L) \\ -q(\delta\varepsilon\alpha_A+\delta\varepsilon\alpha_L+\gamma_A\varepsilon\alpha_L) \end{bmatrix}, \\ x_k = \dfrac{k^2q\delta(\delta+\gamma_A)}{G_k} \\ \qquad \times \begin{bmatrix} \delta(\delta+\gamma_A)(kQ+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L) \\ -Q(\delta\varepsilon\alpha_A+\delta\varepsilon\alpha_L+\gamma_A\varepsilon\alpha_L) \end{bmatrix}, \end{cases} \tag{12}$$

where

$$\begin{aligned} G_k &= \delta^2(\delta+\gamma_A)^2(kq+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L) \\ &\quad \times (kQ+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L) \\ &\quad - k^2qQ\left[\delta\varepsilon\alpha_A+\varepsilon\alpha_L(\delta+\gamma_A)\right]^2. \end{aligned}$$

Replacing Q and q with the above equation respectively, we get

$$\begin{aligned} 1 &= \left[\rho_A\varepsilon\alpha_A\delta+\rho_L\varepsilon\alpha_L(\delta+\gamma_A)\right] \\ &\quad \times \sum_{i=1}^{n} P_i\frac{i^2}{G_i}\delta(\delta+\gamma_A)(kq+\delta+\gamma_I+\varepsilon\alpha_A+\varepsilon\alpha_L) \end{aligned}$$

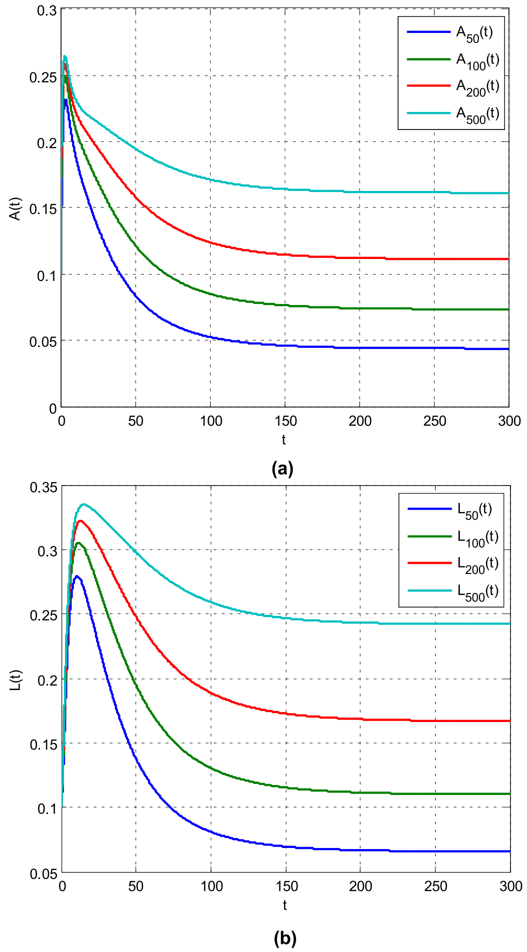

FIGURE 5. **The time series and orbits of the acknowledged or the loath individuals with $R_0 > 1$ and initial values $A(0) = 0.1, L(0) = 0.1, k = 50, 100, 200, 500$. (a) Acknowledged individuals A(t). (b) Loath individuals L(t).**

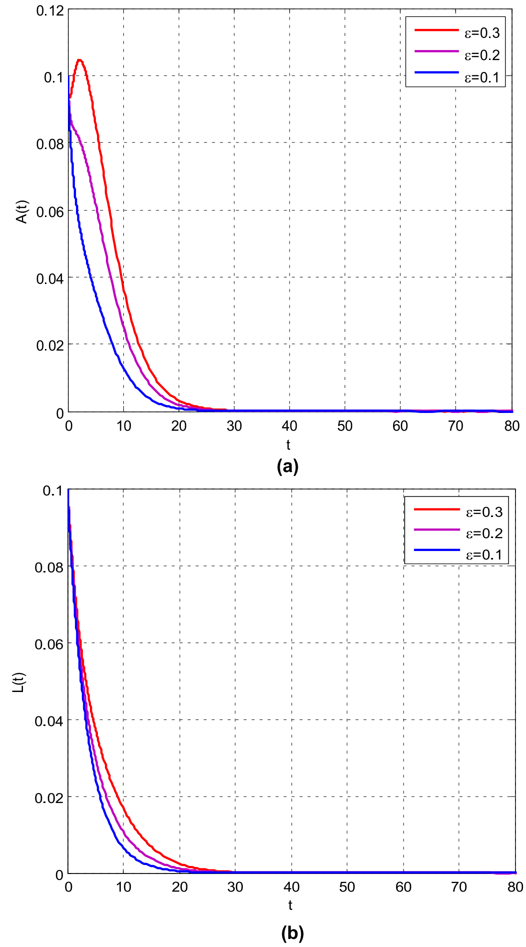

FIGURE 6. **Prevalence $A_{100}(t), L_{100}(t)$ versus t corresponding to different ε with $R_0 < 1$ and initial values $A_{100}(0) = 0.1, L_{100}(0) = 0.1$. (a) Acknowledged individuals A(t). (b) Loath individuals L(t).**

$$\begin{aligned} &- [\rho_A \varepsilon \alpha_A \delta + \rho_L \varepsilon \alpha_L (\delta + \gamma_A)] \\ &\times \sum_{i=1}^{n} P_i \frac{i^2}{G_i} q (\delta \varepsilon \alpha_A + \delta \varepsilon \alpha_L + \gamma_A \varepsilon \alpha_L), \\ 1 = &[\rho_A \varepsilon \alpha_A \delta + \rho_L \varepsilon \alpha_L (\delta + \gamma_A)] \\ &\times \sum_{i=1}^{n} P_i \frac{i^2}{G_i} \delta (\delta + \gamma_A)(kQ + \delta + \gamma_I + \varepsilon \alpha_A + \varepsilon \alpha_L) \\ &- [\rho_A \varepsilon \alpha_A \delta + \rho_L \varepsilon \alpha_L (\delta + \gamma_A)] \\ &\times \sum_{i=1}^{n} P_i \frac{i^2}{G_i} Q (\delta \varepsilon \alpha_A + \delta \varepsilon \alpha_L + \gamma_A \varepsilon \alpha_L). \end{aligned}$$

Simplifying above equations, we obtain

$$(q-Q) \sum_{i=1}^{n} \frac{P_i}{G_i} i^2 [i\delta (\delta+\gamma_A) - (\delta \varepsilon \alpha_A + \delta \varepsilon \alpha_L + \gamma_A \varepsilon \alpha_L)] \equiv 0.$$

This pushes out that $Q = q$. So,

$$\sum\nolimits_{i=1}^{n} P_i [\rho_A (Y_i - y_i) + \rho_L (Z_i - z_i)] = 0,$$

which is equivalent to $Y_i = y_i$ and $Z_i = z_i$ for $1 \leq i \leq n$. According to the equations of the system (10) and (11), it follows that

$$\lim_{t\to\infty} I_k(t) = X_k = x_k, \quad \lim_{t\to\infty} A_k(t) = Y_k = y_k, \\ \lim_{t\to\infty} L_k(t) = Z_k = z_k.$$

Finally, by substituting $Q = q$ into the equation of the system (12), according to the equation of the system (11), we found that $X_k = I_k^*$, $Y_k = A_k^*$ and $Z_k = L_k^*$. The proof is completed, we can know that there always exists the WOM-prevailing equilibrium when $R_0 > 1$.

IV. NUMERICAL SIMULATIONS

This section gives the analytical results by numerical simulations. The system of *SIALS* is formulated on scale-free networks with $P(k) = \omega k^{-3}$, where the parameter ω satisfies $\sum_{k=1}^{n} \omega k^{-3} = 1$, $n = 1000$. And we show the analysis of the basic reproductive number R_0.

In Figure 2, assume that the parameters are chosen as $\delta = 0.2$, $\rho_A = 0.8$, $\rho_L = 0.1$, $\alpha_A = 0.2$, $\alpha_L = 0.2$, $\gamma_A = 0.2$, $\gamma_I = 0.2$, $\varepsilon = 0.2$, thus the basic reproductive

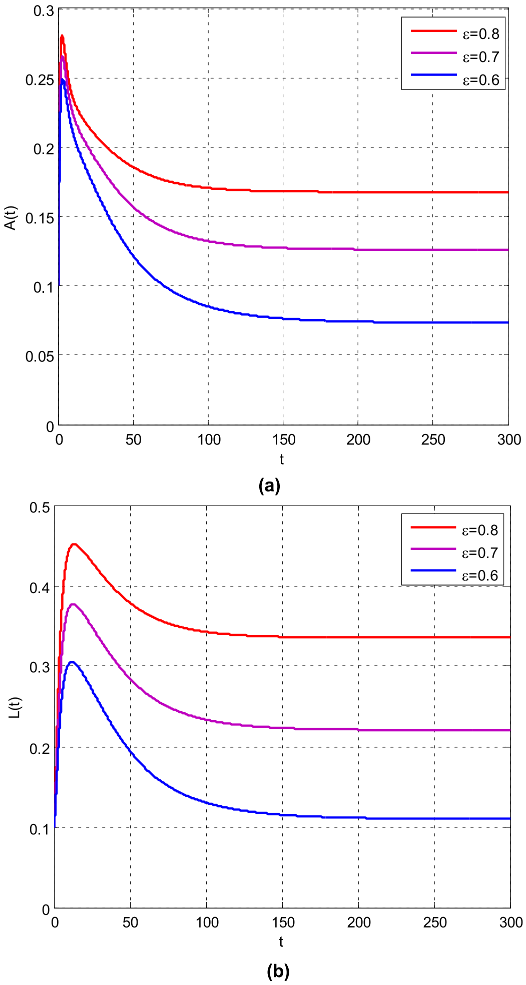

FIGURE 7. Prevalence $A_{100}(t), L_{100}(t)$ versus t corresponding to different ε with $R_0 > 1$ and initial values $A_{100}(0) = 0.1, L_{100}(0) = 0.1$. (a) Acknowledged individuals A(t). (b) Loath individuals L(t).

number $R_0 = 0.9480 < 1$. We can see that there is almost no transmission of WOM when $R_0 < 1$, which means that WOM communication will eventually disappear. It also suggests that the WOM-disappearance equilibrium E_0 is globally asymptotically stable when $R_0 < 1$.

In Figure 3, assume that the parameters are chosen as $\delta = 0.2$, $\rho_A = 0.8$, $\rho_L = 0.5$, $\alpha_A = 0.5$, $\alpha_L = 0.5$, $\gamma_A = 0.2$, $\gamma_I = 0.2$, $\varepsilon = 0.6$, thus the basic reproductive number $R_0 = 6.1433 > 1$. The Figure 3 shows that WOM communication is permanent on scale-free networks when $R_0 > 1$.

In Figure 4(a) and (b), we choose $\delta = 0.2$, $\rho_A = 0.8$, $\rho_L = 0.1$, $\alpha_A = 0.2$, $\alpha_L = 0.2$, $\gamma_A = 0.2$, $\gamma_I = 0.2$, $\varepsilon = 0.2$, and thus $R_0 = 0.9480 < 1$. The Figure 4 describe the time series of the acknowledged individuals $A(t)$ and loath individuals $L(t)$ with different degree. Apparently, we can see that when $R_0 < 1, A(t)$ and $L(t)$ both grow to zero, i.e., WOM communication will ultimately disappear.

In Figure 5(a) and (b), we choose $\delta = 0.2$, $\rho_A = 0.8$, $\rho_L = 0.5$, $\alpha_A = 0.5$, $\alpha_L = 0.5$, $\gamma_A = 0.2$, $\gamma_I = 0.2$, $\varepsilon = 0.6$, and thus $R_0 = 6.1433 > 1$. The Figure 5 describe the time series of the acknowledged individuals $A(t)$ and loath individuals $L(t)$ with different degree. Apparently, we can see that when $R_0 > 1$, $A(t)$ and $L(t)$ grow to a positive constant respectively, i.e., WOM communication is permanent. What's more, we also found that the level of the WOM-prevailing increases with the increasing of degree number k.

In Figure 6(a) and (b), we choose $\delta = 0.3$, $\rho_A = 0.6$, $\rho_L = 0.1$, $\alpha_A = 0.4$, $\alpha_L = 0.2$, $\gamma_A = 0.3$, $\gamma_I = 0.2$, and thus $R_0 < 1$. The Figure 6 describe the time series of the acknowledged individuals $A(t)$ and loath individuals $L(t)$ with different probability ε of people's interest in evaluation of purchasing. Apparently, we can see that when $R_0 < 1$, a smaller ε can accelerate the disappearance of WOM communication.

In Figure 7(a) and (b), we choose $\delta = 0.2$, $\rho_A = 0.8$, $\rho_L = 0.5$, $\alpha_A = 0.5$, $\alpha_L = 0.5$, $\gamma_A = 0.2$, $\gamma_I = 0.2$, and thus $R_0 > 1$. The Figure 7 describe the time series of the acknowledged individuals $A(t)$ and loath individuals $L(t)$ with different probability ε. Apparently, we can see that when $R_0 > 1$, $A(t)$ and $L(t)$ both grow to a positive constant with the increasing of parameter ε. It can be seen that the larger ε can lead to the large value of WOM communication level.

V. CONCLUSION

In this paper, considering the comment mechanism and heterogeneity of online social networks, we have proposed a new *SIALS* model to illustrate the spreading of WOM communication processes with both acknowledged and loath comments on the social networks. Through establishing *SIALS* model, we analyzed the spreading dynamics of WOM communication in detail. We determined that the spreading dynamics of the model depend on the basic reproductive number R_0. And we obtain the conclusion that if $R_0 < 1$, the WOM-disappearance equilibrium E_0 is globally asymptotical stability, i.e., WOM communication will disappear regardless of the initial situation of the infected individuals. If $R_0 > 1$, WOM communication is permanent and globally stable, which means that the acknowledged and loath comments will exist and make WOM communication be a universal phenomenon. Furthermore, we investigate the impact of the comment parameter ε, the probability of people's interest in evaluation of purchasing. Interestingly, the larger ε can promote the spreading of WOM information, namely, increasing the comment probability is conducive to the spreading of WOM communication. The study has a vital significance in studying the spreading dynamics of WOM communication in the heterogeneous network. Our results are useful for promoting the spreading of WOM communication on scale-free networks.

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WENJIN LIU received the B.S. degree from the School of Electronic Information, Yangtze University, in 2017, where she is currently pursuing the master's degree. Her current research interests include complex network theory and the analysis of propagation dynamics.

TAO LI was born in Dezhou, China, in 1974. He received the M.S. degrees in signal acquisition and processing from Yangtze University, Jingzhou, China, in 2004, and the Ph.D. degree in control theory and control engineering from the Huazhong University of Science and Technology, Wuhan, China, in 2007. Since 2007, he has been with the Electronic and Information School, Yangtze University, where he is currently a Professor, a Master Instructor, and an Associate Dean of the Electronic and Information School. His research interests include automatic detection and control, multi-agent system, and complex network theory and application.

XIONGDING LIU graduated from the Department of Automation, School of Electronics and Information, Yangtze University, in 2017. He is currently a graduate student at Yangtze University. His main research interests include complex network spreading dynamics, multi-agent collaborative control, and programmable logic controllers.

HAO XU received the B.S. degree in electrical engineering and automation from Yangtze University in 2017. In 2017, he entered the Yangtze University to pursue the master's degree. He mainly studies complex network spreading dynamics and coordinated control of multi-agent systems.